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# Modeling Indeterminacy In Decision Situations

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## Abstract

We present theoretical foundations and computational procedures of a theory for analysing decisions under risk, when the available information is vague and imprecise. The impreciseness is expressed by a set of global distributions  $T$  over a space  $S$ , where the latter represents the classes of all probability and utility measures over a set of discrete outcomes. We show how local distributions, i.e. distributions over projections of  $S$  on various subspaces of  $S$ , can be derived from  $T$  and introduce consistency measures expressing the extent into which user-asserted local distributions can be used for defining  $T$ . The evaluation model used is based on the expected utility, but this is not a necessary restriction. The approach allows a decision maker to be as deliberately imprecise as she feels natural, as well as provides her with the means for expressing varying degrees of imprecision in the input sentences.

**Keywords:** Decision Analysis, Decision Theory, Utility Theory, Uncertain Reasoning

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# Modeling Indeterminacy In Decision Situations

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## 1 Background

The requirement of providing numerically precise data when analysing decision problems has often been considered unrealistic in real-life situations, and a number of models with representations allowing imprecise statements have been suggested. Some of them use standard probability theory while others contain some specialised formalism. Already in [Choquet, 1953/54] the concept of capacities was introduced, and later these ideas were studied in connection with probability theory [Huber, 1973, Huber and Strassen, 1973]. Logical approaches have also been used for providing methods for how to deal with sentences with upper and lower probabilities [Nilsson, 1986]. Belief states have been defined by interval-valued probability functions by means of classes of probability measures, and integrated in classical probability theory [Good, 1962, Smith, 1961]. [Dempster, 1967] investigated the properties of multi-valued mappings and defined upper and lower probabilities in terms of these. The results of Dempster were further developed into a non-Bayesian approach for quantifying subjective judgements [Shafer, 1976].<sup>1</sup> Fuzzy set theory is a widespread approach to relaxing the requirement of numerically precise data and providing a more realistic model of the vagueness in subjective estimates of values and probabilities, see, e.g., [Chen and Hwang, 1994, Lai and Hwang, 1994]. These approaches allow, among other features, the decision maker to model and evaluate a decision situation in vague linguistic terms and introduce various (often complicated) rules for aggregating this information. The measures defined are local and it is often hard to get an intuitive understanding of the global meaning of the various combinations of these. An important difference between fuzzy approaches and the approach presented in this paper is that the latter introduces global belief distributions with weak restrictions. We also show how to derive admissible classes of local distributions from sets of global distributions. This makes it possible to investigate the restrictions that have to be imposed on user-asserted local distributions depending on which information a decision maker has access to and provides. Furthermore, fuzzy approaches are restricted in the sense that they do not handle all qualitative aspects such as, e.g., comparisons between different components involved in many decision situations.

Quite general approaches for how to evaluate imprecise decision situations are investigated in [Gärdenfors and Sahlin, 1982, 1983, Levi, 1974, 1980]. The authors consider

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<sup>1</sup>As has been pointed out, the Dempster-Shafer representation seems to be unnecessarily strong with respect to interval representation [Weichselberger and Pöhlman, 1990].

global distributions of beliefs, but restrict themselves to the probability case and, like the fuzzy models, interval representations. Another limitation is that they neither investigate the relation between global and local distributions nor introduce methods for determining the consistency of user-asserted sentences. The latter may in real-life decision situations be crucial, since the only information at hand in a decision situation may be local because most agents are not able to perceive their global beliefs over, for instance, a 53-dimensional probability base (which is actually a quite tricky task). The same criticism applies to [Hodges and Lehmann, 1952, Hurwicz, 1951, Wald, 1950]. The work by [Danielson and Ekenberg, 1997ab, Ekenberg, et al., 1996, 1997, Malmnäs, 1994] is restricted in the sense that no distributions over the intervals are taken into account.

The main contribution of this work is that it provides theoretical foundations as well as computational procedures of a theory for analysing decisions under risk, when the available information is vague and imprecise. This impreciseness is expressed by a set of global distributions  $T$  over a space  $S$ , where the latter represents the classes of all probability and utility measures over a set of discrete outcomes. We show how local distributions, i.e. distributions over projections of  $S$  on various subspaces of  $S$ , can be derived from  $T$  and introduce consistency measures expressing into which extent explicit local distributions can be used for defining  $T$ . One further important aspect of the theory is that it provides an increased geometrical understanding of which beliefs distributions express and how they are related to each other.

The next two sections describe how impreciseness can be modelled and discuss some general properties of global belief distributions. In particular, it is described how global belief distributions can be defined over a space and in what sense such distributions can define solution sets to a set of constraints, and how classes of admissible local belief distributions can be derived from projections of global distributions. Or conversely, what restrictions to be imposed on a subset  $K$  of local distributions given a set  $L$  of distributions so that  $K \cup L$  defines a global belief distribution. Section 4 treats evaluation with respect to the representation in sections 2 and 3. The expected utility is generalised, and it is proved how the computational complexity can be considerably reduced given that the local distributions fulfill certain consistency requirements. General forms of sensitivity analyses are also discussed.

## 2 Representation

The motivation behind the present work is to extend the expressibility when representing and evaluating vague and numerically imprecise information in decisions situations. To achieve a basic intuition of what will be presented below, consider a decision situation consisting of a set of  $n$  alternatives

$$\{\{c_{ij}\}_{j=1,\dots,m_i}\}_{i=1,\dots,n}$$

where each alternative is represented by a set of  $m_i$  consequences. We will refer to the latter as a consequence set. In such a decision situation, numerically imprecise sentences like "the probability of consequence  $c_{11}$  is greater than 40%" or comparative sentences like "consequence  $c_{11}$  is preferred to consequence  $c_{12}$ " occur. These sentences can be represented in a numerical format [Danielson and Ekenberg, 1997a]. Examples of vague sentences in that model are: "The consequence  $c_{ij}$  is probable" or "The event  $c_{ij}$  or  $c_{ik}$  is possible". Such sentences are represented by suitable intervals. Another kind of sentences are interval sentences of the form: "The probability of  $c_{ij}$  lies between the numbers  $a_k$  and  $b_k$ ", which are translated to  $p_{ij} \in [a_k, b_k]$ . Finally, comparative sentences are of the

form: "The probability of  $c_{ij}$  is greater than the probability of  $c_{kl}$ ". Such a sentence is translated into an inequality  $p_{ij} \geq p_{kl}$ . Each statement is thus represented by one or more constraints. The conjunction of constraints of the types above, together with  $\sum_{j=1}^{m_i} p_{ij} = 1$  for each consequence set  $\{c_{ij}\}_{j=1, \dots, m_i}$  involved, is a probability base ( $\mathcal{P}$ ). A value base ( $\mathcal{V}$ ) consists of similar translations of vague and numerically imprecise value estimates. In a sense, a probability base can be interpreted as constraints defining the set of all possible probability measures. In the terms of [Gärdenfors and Sahlin, 1982], it defines a set of all epistemologically possible probability distributions.

**Example 1:** The solution set to the probability base  $\{p_{11} + p_{12} + p_{13} = 1, p_{11} \geq 0, p_{11} \leq 0.6, p_{12} \geq 0.3, p_{12} \leq 0.5, p_{13} \geq 0.1, p_{13} \leq 0.5\}$  is a polytope. Each vector in the polytope corresponds to a probability distribution over the consequence set  $\{c_{11}, c_{12}, c_{13}\}$ . Thus, the polytope is a subspace of the space of all possible probability distributions over  $\{c_{11}, c_{12}, c_{13}\}$  with respect to the probability base.<sup>2</sup>

However, a decision maker does not necessarily believe with the same intensity in all the epistemologically possible probability distributions  $E$ . To enable a refinement of the model to allow for a differentiation of distributions in this respect, a global distribution expressing various beliefs can be defined over the set  $E$ .

**Example 2:** We can use a function  $g$  representing<sup>3</sup> beliefs of the possible probability distributions with respect to the decision situation in example 1. Note that  $g(x) > 0$  when  $x$  is an epistemologically possible probability distribution, i.e. is a vector in the polytope in Example 1.

Similar belief distributions can be defined over a set of value distributions, expressing epistemologically possible value distributions, where the latter is defined over consequence sets, for instance in terms of polytopes in the sense of [Danielson and Ekenberg, 1997a].

In the following subsections, we define and investigate some features of global distributions and how these are related to sets of linear constraints.

## 2.1 Consequence Sets and Bases

The basic entities in the kinds of decision situations we will consider are the sets of consequences involved. Over these sets, different functions can be defined, expressing for instance, classes of probability measures, belief functions in the Dempster–Shafer sense, fuzzy measures, or utility functions.

**Definition 1** *Let  $\Theta$  be a set of outcomes. A consequence set  $C_i = \{c_{ij}\}_{j=1, \dots, m_i}$ , is a set of events such that  $c_{ij} \cap c_{ik} = \emptyset$ , for all  $j \neq k$ , and  $\cup_{j=1}^{m_i} c_{ij} = \Theta$ . A decision situation is a set,  $\{\{c_{ij}\}_{j=1, \dots, m_i}\}_{i=1, \dots, n}$ , where each  $\{c_{ij}\}_{j=1, \dots, m_i}$  is a consequence set.*

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<sup>2</sup>Note that in general a set of probability distributions does not necessarily have to be convex. For instance, given the consequence set  $\{c_{11}, c_{12}, c_{13}\}$ ,  $(p_{11}, p_{12}, p_{13}) \in ([0, 0.2] \times [0.8, 1] \times [0, 0]) \cup ([0.3, 0.6] \times [0.1, 0.2] \times [0.2, 0.6])$  does define a set of probability distributions in the same way as above. The approaches in, for instance, [Danielson and Ekenberg, 1997, Ekenberg, et al., 1996, 1997, Levi, 1974, Malmnäs, 1994] assume convex sets and are for this reason of restricted use in cases where this is an adequate representation.

<sup>3</sup>To simplify the geometrical intuition, the function  $g$  can be defined over the natural projection of plane  $p_{11} + p_{12} + p_{13} = 1$  on the  $p_{11}$ – $p_{12}$  plane.

**Definition 2** Let a consequence set  $C$  be given. By a  $c$ -function over  $C$ , we mean a function  $f : 2^C \rightarrow [0, 1]$ . By a generating mapping over  $C$ , we mean a set of  $c$ -functions over  $C$ , where each  $c$ -function  $f$  has the property  $f(\emptyset) = 0$ .

**Example 3:** Let the consequence set  $C = \{c_{11}, c_{12}, c_{13}\}$  be given. One  $c$ -function over  $C$  is  $\{(\emptyset, 0.0), (\{c_{11}\}, 0.3), (\{c_{12}\}, 0.2), (\{c_{13}\}, 0.5), (\{c_{11}, c_{12}\}, 0.5), (\{c_{11}, c_{13}\}, 0.8), (\{c_{12}, c_{13}\}, 0.7), (\{c_{12}, c_{13}, c_{13}\}, 1.0)\}$ . The images of all  $c$ -functions over  $C$  constitutes a 7-dimensional unity cube. The set of all  $c$ -functions over  $C$  is an example of a generating mapping over  $C$ .

**Definition 3** Let a consequence set  $C_i = \{c_{ij}\}_{j=1, \dots, m_i}$  be given. By the cell generating mapping over  $C_i$ , we mean the unity cube  $[0, 1]^{m_i}$ . This space will be called a cell for  $C_i$ . Below, such a cell will be denoted by  $B = (b_{i1}, \dots, b_{im_i})$  or, for notational convenience,  $B = (b_1, \dots, b_k)$ . By a cell for  $c_{ij}$ , we mean the interval  $[0, 1]$ . Such a cell will be denoted  $B = (b_{ij})$  or  $B = (b_i)$ .

**Example 4:** Let the consequence set  $C = \{c_{11}, c_{12}, c_{13}\}$  be given. A cell for  $C$  is the space  $[0, 1] \times [0, 1] \times [0, 1]$ .

**Definition 4** Let a decision situation  $D = \{\{c_{ij}\}_{j=1, \dots, m_i}\}_{i=1, \dots, n}$  and a cell  $B_i$  for  $\{c_{ij}\}_{j=1, \dots, m_i}$  be given. By a cell for  $D$ , we mean the space defined by  $B_1 \times \dots \times B_n$ . Such a cell will be denoted  $B = (B_1, \dots, B_n)$ .

**Example 5:** Let a decision situation  $D = \{\{c_{11}, c_{12}\}, \{c_{21}, c_{22}, c_{23}\}\}$  be given. A cell for  $D$  is the space  $[0, 1]^5$ .

## 2.2 Global Belief Distributions

As was mentioned above, an agent does not necessarily believe with the same faith in all possible functions that the vectors in a cell define. For instance, when the agent considers a class of probability distributions, a reasonable requirement seems to be that the belief should be 0 in a vector where the mapping does not add up to one. To enable for a differentiation of functions in this respect, a global distribution expressing various beliefs can be defined over a cell.

**Definition 5** Let a cell  $B = (b_1, \dots, b_k)$  be given. By a global belief distribution over  $B$ , we mean a positive distribution<sup>4</sup>  $g$  defined on the cell  $B$  such that

$$\int_B g(x) dV_B(x) = 1,$$

where  $V_B$  is the  $k$ -dimensional Lebesgue measure on  $B$ . The set of all global belief distributions over  $B$  is denoted by  $\text{GBD}(B)$ .

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<sup>4</sup>A distribution on a set  $\Omega$  is a linear functional defined on  $C_0^\infty(\Omega)$  which is continuous with respect to a certain topology.



**Example 6:** Consider a uniform global belief distribution over a cell  $B = (b_1, b_2)$ . One interpretation of this is that we have no information about the consequence sets.

**Example 7:** The functions

$$f(x_1) = \max(0, \min(-100x_1 + 20, 100x_1))$$

and

$$h(x_2) = \max(0, \min(-\frac{100}{3}x_2 + \frac{80}{3}, \frac{200}{3}x_2 - \frac{100}{3}))$$

have graphs consisting of triangles with bases on the axes and area = 1. Then we get a global belief distribution  $g(x_1, x_2) = f(x_1) \cdot h(x_2)$  over a cell  $B = (b_1, b_2)$ .

**Example 8:** Let the functions  $f$  and  $h$  be as in the former example. Then we get a global belief distribution  $g(x_1, x_2)$  over the cell  $B = (b_1, b_2)$  by defining  $g(x_1, x_2) = 0$ , when  $x_1 > x_2$ , and  $g(x_1, x_2) = 2 \cdot f(x_1) \cdot h(x_2)$  otherwise.

In the examples above we saw how to use global belief distributions to represent subsets of a cell. If we want to represent a subset which is of lower dimension than the cell itself we cannot use distributions that are upper bounded since a mass under such a distribution will be 0 while integrating with respect to the Lebesgue measure defined on the cell. Instead, we have to use, for example, the Dirac function  $\delta_p(x)$  which has the property<sup>5</sup>

$$\int_B \delta_p(x) f(x) dx = f(p),$$

and especially, if  $f(x) \equiv 1$  then

$$\int_B \delta_p(x) dx = 1$$

so  $\delta_p(x)$  is a global belief distribution according to the definition above. Here  $f(x)$  is a measurable function. The distribution  $\delta_p(x)$  is called *the Dirac distribution, or Delta function, with pole at the point p*.

Thus, we can use Dirac functions to represent pointwise global belief distributions.

**Example 9:** Let a cell  $B = (b_1)$  be given. Assume that a decision maker has a positive relative belief only in the three points  $p_1 = 0.3$ ,  $p_2 = 0.6$ ,  $p_3 = 0.7$ . Further assume that she has a relative belief in these points as 0.1, 0.4, 0.5. To represent this, let  $f$  be defined by

$$f(x) = \begin{cases} 0.1 & \text{if } x = 0.3 \\ 0.4 & \text{if } x = 0.6 \\ 0.5 & \text{if } x = 0.7 \\ 0 & \text{otherwise} \end{cases}$$

The global belief function  $g$  can be defined by

$$g(x) = f(x)(\delta_{p_1}(x) + \delta_{p_2}(x) + \delta_{p_3}(x))$$

Now,

$$\int_B g(x) dx = 1$$

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<sup>5</sup>For a detailed treatment, cf. for example [Friedlander, 1982].

and, consequently,  $g$  is a global belief distribution.<sup>6</sup>

In general, for every measurable subset  $E$  in a  $k$ -dimensional space  $B$  there is a distribution  $g_E$  with mass equal to 1 on  $E$  and 0 otherwise (cf. [Friedlander, 1982]).

**Definition 6** *Let  $A$  be a subset of a cell  $B$ , and let  $f \in \text{GBD}(A)$ . The natural extension  $\tilde{f}_A(x)$  of  $f$  with respect to  $A$  is defined by*

$$\tilde{f}_A(x) = \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

**Definition 7** *Let  $A$  be a subset of  $B$ . A distribution  $g_A \in \text{GBD}(B)$ , is called a characteristic distribution for  $A$  in  $B$ , if*

$$f(p) = \int_B \delta_p(x) \tilde{f}_A(x) g_A(x) dV_B(x),$$

for every  $f \in \text{GBD}(A)$ , and for every point  $p$  in  $A$ .

From distribution theory follows that for every measurable subset  $A$  in a cell  $B$ , there exists a characteristic distribution for  $A$  in  $B$ . It also trivially follows that  $\tilde{f}_A(x) \cdot g_A(x) \in \text{GBD}(B)$  and equals 0 outside  $A$ .

**Example 10:** Let a cell  $B = (b_1, b_2, b_3)$  be given. Let  $A$  denote the subset of  $B$ , where  $x_1 + x_2 + x_3 = 1$ , and let  $f(x_1, x_2) = x_1 \cdot x_2$  be defined on  $A$ . Then  $f \in \text{GBD}(A)$  with respect to the 2-dimensional Lebesgue measure on  $A$ , but  $\tilde{f}_A(x) \notin \text{GBD}(B)$ , because  $\int_B \tilde{f}_A(x) dV_B(x) = 0$ . However,  $\int_B \tilde{f}_A(x) g_A(x) dV_B(x) = 1$ , so  $\tilde{f}_A(x) g_A(x) \in \text{GBD}(B)$ , and  $\tilde{f}_A(x_1, x_2, x_3) g_A(x_1, x_2, x_3) = 0$ , except when  $x_1 + x_2 + x_3 = 1$ . To put this informally;  $\tilde{f}_A \cdot g_A$  represents the same proportional belief over  $B$  as  $f$  does over  $A$ .

### 2.3 Constraints

It should be noted that one property of a global belief distribution is that it in some sense defines the solution set to a set of constraints.

**Definition 8** *Let a cell  $B = (b_1, \dots, b_k)$  be given. We will use the term constraints for the union of the following:<sup>7</sup>*

- *I-constraints are constraints on the form  $a \geq x_i$  or  $a \leq x_i$ , where  $a$  is a real number in  $[0, 1]$ , and  $x_i$  is a variable.*
- *L-constraints are constraints on the form  $\sum x_i = a$ , where  $a$  is a real number in  $[0, 1]$ .*

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<sup>6</sup>As can be seen from the example, it is sufficient that a decision maker expresses her relative belief by the function  $f$  when the global belief distribution represents a subset with dimension zero. A similar result is true for any subset of a lower dimension than the cell. Cf. Example 10 below.

<sup>7</sup>Naturally, the set of constraints can be more generally defined, but these three are sufficient for the purposes of this work.

- $C$ -constraints are constraints on the form  $x_i \leq x_j + a$ , where  $a$  is a real number in  $[0, 1]$ .

**Example 11:** An example of a solution set to a set of constraints is the polytope  $\{(x_1, x_2, x_3) : 1 \geq x_1 > x_2 \geq 0, x_1 + x_2 + x_3 = 1, 0.7 \geq x_3 \geq 0.1\}$ .

**Definition 9** Let a cell  $B = (b_1, \dots, b_k)$  and a distribution  $g$  over  $B$  be given. The support of  $g$  ( $\text{supp } g$ ) is the closure of the set  $\{(x_1, \dots, x_k) : g(x_1, \dots, x_k) > 0\}$ .

**Example 12:** Let a cell  $B = (b_1, b_2)$  be given. Define a global belief distribution  $g$ , such that  $g(x_1, x_2) = 0$ , when  $x_1 > x_2$ , and  $g(x_1, x_2) = 2 \cdot f(x_1) \cdot h(x_2)$  otherwise.  $\text{supp } g$  is the 2-dimensional polytope  $\{(x_1, x_2) : 1 \geq x_1 > x_2 \geq 0\}$ .

**Definition 10** Let a cell  $B = (b_1, \dots, b_k)$  and a set  $C$  of constraints in  $x_1, \dots, x_k$  be given. The set of solution vectors to  $C$  constitutes the solution set for  $C$ , and will be denoted by  $s(C)$ . If there is a non-empty solution set for  $C$ , it is consistent. Otherwise  $C$  is inconsistent.  $\text{Sup}(\{n_i : (n_1, \dots, n_i, \dots, n_k) \in s(C)\})$  will be denoted  $\text{Sup}_i(C)$ .  $\text{Inf}_i(C)$  is defined analogously.

As explained in the introduction to section 2, linear constraints can be used to model vague and numerically imprecise probability- and value statements. The introduction of global belief distributions over cells generalises the concept of probability- and value bases.

**Example 13:** A value base  $V$  can be defined through a global belief distribution. Given a cell  $V = (v_1, v_2)$  and a distribution  $g_V$  over  $V$  defined by  $g_V(v_1, v_2) = 6 \cdot \max(v_1 - v_2, 0)$ . Then  $g_V \in \text{GBD}(V)$ , and  $\text{supp } g_V = \{(v_1, v_2) : 0 \leq v_i \leq 1 \ \& \ v_1 < v_2\}$ .  $\text{Sup}_1(V) = \text{Sup}_2(V) = 1$ .  $\text{Inf}_1(V) = \text{Inf}_2(V) = 0$ .

### 3 Local Belief Distributions

This section investigates relationships between global and local distributions and introduces measures for determining the consistency of user-asserted sentences. Such relationships are important, since the only information at hand in a decision situation may be local. We show how classes of admissible local belief distributions can be derived from projections of global distributions, and what restrictions to be imposed on a subset  $K$  of local distributions given a set  $L$  of distributions so that  $K \cup L$  defines a global belief distribution.

**Definition 11** Let a cell  $B = (b_1, \dots, b_k)$  be given. By a local belief distribution over  $B$ , we mean a positive distribution  $f$  defined on the cell  $b_i$  such that

$$\int_{b_i} f(x_i) dV_{b_i}(x_i) = 1,$$

where  $V_{b_i}$  is the Lebesgue measure on  $b_i$ . The set of all local belief distributions over  $b_i$  is denoted by  $\text{LBD}(b_i)$ .

### 3.1 Centroids

Local belief distributions over the axes of a cell  $B$  can be derived from a global belief distribution over  $B$ .

**Definition 12** *Let a cell  $B = (b_1, \dots, b_k)$  and  $F \in \text{GBD}(B)$  be given. Let*

$$f_i(x_i) = \int_{B_i^-} F(x) dV_{B_i^-}(x)$$

where  $B_i^- = (b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_k)$ . We say that  $f_i(x_i)$  is derived from  $F$ .

**Example 14:** Let a cell  $B = (b_1, b_2)$  be given, and let  $F(x_1, x_2) = x_1 + x_2$ . The derived local belief distributions from  $F$  are

$$\begin{aligned} f_1(x_1) &= \int_0^1 (x_1 + x_2) dx_2 = \left[ x_1 x_2 + \frac{1}{2} x_2^2 \right]_0^1 = x_1 + \frac{1}{2} \\ f_2(x_2) &= \int_0^1 (x_1 + x_2) dx_1 = x_2 + \frac{1}{2}. \end{aligned}$$

In section 4, we will show how the use of centroids logarithmically reduces the computational complexity in the evaluations of a generalised expected utility. Intuitively, the centroid of a distribution is a point in space where some of the geometrical properties of the distribution can be regarded as concentrated.

**Definition 13** *Let a cell  $B = (b_1, \dots, b_k)$  and  $g_B \in \text{GBD}(B)$  be given. The centroid of  $g_B$  is the point  $x_{g_B} = (\beta_1, \dots, \beta_k)$  in  $B$  whose  $i$ :th component is*

$$\beta_i = \int_B x_i \cdot g_B(x) dV_B(x).$$

**Definition 14** *Let a cell  $B = (b_1, \dots, b_k)$  and  $f_{b_i} \in \text{LBD}(b_i)$  be given. The centroid of  $f_{b_i}$  is the point in  $b_i$  defined by*

$$x_{f_{b_i}} = \int_{b_i} x_i \cdot f_{b_i}(x_i) dV_{b_i}(x_i).$$

Centroids are invariant under projections on the local cells in the sense that the projection of the centroid on the local cell has the same coordinates as the centroid of the corresponding derived local belief distribution. Lemma 1 shows that a product of local belief distributions has the same centroid as the distribution of a product. Moreover, given a global belief function  $G$ , Lemma 2 shows that the only possible product is the product of the derived distributions from  $G$ .

**Lemma 1** *Let a cell  $B = (b_1, \dots, b_k)$  and  $F \in \text{GBD}(B)$  be given. Let  $f_i(x_i)$  be derived from  $F$ . Furthermore, let*

$$G(x_1, \dots, x_k) = f_1(x_1) \cdot \dots \cdot f_k(x_k).$$

*Then*

(i)  $f_i(x_i) \in \text{LBD}(b_i)$ ,  $i = 1, \dots, k$ .

(ii)  $G \in \text{GBD}(B)$

(iii)  $x_G = x_F$

(iv) If  $x_G = (\alpha_1, \dots, \alpha_k)$  then  $\alpha_i = x_{f_{b_i}}$ .

**Proof:**

(i): All we need to show is that  $\int_{b_i} f_i(x_i) dx_i = 1$ . According to the definition of  $f_i$ , we get

$$\int_{b_i} f_i(x_i) dx_i = \int_{b_i} \int_{B_i^-} F(x) dV_{B_i^-}(x) dx_i = \int_B F(x) dV_B(x) = 1.$$

(ii):  $\int_B G(x) dV_B(x) = \int_B f_1(x_1) \dots f_k(x_k) dx_1 \dots dx_k = \left( \int_{b_1} f_1(x_1) dx_1 \right) \dots \left( \int_{b_k} f_k(x_k) dx_k \right) = 1 \dots 1$ .

(iii) and (iv): Let  $x_G = (\alpha_1, \dots, \alpha_k)$ , and  $x_F = (\beta_1, \dots, \beta_k)$ . We show that  $\alpha_1 = x_{f_{b_1}} = \beta_1$ . That  $\alpha_i = x_{f_{b_i}} = \beta_i$  for  $i = 2, \dots, k$ , follows by analogous calculations.

$\alpha_1 = x_{f_{b_1}}$ , since

$$\begin{aligned} \alpha_1 &= \int_B x_1 \cdot G(x) dV_B(x) \\ &= \int_{b_1} \left( x_1 \cdot f_1(x_1) \left( \int_{b_2} f_2(x_2) dx_2 \right) \dots \left( \int_{b_k} f_k(x_k) dx_k \right) \right) dx_1 \\ &= \int_{b_1} (x_1 \cdot f_1(x_1) \cdot 1 \dots 1) dx_1 = x_{f_{b_1}} \end{aligned}$$

and  $x_{f_{b_1}} = \beta_1$  since

$$\int_{b_1} x_1 \cdot f_1(x_1) dx_1 = \int_{b_1} x_1 \int_{B_1^-} F(x) dV_{B_1^-}(x) dx_1 = \int_B x_1 \cdot F(x) dV(x) = \beta_1,$$

because  $f_i(x_i) = \int_{B_i^-} F(x) dV_{B_i^-}(x)$ .

**Example 15:** Let a cell  $B = (b_1, b_2)$  be given, and let  $F(x_1, x_2) = x_1 + x_2$ . Then  $F \in \text{GBD}(B)$  since

$$\begin{aligned} \int_0^1 \int_0^1 (x_1 + x_2) dx_1 dx_2 &= \int_0^1 \left[ \frac{1}{2} x_1^2 + x_1 x_2 \right]_0^1 dx_2 = \int_0^1 \frac{1}{2} + x_2 dx_2 \\ &= \left[ \frac{1}{2} x_2 + \frac{1}{2} x_2^2 \right]_0^1 = \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

The corresponding derived local belief distributions are as in the example above.

$$\begin{aligned} f_1(x_1) &= \int_0^1 (x_1 + x_2) dx_2 = x_1 + \frac{1}{2} \\ f_2(x_2) &= \int_0^1 (x_1 + x_2) dx_1 = x_2 + \frac{1}{2}. \end{aligned}$$

We can now check that the function

$$G(x_1, x_2) = f_1(x_1) \cdot f_2(x_2) = \left(x_1 + \frac{1}{2}\right) \cdot \left(x_2 + \frac{1}{2}\right)$$

is a global belief distribution on  $G$ :

$$\begin{aligned} \int_0^1 \int_0^1 \left(x_1 + \frac{1}{2}\right) \cdot \left(x_2 + \frac{1}{2}\right) dx_1 dx_2 &= \int_0^1 \left(x_1 + \frac{1}{2}\right) dx_1 \cdot \int_0^1 \left(x_2 + \frac{1}{2}\right) dx_2 \\ &= \left[\frac{1}{2}x_1^2 + \frac{1}{2}x_1\right]_0^1 \cdot \left[\frac{1}{2}x_2^2 + \frac{1}{2}x_2\right]_0^1 \\ &= \left(\frac{1}{2} + \frac{1}{2}\right) \cdot \left(\frac{1}{2} + \frac{1}{2}\right) = 1. \end{aligned}$$

Denote the centroid  $x_G = (\alpha_1, \alpha_2)$ . Then

$$\begin{aligned} \alpha_1 &= \int_0^1 \int_0^1 x_1 \left(x_1 + \frac{1}{2}\right) \cdot \left(x_2 + \frac{1}{2}\right) dx_1 dx_2 \\ &= \int_0^1 x_1 \left(x_1 + \frac{1}{2}\right) dx_1 \cdot \int_0^1 \left(x_2 + \frac{1}{2}\right) dx_2 \\ &= \left[\frac{1}{3}x_1^3 + \frac{1}{4}x_1^2\right]_0^1 \cdot \left[\frac{1}{2}x_2^2 + \frac{1}{2}x_2\right]_0^1 \\ &= \left(\frac{1}{3} + \frac{1}{4}\right) \cdot 1 = \frac{7}{12}. \end{aligned}$$

In the same way we get  $\alpha_2 = \frac{7}{12}$ . We can now verify that  $x_G = x_F$ . If  $x_F = (\beta_1, \beta_2)$  then

$$\begin{aligned} \beta_1 &= \int_0^1 \int_0^1 x_1 \cdot (x_1 + x_2) dx_1 dx_2 = \int_0^1 \left[\frac{1}{3}x_1^3 + \frac{1}{2}x_1^2 x_2\right]_0^1 dx_2 = \int_0^1 \frac{1}{3} + \frac{1}{2}x_2 dx_2 \\ &= \left[\frac{1}{3}x_2 + \frac{1}{4}x_2^2\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}. \end{aligned}$$

By the same calculations  $\beta_2 = \frac{7}{12}$ . We can also check that the coordinates  $(7/12, 7/12)$  are the same as the corresponding values of  $x_{f_1}$  and  $x_{f_2}$  respectively:

$$x_{f_1} = \int_0^1 x_1 \cdot f_1(x_1) dx_1 = \int_0^1 x_1 \cdot (x_1 + 1/2) dx_1 = \left[\frac{1}{3}x_1^3 + \frac{1}{4}x_1^2\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12},$$

and similarly  $x_{f_2} = \frac{7}{12}$ .

**Lemma 2** *Let a cell  $B = (b_1, b_2, \dots, b_k)$  be given, and consider an arbitrary set of local belief distributions  $f_i, g_i \in \text{LBD}(b_i)$ . Local belief distributions generating a particular global belief distribution are then unique in the sense that*

$$f_1(x_1) \cdot \dots \cdot f_k(x_k) = g_1(x_1) \cdot \dots \cdot g_k(x_k) \text{ iff } f_i = g_i \text{ for all } i = 1, \dots, k.$$

**Proof:** Assume that  $f_i, g_i \in \text{LBD}(b_i)$  so that

$$\int_0^1 f_i(x_i) dx_i = \int_0^1 g_i(x_i) dx_i = 1$$

and assume that  $f_1(x_1) \cdot \dots \cdot f_k(x_k) = g_1(x_1) \cdot \dots \cdot g_k(x_k)$ . Then

$$\begin{aligned} &\int_{B_i^-} f_1(x_1) \cdot \dots \cdot f_{i-1}(x_{i-1}) \cdot f_i(x_i) \cdot f_{i+1}(x_{i+1}) \cdot \dots \cdot f_k(x_k) dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_k \\ &= \int_{B_i^-} g_1(x_1) \cdot \dots \cdot g_{i-1}(x_{i-1}) \cdot g_i(x_i) \cdot g_{i+1}(x_{i+1}) \cdot \dots \cdot g_k(x_k) dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_k \end{aligned}$$

so that

$$\begin{aligned} f_i(x_i) \int_{B_i^-} f_1(x_1) \cdot \dots \cdot f_{i-1}(x_{i-1}) \cdot f_{i+1}(x_{i+1}) \cdot \dots \cdot f_k(x_k) dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_k \\ = g_i(x_i) \int_{B_i^-} g_1(x_1) \cdot \dots \cdot g_{i-1}(x_{i-1}) \cdot g_{i+1}(x_{i+1}) \cdot \dots \cdot g_k(x_k) dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_k \end{aligned}$$

which gives  $f_i(x_i) = g_i(x_i)$  since  $\int_0^1 f_j(x_j) dx_j = \int_0^1 g_j(x_j) dx_j = 1$ ,  $j = 1, \dots, k$ .

From the proof of Lemma 1 (ii) follows that a product of local distributions is a global distribution. Lemma 2 implies that this product is unique. Moreover, the following lemma strengthens this result by saying that if a global belief distribution  $G$  is a product of local belief distributions, then the factors are the derived local distributions from  $G$ .

**Lemma 3** *Let a cell  $B = (b_1, \dots, b_k)$  be given. Let  $G(x_1, \dots, x_k) = g_1(x_1) \cdot \dots \cdot g_k(x_k)$ , where  $g_i \in \text{LBD}(b_i)$ . Then  $g_i(b_i)$  is derived from  $G$ .*

**Proof:**

$$\begin{aligned} \int_{B_i^-} g_1(x_1) \cdot \dots \cdot g_k(x_k) dV_{B_i^-}(x) &= \int_B g_1(x_1) \cdot \dots \cdot g_k(x_k) dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_k \\ &= g_i(x_i) \cdot \left( \int_{b_1} g_1(x_1) dx_1 \right) \cdot \\ &\quad \dots \cdot \left( \int_{b_{i-1}} g_{i-1}(x_{i-1}) dx_{i-1} \right) \cdot \left( \int_{b_{i+1}} g_{i+1}(x_{i+1}) dx_{i+1} \right) \cdot \dots \cdot \left( \int_{b_k} g_k(x_k) dx_k \right) \\ &= g_i(x_i) \cdot \dots \cdot 1 \end{aligned}$$

### 3.2 Relations Between Constraints and Belief Distributions

Of particular interest is to what extent local belief distributions can combine to a global belief distribution, so that the global distribution in some sense represents the local belief distributions as well as a set of constraints imposed on the decision situation.

**Definition 15** *Let a cell  $B = (b_1, \dots, b_k)$  and a consistent set  $C$  of constraints in  $B$  be given. The global belief distribution  $F$  is called  $C$ -admissible iff*

$$x \text{ is a solution vector to } C \text{ iff } x \in \text{supp } F.$$

Furthermore, a set of local belief distributions will be called  $I_C$ -admissible if the support of the distributions are congruent with the solution set to a set  $C$  of  $I$ -constraints.

**Definition 16** *Let a cell  $B = (b_1, \dots, b_k)$  and a consistent set  $C$  of  $I$ -constraints in  $B$  be given. A set  $L = \{f_i(x_i) \in \text{LBD}(b_i)\}_{i=1, \dots, k}$  of local belief distributions is called  $I_C$ -admissible iff*

$$(x_1, \dots, x_k) \text{ is a solution vector to } C \text{ iff } x_i \in \text{supp } f_i \text{ for all } i = 1, \dots, k.$$

Usually a decision maker has access only to local information and a set of relations between different parameters and, consequently, has no explicit idea about the global distribution. In many cases, it may be that the only accessible relations between the local distributions are in terms of constraints. When only  $I$ -constraints are considered, this set of distributions can be used to define a global belief which support is positive only on the polytope defined by the constraints.

**Theorem 1** *Let a cell  $B = (b_1, \dots, b_k)$  be given. Let  $C$  be a consistent set of  $I$ -constraints such that  $s(C) \subseteq B$ , and let a set  $\{g_i(x_i)\}_{i=1, \dots, k}$  such that  $g_i \in \text{LBD}(b_i)$  be given. Then there exists  $F \in \text{GBD}(B)$ , such that  $F$  is  $C$ -admissible iff  $g_i(x_i)$  is  $I_C$ -admissible for every  $i = 1, \dots, k$ .*

**Proof:** Define  $F = g_1 \cdot \dots \cdot g_k$ . Then from Lemma 3 follows that  $f_i = g_i$ , where  $f_i$  is derived from  $F$ . From Lemma 1 follows that  $F \in \text{GBD}(B)$ . Furthermore,  $F$  is  $I_C$ -admissible iff

$$x \in s(C) \iff x \in \text{supp } F.$$

However,  $F = 0$  iff  $g_i = 0$  for some  $i$ , i.e.  $(x_1, \dots, x_k) \notin \text{supp } F$  iff  $x_i \notin \text{supp } g_i$  for some  $x_i \in (x_1, \dots, x_k)$ . This is equivalent to that the set  $\{g_i(b_i)\}_{i=1, \dots, k}$  is  $I_C$ -admissible.

### 3.3 Congruency

If the decision maker is able to define a set of local belief distributions and a set of  $L$ - and  $C$ -constraints describing the decision problem, these must be congruent in a certain respect. Given a set of constraints, a decision maker is restricted concerning which combinations of local belief distributions that are possible to impose, if she wants to be consistent in a reasonable sense.<sup>8</sup> This is expressed by the following definition.

**Definition 17** *Let a cell  $B = (b_1, \dots, b_k)$  and a consistent set  $C$  of constraints in  $B$  be given. A set  $L = \{f_i(x_i) \in \text{LBD}(b_i)\}_{i=1, \dots, k}$  is called  $C$ -admissible iff the vector  $(x_{f_1}, \dots, x_{f_k})$  is in the solution set for  $C$ , where  $x_{f_i}$  denotes the centroid of  $f_i$ .*

**Theorem 2** *Let a cell  $B = (b_1, \dots, b_k)$  and a consistent set  $C$  of constraints, such that  $s(C) \subseteq B$  be given. Let  $G$  be a  $C$ -admissible global distribution and let  $g_i(b_i), i = 1, \dots, k$ , be derived from  $G$ . Then  $\{g_i(x_i)\}_{i=1, \dots, k}$  is a  $C$ -admissible set of local belief distributions. Furthermore, if  $F = g_1 \cdot \dots \cdot g_k$ , then  $x_F = x_G$ .*

**Proof:** The second part of the theorem follows immediately from Lemma 1, i.e.  $x_F = (x_{f_1}, \dots, x_{f_k}) = (x_{g_1}, \dots, x_{g_k}) = x_G$ . The first part of the theorem follows from Lemma 1 and the observation, from standard convexity theory, that the solution set to a set of linear constraints is convex.

Theorem 1 and 2 imply that if a decision maker defines a set of local belief distributions describing a problem, and if these are admissible w.r.t. the constraints involved, a global belief distribution can be determined. This distribution has the property of having the

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<sup>8</sup>As will be seen in section 4, for the purpose of evaluating a generalised expected value, a sufficient condition on the distributions is that the vectors of centroids of the user-asserted local belief distributions are consistent with the constraints.



same centroid (and the same support relative to the local belief distributions) as any global belief distribution from which the user-asserted local belief distributions can be derived.

When there are linear dependencies involved (as an effect of a set of  $L$ -constraints), then the sets of all local belief distributions are, in a sense, determined by a subset of such a set. We will say that this subset is complete.

**Definition 18** *Let a cell  $B = (b_1, \dots, b_k)$  be given. Let  $C$  be a set of constraints in  $B$ , and let  $L = \{f_i(x_i)\}_{i=1, \dots, k}$  where  $f_i(x_i) \in \text{LBD}(b_i)$ . A subset  $\{f_i(x_i)\}_{i=t_1, \dots, t_s}$  of  $L$  is  $B$ -complete if the dimension of  $(b_{t_1}, \dots, b_{t_s})$  is equal to the dimension of the solution set to  $C$ . A  $B$ -complete subset  $\{f_i(x_i)\}_{i=t_1, \dots, t_s} \subseteq L$  is  $C_{COMP}^B$ -admissible if the vector  $(\alpha_1, \dots, \alpha_s) \in s(C)$ , where  $\alpha_i = x_{f_{t_i}}$  is the centroid of  $f_{t_i}$ .*

Theorem 2 implies the following corollary, which says that if  $\{f_i(x_i)\}_{i=t_1, \dots, t_s}$  is a  $C_{COMP}^B$ -admissible set, then the admissible local distributions over  $(b_1, \dots, b_k)$  outside  $(b_{t_1}, \dots, b_{t_s})$  are determined w.r.t. their centroids.

**Corollary 1** *Let a cell  $B = (b_1, \dots, b_k)$  be given. Let a set  $C$  of constraints in  $B$ , such that  $s(C) \subseteq B$ , and let a set  $L = \{g_i(x_i)\}_{i=1, \dots, k}$ , such that  $g_i \in \text{LBD}(b_i)$ , be given. Furthermore, let  $F(x) = g_1(x_1) \cdot \dots \cdot g_k(x_k)$ . Then  $x_F \in s(C)$  iff all  $B$ -complete subsets of  $L$  are  $C_{COMP}^B$ -admissible.*

When a decision maker asserts local belief distributions, they do not always fulfill the admissibility described above. One option for the decision maker in such cases is to select a  $C_{COMP}^B$ -admissible subset of local distributions and then calculate the remaining ones.

However, the decision maker may want to know how the set of originally asserted distributions deviate from the calculated ones. A measure of this deviation could be the difference with respect to the centroids.

**Definition 19** *Let a cell  $B = (b_1, \dots, b_k)$  be given. Let  $F$  be an admissible global belief distribution over  $B$ . Let  $A = \{f_i(x_i)\}_{i=1, \dots, k}$  be the set of derived local belief distributions from  $F$ , and let  $G = \{g_i(x_i)\}_{i=1, \dots, k}$  be a set of (user-asserted) local distributions. The inconsistency of  $g_i(x_i)$  relative to  $f_i(x_i)$  is  $|x_{g_i} - x_{f_i}|$ . The inconsistency of  $G$  relative to  $A$  is  $\sum_{i=1}^k |x_{g_i} - x_{f_i}|$ .*

This measure is in some sense a mean. To investigate in what extent the distributions are separate, a measure corresponding to the variance can be introduced.

**Definition 20** *Let a cell  $B = (b_1, \dots, b_k)$  be given, and let  $F$  be an admissible global belief distribution over  $B$ . Let  $A = \{f_i(x_i)\}_{i=1, \dots, k}$  be the set of derived local belief distributions from  $F$ , and let  $G = \{g_i(x_i)\}_{i=1, \dots, k}$  be a set of local distributions. The variance of  $g_i(x_i)$  relative to  $f_i(x_i)$  is  $\int_{b_i} |g_i(x_i) - f_i(x_i)| dx_i$ . The variance of  $G$  relative to  $A$  is  $\sum_{i=1}^k \int_{b_i} |g_i(x_i) - f_i(x_i)| dx_i$ .*

## 4 Evaluation

The evaluation principle treated in this section is based on the principle of maximising the expected value. Given a decision situation  $D$ , let  $\mathcal{P}$  and  $\mathcal{V}$  be a probability base and a value base for  $D$ , respectively. The expected value  $E(C_i)$  denotes the expression  $\sum_{j=1}^{m_i} p_{ij} v_{ij}$ ,

where  $p_{ij}$  and  $v_{ij}$  are variables in  $\mathcal{P}$  and  $\mathcal{V}$ . To evaluate the expected value [Danielson and Ekenberg, 1997a, Ekenberg and Danielson, 1994, Ekenberg, et al., 1996, Malmnäs, 1994] investigates the set  $\{E(C_i)\}_{i=1,\dots,n} \cup \mathcal{P} \cup \mathcal{V}$  in a variety of respects. However, in the present framework, distributions are included, and we will suggest how these can be taken into account in the evaluations.

#### 4.1 Generalised Mean Values

We will now describe how belief functions can be imposed on evaluations of a generalised expected mean value.

First, we define a decision scenario as containing a number of global belief distributions. Informally, these express various beliefs in vectors in subsets of the solution sets to probability-, and utility bases. For a given consequence set, there is one global belief distribution for the probabilities, and one global belief distribution for the utilities, with respect to this set.

**Definition 21** *A decision scenario is a structure  $(D, P, V, \{\mathbf{p}_i\}_{i=1,\dots,n}, \{\mathbf{v}_i\}_{i=1,\dots,n})$ , where*

- *$D$  is a decision situation  $\{\{c_{ij}\}_{j=1,\dots,m_i}\}_{i=1,\dots,n}$ .*
- *$P = (p_{11}, p_{12}, \dots, p_{mn})$  is a cell.*
- *$V = (v_{11}, v_{12}, \dots, v_{mn})$  is a cell.*
- *$\mathbf{p}_i$  is a global belief distribution over the cell  $P_i = (p_{i1}, \dots, p_{im_i})$  such that  $\mathbf{p}_i(x) = 0$ , when  $\sum_{j=1}^{m_i} p_{ij} \neq 1$ .*
- *$\mathbf{v}_i$  is a global belief distribution over the cell  $V_i = (v_{i1}, \dots, v_{im_i})$ .*

The next definition suggests a generalised expected value. This is summed over all possible expected values weighted by the global belief distributions over the solution sets to the probability-, and utility bases.

**Definition 22** *Let a decision scenario  $(\{C_i\}_{i=1,\dots,n}, P, V, \{\mathbf{p}_i\}_{i=1,\dots,n}, \{\mathbf{v}_i\}_{i=1,\dots,n})$  be given. The expression*

$$\int_{P_i \times V_i} \left( \sum_{j=1}^{m_i} x_{ij} y_{ij} \right) \mathbf{p}_i(x_{i1}, \dots, x_{im_i}) \cdot \mathbf{v}_i(y_{i1}, \dots, y_{im_i}) dV(x_{i1}, \dots, x_{im_i}) dV(y_{i1}, \dots, y_{im_i})$$

*is called the generalised expected value for  $C_i$ , and is denoted by  $G(C_i)$ .*

The next theorem shows how the generalised expected value of a consequence set can be calculated by using only the centroids of the global belief distributions.

**Theorem 3** *Let a decision scenario  $(D, P, V, \{\mathbf{p}_i\}_{i=1,\dots,n}, \{\mathbf{v}_i\}_{i=1,\dots,n})$  be given. Then*

$$G(C_i) = \langle x_{\mathbf{p}_i}, x_{\mathbf{v}_i} \rangle,$$

*where  $\langle x, y \rangle$  is the standard inner product of  $x$  and  $y$ .*

**Proof:** Let  $x_{\mathbf{p}_i} = (\alpha_{i1}, \dots, \alpha_{im_i})$  in  $P_i$  and let  $x_{\mathbf{v}_i} = (\beta_{i1}, \dots, \beta_{im_i})$  in  $V_i$ . Since  $\mathbf{p}_i \in \text{GBD}(P_i)$ ,  $\mathbf{v}_i \in \text{GBD}(V_i)$ , then by definition 13

$$\int_{P_i} x_{ij} \cdot \mathbf{p}_i dV(x_{i1}) \cdots dV(x_{im_i}) = \alpha_{ij}.$$

Analogously we get

$$\int_{V_i} y_{ij} \cdot \mathbf{v}_i dV(y_{i1}) \cdots dV(y_{im_i}) = \beta_{ij}.$$

Thus, by the independence of  $P_i$  and  $V_i$  we get

$$\int_{P_i \times V_i} x_{ij} \cdot y_{ij} \cdot \mathbf{p}_i(x_{i1}, \dots, x_{im_i}) \cdot \mathbf{v}_i(y_{i1}, \dots, y_{im_i}) dV(x_{i1}, \dots, x_{im_i}) dV(y_{i1}, \dots, y_{im_i}) = \alpha_{ij} \cdot \beta_{ij}.$$

Thus

$$G(C_i) = \alpha_{i1} \cdot \beta_{i1} + \dots + \alpha_{im_i} \cdot \beta_{im_i} = \langle x_{\mathbf{p}_i}, x_{\mathbf{v}_i} \rangle.$$

However, as mentioned above, usually a decision maker does only have access to local information and a set of relations between different parameters, and has no explicit idea about the global distributions. If the decision maker is able to define a set of local belief distributions describing the decision problem, and if these are congruent with the constraints involved, global belief distributions over  $s(\mathcal{P})$  and  $s(\mathcal{V})$  can be determined. These can be included in the formula expressing the generalised value of  $C_i$ , and Theorem 4 shows that this is equal to the generalised expected value of  $C_i$  involving any global belief distribution, which have a positive support on the solutions sets  $s(\mathcal{P})$  and  $s(\mathcal{V})$  only, and from which the local belief distributions is derived.

**Definition 23** *A potential decision scenario is a structure*

$$(D, P, V, \{\{f_{p_{ij}}\}_{j=1, \dots, m_i}\}_{i=1, \dots, n}, \{\{f_{v_{ij}}\}_{j=1, \dots, m_i}\}_{i=1, \dots, n}, C),$$

where

- $D$  is a decision situation  $\{\{c_{ij}\}_{j=1, \dots, m_i}\}_{i=1, \dots, n}$ .
- $P = (p_{11}, p_{12}, \dots, p_{mn})$  is a cell.
- $V = (v_{11}, v_{12}, \dots, v_{mn})$  is a cell.
- $f_{p_{ij}} \in \text{LBD}(p_{ij})$ .
- $f_{v_{ij}} \in \text{LBD}(v_{ij})$ .
- $C$  is the sets  $\{C_{P_i}\}_{i=1, \dots, n}$  and  $\{C_{V_i}\}_{i=1, \dots, n}$ , where  $C_{P_i}$  is a set of constraints in the  $p_{ij}$  variables, and  $C_{V_i}$  is a set of constraints in the  $v_{ij}$  variables.

**Theorem 4** *Let a potential decision scenario*

$$(D, P, V, \{\{f_{p_{ij}}\}_{j=1, \dots, m_i}\}_{i=1, \dots, n}, \{\{f_{v_{ij}}\}_{j=1, \dots, m_i}\}_{i=1, \dots, n}, C) \text{ be given.}$$

If  $p_i \in \text{GBD}(P_i)$  is  $C_{P_i}$ -admissible,  $v_i \in \text{GBD}(V_i)$  is  $C_{V_i}$ -admissible, and  $f_{p_{ij}}$  and  $f_{v_{ij}}$  are derived from  $p_i$  and  $v_i$  respectively. Then  $\{f_{p_{ij}}\}_{j=1,\dots,m_i}$  are  $C_{P_i}$ -admissible,  $\{f_{v_{ij}}\}_{j=1,\dots,m_i}$  are  $C_{V_i}$ -admissible, and

$$\begin{aligned} \langle x_{p_i}, x_{v_i} \rangle &= \\ &\int_{P_i \times V_i} \left( \sum_{i=1}^{m_i} x_{ij} y_{ij} \right) f_{p_{i1}}(x_{i1}) \cdots f_{p_{im_i}}(x_{im_i}) f_{v_{i1}}(y_{i1}) \cdots f_{v_{im_i}}(y_{im_i}) dV(x_{i1}, \dots, x_{im_i}) dV(y_{i1}, \dots, y_{im_i}) \\ &= \langle (x_{f_{p_{i1}}}, \dots, x_{f_{p_{im_i}}}), (y_{f_{v_{i1}}}, \dots, y_{f_{v_{im_i}}}) \rangle, \end{aligned}$$

where  $\langle x, y \rangle$  is the standard inner product of  $x$  and  $y$ .

**Proof:** The first part is a direct consequence of theorem 2. The second part follows from Lemma 1.

## 5 Concluding Remarks

We have presented theoretical foundations and computational procedures of a theory for analysing decision situations including probability- and value estimates, when the available information is indeterminate. The approach allows a decision maker to be as deliberately imprecise as she feels is natural as well as provides her with the means for expressing varying degrees of imprecision in the input sentences. The main idea is that impreciseness is expressed by global belief distributions, expressing relative beliefs in different values. However, in many cases the only accessible information might be in terms of local beliefs and sets of relations. Therefore, we also investigate how local distributions, i.e. belief distributions over various subspaces of the solution sets to the probability-, and utility bases, can combine to global belief distributions that, in some reasonable sense, correspond to the local information as well as to the relations. Furthermore, we investigate which properties user-asserted local belief distributions should have to be congruent with the constraints involved, and how new local distributions can be derived from complete sets of local belief distributions. Using this, we are also able to measure the degree of inconsistency of a set of user-asserted local distributions. The evaluation model used herein is based on the principle of maximising of the expected utility, but this is not a necessary restriction, and further work will include how other decision rules can be generalised to evaluate decision situations modelled in the framework. We will also generalise our earlier results on contractions to take account of various aspects on sensitivity analyses. The model presented herein is based on finite sets of consequences, and another line of research is to generalise the model into the infinite case.

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