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On Stochastic Dominance and Mean-Semideviation Models

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Abstract

We analyse relations between two methods frequently used for modeling the choice among uncertain outcomes: stochastic dominance and mean–risk approaches. The concept of α-consistency of these approaches is defined as the consistency within a bounded range of mean–risk trade-offs. We show that mean–risk models using central semideviations as risk measures are α-consistent with stochastic dominance relations of the corresponding degree if the trade-off coefficient for the semideviation is bounded by one.

Key Words: Decisions under risk, Stochastic dominance, Mean–risk models, Portfolio optimization.

Contents

On Stochastic Dominance and Mean-Semideviation Models

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1 Introduction

Uncertainty is the key ingredient in many decision problems. Financial planning, cancer screening and airline scheduling are just few examples of areas in which ignoring uncertainty may lead to inferior or simply wrong decisions. There are many ways to model uncertainty; one that proved particularly fruitful is to use probabilistic models.

We consider decisions with real–valued outcomes, such as return, net profit or number of lives saved. Although we sometimes discuss implications of our analysis in the portfolio selection context, we do not assume any specificity related to this or any other application.

Whatever the application, the fundamental question is how to compare uncertain outcomes. This has been the concern of many authors and will remain our concern in this paper. The general assumption that we make is that larger outcomes are preferred over smaller outcomes.

Two methods are frequently used for modeling the choice among uncertain prospects: *stochastic dominance* (Whitmore and Findlay, 1978; Levy, 1992) and *mean–risk* approaches (e.g., Markowitz, 1987). The first one is based on an axiomatic model of risk averse preferences, but does not provide a simple computational recipe. It is, in fact, a multiple criteria model with a continuum of criteria.

The second approach quantifies the problem in a lucid form of two criteria: the *mean*, that is the expected outcome, and the *risk*—a scalar measure of the variability of outcomes. The mean–risk model is appealing to decision makers and allows a simple trade–off analysis, analytical or geometrical. On the other hand, the mean–risk approach is unable to model the entire richness of risk-averse preferences. Moreover, for typical dispersion statistics used as risk measures, the mean–risk approach may lead to obviously inferior solutions.

The seminal portfolio optimization model of Markowitz (1952) uses the variance as the risk measure. It is, in general, not consistent with the stochastic dominance rules; the use of the semivariance rather than variance was already recommmended by Markowitz (1959) himself. Porter (1974) showed that a fixed target semivariance as the risk measure makes the mean–risk model consistent with the stochastic dominance. This approach was extended by Fishburn (1977) to more general risk measures associated with outcomes below some fixed target.

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Our aim is to develop relations between the stochastic dominance and mean–risk approaches that use more natural measures of risk, associated with all underachievemets below the mean. Therefore, we focus our analysis on the central semideviations:

$$\bar{\delta}_x^{(k)} = \left(\int_{-\infty}^{\mu_x} (\mu_x - \xi)^k \, P_x(d\xi) \right)^{1/k}, \quad k = 1, 2, \ldots, \tag{1}$$

where P_x denotes the probability measure induced by the random variable x on the real line, and $\mu_x = \mathbb{E}\{x\} = \int \xi \, P_x(d\xi)$. In particular, (1) for $k = 1$ represents the *absolute semideviation*

$$\bar{\delta}_x^{(1)} = \bar{\delta}_x = \int_{-\infty}^{\mu_x} (\mu_x - \xi) \, P_x(d\xi) = \frac{1}{2} \int_{-\infty}^{\infty} |\xi - \mu_x| \, P_x(d\xi), \tag{2}$$

and for $k = 2$ the *standard semideviation*:

$$\bar{\delta}_x^{(2)} = \bar{\sigma}_x = \left(\int_{-\infty}^{\mu_x} (\mu_x - \xi)^2 \, P_x(d\xi) \right)^{1/2}. \tag{3}$$

We shall show that mean–risk models using semideviations as risk measures are consistent with the stochastic dominance order, if the mean–risk trade-off is bounded by one.

In Section 2 we recall the notion of stochastic dominance and establish its basic properties. Section 3 develops new necessary conditions of stochastic dominance. In Section 4 we use these conditions to establish relations between stochastic dominance and mean–risk models, and in Section 5 we present simple sufficient conditions of stochastic efficiency. Finally, we have a conclusions section.

2 Stochastic dominance

Stochastic dominance is based on an axiomatic model of risk averse preferences (Fishburn, 1964). It originated from the majorization theory for the discrete case (Hardy, Littlewood and Polya, 1934; Marshall and Olkin, 1979) and was later extended to general distributions (Hanoch and Levy, 1969; Rothschild and Stiglitz, 1969). Since that time it has been widely used in economics and finance (see Bawa, 1982; Levy, 1992 for numerous references). In the stochastic dominance approach random variables are compared by the pointwise comparison of their distribution functions $F^{(k)}$.

For a real random variable x the first function $F_x^{(1)}$ is the right–continuous cumulative distribution function

$$F_x^{(1)}(\eta) = F_x(\eta) = \int_{-\infty}^{\eta} P_x(d\xi) = \mathbb{P}\{x \le \eta\} \quad \text{for } \eta \in \mathbb{R}. \tag{4}$$

The kth function $F_x^{(k)}$ (for $k = 2, 3, \ldots$) is defined as

$$F_x^{(k)}(\eta) = \int_{-\infty}^{\eta} F_x^{(k-1)}(\xi) \, d\xi \quad \text{for } \eta \in \mathbb{R}. \tag{5}$$

The relation of the *kth degree stochastic dominance* (kSD) is understood in the following way:

$$x \succeq_{(k)} y \quad \Leftrightarrow \quad F_x^{(k)}(\eta) \le F_y^{(k)}(\eta) \quad \text{for all } \eta \in \mathbb{R}. \tag{6}$$

The corresponding strict dominance relation $\succ_{(k)}$ is defined by the standard rule

$$x \succ_{(k)} y \quad \Leftrightarrow \quad x \succeq_{(k)} y \quad \text{and} \quad y \not\succeq_{(k)} x. \tag{7}$$

Thus, we say that x *dominates* y *by the kSD rules* ($x \succ_{(k)} y$), if $F_x^{(k)}(\eta) \le F_y^{(k)}(\eta)$ for all $\eta \in R$, with at least one strict inequality.

Clearly, $x \succeq_{(k-1)} y$ implies $x \succeq_{(k)} y$ and $x \succ_{(k-1)} y$ implies $x \succ_{(k)} y$, provided that the kth degree function $F_x^{(k)}$ is well defined.

We shall employ a sightly more general approach to the topic. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an abstract probability space, and let $\mathbb{E}x = \int x(\omega)\,\mathbb{P}(d\omega)$ denote the expected value of a random variable x. The space of real random variables x such that $\mathbb{E}\{|x|^k\} < \infty$ is denoted, as usual, $\mathcal{L}_k(\Omega, \mathcal{F}, \mathbb{P})$ (we frequently write simply $\mathcal{L}_k$). The norm in $\mathcal{L}_k$ is defined as

$$\|x\|_k = \Big(\mathbb{E}\{|x|^k\}\Big)^{1/k}.$$

The distribution functions (5) are closely related to the norms in the spaces $\mathcal{L}_k$.

Proposition 1 *Let $k \ge 1$ and $x \in \mathcal{L}_k$. Then for all $\eta \in \mathbb{R}$*

$$F_x^{(k+1)}(\eta) = \frac{1}{k!}\int_{-\infty}^{\eta} (\eta-\xi)^k \, P_x(d\xi) = \frac{1}{k!}\|\max(0,\eta-x)\|_k^k.$$

Proof. If $k = 0$, the left equation follows directly from (4) (with the convention $0! = 1$). Assuming that it holds for $k-1$ we shall show it for k. We have

$$\begin{aligned} F_x^{(k+1)}(\zeta) &= \frac{1}{(k-1)!}\int_{-\infty}^{\zeta}\left(\int_{-\infty}^{\eta}(\eta-\xi)^{k-1}\,P_x(d\xi)\right)d\eta \\ &= \frac{1}{(k-1)!}\int_{-\infty}^{\zeta}\left(\int_{\xi}^{\zeta}(\eta-\xi)^{k-1}\,d\eta\right)P_x(d\xi), \end{aligned}$$

where the order of integration could be changed by Fubini's theorem (see, e.g., Billingsley, 1995). Evaluation of the integral with respect to η gives the result for k. □

Remark 1 Proposition 1 allows to define the distribution functions $F_x^{(\kappa)}$ and the corresponding dominance relations for arbitrary real $\kappa > 0$:

$$F_x^{(\kappa+1)}(\eta) = \frac{1}{\Gamma(\kappa+1)}\|\max(0,\eta-x)\|_\kappa^\kappa,$$

where $\Gamma(\cdot)$ denotes the Euler's gamma function. In the sequel, however, we shall consider only integer κ.

Equation (1) and Proposition 1 imply the following observation.

Corollary 1 *Let $k \ge 1$ and $x \in \mathcal{L}_k$. Then* $\bar{\delta}_x^{(k)} = \Big(k!F_x^{(k+1)}(\mu_x)\Big)^{1/k}$.

It is also clear that the functions $F^{(k)}$ are nondecreasing and convex, but the convexity property can be strengthened substantially.

Proposition 2 *Let $k \ge 1$ and $x \in \mathcal{L}_k$. Then for all $a, b \in \mathbb{R}$ and all $t \in [0,1]$ one has*

$$F_x^{(k+1)}((1-t)a+tb) \le \left((1-t)\Big(F_x^{(k+1)}(a)\Big)^{1/k} + t\Big(F_x^{(k+1)}(b)\Big)^{1/k}\right)^k. \qquad (8)$$

Proof. Let $t \in [0,1]$. Consider the random variables $A = \max(0, a-x)$, $B = \max(0, b-x)$, and $U = \max(0, (1-t)a + tb - x)$. By the convexity of the function $z \to \max(0, z-x)$, with probability one

$$0 \le U \le (1-t)A + tB.$$

Therefore,

$$\|U\|_k \le \|(1-t)A + tB\|_k \le (1-t)\|A\|_k + t\|B\|_k,$$

where we used the triangle inequality for $\|\cdot\|_k$.

By Proposition 1, $k!F^{(k+1)}(a) = \|A\|_k^k$. Similarly, $k!F^{(k+1)}(b) = \|B\|_k^k$, and $k!F^{(k+1)}((1-t)a + tb) = \|U\|_k^k$. Substitution into the last inequality yields the required result. □

In a similar way we can prove the following properties.

Proposition 3 *Let $k \ge 1$ and $x, y \in \mathcal{L}_k$. Then for all $\eta \in \mathbb{R}$ and all $t \in [0,1]$ one has*

$$F^{(k+1)}_{(1-t)x+ty}(\eta) \le \Big((1-t)\Big(F^{(k+1)}_x(\eta)\Big)^{1/k} + t\Big(F^{(k+1)}_y(\eta)\Big)^{1/k}\Big)^k. \tag{9}$$

Proof. We define the random variables $A = \max(0, \eta - x)$, $B = \max(0, \eta - y)$, and $U = \max(0, \eta - (1-t)x - ty)$, and proceed exactly as in the proof of Proposition 2. □

Proposition 4 *Let $k \ge 1$ and $x, y \in \mathcal{L}_k$. Then for all $t \in [0,1]$ one has*

$$\bar{\delta}^{(k)}_{(1-t)x+ty}(\eta) \le (1-t)\bar{\delta}^{(k)}_x + t\bar{\delta}^{(k)}_y. \tag{10}$$

Proof. Define $A = \max(0, \mu_x - x)$, $B = \max(0, \mu_y - y)$, and $U = \max(0, (1-t)\mu_x + t\mu_y - (1-t)x - ty)$, and proceed as in the proof of Proposition 2. □

3 Necessary conditions of stochastic dominance

The simplest necessary condition of the kth degree stochastic dominance establishes the corresponding inequality for the expected values (Fishburn, 1980).

Proposition 5 *Let $k \ge 1$ and $x, y \in \mathcal{L}_k$. If $x \succeq_{(k+1)} y$, then $\mu_x \ge \mu_y$.*

Our objective is to develop necessary conditions that involve central semideviations. At first we establish some technical results.

Lemma 1 *Let $k \ge 1$ and $x \in \mathcal{L}_k$. Then*

$$\Big(i!F^{(i+1)}_x(\eta)\Big)^{1/i} \le \Big(k!F^{(k+1)}_x(\eta)\Big)^{1/k}\Big(\mathbb{P}\{x < \eta\}\Big)^{1/i-1/k} \quad \text{for } i = 1, \ldots, k.$$

Proof. We have

$$i!F^{(i+1)}_x(\eta) = \mathbb{E}\Big\{(\max(0, \eta - x))^i\Big\} = \mathbb{E}\Big\{(\max(0, \eta - x))^i \cdot 1\!\!1_{x<\eta}\Big\},$$

where $1\!\!1_{x<\eta}$ denotes the indicator function of the event $\{x < \eta\}$.

Define $A = (\max(0, \eta - x))^i$, $B = 1\!\!1_{x<\eta}$, $p = k/i$ and $q = k/(k-i)$. From Hölder's inequality $\mathbb{E}\{AB\} \le \|A\|_p \|B\|_q$ (see, e.g., Billingsley, 1995) we obtain

$$\begin{aligned} i!F^{(i+1)}_x(\eta) &\le \Big\|(\max(0, \eta - x))^i\Big\|_{k/i}^{i/k} \cdot \Big\|1\!\!1_{x<\eta}\Big\|_{k/(k-i)}^{(k-i)/k} \\ &= \|\max(0, \eta - x)\|_k^i \Big(\mathbb{P}\{x < \eta\}\Big)^{(k-i)/k}. \end{aligned}$$

Raising both sides to the power $1/i$ we obtain the result. □

Lemma 2 *Let $k \geq 1$, $x, y \in \mathcal{L}_k$ and let $x \succeq_{(k+1)} y$. Then*
(i) $\Big(i! F_x^{(i+1)}(\mu_y)\Big)^{1/i} \leq \bar{\delta}_y^{(k)} \Big(\mathbb{P}\{x < \mu_y\}\Big)^{1/i-1/k}$ *for all* $i = 1, \ldots, k$;
(ii) *if* $\bar{\delta}_y^{(k)} > 0$, *then* $\Big(i! F_x^{(i+1)}(\mu_y)\Big)^{1/i} < \bar{\delta}_y^{(k)}$ *for all* $i = 1, \ldots, k-1$.

Proof. By Lemma 1 and the dominance,

$$\begin{aligned}
\Big(i! F_x^{(i+1)}(\mu_y)\Big)^{1/i} &\leq \Big(k! F_x^{(k+1)}(\mu_y)\Big)^{1/k} \Big(\mathbb{P}\{x < \mu_y\}\Big)^{1/i-1/k} \\
&\leq \Big(k! F_y^{(k+1)}(\mu_y)\Big)^{1/k} \Big(\mathbb{P}\{x < \mu_y\}\Big)^{1/i-1/k} \\
&= \bar{\delta}_y^{(k)} \Big(\mathbb{P}\{x < \mu_y\}\Big)^{1/i-1/k}, \qquad (11)
\end{aligned}$$

for $i = 1, \ldots, k$, which completes the proof of (i). To prove (ii), note that Proposition 5 implies that $\mathbb{P}\{x < \mu_y\} \leq \mathbb{P}\{x < \mu_x\} < 1$. □

We are now ready to state the main result of this section.

Theorem 1 *Let $k \geq 1$ and $x, y \in \mathcal{L}_k$. If $x \succeq_{(k+1)} y$ then $\mu_x \geq \mu_y$ and*

$$\mu_x - \bar{\delta}_x^{(k)} \geq \mu_y - \bar{\delta}_y^{(k)},$$

where the last inequality is strict whenever $\mu_x > \mu_y$.

Proof. By (5) and (6),

$$F_x^{(k+1)}(\mu_x) = F_x^{(k+1)}(\mu_y) + \int_{\mu_y}^{\mu_x} F_x^{(k)}(\xi)\, d\xi \leq F_y^{(k+1)}(\mu_y) + \int_{\mu_y}^{\mu_x} F_x^{(k)}(\xi)\, d\xi. \qquad (12)$$

Let $k > 1$. Owing to Proposition 5, $\mu_x \geq \mu_y$, and the assertion needs to be proved only in the case of $\bar{\delta}_x^{(k)} > \bar{\delta}_y^{(k)}$. The integral on the right hand side of (12) can be estimated by Proposition 2:

$$\begin{aligned}
\int_{\mu_y}^{\mu_x} F_x^{(k)}(\xi)\, d\xi &= (\mu_x - \mu_y) \int_0^1 F_x^{(k)}((1-t)\mu_y + t\mu_x)\, dt \\
&\leq (\mu_x - \mu_y) \int_0^1 \bigg((1-t)\Big(F_x^{(k)}(\mu_x)\Big)^{1/(k-1)} + t\Big(F_x^{(k)}(\mu_y)\Big)^{1/(k-1)}\bigg)^{k-1} dt.
\end{aligned}$$

Using Lemmas 1 and 2 (with $i = k-1$), and integrating we obtain:

$$\begin{aligned}
\int_{\mu_y}^{\mu_x} F_x^{(k)}(\xi)\, d\xi &\leq \frac{\mu_x - \mu_y}{(k-1)!} \int_0^1 \Big((1-t)\bar{\delta}_x^{(k)} (\mathbb{P}\{x < \mu_x\})^{1/k(k-1)} \\
&\qquad + t\bar{\delta}_y^{(k)} (\mathbb{P}\{x < \mu_y\})^{1/k(k-1)}\Big)^{k-1} dt \\
&\leq \frac{\mu_x - \mu_y}{(k-1)!} \int_0^1 \Big((1-t)\bar{\delta}_x^{(k)} + t\bar{\delta}_y^{(k)}\Big)^{k-1} dt \\
&= \frac{\mu_x - \mu_y}{k!} \cdot \frac{(\bar{\delta}_x^{(k)})^k - (\bar{\delta}_y^{(k)})^k}{\bar{\delta}_x^{(k)} - \bar{\delta}_y^{(k)}}. \qquad (13)
\end{aligned}$$

Substitution into (12) and simplification with the use of Corollary 1 yield

$$\bar{\delta}_x^{(k)} - \bar{\delta}_y^{(k)} \leq \mu_x - \mu_y, \qquad (14)$$

which was set out to prove.

We shall now prove that (14) is strict, if $\mu_x > \mu_y$. Suppose that $\bar{\delta}_y^{(k)} > 0$. By virtue of Lemma 2(ii), inequality (13) is strict, which makes (14) strict, too.

If $\mu_x > \mu_y$ and $\bar{\delta}_y^{(k)} = 0$, we must have $\mathbb{P}\{x < \mu_y\} = 0$, so

$$\bar{\delta}_x^{(k)} \le \mathbb{P}\{x < \mu_x\}^{1/k}(\mu_x - \mu_y) < \mu_x - \mu_y.$$

and (14) is strict again.

If $k = 1$ the integral on the right hand side of (12) can be simply bounded by $\mu_x - \mu_y$, and we get (14) in this case, too. Moreover, $F_x(\xi) < 1$ for $\xi < \mu_x$, and the inequality is strict whenever $\mu_y < \mu_x$. □

Since the dominance $x \succeq_{(k+1)} y$ implies $x \succeq_{(m)} y$ for all $m \ge k+1$ such that $F_x^{(m)}$ is well-defined, we obtain the following corollary.

Corollary 2 *If $x \succeq_{(k+1)} y$ for $k \ge 1$, then $\mu_x \ge \mu_y$ and $\mu_x - \bar{\delta}_x^{(m)} \ge \mu_y - \bar{\delta}_y^{(m)}$ for all $m \ge k$ such that $\mathbb{E}\{|x|^m\} < \infty$.*

In the special case of the second degree stochastic dominance our results have a useful graphical interpretation. For a random outcome x having a bounded variance we consider the graph of the function $F_x^{(2)}$: the Outcome-Risk (O-R) diagram (Figure 1). By Corollary 1, the first two semimoments are easily identified in the O-R diagram: the absolute semideviation $\bar{\delta}_x = \bar{\delta}_x^{(1)}$ is the value $F_x^{(2)}(\mu_x)$, and the semivariance $\bar{\sigma}_x^2 = (\bar{\delta}_x^{(2)})^2$ is the doubled area below the graph from $-\infty$ to μ_x. We also have a manifestation of the Lyapunov inequality $\bar{\sigma}_x \ge \bar{\delta}_x$ (Lemma 1 with $\eta = \mu_x$, $k = 2$ and $i = 1$), because the shaded area contains the triangle with the vertices $(\mu_x, 0)$, $(\mu_x, \bar{\delta}_x)$ and $(\mu_x - \bar{\delta}_x, 0)$.

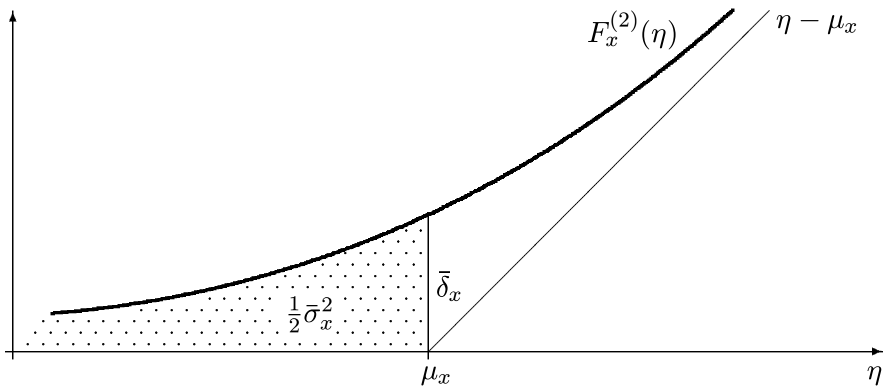

Figure 1: The O-R diagram and semimoments

Now consider two random variables x and y in a common O-R diagram. If $x \succeq_{(2)} y$ then Theorem 1 implies that $\mu_x - \bar{\delta}_x^{(1)} \ge \mu_y - \bar{\delta}_y^{(1)}$. This is obvious from Figure 2, because the slope of $F_x^{(2)}$ does not exceed one.

But we also have a more sophisticated relation illustrated in Figure 3. The area below $F_x^{(2)}$ to the left of μ_x, equal to $\frac{1}{2}\bar{\sigma}_x^2$, is not larger than the area below $F_y^{(2)}$ to the left of μ_y, increased by the area of the trapezoid with the vertices: $(\mu_y, 0)$, $(\mu_y, \bar{\delta}_y)$, $(\mu_x, 0)$, and $(\mu_x, \bar{\delta}_x)$. Employing the Lyapunov inequalities $\bar{\delta}_x \le \bar{\sigma}_x$ and $\bar{\delta}_y \le \bar{\sigma}_y$, we obtain a graphical

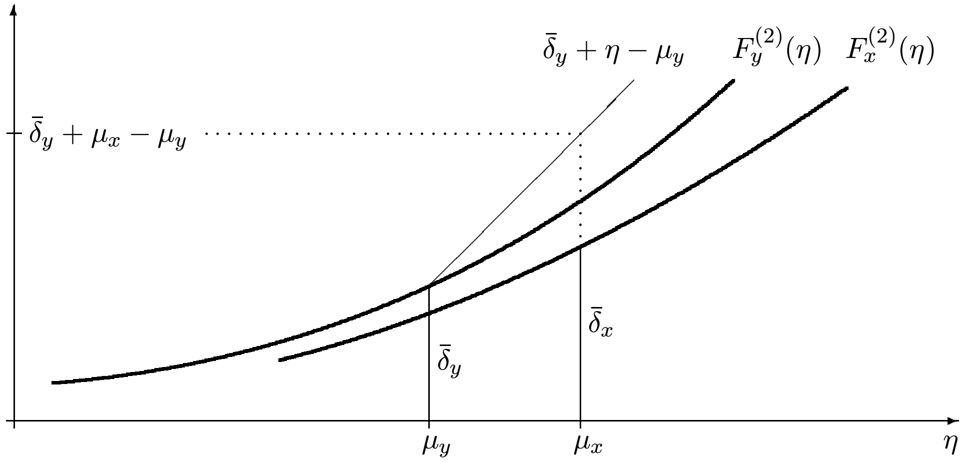

Figure 2: First necessary condition: $x \succeq_{(2)} y \;\Rightarrow\; \bar{\delta}_x \leq \bar{\delta}_y + \mu_x - \mu_y$

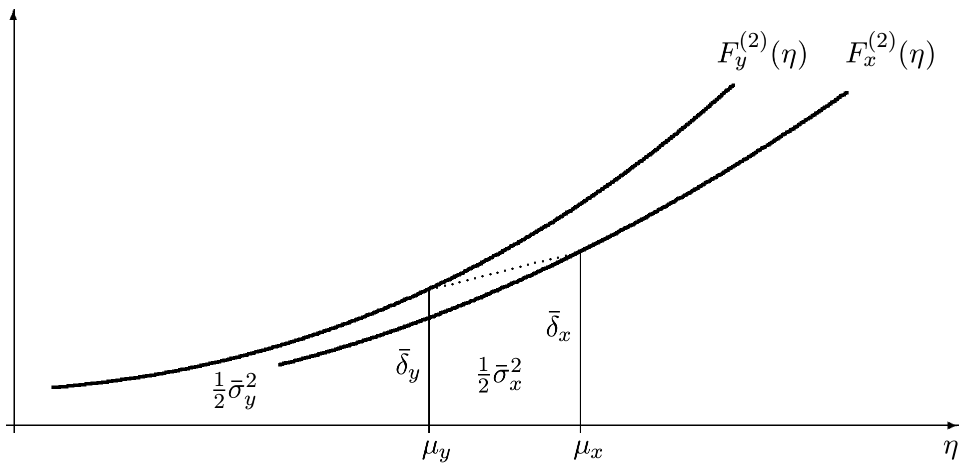

Figure 3: Second necessary condition: $x \succeq_{(2)} y \;\Rightarrow\; \frac{1}{2}\bar{\sigma}_x^2 \leq \frac{1}{2}\bar{\sigma}_y^2 + \frac{1}{2}(\mu_x - \mu_y)(\bar{\delta}_x + \bar{\delta}_y)$

proof of Corollary 2 for $k = 1$ and $m = 2$. For more details on the properties of the O-R diagram the reader is referred to (Ogryczak and Ruszczyński, 1997).

A careful analysis of the proof of Theorem 1 reveals that its assertion can be slightly strengthened. Indeed, estimating in (13) the quantities $\mathbb{P}\{x < \mu_y\}$ and $\mathbb{P}\{x < \mu_x\}$ from above by some constant ρ,

$$\mathbb{P}\{x < \mu_x\} \le \rho \le 1,$$

we obtain the necessary condition

$$\rho^{1/k}\mu_x - \bar{\delta}_x^{(k)} \ge \rho^{1/k}\mu_y - \bar{\delta}_y^{(k)}.$$

Unfortunately, it does not possess the separability properties of the assertion of Theorem 1, because the right hand side contains a factor dependent on x. In the special case of *symmetric* distributions, however, we can use $\rho = 1/2$. We can also use central deviations

$$\delta_x^{(k)} = \left(\int_{-\infty}^{\infty} |\mu_x - \xi|^k \, P_x(d\xi) \right)^{1/k} = 2^{1/k}\bar{\delta}_x^{(k)},$$

and obtain a stronger necessary condition.

Corollary 3 *If x and y are symmetric random variables and $x \succeq_{(k+1)} y$ for $k \ge 1$, then $\mu_x \ge \mu_y$ and $\mu_x - \delta_x^{(m)} \ge \mu_y - \delta_y^{(m)}$ for all $m \ge k$ such that $\mathbb{E}\{|x|^m\} < \infty$.*

4 Mean–semideviation models

Mean–risk approaches are based on comparing two scalar characteristics (summary statistics) of each outcome: the expected value μ and some measure of risk r. The weak relation of mean–risk dominance is defined as follows

$$x \succeq_{\mu/r} y \quad \Leftrightarrow \quad \mu_x \ge \mu_y \quad \text{and} \quad r_x \le r_y.$$

The corresponding strict dominance relation $\succ_{\mu/r}$ is defined in the standard way (7). Thus we say that x *dominates* y *by the* μ/r *rules* ($x \succ_{\mu/r} y$), if $\mu_x \ge \mu_y$ and $r_x \le r_y$ where at least one inequality is strict.

An important advantage of mean–risk approaches is the possibility to perform a pictorial trade-off analysis. Having assumed a trade-off coefficient $\lambda \ge 0$ one may directly compare real values of $\mu_x - \lambda r_x$ and $\mu_y - \lambda r_y$. This approach is consistent with the mean–risk dominance in the sense that

$$x \succeq_{\mu/r} y \quad \Rightarrow \quad \mu_x - \lambda r_x \ge \mu_y - \lambda r_y \quad \text{for all } \lambda \ge 0. \tag{15}$$

Therefore, an outcome that is inferior in terms of $\mu - \lambda r$ for some $\lambda \ge 0$ cannot be superior by the mean–risk dominance relation.

A mean–risk model is said to be *consistent* with the stochastic dominance relation of degree k if

$$x \succeq_{(k)} y \quad \Rightarrow \quad x \succeq_{\mu/r} y. \tag{16}$$

Such a consistency would be highly desirable, because it would allow us to search for stochastically non-dominated solutions by the rule:

$$x \succ_{\mu/r} y \quad \Rightarrow \quad y \not\succeq_{(k)} x,$$

or by the implied rule involving the scalarized objective:

$$\mu_x - \lambda r_x > \mu_y - \lambda r_y \quad \text{for some } \lambda \geq 0 \quad \Rightarrow \quad y \not\succeq_{(k)} x.$$

We would then know that using simplified aggregate measures of form $\mu - \lambda r$ would not lead to solutions that are inferior in terms of stochastic dominance.

A natural question arises: can mean–risk models be consistent with the stochastic dominance relation?

The most commonly used risk measure is the variance (see Markowitz, 1987). Unfortunately, the resulting mean–risk model is not, in general, consistent with stochastic dominance. The use of fixed-target risk measures is a possible remedy, because stochastic dominance relations are based on norms of fixed-target underachievents (Proposition 1).

We shall try to address the question in a different way. We modify the concept of consistency to accomodate scalarizations, and we use central semideviations as risk measures.

Definition 1 We say that a mean–risk model is *α-consistent* with the kth degree stochastic dominance relation if

$$x \succeq_{(k)} y \quad \Rightarrow \quad \mu_x \geq \mu_y \quad \text{and} \quad \mu_x - \alpha\, r_x \geq \mu_y - \alpha\, r_y, \tag{17}$$

where α is a non-negative constant.

By virtue of (15), the consistency in the sense of (16) implies α-consistency for all $\alpha \geq 0$. Moreover,

$$\mu_x \geq \mu_y \quad \text{and} \quad \mu_x - \alpha\, r_x \geq \mu_y - \alpha\, r_y \quad \Rightarrow \quad \mu_x - \lambda r_x \geq \mu_y - \lambda r_y \text{ for all } 0 \leq \lambda \leq \alpha \tag{18}$$

(combine the inequalities at the left side with the weights $1 - \lambda/\alpha$ and λ/α). Thus α-consistency implies λ-consistency for all $\lambda \in [0, \alpha]$. It still guarantees that the mean–risk analysis leads us to non-dominated results in the sense that

$$\mu_x - \lambda r_x > \mu_y - \lambda r_y \quad \text{for some } 0 \leq \lambda \leq \alpha \quad \Rightarrow \quad y \not\succeq_{(k)} x.$$

With these general definitions we can return to our main question: how can the risk measure be defined to maintain α-consistency with the stochastic dominance order? The answer follows immediately from Theorem 1.

Theorem 2 *In the space $\mathcal{L}_k(\Omega, \mathcal{F}, \mathbb{P})$ the mean–risk model with $r = \bar{\delta}^{(k)}$ is 1-consistent with all stochastic dominance relations of degrees $1, \ldots, k+1$.*

Proof. By Theorem 1,

$$\mu_x - \bar{\delta}_x^{(k)} > \mu_y - \bar{\delta}_y^{(k)} \quad \Rightarrow \quad y \not\succeq_{(k+1)} x.$$

The implication $y \not\succeq_{(i+1)} x \;\Rightarrow\; y \not\succeq_{(i)} x$, $i = k, \ldots, 1$, completes the proof. □

In the special case of $k = 1$ we conclude that the mean–absolute deviation model of Konno and Yamazaki (1991) is $\frac{1}{2}$-consistent with the first and second degree stochastic dominance. Indeed, the absolute deviation $\delta^{(1)} = 2\bar{\delta}^{(1)}$, and Theorem 2 implies the result.

For $k = 2$ we see that the use of the central semideviation as the risk measure (instead of the variance in the Markowitz model) guarantees 1-consistency with stochastic dominance relations of degrees one, two and three.

The constant $\alpha = 1$ in Theorem 2 cannot be increased for general distributions, as the following example shows: $\mathbb{P}\{x = 0\} = (1+\varepsilon)^{-k}$, $\mathbb{P}\{x = 1\} = 1 - (1+\varepsilon)^{-k}$, and $y = 0$.

Obviously $x \succ_{(k+1)} y$, but for each $\alpha > 1$ we can find $\varepsilon > 0$ for which $\mu_x - \alpha\bar{\delta}_x^{(k)} < 0 = \mu_y - \alpha\bar{\delta}_y^{(k)}$.

For symmetric distributions we can use Corollary 3 to get a wider range of trade-offs for semideviations, which allows us to replace semideviations with the corresponding deviations.

Corollary 4 *In the class of symmetric random variables in $\mathcal{L}_k(\Omega, \mathcal{F}, \mathbb{P})$ the mean–risk model with $r = \delta^{(k)}$ is 1-consistent with all stochastic dominance relations of degrees $1, \ldots, k+1$.*

5 Stochastic efficiency in a set

Comparison of random variables is usually related to the problem of choice among risky alternatives in a given feasible (attainable) set Q. For example, in the simplest problem of portfolio selection (Markowitz, 1987) the feasible set of random variables is defined as all convex combinations of a given collection of investment opportunities (securities). A feasible random variable $x \in Q$ is called *efficient* by the relation $\succeq$ if there is no $y \in Q$ such that $y \succ x$. Consistency (17) leads to the following result.

Proposition 6 *If the mean–risk model is α-consistent with the kth degree stochastic dominance relation with $\alpha > 0$, then, except for random variables with identical μ and r, every random variable that is maximal by $\mu - \lambda r$ with $0 < \lambda < \alpha$ is efficient by the kth degree stochastic dominance rules.*

Proof. Let $0 < \lambda < \alpha$ and $x \in Q$ be maximal by $\mu - \lambda r$. Suppose that there exists $z \in Q$ such that $z \succ_{(k)} x$. Then from (17) we obtain

$$\mu_z \geq \mu_x,$$

and

$$\mu_z - \alpha r_z \geq \mu_x - \alpha r_x.$$

Due to the maximality of x,

$$\mu_z - \lambda r_z \leq \mu_x - \lambda r_x.$$

All these relations may be true only if they are satisfied as equations; otherwise, combining the first two with weights $1 - \lambda/\alpha$ and λ/α we obtain a contradiction with the third one. Consequently, $\mu_z = \mu_x$ and $r_z = r_x$. □

It follows from Proposition 6 that for mean–risk models satisfying (17), an optimal solution of problem

$$\max\{\mu_x - \lambda\, r_x \;:\; x \in Q\}$$

with $0 < \lambda < \alpha$, is efficient by the kSD rules, provided that it has a unique pair (μ_x, r_x) among all optimal solutions. In the case of nonunique pairs, however, we only know that the optimal set *contains* a solution which is efficient by the kSD rules.

Combining Proposition 6 and Theorem 2 we obtain the following sufficient condition of stochastic efficiency.

Theorem 3 *If $\widehat{x}$ is the unique solution of the problem*

$$\max\{\mu_x - \lambda\, \bar{\delta}_x^{(k)} \;:\; x \in Q\} \tag{19}$$

for some $\lambda \in (0, 1]$ and $k \geq 1$, then it is efficient by the rules of stochastic dominance of degrees $1, \ldots, k+1$.

Proof. It remains to consider the case $\lambda = 1$. Suppose that there exists $z \in Q$ such that $z \succ_{(k)} \widehat{x}$. Then from Theorem 2, $\mu_z \geq \mu_x$ and

$$\mu_z - \lambda\ \bar{\delta}_z^{(k)} \geq \mu_{\widehat{x}} - \lambda\ \bar{\delta}_{\widehat{x}}^{(k)}.$$

Since $\widehat{x}$ is the unique maximal solution of (19), $z = \widehat{x}$. □

Theorem 3 can be extended on risk measures defined as convex combinations of semideviations of various degrees. By applying Theorem 2 for $i = k, \ldots, m$ the following sufficient condition of stochastic efficiency can be obtained.

Corollary 5 *If $\widehat{x}$ is the unique solution of the problem*

$$\max\{\mu_x - \sum_{i=k}^{m} \lambda_i \bar{\delta}_x^{(i)} \ : \ x \in Q\} \tag{20}$$

for $m \geq k \geq 1$ such that $\mathbb{E}\{|x|^m\} < \infty$, and some $\lambda_i \geq 0$ satisfying $0 < \sum_{i=k}^{m} \lambda_i \leq 1$, then it is efficient by the rules of stochastic dominance of degrees $1, \ldots, k+1$.

Owing to Corollaries 3 and 4, for symmetric distributions we can use the central deviation $\delta_x^{(i)}$ as the risk measures in (20).

It is practically important that the simplified objective functionals in (19) and (20) are concave (see Proposition 4).

6 Conclusions

The stochastic dominance relation $x \succeq_{(k+1)} y$ is rather strong and difficult to verify: it is an inequality of two distribution functions, $F_x^{(k+1)} \leq F_y^{(k+1)}$. The necessary conditions of Section 3 establish useful relations:

$$\mu_x - \lambda\bar{\delta}_x^{(k)} \geq \mu_y - \lambda\bar{\delta}_y^{(k)}, \quad \text{for all } \lambda \in [0, 1],$$

that follow from the dominance (μ_x and $\bar{\delta}_x^{(k)}$ denote the expectation and the kth central semideviation of x).

This allows us to relate stochastic dominance to mean–risk models with the risk represented by the kth central semideviation $\bar{\delta}_x^{(k)}$. The key observation is that maximizing simplified objective functionals of the form

$$\mu_x - \lambda\bar{\delta}_x^{(k)},$$

where $\lambda \in (0, 1)$, we obtain solutions which are efficient in terms of stochastic dominance. This may help to quickly identify prospective candidates in complex decision problems under uncertainty.

Still, sufficient conditions of stochastic dominance in some specific classes of decision problems under uncertainty require more attention. We hope to address these issues in the near future.

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