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The Rise of Complex Beliefs Dynamics

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Preface

We prove that complex beliefs dynamics may emerge in linear stochastic models as the outcome of bounded rationality learning. If agents believe in a misspecified law of motion (which is correctly specified at the Rational Expectations Equilibria of the model) and update their beliefs observing the evolving economy, their beliefs can follow in the limit a beliefs cycle which is not a self-fulfilling solution of the model. The stochastic process induced by the learning rule is analyzed by means of an associated ordinary differential equation (ODE). The existence of a uniformly asymptotically stable attractor for the ODE implies the existence of a beliefs attractor, to which the learning process converges. We prove almost sure convergence by assuming that agents employ a projection facility and convergence with positive probability dropping this assumption. The rise of a limit cycle and of even more complex attractors is established in some monetary economics models assuming that agents update their beliefs with the Recursive Ordinary Least Squares and the Least Mean Squares algorithm.

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The Rise of Complex Beliefs Dynamics

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1 Introduction

A large literature has grown up in the last two decades on the emergence of complex dynamics in nonlinear deterministic economic models, see for example [Grandmont, 1985, Grandmont, 1987, Bodrin and Woodford, 1990, Guesnerie and Woodford, 1992]. Complex dynamics have been obtained as the outcome of dynamical optimization problems under the assumption of perfect foresight (agents have full knowledge of the economic model) or by modeling the agents behavior on the basis of some behavioral assumption, on the two approaches see for example [Boldrin and Montrucchio, 1986] and [Day, 1994]. No results have been established in a stochastic environment until now; the stochastic and the complex dynamics view of the world are perceived to be antithetic both in a theoretical and in an empirical perspective, see [Brock, 1987].

In this paper we show that complex dynamics (non perfect foresight trajectories) may emerge in linear stochastic models as the outcome of agents' bounded rationality learning. Complex dynamics do not concern directly the state variables of the model, but the agents beliefs (beliefs complex dynamics). Complexity is not due to an intrinsic complexity of the economic model but it is the result of the interaction between agents' learning and the evolution of the economy.

The models considered in this paper are linear in the state variables and in the agents' expectations; nonlinearities come from the assumption of bounded rationality: agents do not know the complete economic model, they form their expectations believing in a linear misspecified model which is correct only at a perfect foresight solution, outside that solution the actual law of motion of the economy may be highly nonlinear in agents beliefs; the analysis is developed in the framework of Self Referential Linear Stochastic (SRLS) models, see [Marcet and Sargent, 1989b], and can be easily extended to models with hidden state variables and private information, see [Marcet and Sargent, 1989a].

Observing the evolving economy, agents update their beliefs with a recursive information processing rule. Many recursive learning mechanisms have been proposed in the li-

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terature: iterative expectations, [Canio, 1979], recursive ordinary least squares (ROLS), [Marcet and Sargent, 1989b], ordinary least squares with finite memory, [Grandmont and Laroque, 1991], least mean squares (LMS), [Barucci and Landi, 1995b]. These learning procedures are characterized by different information processing rules and memory capacities: the ROLS algorithm is the best linear unbiased estimator and is characterized by a long memory, the LMS algorithm is a steepest descent gradient algorithm with very short memory and a strong behavioral interpretation.

The learning process may lead to different outcomes: agents' beliefs can converge to a point, diverge to infinity, or converge to a beliefs cycle. For the first three learning mechanisms mentioned above only a REE can be the limit point, LMS learning may also converge to a set of non-RE beliefs, see [Barucci and Landi, 1995b]. Convergence to a perfect foresight beliefs cycle has been studied in a deterministic framework assuming that agents are rationally bounded but know the existence of a cycle and form their beliefs consistently; for example, if a perfect foresight cycle of order k exists then the agents use an adaptive scheme of order k and form expectations by looking back k periods, see [Guesnerie and Woodford, 1991, Marimon and Sunder, 1993, Marimon et al., 1993]. With the same approach convergence of learning to a cycle has also been shown for nonlinear stochastic models in [Evans and Honkapohja, 1995b]. For all the three limit behaviors the average forecasting error either converges or diverges to infinity: if convergence to a REE or to a perfect foresight cycle is obtained then the average forecasting error converges to zero, if convergence to a set of non-RE beliefs is obtained then the average forecasting error converges to a value different from zero. If the learning process diverges (beliefs go to infinity), the square forecasting error goes to infinity, too.

This kind of results contrasts with empirical and experimental evidences. Experimental economics has shown that the agents' forecasting error may not go to zero but remain bounded with an erratic behavior over a long time period, see [Marimon and Sunder, 1993, Marimon et al., 1993]. This suggests that "adaptive learning might generate endogenously complex nonlinear trajectories, along which forecasting errors would never vanish" [Grandmont and Laroque, 1991, pag. 248], see also [Evans, 1985] on this point. The claim has not been demonstrated in a stochastic framework, such a result has been only obtained in a deterministic nonlinear hyperinflation model, see [Bullard, 1994]. Our analysis concerns linear stochastic models. The emergence of complex beliefs dynamics is investigated by means of the *ordinary differential equation* (ODE) associated with the learning dynamics. New results obtained in the stochastic approximation literature have shown that the ODE is not only relevant for the analysis of the local convergence of a learning mechanism to a stationary solution but also to detect the existence of limit cycles or even more complex dynamics: every uniformly asymptotically stable attractor of the ODE is contained in the attainable limit sets of the corresponding stochastic process, see [Benaim and Hirsch, 1994, Posch, 1994]. We prove almost sure convergence employing the projection facility and convergence with positive probability employing results in [Benveniste et al., 1990] with an approach similar to the one in [Evans and Honkapohja, 1994a].

With respect to the literature on complex dynamics in economics, the main features of the analysis developed in this paper are:

- linearity of the model at the REE and in disequilibrium with respect to state variables and agents expectations,
- bounded rationality,
- stochastic environment,
- complex dynamics concern beliefs rather than directly the economic state variables.

The nice thing of our analysis is that complex dynamics do not arise because of a simple behavioral assumption or of a maximizing behavior under perfect foresight. Both, the two assumptions have been deeply criticized in the literature. In what follows we are in the middle: agents are not fully rational, i.e. they do not know the complete economic model, they learn from observations by means of an information processing rule. So the result is that agents endogenously learn to "believe" in complex beliefs dynamics.

The paper has both methodological aspects (the application of new results obtained in the stochastic approximation literature to economic theory) and theoretical aspects (the analysis of complex beliefs dynamics in some macroeconomic models). To comply with these two goals, we present in Section 2 stochastic approximation results needed in our analysis and we analyze in Section 3 some macroeconomic models. As we show in Section 3.4, the analysis can also be extended to a class of nonlinear models obtained in the overlapping generations framework. In Appendix A we present the class of economic models considered in this paper and the two learning algorithms employed (the ROLS and the LMS algorithm). In Appendix B we report the technical conditions on the stochastic process and on the learning process needed in our analysis referring mainly to [Evans and Honkapohja, 1994a].

2 The Rise of Complex Beliefs Dynamics in Linear Stochastic Models

To determine the existence of complex dynamics in a learning model, we study the system of differential equations associated with the learning algorithm, i.e. (30) for ROLS learning and (35) for LMS learning; from the existence of a uniformly asymptotically stable attractor for the system of differential equations we deduce the existence of an attractor for the stochastic approximation process describing the learning rule. The results established in this section are based on results contained in [Benveniste et al., 1990, Evans and Honkapohja, 1994a, Benaim and Hirsch, 1995]. Both learning processes considered in our analysis are stochastic approximation processes of the type

$$\theta_t = \theta_{t-1} + \eta_t H(\theta_{t-1}, x_t) \quad (1)$$
$$x_t = F(\theta_{t-1})x_{t-1} + G(\theta_{t-1})w_{t-1} \quad (2)$$

where $\theta_t \in I\!R^d$, $\eta_t \in I\!R^+$, $x_t \in I\!R^k$, w_t is an i.i.d. noise vector, $H(\cdot,\cdot)$ is a function, and $F(\cdot), G(\cdot)$ are matrices of appropriate dimension. Let D be a fixed subset of $\Re^d$. Following [Evans and Honkapohja, 1994a], we require the process to satisfy conditions A1-A3 and B1-B2 (see Appendix B). Under these conditions the process x_t is asymptotically stationary and thus, there is a locally Lipschitz continuous function $h(\theta)$ such that

$$h(\theta) = \lim_{t\to\infty} E(H(\theta, \bar{x}_t))$$

where $\bar{x}_t = F(\theta)\bar{x}_{t-1} + G(\theta)w_{t-1}$. The limit sets of the process θ_t can be characterized by the corresponding ODE

$$\dot{\theta} = h(\theta), \quad \theta \in D. \tag{3}$$

For ROLS learning this differential equation is given by (30), for LMS learning by (35). An important notion to describe the limit sets of the stochastic process is that of *chain recurrence.*

Definition 2.1 *Let $\Phi_t(\theta)$ denote the flow induced by (3). A point $\theta \in I\!R^d$ is called (δ, T)* recurrent *if $\delta > 0, T > 0$ and there is an integer k, points $\xi_j \in \Re^d$, $0 \le j \le k$, and numbers $t_j > T$, $0 \le j \le k-1$ such that: $||\xi_0 - \theta|| < \delta, \xi_k = \theta$ and $||\Phi_{t_j}(\xi_j) - \xi_{j+1}|| < \delta, j = 0, \ldots, k-1$. If θ is (δ, T) recurrent for all $\delta > 0, T > 0$ then θ is called chain recurrent.*

Let $R(h)$ denote the set of chain recurrent points of (3). Combining results in [Evans and Honkapohja, 1994a] and [Benaim, 1993] we get the following theorem:

Theorem 2.1 *Let $\Gamma \subset D$ be a compact uniformly asymptotically stable set of (3). Suppose that assumptions A and B in Appendix B are satisfied on D. Denote the domain of attraction of Γ by N and let $U \subset N \cap D$ be an open neighborhood of Γ.*

Then there exists a neighborhood $V \subset U$ of Γ and constants B_1, s such that for a set of initial conditions $(t_0 = n, \theta_{t_0} \in V, x_{t_0} = x)$ we have

(a)

$$P(\min_{\theta_k \notin U}\{k \ge n\} < \infty) < B_1(1 + |x|^s)J(n)$$

where $J(n)$ is a positive decreasing sequence with $\lim_{n\to\infty} J(n) = 0$.

(b) Let $\mathcal{A}$ be the event such that the ω-limit of the process θ_t in (1) is a compact, invariant subset of $R(h)$. Then

$$P(\{\min_{\theta_k \notin U}\{k \ge n\} < \infty\} \cup \mathcal{A}) = 1.$$

Proof:
(a) is an immediate consequence of Theorem 1 in [Evans and Honkapohja, 1994a]. The existence of the required Ljapunov functions follows since Γ is uniformly asymptotically stable. In [Evans and Honkapohja, 1994a] the inequality (a) has been proved for the case where the asymptotically stable set is just a point.

(b) By step (ii) of the proof of Theorem 1 in [Evans and Honkapohja, 1994a] on the set $\{\min_{\theta_k \notin U}\{k \geq n\} = \infty\}$ the process θ_t can be written as

$$\theta_{t+1} = \theta_t + \eta_{t+1} h(\theta) + \epsilon_t$$

where $\sum_{t=1}^{\infty} \epsilon_t$ converges almost surely.

Thus, we can apply Theorem 1 in [Benaim, 1993] and (b) follows.□

For a discussion of chain recurrence see [Benaim, 1993]. If (3) defines a flow in $\Re^2$ which has isolated equilibria then the chain recurrent sets consist only of fixed points, periodic orbits and orbit chains of the flow, see [Benaim and Hirsch, 1994, Theorem 1.6]. Thus, if Γ contains only the fixed point θ^* whose domain of attraction is N (and no periodic orbits or orbit chains) then the process almost surely either leaves the domain of attraction or converges to θ^*. If Γ contains a limit cycle with orbit ρ and domain of attraction N (and no fixed points) then almost surely the ω-limit of all paths that stay forever in N is ρ: since ρ is the only chain recurrent set, the ω-limit set for all paths that stay in N is contained in ρ, but as the ω-limit set is invariant it contains ρ and hence the $\omega-$limit for the paths in N and ρ are identical. If Γ contains a general attractor, we know from Theorem 2.1 that the ω-limit set of each path that does not leave U is almost surely an invariant subset of $R(h)$. Thus, if the system of differential equations in (3) has e.g. a strange attractor it will be contained in the attractor of the stochastic process.

In the following we consider the case of limit cycles. First of all we need the following definition:

Definition 2.2 *A deterministic sequence $\bar{\theta}_t \in I\!R^d$ is called* asymptotically cycling *if its ω-limit is a closed curve.*

From Theorem 2.1 and the above arguments we get the following corollary:

Corollary 2.1 *Assume that the flow induced by (3) admits a uniformly asymptotically stable limit cycle ρ. Let U denote a neighborhood of ρ which is a subset of the domain of attraction of ρ and such that conditions A and B in Appendix B hold on U. Then*

1. *there exist constants B_2, s and a neighborhood $V \subset U$ of ρ such that for the process (1)-(2) with initial conditions $(t_0 = n, \theta_{t_0} \in V, x_{t_0} = x)$*

$$P(\theta_t \text{ is asymptotically cycling}) \geq 1 - B_2(1 + |x|^s) J(n),$$

where $J(n)$ is a positive decreasing sequence with $\lim_{n\to\infty} J(n) = 0$;

2. *there exists a neighborhood $V \subset U$ of ρ such that if we modify the process (1)-(2) by introducing a projection facility such that whenever the process θ_t leaves the set V it is moved back to V, then for all initial conditions $(\theta_{t_0} \in V, t_0 > 0, x_{t_0} = x)$ the process θ_t is almost surely asymptotically cycling.*

Finally we consider a non generic case. The flow induced by (3) has a continuum of periodic orbits as for example in a system of linear differential equations with pure imaginary eigenvalues or in the Lotka-Volterra framework. In this context the concept of chain recurrence is not applicable. Instead, we use the notion of *invariant of motion* to prove cycling.

Theorem 2.2 *Let $\theta_t \in I\!R^2$ and in addition to conditions A and B in Appendix B assume that $H(\cdot,\cdot)$ and w_t are bounded on D, that x_t is stationary, and η_t is of order $1/t$. Assume the flow of $h(\theta)$ has a continuum of cycles around a fixed point θ^* and an invariant of motion $Q(\theta) : \Re^d \rightarrow \Re$ exists satisfying*

1. *$Q \in C^2(D)$ and the second derivatives are bounded;*
2. *$\langle \nabla Q, h \rangle = 0, \quad \forall x \in D$;*
3. *$Q(\theta) \geq 0, \quad \forall x \in D$;*
4. *θ^* is a global strict minimum point of $Q(\cdot)$ and the only critical point.*

Then there is a neighborhood V of θ^* and an $n > 0$ such that, for all initial conditions $t_0 > n$ and $\theta_{t_0} \in V$, θ_t is asymptotically cycling with positive probability.

Proof:
This is a consequence of Propositions 4 and 5 in [Posch, 1994] and of Theorem 1 in [Benaim, 1993].□

One can actually show that each open set of periodic orbits is attained in the limit with positive probability (see [Posch, 1994]).

3 Some Economic Models

In this section we discuss some economic models where agents' bounded rationality learning leads to non perfect foresight complex beliefs dynamics; specifically we show that the beliefs dynamics may be characterized by a continuum of cycles or an asymptotically stable limit cycle to which agents' beliefs converge.

The first example (Section 3.1) is a very simple model which leads to a system of linear differential equations for agents beliefs. For a particular set of parameters there exists a continuum of cycles. In this (non generic) case the learning process converges to a cycle which is randomly selected. In the second example (an open economy model) we show beliefs convergence to a limit cycle, which we determine using the Theorem of Hopf (Section 3.2). In the third example (Section 3.3) we show the presence of a limit cycle in a model with forward looking expectations, agents take expectations of a random variable with one, two and three steps ahead. In Section 3.4 we extend the analysis to a class of nonlinear models which are quite common in the overlapping generations literature.

The above analysis can be developed both for ROLS and LMS learning. However, since the system of differential equations associated with ROLS learning is of dimension higher than the one associated with LMS learning, it is much harder to analyze its phase portrait, and thus to prove the existence of cycles for ROLS learning.

3.1 Example I

Let us consider the following model

$$\begin{aligned} p_t &= \alpha E_t(p_{t+1}) + \gamma E_t(d_{t+1}) + v_t \\ d_t &= \lambda + \delta E_t(p_{t+1}) + \psi(E_t(d_{t+1}) - \bar{d}) + w_t, \end{aligned}$$

where v_t and w_t are two bounded white noise variables and E_t denotes expectation taken by the agents at time t. In the following we assume $\alpha, \gamma, \lambda, \psi > 0$, $\delta < 0$ and $\bar{d} > 0$. Agents believe in the following misspecified law of motion (*perceived* law of motion):

$$\begin{aligned} p_t &= \beta_{0t} + \omega_{1t}, \\ d_t &= \beta_{1t} + \omega_{2t}. \end{aligned}$$

Thus, agents believe that the two random variables are two constants plus noise. Inserting agents expectations in the system we obtain the following *actual* law of motion:

$$\begin{aligned} p_t &= \alpha\beta_{0t} + \gamma\beta_{1t} + v_t \\ d_t &= \lambda + \delta\beta_{0t} + \psi(\beta_{1t} - \bar{d}) + w_t. \end{aligned}$$

The analysis can be developed in the SRLS models framework by setting

$$z_t = [p_t, d_t, 1]^T, \ z_{1t} = [p_t, d_t]^T, \ z_{2t} = 1, \ u_t = [v_t, w_t]^T$$

$$\mathcal{B} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, \ T(\mathcal{B})^T = \begin{bmatrix} \alpha\beta_0 + \gamma\beta_1 \\ \lambda + \delta\beta_0 + \psi(\beta_1 - \bar{d}) \end{bmatrix}, \ V(\mathcal{B}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A(\mathcal{B})^T = [0, 0, 1]$$

and $B(\mathcal{B})$ is a null vector. The model has only one REE $\mathcal{B}^*$ which is given by

$$\begin{bmatrix} \beta_0^* = \frac{\gamma(\psi\bar{d} - \lambda)}{\gamma\delta + (\psi - 1)(1-\alpha)} \\ \beta_1^* = \frac{(1-\alpha)(\psi\bar{d} - \lambda)}{\gamma\delta + (\psi - 1)(1-\alpha)} \end{bmatrix}.$$

The system of differential equations to be studied to determine the limit behavior of LMS learning is linear and is given by

$$\frac{d\mathcal{B}}{dt} = \begin{bmatrix} (\alpha - 1)\beta_0 + \gamma\beta_1 \\ \lambda + \delta\beta_0 + (\psi - 1)\beta_1 - \psi\bar{d} \end{bmatrix}. \tag{4}$$

Let us remark that the differential equation for ROLS learning is similar to the one of LMS learning; we have only to add one dimension for the updating of the information matrix R.

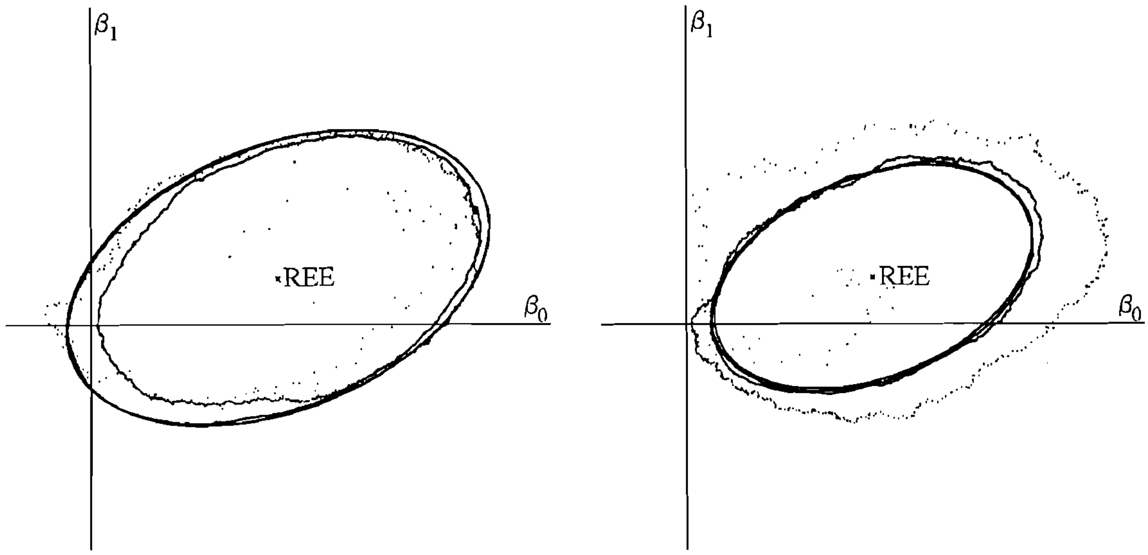

Figure 1: Two simulations of learning in Example I.

Since the differential equation for R is decoupled and has a globally asymptotically stable fixed point, the limit dynamics of ROLS learning can be reduced to the plane (β_0, β_1), where it is described by the system of differential equations in (4).

For $\psi = 2 - \alpha$ and $(\alpha - 1)(\psi - 1) - \delta\gamma > 0$ the Jacobian J of (4) evaluated at the REE has pure imaginary eigenvalues. Thus, for these parameter values the solutions of (4) are the REE, and a continuum of cycles surrounding it.

An invariant of motion for this system is given by $Q(\beta_0, \beta_1) = (\mathcal{B} - \mathcal{B}^*)^T M^T M(\mathcal{B} - \mathcal{B}^*)$, where M is the inverse of the 2×2 matrix consisting of the imaginary and real part of the eigenvectors of J. Since for fixed $\mathcal{B}$ the state variables p_t, d_t are stationary, we can apply Theorem 2.2. Hence, the process is with positive probability asymptotically cycling.

To the above model the following interpretation can be given: think of p_t and d_t respectively as the price and the dividend of a stock at time t. The first equation represents the classical no-arbitrage condition with $0 < \alpha = \gamma < 1$. The second equation describes the firm dividends policy: the firm deciding at time t the amount of dividends to be paid considers both the agents expectations of the dividends and of the price of the stock in the next period. If agents expect a dividend larger than $\overline{d}$ then the firm accommodates this opinion raising the dividends, $\psi > 0$, if the agents expect an increase in the stock price (a positive capital gain) then the firm decides to decrease the amount of dividends to be paid.

Figure 1 shows two runs of the learning process with the parameters $\alpha = 0.5$, $\psi = 1.5$, $\delta = -1, \gamma = 2$, $\lambda = 3$, $d = 1$, v_t, w_t are two uniformly distributed random variables on the interval $[-3, 3]$ and the process is started at the REE with $t_0 = 10$. Notice, that at each run a different cycle is selected.

3.2 Example II

Let us consider the following open economy model analyzed in [McCallum, 1989, Chapter 14]

$$B = b_1[E_{t-1}(e_{t+1} - e_t) - E_{t-1}(p_{t+1} - p_t)] + b_2(p_t - e_t) + v_t$$
$$-p_t = C + c_2 E_{t-1}(e_{t+1} - e_t) + \epsilon_t.$$

where v_t, ϵ_t are independent white noise variables, satisfying condition B1 in Appendix B. E_{t-1} denotes expectation taken by the agents at time $t-1$. The model is a standard open economy IS/LM model, p_t is the log of the domestic money price of domestic goods, e_t is the log of the home country exchange rate. With respect to the formulation in [McCallum, 198?] we assume that agents take expectations on the basis of the information set at time $t-1$. Let us assume that the agents believe in the law of motion

$$\begin{aligned} p_t &= \beta'_{0t} + \beta_{0t} e_{t-1} + \omega_{1t}, \\ e_t &= \beta'_{1t} + \beta_{1t} e_{t-1} + \omega_{2t} \end{aligned}$$

where ω_{1t}, ω_{2t} are white noise. The dynamics of the model is described in terms of the SRLS models framework as follows

$$z_t = [p_t, e_t, 1]^T,\ z_{1t} = [p_t, e_t]^T,\ z_{2t} = [e_t, 1]^T,\ u_t = [v_t, \epsilon_t, 0]^T,\ \mathcal{B}^T = \begin{bmatrix} \beta_0 & \beta'_0 \\ \beta_1 & \beta'_1 \end{bmatrix}$$

$$T(\mathcal{B})^T = \begin{bmatrix} -c_2(\beta_1^2 - \beta_1) & -C - c_2\beta'_1\beta_1 \\ \frac{b_1[\beta_1^2 - \beta_1 - \beta_0^2 + \beta_0]}{b_2} - c_2(\beta_1^2 - \beta_1) & \frac{b_1[\beta'_1\beta_1 - \beta'_0\beta_0] - B}{b_2} - C - c_2\beta'_1\beta_1 \end{bmatrix}$$

$$V(\mathcal{B})^T = \begin{bmatrix} \frac{1}{b_2} & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$A(\mathcal{B}) = [0, 0, 1]^T,\ B(\mathcal{B}) = [0, 0, 0]^T.$

We prove existence of a non perfect foresight beliefs limit cycle for LMS learning assuming $C = B = 0$; For C and B non zero the analysis can only be done numerically. With $B = C = 0$ the system simplifies essentially: the vector z_{2t} becomes a scalar and the perceived law of motion becomes

$$\begin{aligned} p_t &= \beta_{0t} e_{t-1} + \omega_{1t}, \\ e_t &= \beta_{1t} e_{t-1} + \omega_{2t}. \end{aligned}$$

Thus, the beliefs matrix $\mathcal{B}$ is reduced to a two dimensional vector and we have

$$z_t = z_{1t} = [p_t, e_t]^T,\ z_{2t} = e_t,\ u_t = [v_t, \epsilon_t]^T,\ \mathcal{B}^T = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix},$$

$$T(\mathcal{B})^T = \begin{bmatrix} -c_2(\beta_1^2 - \beta_1) \\ \frac{b_1[\beta_1^2-\beta_1-\beta_0^2+\beta_0]}{b_2} - c_2(\beta_1^2 - \beta_1) \end{bmatrix},$$

$$V(\mathcal{B})^T = \begin{bmatrix} \frac{1}{b_2} & -1 \\ 0 & 1 \end{bmatrix},$$

$A(\mathcal{B})$, $B(\mathcal{B})$ are null matrices. By a simple computation we have $M_{z_2}(\mathcal{B}) = \frac{\sigma_v^2}{1-T_2(\mathcal{B})^2}$, where σ_v^2 denotes the variance of v_t and $T_2(\mathcal{B})$ is the second component of $T(\mathcal{B})$. The system of differential equations associated with the LMS learning process (33) is (35), and in this particular case we have

$$\begin{aligned} \dot{\beta}_0 &= M_{z_2}(\mathcal{B})\,[-c_2(\beta_1^2 - \beta_1) - \beta_0] \\ \dot{\beta}_1 &= M_{z_2}(\mathcal{B})\,\left[d\,[\beta_1^2 - \beta_1 - \beta_0^2 + \beta_0] - c_2(\beta_1^2 - \beta_1) - \beta_1\right], \end{aligned} \tag{5}$$

where $d = \frac{b_1}{b_2}$. The model has four REEs

$$\mathcal{B}^* = \left(\frac{\Gamma_1 - \Gamma_2}{2\,d\,c_2}, \frac{\Gamma_3 + \Gamma_2}{2\,d\,c_2}\right) \tag{6}$$

$$\mathcal{B}^{**} = \left(\frac{\Gamma_1 + \Gamma_2}{2\,d\,c_2}, \frac{\Gamma_3 - \Gamma_2}{2\,d\,c_2}\right) \tag{7}$$

$$\mathcal{B}^{***} = \left(1 - \frac{1}{c_2}, 1 - \frac{1}{c_2}\right) \tag{8}$$

$$\mathcal{B}^{****} = (0,\,0) \tag{9}$$

where

$$\Gamma_1 = 2\,c_2 + d\,(c_2 - 1), \quad \Gamma_2 = \sqrt{d\,[-4\,c_2 + d\,(1 - 2\,c_2 + c_2^2)]}, \quad \Gamma_3 = d\,(1 + c_2).$$

Let $D_s = \{\mathcal{B} \in I\!R^2 |\, |T_2(\mathcal{B})| < 1\}$ denote the set where e_t is an asymptotically stationary stochastic process. $M_{z_2}(\mathcal{B})$ is well defined on D_s and positive. On D_s the system (5) can be analyzed dropping the positive scalar $M_{z_2}(\mathcal{B})$ since this does not change the phase portrait, see [Hofbauer and Sigmund, 1988, pag. 92]. Thus, we consider the system

$$\begin{aligned} \dot{\beta}_0 &= -c_2(\beta_1^2 - \beta_1) - \beta_0 \\ \dot{\beta}_1 &= d\,[\beta_1^2 - \beta_1 - \beta_0^2 + \beta_0] - c_2(\beta_1^2 - \beta_1) - \beta_1. \end{aligned} \tag{10}$$

We will prove existence of a limit cycle for (10), using the Theorem of Hopf. To this end, we consider the Jacobian of (5) evaluated at a point $\mathcal{B} \in I\!R^2$ obtaining

$$J_{\mathcal{B}} = \begin{bmatrix} -1 & c_2\,(1 - 2\,\beta_1) \\ d\,(1 - 2\,\beta_0) & -1 - d + c_2 + 2\,(d - c_2)\,\beta_1 \end{bmatrix}.$$

Evaluating $J_{\mathcal{B}}$ at the third and the fourth REE, it is easy to see that the eigenvalues are always real, and therefore no Hopf bifurcation can occur. However, for the first REE things are different, and the following Proposition can be stated:

Proposition 3.1 *Let* $k > 1$, $d_h = \frac{1}{k}\left(1 - \frac{\sqrt{1-4k+6k^2-7k^4+4k^5}}{(1-k)^2}\right)$. *Then for all sufficiently small* $\epsilon > 0$, $c_2 = k\,d_h$, $d = d_h - \epsilon$ *the ODE (10) has a uniformly asymptotically stable limit cycle around the REE* $\mathcal{B}^*$ *such that in a neighborhood of the limit cycle, conditions A and B (see Appendix B) are satisfied and thus Corollary 2.1 applies. Hence, beliefs updated with the LMS algorithm converge given appropriate initial conditions to the limit cycle with positive probability.*

Proof.
The Jacobian of the system evaluated at $\mathcal{B}^*$ is

$$J_{\mathcal{B}^*} = \begin{bmatrix} -1 & -1 - \frac{\Gamma_2}{d} \\ -2 + \frac{d+\Gamma_2}{c_2} & -2 + \frac{d}{c_2} + \Gamma_2\left(\frac{1}{c_2} - \frac{1}{d}\right) \end{bmatrix}.$$

We want to find parameter values c_2, d such that $J_{\mathcal{B}^*}$ has purely imaginary eigenvalues, i.e. $\mathrm{Trace}\,(J_{\mathcal{B}^*}) = 0$ and $\mathrm{Det}\,(J_{\mathcal{B}^*}) > 0$. To simplify calculations we set $c_2 = k\,d$. For the determinant then we get

$$Det\,(J_{\mathcal{B}^*}) = \frac{\Gamma_2}{-d}\left(1 - \frac{1}{k} - \frac{\Gamma_2}{k\,d}\right).$$

Thus, for $k > 1$ and $d < 0$ the determinant is strictly positive and $c_2 < d < 0$. Setting the trace equal to zero we obtain

$$\Gamma_2 = \frac{d\,(3c_2 - d)}{d - c_2}. \tag{11}$$

Let us remark that the right hand side has to be positive, this happens if $c_2 < d < 0$. Take squares in (11) and divide the whole equation by $c_2 d$. It follows that the trace of $J_{\mathcal{B}^*}$ is equal to zero if $c_2 < d < 0$ and

$$d^3\,(2 - c_2) + d^2\,(2\,c_2^2 - 4\,c_2) + d\,(2\,c_2^2 - c_2^3) + 4\,c_2^2 = 0,$$

substituting $c_2 = k\,d$, we obtain

$$d^2\,[4\,k^2 + d\,(2 - 4\,k + 2\,k^2) + d^2\,(-k + 2\,k^2 - k^3)] = 0. \tag{12}$$

Thus, we have only to solve a second order polynomial obtaining the solution

$$d_h(k) = \frac{1}{k}\left(1 - \frac{\sqrt{1 - 4\,k + 6\,k^2 - 7\,k^4 + 4\,k^5}}{(1-k)^2}\right)$$

The second root of (12) is positive and therefore it cannot satisfy the conditions for the determinant to be positive and the trace equal to zero. It easily follows that for all $k > 1$, $d_h(k) < 0$ holds. Thus, for all $k > 1$ and $d = d_h(k), c_2 = k\,d_h$ the eigenvalues of $J_{\mathcal{B}^*}$ are imaginary. Since for $k > 1$ and $d_h(k) < 0$ we have

$$\frac{\mathrm{d\ Trace}(J)}{\mathrm{d}\,d} = \frac{d_h\,(k-1)\,(1 - d_h\,k)}{\Gamma_2} < 0, \tag{13}$$

from the Theorem of Hopf it follows that at these parameter values a Hopf bifurcation occurs.

Let $\mathcal{B}^*(k)$ denote the REE $\mathcal{B}^*$ in (6) for the parameter values $d = d_h(k)$, $c_2 = k\, d_h(k)$. To determine if the resulting periodic orbit is stable, we apply the normal form calculation given in [Guckenheimer and Holmes, 1983, pag. 152]. To this end, we make a change of coordinates such that $\mathcal{B}^*(k)$ is moved to the origin and such that the differential equation (10) takes the form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -\mu \\ \mu & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix},$$

where μ is the imaginary part of the eigenvalues of $J_{\mathcal{B}^*(k)}$. Thus, we set

$$\begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} \beta_0 + \beta_0^* \\ \beta_1 + \beta_1^* \end{bmatrix},$$

where A is a matrix consisting of the imaginary and real part of the eigenvectors of $J_{\mathcal{B}^*(k)}$. The Hopf bifurcation at $k > 1$ is stable if for the resulting functions f and g

$$a(k) := f_{xy}\,(f_{xx} + f_{yy}) - g_{xy}\,(g_{xx} + g_{yy}) - f_{xx}\,g_{xx} + f_{yy}\,g_{yy} < 0 \tag{14}$$

where the subscripts denote partial derivatives, which are evaluated at $\mathcal{B}^*(k)$. A plot of $a(k)$ (which is a quite complicated function of k), shows that (14) holds for all $k > 1$.

Hence, we have proved that for all $k > 1$ and a sufficiently small $\epsilon > 0$ there is an asymptotically stable periodic orbit around the REE $\mathcal{B}^*(k)$ for $d = d_h(k) - \epsilon$ and $c_2 = k\, d_h(k)$.

To transfer the results we derived for (10) to the original differential equation (5) we have to assure that $\mathcal{B}^*(k) \in D_s$. Again, a plot of $\beta_1^*(k)$ as function of k shows that this is the case for all $k > 1$. For sufficiently small ϵ also the limit cycle lies in D_s. It follows that in a neighborhood of the limit cycle, the stationarity condition $B1$ is satisfied. The other conditions of Appendix B are trivially satisfied. Thus, we can apply Corollary 2.1 to see that beliefs tend to the limit cycle. □

The parameter restrictions require $d < 0$. This implies that $b_2 < 0$ and $b_1 > 0$, i.e. the IS curve is downward sloping (the marginal propensity to save exceeds the marginal propensity to invest, see [Sargent, 1987, pag. 54]). So we have that if both the IS and the LM curve are downward sloping then agents learn to believe in a non perfect foresight beliefs limit cycle.

Figure 2 shows a simulation of the learning process with the parameters $b_1 = 0.609808$, $b_2 = -1$, $c_2 = -6.09808$, v_t, w_t are two uniformly distributed random variables on the interval $[-1, 1]$ and the initial conditions are $(\beta_{0t_0}, \beta_{1t_0}, t_0) = (\mathcal{B}^*, 500)$. In the graph we moved the REE $\mathcal{B}^*$ to the origin. (For technical reasons the plot was generated using the learning process till $t = 3 \cdot 10^6$ and then continued by the solution of the ODE).

3.3 Example III

Let us consider the following model

$$\begin{aligned} x_t &= a_0 x_{t-1} + a_1 E_{t-1}(x_t) + a_2 E_{t-1}(y_t) + a_3 E_{t-1}(x_{t+1}) + a_4 E_{t-1}(x_{t+2}) + v_t \\ y_t &= b_1 E_{t-1}(x_t) + b_2 E_{t-1}(y_t) + w_t, \end{aligned}$$

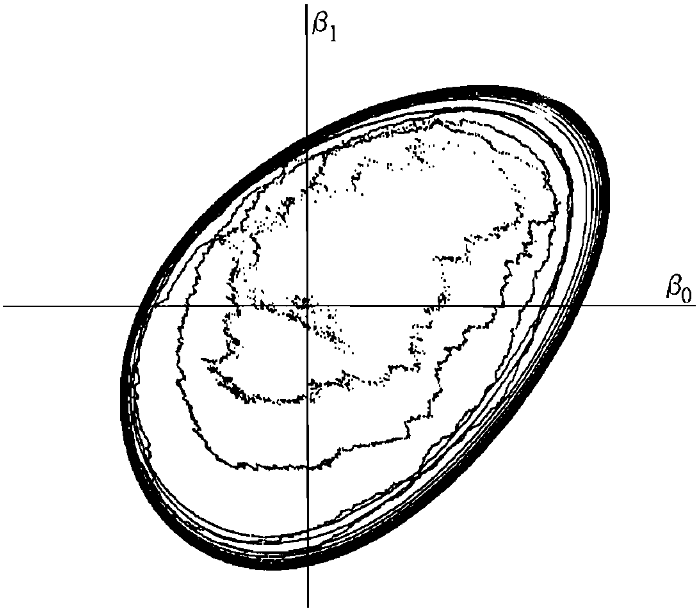

Figure 2: A simulation of learning in Example II.

where v_t and u_t are two white noise variables satisfying condition B1 in Appendix B and E_{t-1} denotes expectation taken by the agents at time $t-1$. Agents believe in the following (misspecified) law of motion:

$$x_t = \beta_{0t}x_{t-1} + \omega_{1t}$$
$$y_t = \beta_{1t}x_{t-1} + \omega_{2t}.$$

Inserting the agents' perceived law of motion in the above system we obtain

$$x_t = a_0 x_{t-1} + a_1\beta_{0t}x_{t-1} + a_2\beta_{1t}x_{t-1} + a_3\beta_{0t}^2 x_{t-1} + a_4\beta_{0t}^3 x_{t-1} + v_t$$
$$y_t = b_1\beta_{0t}x_{t-1} + b_2\beta_{1t}x_{t-1} + w_t.$$

The example can be analyzed in the SRLS models framework by setting

$$z_t = [x_t, y_t]^T,\ z_{1t} = [x_t, y_t]^T,\ z_{2t} = x_t,\ u_t = [v_t, w_t]^T$$

$$\mathcal{B} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix},\ T(\mathcal{B})^T = \begin{bmatrix} a_0 + a_1\beta_0 + a_2\beta_1 + a_3\beta_0^2 + a_4\beta_0^3 \\ b_1\beta_0 + b_2\beta_1 \end{bmatrix},\ V(\mathcal{B}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and $A(\mathcal{B})$, $B(\mathcal{B})$ are null matrices. For $M_{z_2}(\mathcal{B})$ we have

$$M_{z_2}(\mathcal{B}) = \frac{\sigma_v^2}{1 - (a_0 + a_1\beta_0 + a_2\beta_1 + a_3\beta_0^2 + a_4\beta_0^3)^2}.$$

The REE are obtained as solutions of the third order polynomial equation

$$a_0 + (a_1 - \frac{a_2 b_1}{b_2} - 1)\beta_0 + a_3\beta_0^2 + a_4\beta_0^3 = 0.$$

The behavior of LMS learning can be studied by means of the following system

$$\frac{d\mathcal{B}}{dt} = M_{z_2}(\mathcal{B}) \begin{bmatrix} a_0 + a_1\beta_0 + a_2\beta_1 + a_3\beta_0^2 + a_4\beta_0^3 - \beta_0 \\ b_1\beta_0 + b_2\beta_1 - \beta_1 \end{bmatrix}. \tag{15}$$

The existence of a limit cycle depends on the parameters of the model. Here we consider the case $a_0 = a_3 = 0$, such that the origin is a REE. If $a_0 \neq 0$ the system can be reduced to the former case by a change of coordinates. The proposition also holds for $a_3 \neq 0$ sufficiently small in absolute value, as we can observe by continuity arguments.

Proposition 3.2 *Let the parameters of the model satisfy the following conditions:*

(a) $a_0 = 0,\ a_1 > 0,\ a_2 > 0,\ a_3 = 0,\ a_4 < 0,\ b_1 < 0,\ b_2 < 1,$

(b) $a_1 + b_2 > 2,\ (a_1 - 1)(b_2 - 1) - b_1 a_2 > 0,$

(c) $\frac{-a_1^3}{a_4} < \frac{27}{16},$

(d) There exists a $\bar{c} > 0$ *such that*

$$\bar{c} > \max\left\{ \frac{(a_1 - 1)\, b_1}{2\, a_4}, \frac{b_1\, (b_2 - a_1)^2}{8\, a_4\, (b_2 - 1)} \right\}, \tag{16}$$

$$\bar{c} < \frac{1}{8\, a_2}, \tag{17}$$

$$\frac{1}{2} > \sqrt{\frac{2\,\bar{c}}{-b_1}} \left| a_1 - \frac{2\,\bar{c}\, a_4}{b_1} \right|. \tag{18}$$

Then the ODE (15) has only one fixed point. It is unstable and surrounded by a uniformly asymptotically stable limit cycle. In a neighborhood of the limit cycle, conditions A and B (see Appendix B) are satisfied and thus Corollary 2.1 applies. Hence, beliefs updated with the LMS algorithm converge with positive probability given appropriate initial conditions to the limit cycle.

Proof.
The proof is in two steps.

Step I

Let $s(\mathcal{B}) = |a_1\beta_0 + a_2\beta_1 + a_4\beta_0^3|$. Consider the restriction of the ODE to the set $D_s = \{\mathcal{B} \in I\!R^2 : s(\mathcal{B}) < 1\}$. By condition (a) $a_0 = a_3 = 0$ and thus $s(\mathcal{B})$ is just the absolute value of the first component of $T(\mathcal{B})$. Hence, on D_s the process z_{2t} is asymptotically stationary. We show that there exists a uniformly asymptotically stable (and thus, positively invariant) compact subset Γ of D_s.

Define the function

$$v(\mathcal{B}) = \frac{1}{2}(-b_1\, \beta_0^2 + a_2\, \beta_1^2). \tag{19}$$

By condition (a) $(0,0)$ is a global minimum and the only critical point of $v(\mathcal{B})$. Thus, for $c \in I\!R^+$ the sets $v^{-1}([0,c])$ are neighborhoods of $(0,0)$ that shrink with decreasing c. We show that for $\bar{c}$ satisfying condition (d) we have

(i) $\Gamma := v^{-1}([0,\bar{c}]) \subset \{\mathcal{B} : s(\mathcal{B}) < 1\}$,

(ii) the set Γ is positively invariant for the flow defined by (15).

ad (i) Since $s(\mathcal{B}) \le |a_1\beta_0 + a_4\beta_0^3| + |a_2\beta_1| =: |s_1(\beta_0)| + |s_2(\beta_1)|$ it is sufficient to show that for all $\mathcal{B} \in \Gamma$ we have

$$|s_1(\beta_0)| < \frac{1}{2}, \tag{20}$$

$$|s_2(\beta_1)| < \frac{1}{2}. \tag{21}$$

For all $\mathcal{B} \in \Gamma$ we have

$$\beta_0 \in \left[-\sqrt{\frac{2\,\bar{c}}{-b_1}}, \sqrt{\frac{2\,\bar{c}}{-b_1}}\right] =: [\beta_c, \beta^c], \quad \beta_1 \in \left[-\sqrt{\frac{2\,\bar{c}}{a_2}}, \sqrt{\frac{2\,\bar{c}}{a_2}}\right].$$

Now for all such β_1 we have

$$|s_2(\beta_1)| < a_2\sqrt{\frac{2\,\bar{c}}{a_2}} < 1/2,$$

where the last inequality follows from (17). To show (20), note that by (18) $|s_1(\beta_c)| = |s_1(\beta^c)| = \sqrt{\frac{2\,\bar{c}}{-b_1}}\left|a_1 - \frac{2\,\bar{c}\,a_4}{b_1}\right| < \frac{1}{2}$. Additionally, $|s_1(\beta_1)|$ has only one local maximum on $I\!R$. There it takes the value $\sqrt{\frac{-4\,a_1^3}{27\,a_4}}$ which is less than $\frac{1}{2}$ by (c). Thus, (20) holds.

ad (ii)

Let $\frac{dv}{dt}(\mathcal{B})$ denote the time derivative of the function (19) along the solutions of (15)

$$\frac{dv}{dt}(\mathcal{B}) = \langle \nabla v, M_{z_2}(\mathcal{B})\,(T(\mathcal{B}) - \mathcal{B})\rangle \tag{22}$$

$$= M_{z_2}(\mathcal{B})\,[-\beta_0^2\,b_1(a_1 - 1 + a_4\,\beta_0^2) + a_2\,(b_2 - 1)\beta_1^2]. \tag{23}$$

To prove the invariance of Γ we show that on the boundary of Γ $\frac{dv}{dt} < 0$ holds. For all $\mathcal{B} \in D_s$ we have $M_{z_2}(\mathcal{B}) > 0$. Thus, to study the sign of $\frac{dv}{dt}(\mathcal{B})$ on D_s it suffices to consider the right factor of (23). The boundary of Γ can be written as $V_c = \left\{\left(\beta_0, \pm\sqrt{\frac{2\,\bar{c}+b_1\,\beta_0^2}{a_2}}\right) \mid \beta_0 \in [\beta_c, \beta^c]\right\}$. Thus, the term $\frac{1}{M_{z_2}(\mathcal{B})}\frac{dv}{dt}$ on V_c is given by

$$f(\beta_0) := -b_1\beta_0^2\,(a_1 - b_2 + a_4\,\beta_0^2) + 2\,(b_2 - 1)\,\bar{c} \tag{24}$$

where $\beta_0 \in [\beta_c, \beta^c]$. Now, by (16) we have $f(\beta_c) = f(\beta^c) = 2\bar{c}(a_1 - 1 - \frac{2a_4\bar{c}}{b_1}) < 0$. The function $f(\beta_0)$ has two local maxima on $I\!R$. At both of them $f(\beta_0)$ takes the value $\frac{b_1(-a_1+b_2)^2}{4a_4} + 2(b_2 - 1)\bar{c}$ which again by (16) is less than zero. Thus, (24) and hence $\frac{dv}{dt}$ is negative for all β_0 defining the first coordinate of a point in V_c.

Step II

Existence of cycles. First, note that the origin is the only fixed point of (15). The fixed points are the roots of a polynomial of order three. By conditions (a) and (b) it follows straight forward that (15) has only one real root (which corresponds to the fixed point $(0,0)$).

Since on the positively invariant set Γ $M_{z_2}(\mathcal{B}) > 0$ (see Step I), the system (15) can be analyzed dropping the positive scalar $M_{z_2}(\mathcal{B})$, this does not change in fact the phase portrait, see [Hofbauer and Sigmund, 1988, pag. 92]. The Jacobian of the simplified system evaluated at the origin is

$$\begin{bmatrix} a_1 - 1 & a_2 \\ b_1 & b_2 - 1 \end{bmatrix}.$$

Thus, by condition (b) the fixed point is a source. Hence, Γ is a compact positively invariant set containing a source. Since the origin is the only fixed point, by the Poincarè-Bendixson theorem there exists a periodic orbit in Γ. Since the differential equation is analytical, outside a neighborhood of the fixed point also the associated Poincarè mapping is analytical. Hence, there are only finitely many periodic orbits. By a counting argument it follows that one of them has to be asymptotically stable.

Conditions A and B1 are trivially satisfied on Γ. Since $M_{z_2}(\mathcal{B}) < 1$ on Γ also condition B2 holds.□

Let us remark that the classical Liénard equation and Van der Pool equation can be obtained in (15) by choosing proper parameters.

The model can be thought as an extension of the model analyzed in [Evans and Honkapohja, 1994b]. To the scalar model in [Evans and Honkapohja, 1994b] we have added a second variable, y_t, which can be interpreted as a sunspot driven by the agents' expectation for its future value and with the feature that it affects positively via its expected value the evolution of x_t which in turns has a negative effect via its expectation on the evolution of y_t.

In the Dornbush overshooting model framework, [Dornbusch, 1976], analyzed under bounded rationality in [Evans and Honkapohja, 1994b, Section 5], the variable x_t is the price level, the second variable y_t can be interpreted as a sunspot about the health of the government finance, a sunspot which is driven by the expectation that the agents have about its future level, negatively affected by the agents' expected price level and with a positive effect on the price level via its expected value.

Figure 3 shows a simulation of the learning process with the parameters $a_1 = 1.2$, $a_2 = 0.12$, $a_3 = 0.1$, $a_4 = -3$, $b_1 = -1.5$, $b_2 = 0.9$, v_t, w_t are two uniformly distributed random variables on the interval $[-3, 3]$ and the initial conditions are $(\beta_{0t_0}, \beta_{1t_0}, t_0) = (0, 0, 50)$. (For technical reasons the plot was generated using the learning process till $t = 3\,10^6$ and then continued by the solution of the ODE). On the left there is a plot of a solution of the corresponding ODE starting close to the fixed point (0,0).

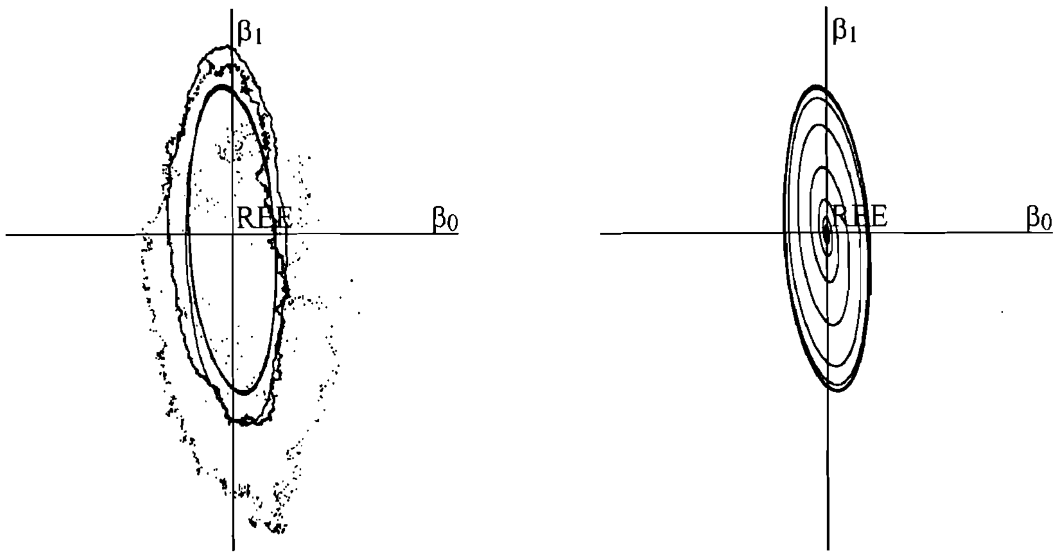

Figure 3: A simulation of learning in Example III and the corresponding ODE.

3.4 Example IV

The analysis can be extended to stochastic nonlinear models of the type analyzed in [Evans and Honkapohja, 1995a, Evans and Honkapohja, 1995b]. Let us consider the class of nonlinear models

$$y_t = E_t[F(y_{t+1})] + v_t$$

where y_t is a random vector of dimension n, v_t is a vector of bounded i.i.d random variables and $F(\cdot)$ is a nonlinear continuously differentiable function. With respect to the analysis in [Evans and Honkapohja, 1995a, Evans and Honkapohja, 1995b] which presents only a scalar model, we have augmented the dimension: in one dimensional models we cannot observe limit cycles or other complex dynamics as for example strange attractors. In what follows we concentrate our attention on steady state solutions. We introduce the notation

$$Y(\theta) = E_t[F(\theta + w)]$$

where $\theta \in I\!R^n$.

A Rational Steady State solution takes the form

$$y_t = \overline{\theta} + v_t \text{ with } \overline{\theta} \in I\!R^n \text{ such that } \overline{\theta} = Y(\overline{\theta}).$$

Let us assume that agents believe in the following misspecified law of motion

$$y_t = \theta_{t-1} + v_t,$$

according to [Evans and Honkapohja, 1995a] the ROLS and the LMS updating rule are described by the following nonlinear stochastic difference equation

$$\theta_t = \theta_{t-1} + t^{-1}[F(\theta_{t-1} + v_t) - \theta_{t-1}].$$

This is again a stochastic process of the type studied in Section 2 and thus its limit behavior is described by the following system of differential equations

$$\dot{\theta} = Y(\theta) - \theta \tag{25}$$

and the results of Section 2 apply. Depending on the model and on the dimension of the vector of state variables a large variety of dynamics can emerge in (25). Depending on $F(\cdot)$ and in particular on $Y(\cdot)$, if the state vector is of dimension 2 then we can observe limit cycles, if the state vector is of dimension 3 then we can even observe chaotic dynamics.

4 Conclusions

In the literature, complex dynamics have been obtained in deterministic economic models assuming that agents have complete knowledge of the economy or assuming that they follow a naive behavior. In this paper we have proved the existence of complex dynamics in stochastic linear models with forward looking expectations assuming that agents are rationally bounded; the models are not characterized *per se* by complex dynamics, agents learn to believe in complex beliefs dynamics because they learn from the evolution of the economy and they believe in a misspecified economic model. Complex dynamics concern agents' beliefs rather than the state variables of the model and are induced by bounded rationality learning.

We have shown the rise of complex beliefs dynamics in some macroeconomic models. Specifically, we have shown that some open economy macroeconomics models, as well as overlapping generations models, are characterized by non perfect foresight complex dynamics.

We think the result obtained in this paper offer new perspectives to the analysis of complex dynamics in economics: it is not necessary to assume a deterministic environment to obtain complex dynamics, also in a stochastic environment we can observe them; complex dynamics concern agents' beliefs rather than economic variables.

A Bounded Rationality Learning in Linear Rational Expectations Models

The class of models analyzed in this paper is the one of SRLS models, the models are linear in economic variables and in agents expectations and are characterized by linear REE; for non-RE agents beliefs, the law of motion of the economic model may be highly nonlinear in agents beliefs. Following [Marcet and Sargent, 1989b], we describe the economic variables at time t by an n dimensional vector of random variables z_t. We denote by two subvectors of z_t the set of economic variables that agents are interested in, $z_{1t} \in I\!R^{n_1}$, and the set of economic variables, $z_{2t} \in I\!R^{n_2}$, that agents think are relevant to predict the first subvector of variables. The vector z_t can be written, without loss of generality, as follows

$$z_t = \begin{pmatrix} z_{1t} \\ z_{1t}^c \end{pmatrix} = \begin{pmatrix} z_{2t}^c \\ z_{2t} \end{pmatrix}$$

where the superscript c expresses the complement with respect to z_t. As in [Marcet and Sargent, 1989b], we assume that the agents' *perceived* law of motion is linear and is expressed as

$$z_{1t} = \mathcal{B}_t^T z_{2(t-1)} + \omega_t \tag{26}$$

where $\mathcal{B}_t \in I\!R^{n_2 \times n_1}$ is the parameter matrix representing agents' beliefs and ω_t is a white noise component. The law of motion perceived by the agents in (26) causes the *actual* law of motion for the vector z_t to be given in a general setting by

$$z_t = \begin{bmatrix} z_{1t} \\ z_{1t}^c \end{bmatrix} = \begin{bmatrix} 0 & T(\mathcal{B}_t)^T \\ & A(\mathcal{B}_t)^T \end{bmatrix} \cdot \begin{bmatrix} z_{2(t-1)}^c \\ z_{2(t-1)} \end{bmatrix} + \begin{bmatrix} V(\mathcal{B}_t)^T \\ B(\mathcal{B}_t)^T \end{bmatrix} \cdot u_t \tag{27}$$

where $u_t \in I\!R^n$ is a white noise vector. Given the economic model, z_{1t}, z_{2t} and the operators $T(\cdot)$, $A(\cdot)$, $B(\cdot)$, $V(\cdot)$ are defined. Assuming that the agents' *perceived* law of motion is given by (26), the *instantaneous* forecasting error is $\varepsilon_t = z_{1t} - z_{1t}^e$, where z_{1t}^e is the expected value of z_{1t} according to (26). A REE is a fixed point for $T(\mathcal{B})$: $\mathcal{B}^*$ such that $T(\mathcal{B}^*) = \mathcal{B}^*$. Note that the data generating process in (27) does not imply that z_t is a stationary process. As in [Marcet and Sargent, 1989b], we restrict our attention to the beliefs set D_s for which the stochastic process is stationary. To take into account the case of the presence of constants in the perceived law of motion, the set D_s is defined as the set of beliefs for which the above stochastic process, rewritten properly as $z_t = L(\mathcal{B}_t)z_{t-1} + E(\mathcal{B}_t)u_t$, is stationary, on this point see [Chang et al., 1995].

The learning mechanisms considered in our analysis are ROLS learning and LMS learning.

Let $\{\alpha_t\}$ be a positive, non-decreasing sequence of real numbers, with $\alpha_t \to 1$ as $t \to \infty$. Define $\bar{\mathcal{B}}_{t+1}$ and $\bar{R}_{t+1}$ as

$$\begin{aligned} \bar{\mathcal{B}}_{t+1} &= \mathcal{B}_t + \tfrac{\alpha_{t+1}}{t+1} R_t^{-1} \left\{ z_{2_{t-1}} z_{2_{t-1}}^T \left[T(\mathcal{B}_t) - \mathcal{B}_t \right] + z_{2_{t-1}} u_t^T V(\mathcal{B}_t) \right\} \\ \bar{R}_{t+1} &= R_t + \tfrac{\alpha_{t+1}}{t+1} \left(z_{2_t} z_{2_t}^T - \tfrac{R_t}{\alpha_{t+1}} \right) \end{aligned} \tag{28}$$

If $\alpha_i = 1 \; \forall i$, then the ROLS algorithm is obtained, otherwise the Weighted ROLS algorithm is obtained. A projection facility is needed to ensure almost sure convergence, see [Marcet and Sargent, 1989b]. Let $D_2 \subset D_1 \in I\!\!R^{n_1 \times (n_2)^3}$, the algorithm generating beliefs becomes

$$(\mathcal{B}_{t+1}, R_{t+1}) = \begin{cases} (\bar{\mathcal{B}}_{t+1}, \bar{R}_{t+1}) & \text{if } (\bar{\mathcal{B}}_{t+1}, \bar{R}_{t+1}) \in D_1 \\ \text{some value in } D_2 & \text{if } (\bar{\mathcal{B}}_{t+1}, \bar{R}_{t+1}) \notin D_1 \end{cases} \tag{29}$$

where the set D_2 is closed and D_1 is open and bounded so that if $\mathcal{B} \in D_s$ then $(R, \mathcal{B}) \in D_1$.

The learning algorithm defined in (28)-(29) applied to SRLS models has been studied by means of the Ljung's method, see [Ljung, 1977]. Under some assumptions, the method associates with the learning scheme (28)-(29) an ordinary differential equation that almost surely mimics the behavior of $(\mathcal{B}_{t+1}, R_{t+1})$ as $t \to \infty$:

$$\left(\frac{d}{dt}\right) \begin{bmatrix} \mathcal{B} \\ R \end{bmatrix} = \begin{bmatrix} R^{-1} M_{z_2}(\mathcal{B}) \left[T(\mathcal{B}) - \mathcal{B}\right] \\ M_{z_2}(\mathcal{B}) - R \end{bmatrix} \tag{30}$$

where $M_{z_2}(\mathcal{B}) = \lim_{t \to \infty} E\{z_{2t} z_{2t}^T\}$. The sets D_1 and D_2 are chosen such that trajectories of the differential equation in (30) with initial condition $(\mathcal{B}_0, R_0) \in D_2$ never leave the closed set D_1.

The fixed points of the differential equation in (30) correspond for the first n_1 rows to the REE of the SRLS model in (27). Stability of the differential equation at $(\mathcal{B}^*, M_{z_2}(\mathcal{B}^*))$ means that learning based on the ROLS algorithm converges *almost surely* to the REE, thanks to the projection facility. On the other side, instability means that ROLS learning does not converge to the REE, see [Marcet and Sargent, 1989b]. The stability of the system in (30) can be analyzed locally in a neighborhood of a fixed point by means of the following simpler system of differential equations

$$\frac{d\mathcal{B}}{dt} = T(\mathcal{B}) - \mathcal{B} \; . \tag{31}$$

The *Least Mean Squares* algorithm is the simplest learning mechanism developed in the adaptive control/signal processing literature, see [Widrow, 1971, Widrow and Stearns, 1985]. The application of the LMS algorithm as a learning mechanism and the proofs of the results reported below are provided in [Barucci and Landi, 1995b]. The LMS algorithm is a procedure which updates the beliefs matrix $\mathcal{B}_t$ to minimize the error variance function

$$\xi(\mathcal{B}) = E\{\varepsilon \cdot \varepsilon^T\}$$

where $E\{\cdot\}$ is the expectation operator and $\xi(\mathcal{B}) \in R^{n_1 \times n_1}$. Assuming that the error components are not correlated, the matrix function $\xi(\mathcal{B})$ is diagonal, its i-th component ($\xi_i(\beta_i) \in I\!\!R$) is the expected square of the i-th component of the error forecasting vector ε,

$$\xi_i(\beta_i) = E\{\varepsilon_i^2\} \; i = 1, \ldots, n_1 \; ,$$

the minimization of $\xi(\mathcal{B})$ corresponds to the minimization of the sum of the $\xi_i(\beta_i)$, $i = 1, \ldots, n_1$. Because of the absence of correlation among errors, the LMS algorithm for the matrix $\mathcal{B}$ can be defined with respect to each component of z_1 and therefore to each column $\beta_i \in I\!R^{n_2}$ of $\mathcal{B}$ minimizing the i-th component of the function $\xi(\mathcal{B})$.

The LMS algorithm looks for a minimum point of the MSE function according to the *steepest descent* gradient procedure taking at time t the scalar $\varepsilon_{t,i}^2$ as an estimate of $\xi_i(\beta_i)$, $i = 1, \ldots, n_1$. Therefore the LMS updating rule for the i-th column $\beta_{t,i}$ of $\mathcal{B}_t$ is

$$\beta_{t+1,i} = \beta_{t,i} - 2\eta_t \varepsilon_{t,i} \hat{\nabla}_t \varepsilon_{t,i} \ . \tag{32}$$

where η_{t+1} is a decreasing function of t. Let us notice that agents are not able to compute the "true" gradient because they do not know the "true" law of motion of the model,i.e. $T(\mathcal{B})$, they know only that their estimate enters linearly the error expression. For the analysis of the algorithm with an exact computation of gradient see [Barucci and Landi, 1995a]. The LMS algorithm for the SRLS model in (27) implies the following updating rule

$$\bar{\beta}_{t+1,i} \ = \beta_{t,i} + 2\eta_t \left[z_{2(t-1)} z_{2(t-1)}^T \left(T_i(\beta_{t,i}) - \beta_{t,i} \right) + z_{2(t-1)} u_t^T V_i(\beta_{t,i}) \right] \ . \tag{33}$$

As in [Marcet and Sargent, 1989b] we invoke the projection facility, let us define the sets $D_2 \subset D_1 \subset I\!R^{n_2 \times n_1}$. The algorithm for generating beliefs $\mathcal{B}_{t+1}$ is

$$\mathcal{B}_{t+1} = \begin{cases} \bar{\mathcal{B}}_{t+1} & \text{if } \bar{\mathcal{B}}_{t+1} \in D_1 \\ \text{some value in } D_2 & \text{if } \bar{\mathcal{B}}_{t+1} \notin D_1 \end{cases} \ . \tag{34}$$

Given some regularity assumptions we have proved in [Barucci and Landi, 1995b] by applying the Ljung theory that the LMS updating rule can be analyzed in the limit by means of the following differential equation

$$\frac{d\mathcal{B}}{dt} = M_{z_2}(\mathcal{B}) \cdot (T(\mathcal{B}) - \mathcal{B}) \, . \tag{35}$$

Let us remark that convergence to a non REE occurs if there exists a $\mathcal{B}^o \in I\!R^{n_2 \times n_1}$ such that $\mathcal{B}^o \neq T(\mathcal{B}^o)$ and $M_{z_2}(\mathcal{B}^o)(T(\mathcal{B}^o) - \mathcal{B}^o) = 0$, that is $M_{z_2}(\mathcal{B}^o)$ is not a full rank matrix. Dropping the assumption of non correlation among the forecasting error components the analysis can still be developed assuming that agents are interested in minimizing independently the n_1 error variances.

B Conditions on the Stochastic Process

A1 η_t is a deterministic non-increasing sequence satisfying $\sum_{t=1}^{\infty} \eta_t = \infty$ and $\sum_{t=1}^{\infty} \eta_t^2 < \infty$.

A2 For any compact subset $Q \subset D$ there are constants C_1 and q_1 such that $\forall \theta \in Q$ and $\forall t$, $|H(\theta, x)| \leq C_1 (1 + |x|^{q_1})$.

A3 For any compact subset $Q \subset D$ the function $H(\theta, x)$ satisfies for all $\theta, \theta' \in Q$ and $x_1, x_2, x \in I\!R^k$ the conditions

(i) $|H(\theta, x_1) - H(\theta, x_2)| \leq L_1 |x_1 - x_2|$,

(ii) $|H(\theta, 0) - H(\theta', 0)| \leq L_2 |\theta - \theta'|$,

(iii) $|\frac{\partial H(\theta,x)}{\partial x} - \frac{\partial H(\theta',x)}{\partial x}| \leq L_2 |\theta - \theta'|$,

for some constants L_1, L_2.

B1 w_t is identically and independently distributed with finite absolute moments, i.e. $E(|w_t|^q) < \infty$ for all $q = 1, 2, 3, \ldots$

B2 For any compact subset $Q \subset D$

$$\sup_{\theta \in Q} |G(\theta)| \leq M \text{ and } \sup_{\theta \in Q} |F(\theta)| \leq q < 1,$$

for some matrix norm $|\cdot|$, and $F(\theta), G(\theta)$ satisfy Lipschitz conditions on Q.

References

[Barucci and Landi, 1995a] Barucci, E. and Landi, L. (1995a). Computing rational expectations equilibria through the least mean squares algorithm. Università di Firenze.

[Barucci and Landi, 1995b] Barucci, E. and Landi, L. (1995b). Least Mean Squares learning in Self–Referential Linear Stochastic models. Technical report, Università di Firenze.

[Benaim, 1993] Benaim, M. (1993). A dynamical system approach to stochastic approximations. Mimeo, Univ. of Berkley.

[Benaim and Hirsch, 1995] Benaim, M. and Hirsch, M. (1995). Dynamics of morse-smale urn processes. Mimeo.

[Benaim and Hirsch, 1994] Benaim, M. and Hirsch, M. W. (1994). Asymptotic pseudotrajectories, chain recurrent flows and stochastic approximations. Dept. of Mathematics, UCL at Berkeley.

[Benveniste et al., 1990] Benveniste, A., Métivier, M., and Priouret, P. (1990). *Adaptive Algorithms and Stochastic Approximations.* Springer Berlin.

[Bodrin and Woodford, 1990] Bodrin, M. and Woodford, M. (1990). Equilibrium models displaying endogenous fluctuations and chaos. *Journal of Monetary Economics*, 25:189–222.

[Boldrin and Montrucchio, 1986] Boldrin, M. and Montrucchio, L. (1986). On the indeterminacy of capital accumulation paths. *Journal of Economic Theory*, 40:26–39.

[Brock, 1987] Brock, W. (1987). Distinguishing random and deterministic systems: Abridged version. In Grandmont, J., editor, *Nonlinear Economic Dynamics.* Academic Press.

[Bullard, 1994] Bullard, J. (1994). Learning equilibria. *Journal of Economic Theory*, 64:468–485.

[Canio, 1979] Canio, S. D. (1979). Rational expectations amd learning from experience. *Quarterly Journal of Economics*, 93:47–58.

[Chang et al., 1995] Chang, M., Chu, C., and Lin, K. (1995). A note on least-squares learning mechanism. *Journal of Economic Dynamics and Control*, 19:1293–1296.

[Day, 1994] Day, R. (1994). *Complex Economic Dynamics.* MIT Press, Cambridge.

[Dornbusch, 1976] Dornbusch, R. (1976). Expectations and exchange rate dynamics. *Journal of Political Economy*, 84:1161–1176.

[Evans and Honkapohja, 1994a] Evans, G. and Honkapohja, S. (1994a). Economic dynamics with learning: New stability results. *Mimeo.*

[Evans, 1985] Evans, G. W. (1985). Expectations stability and the multiple equilibria problem in Rational Expectations models. *Quaterly Journal of Economics*, 100:1217–1233.

[Evans and Honkapohja, 1994b] Evans, G. W. and Honkapohja, S. (1994b). Learning, convergence, and stability with multiple rational expectations equilibria. *European Economic Review*, 38:1071–1098.

[Evans and Honkapohja, 1995a] Evans, G. W. and Honkapohja, S. (1995a). Adaptive learning and expectational stability: an introduction. In Kirman, A. and Salmon, M., editors, *Learning and Rationality in Economics*, Oxford. Blackwell.

[Evans and Honkapohja, 1995b] Evans, G. W. and Honkapohja, S. (1995b). Local convergence of recursive learning to steady states and cycles in stochastic nonlinear models. *Econometrica*, 63:195–206.

[Grandmont, 1985] Grandmont, J. (1985). On endogenous competitive business cycles. *Econometrica*, 53:995–1045.

[Grandmont and Laroque, 1991] Grandmont, J. and Laroque, G. (1991). Economic dynamics with learning: Some instability examples. In Barnett, W., Cornet, B., D'Aspremont, C., J., G., and Mas-Colell, A., editors, *Equilibrium Theory and Applications*. Cambridge University Press.

[Grandmont, 1987] Grandmont, J. E. (1987). *Nonlinear Economic Dynamics*. Academic Press.

[Guckenheimer and Holmes, 1983] Guckenheimer, J. and Holmes, P. (1983). *Non-linear oscillations, Dynamical System, and Bifurcations of vector fields*. Springer Verlag.

[Guesnerie and Woodford, 1991] Guesnerie, R. and Woodford, M. (1991). Stability of cycles with adaptive learning rules. In Barnett, W., Cornet, B., D'Aspremont, C., J., G., and Mas-Colell, A., editors, *Equilibrium Theory and Applications*. Cambridge University Press.

[Guesnerie and Woodford, 1992] Guesnerie, R. and Woodford, M. (1992). Endogenous fluctuations. In Laffont, J., editor, *Advances in Economic Theory*. Cambridge University Press.

[Hofbauer and Sigmund, 1988] Hofbauer, J. and Sigmund, K. (1988). *The Theory of Evolution and Dynamical Systems*. Cambridge University Press.

[Ljung, 1977] Ljung, L. (1977). Analysis of recursive stochastic algorithms. *IEEE Transactions on Automatic Control*, AC-22(4):551–575.

[Marcet and Sargent, 1989a] Marcet, A. and Sargent, T. J. (1989a). Convergence of Least Squares learning in environments with hidden state varaibles and private information. *Journal of Political Economy*, 97:1306–1323.

[Marcet and Sargent, 1989b] Marcet, A. and Sargent, T. J. (1989b). Convergence of Least Squares learning mechanisms in self–referential linear stochastic models. *Journal of Economic Theory*, 48:337–368.

[Marimon et al., 1993] Marimon, R., Spear, S., and Sunder, S. (1993). Expectationally driven market volatility: An experimental study. *Journal of Economic Theory*, 61:74–103.

[Marimon and Sunder, 1993] Marimon, R. and Sunder, S. (1993). Indeterminacy of equilibria in a hyperinflationary world: experimental evidence. *Econometrica*, 61(5):1073–1107.

[McCallum, 1989] McCallum, B. T. (1989). *Monetary Economics.* Maxwell MacMillian International Editions, New York.

[Posch, 1994] Posch, M. (1994). Cycling with a generalized urn scheme and a learning algorithm for 2x2 games. Master's thesis, University of Vienna.

[Sargent, 1987] Sargent, T. (1987). *Macroeconomic Theory.* Academic Press.

[Widrow, 1971] Widrow, B. (1971). Adaptive filters. In Kalman, R. E. and Claris, N. D., editors, *Aspects of Network and Systems Theory*, pages 563–587, New York. Holt and Rinehart and Winston.

[Widrow and Stearns, 1985] Widrow, B. and Stearns, S. (1985). *Adaptive Signal Processing.* Prentice-Hall (Eds.), Englewood Cliffs, N. J.