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DEFINITIONS OF RESILIENCE

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PREFACE

One of the most striking examples of collaboration between projects at IIASA was the informal "Resilience Group," made up of members of the Ecology, Energy and Methodology Projects. Holling's original idea of resilience as a property or measure of an ecological system found unexpected applications to society models considered by the Energy Project; at the same time, the mathematical identification of the resilience concept gave rise to several interesting methodological problems. The collaboration arising out of these common interests proved extremely fruitful to all participants of this informal group. From the group discussions, the author has distilled precise mathematical definitions for the many facets of the resilience concept. This paper shows that the language of differential topology is rich enough to express all the--sometimes diverging--ideas about resilience that came up at IIASA. By necessity, the discussion in this paper takes place on a somewhat technical level; an appendix summarizes the necessary mathematical terms.

SUMMARY

During the past year, several research efforts at IIASA have tried to develop a precise mathematical definition of Holling's very general and rich resilience concept. This paper develops a mathematical language for resilience, using the terms and concepts of differential topology. Central to this treatment is the division of the state space of a system into basins, each containing an attractor. The translation of Holling's concept into this language reads roughly as follows: a system is resilient if, after perturbation, it will still tend to the same attractor as before (or to an "only slightly changed" attractor). The reason for treating changes of state variables and changes of parameters separately is explained. All resilience measures conceived up to now, as defined within this language, are listed as well. The various definitions of resilience are then compared to the well-known concepts of structural stability and of Thom's catastrophe theory. Finally, the author indicates some--in his opinion--important directions for further research into the general resilience concept.

DEFINITIONS OF RESILIENCE

1. INTRODUCTION

The resilience concept, pioneered by C.S. Holling [1], has been the subject of a prolonged discussion at IIASA. Various expressions for a resilience value have been proposed by Holling's group [2], and, in the course of the study of the "New Societal Equations," another possible expression for resilience was computed explicitly [3] and will be used as input to an optimization problem. Nevertheless, apart from [4], no general investigation has been made of the resilience concept in the context of the theory of differentiable dynamical systems. It is the purpose of this paper to give precise, workable definitions of resilience in the language of this theory. This language will turn out to be rich enough to express all different facets of the "resilience concept." As we shall see, Holling's original concept has to be split into two essentially different properties. This distinction has been noted in the resilience discussion for some time under the labels "resilience in state space" vs. "resilience of state space." The present work is restricted to deterministic systems. The problem is first discussed in an informal conceptual way, as a motivation for the rigorous mathematics contained in the second part. An appendix summarizes some mathematical definitions used.

2. The MATHEMATICAL STAGE: DIFFERENTIABLE DYNAMICAL SYSTEMS

Although some attempts have been made to link resilience to "long exit time expectations" in a stochastic approach [5], the discussion so far has been centered on deterministic systems. One natural mathematical language for the description of such systems is the theory of differential dynamical systems, i.e., of differentiable maps and flows on a manifold. (An informal collection of concepts and results of this theory is given in the appendix.) For the definitions of resilience, we therefore assume that we have abstracted a mathematical model of the given system in the following form:

$$X(t) = \phi_t(x) \quad ; \quad x \in M \quad , \quad t = 1,2,\ldots \text{ or } t \in R^+ \quad . \quad (2.1)$$

M denotes the state space of the system, assumed to be a manifold; ϕ_t gives the total dynamic evolution of the system over time, either discrete (as in various ecological models, where the average reproduction time of a species gives a natural time step) or continuous. By choosing a particular functional form for ϕ--or for the differential equation defining it--we perform the "first cut" for our model: we separate the "system" from the "rest of the real world," which will be regarded as unknown perturbations. The second cut involves the separation of state variables and parameters; the dimensionality and interpretation of the state space enter here. This separation can be performed only via a significant difference in time scales; in the language of [14], state variables and parameters must belong to different "strata." Loosely speaking, parameters are allowed to vary in only two ways: either by sudden jumps,[1] such that the state variables can be approximated to be constant during the change, or by slow "adiabatic" changes, such that the system can be assumed to be already in its asymptotic state, as in the applications of catastrophe theory [13]. This distinction will appear in the treatment of resilience of the state space; it reminds one of the sudden and the adiabatic approximation in time-dependent quantum-mechanical perturbation theory.

In this language, a parameter is therefore a variable contained in ϕ, without being a function on the state space M. Later, it will be varied over a parameter manifold P. The semi-group property $\phi_{t+s} = \phi_t \phi_s$ then expresses the autonomy of our system. In the discrete case, this implies immediately $\phi_n = f^n (n \in N)$ with a single map f. We assume all maps to be

[1] "Sudden" and "slow" with respect to a typical time scale of the system itself.

once differentiable (C^1). Further differentiability requirements cannot be made in general: threshold behaviors of reproduction rates, for example, imply discontinuities in the first or higher derivatives as in [6].

In the continuous case, ϕ will of course not be given explicitly, but only through a differential equation $\dot{x}(t) = F(x)$. The "integrated form" (2.1) is introduced for convenience in the definitions. We neglect for the moment difficulties due to incompleteness of F (i.e., not all trajectories can be extended to arbitrary large times). For an example of this problem, see [7].

3. EXPRESSING THE RESILIENCE CONCEPT

In a very crude way, the desired definition could be given in the form: "the system can absorb changes." Those changes, to be sure, are assumed to be sudden and external (controllable or uncontrollable, predictable or random) and therefore not to be included in the mathematical description via ϕ_t. But a question suggests itself immediately: "How large can those changes be?"[2] Thus resilience, as we see it, will be a two-stage concept. First, on the qualitative side, we try to answer the question, "Is a system resilient?" If the answer is yes, then the quantitative side asks, "How resilient is it?" By no means will this question have a unique answer; the consensus stresses the necessity of several "resilience measures." One system could very well be more resilient than another with respect to one measure, and less so with respect to a second one. To define resilience values, we must assume that in the state space M we have a notion of distance, (a metric d) in accordance with one well-established idea of resilience as "distance to

[2] To illustrate this idea, if the change in state variables occurs at time t_o, the evaluation given by ϕ_t continues on from a new x'_{t_o} instead of the x_{t_o} reached through evolution before t_o. To talk about the magnitude of the change, we need some distance between x'_{t_o} and x_{t_o}.

the next point of catastrophic behavior"; or a notion of volume (a measure v), expressing the other main idea of resilience as "size of basins." The identification of this distance or volume will be a non-trivial problem to be solved for each system individually; any identification of distance involves implicit assumptions about the structure of possibly occurring disturbances, any identification of volume involves assumptions of average distributions of points describing the system over the basin. Here we mention only two questions alluded to in [2] and [3]: logarithmic scale vs. linear scale of the perturbations, and the problem of "natural units" if the coordinates in M (the state variables) have different dimensions.

If no coherent scaling of different state variables is possible, we might even have to replace d by a multidimensional distance notion: a family of semi-metrics $(d_1 \ldots d_i)$ measuring the size of jumps in different directions of state space. The various definitions of resilience measures can then be adjusted accordingly.

Apart from this two-stage concept, there is a qualitative distinction when we try to answer the question: "What kind of changes can the system absorb?" Although in [1] Holling treated "changes in state variables" and "changes in parameters" on an equal footing, for a rigorous mathematical treatment we will have to make a distinction. Changing the state of the system at one particular time changes one particular orbit, while changing the functional form of the flow or map through a change in parameters, for instance, involves the whole phase portrait. Given a precise formulation of the statement "the system can absorb," we will then have two concepts of resilience, depending on the nature of the changes.

Two suggestive names for those concepts have been proposed: "resilience in state space" (short for "of a point in phase space"), corresponding to changes of state variables, vs. "resilience of state space" corresponding to changes of parameters. The latter concept will have different aspects as we deal with sudden or adiabatic parameter changes. In this paper we call them

R_1 and R_2 for simplicity.

4. THE ROLE OF ATTRACTORS AND BASINS

We use the picture presented in the appendix: a finite number of attractors A_i are located in basins B_i, separated by separatrices S_j. Let us assume that we have singled out one attractor[3] A_1 as "desirable" (this, of course, is an external value judgment), and that the current state lies in B_1; the system thus tends towards A_1 as $t \to +\infty$. A change will obviously be absorbed if, after the change, the system still tends to "almost" the same region of state space.

Corresponding to the two kinds of resilience discussed above, non-resilient behavior can occur in two ways:

R_1: The sudden jump of the state variables moves the point describing the system across a separatrix into another basin;

R_2: The phase portrait changes in such a way that the system (assumed to have the same values of the state variables as before the changes) now lies in a basin whose attractor is in a different region of state space. We see here, by the way, one difference between sudden and adiabatic changes: a slow "adiabatic" change will not cause the system to tend to a different attractor; rather, it will tend to an "adiabatically changed" one.

5. MATHEMATICAL DEFINITION

We first assume the same situation as in the last section and define the set S as $M - \bigcup_i B_i$ (the union of all separatrices).

Definition (5.1) (R_1):

Given a system $\{\phi_t\}$ in M with a finite number of attractors A_i and basins B_i, it is called resilient in the R_1 sense

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[3]Or, in general, a subset of attractors!

if $M - \bigcup_i B_i$ has measure zero.[4]

This means that almost all initial conditions lead to attractors and small shifts do not disturb the asymptotic behavior since the basins are open sets. This condition is fulfilled, e.g., if the non-wandering set consists only of hyperbolic fixed points and closed orbits (finite in number), or if ϕ satisfies Axiom A and is twice differentiable [8]. There are C^1 counter-examples satisfying Axiom A where the separatrices have positive measure. It is not clear what this mathematical statement would mean in the real world; it could be interpreted as saying that there is a non-zero probability that the system lies arbitrarily close to a basin boundary.

We generalize this concept to deal with continuous families of fixed points, e.g.:

Definition 5.1.a (generalized R_1):

For a point $x \in M$, we denote by $\omega(x)$ the future limit set of $x (y \in \omega(x)$ if there exists a sequence of real numbers $t_i \uparrow \infty$ such that $\phi_{t_i}(x) \mapsto y)$. Then $\{\phi_t\}$ is called generalized R_1 if the map $x \mapsto \omega(x)$ is continuous with the Hausdorff metric[5] on the space of compact subsets of M except on a set of measure zero. This expresses the stability of the asymptotic behavior under disturbances of the initial condition. Of course (5.1.a) contains (5.1) since $\omega(x) \equiv A_i$ for $x \in B_i$.

Turning to resilience measures, we define $r(x) = d(x,S)$ for each point in S. We distinguish different expressions (dependent on the application intended).

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[4]The property of zero measure is independent of the particular distribution of state variables described by, e.g. the volume notion, as long as it has a density.

[5]The Hausdorff distance between the compact sets A and B is defined by $\tilde{d}(A,B) = \max(\sup_{x\in A} \inf_{y\in B} d(x,y), \sup_{y\in B} \inf_{x\in A} d(x,y))$. (There is a point in B at distance $\tilde{d}(A,B)$ from any point in A, and vice versa.)

i) Mean resilience of the basin B_1 (Holling and co-workers):

$$R_M = \int_{B_1} r(x)d\mu(x) \quad , \tag{5.2}$$

with some probability measure μ.

This concept is useful in some ecological applications--e.g., where one is dealing with an ensemble of systems--and has been used by Holling's group.

ii) Trajectory resilience. Here we focus on one particular initial condition, so that the resilience value is a function of x. Häfele in [9] proposed "average resilience:"

$$R_{av} = 1/\int_0^t \frac{dt}{|\dot{x}(t)|r(\phi_t x)} \quad ; \tag{5.3}$$

and the author [7] proposed "minimal resilience:"

$$R_{min}(x) = \min_{t \geq 0} r(x_t) \quad , \tag{5.4}$$

e.g., in a normative approach to standard setting. The last expression has been tabulated for the model in [3] and is being used as input to an optimization program.

Expressions such as those in (i) have been called resilience numbers, those in (ii) resilience functions, since they still are functions over state space. Another possible resilience number could be

iii) Volume resilience: (5.5)

$$R_V = v(B_1) \quad ,$$

v being the volume on state space as in Section 2. This number may be less significant due to the fact that in higher-dimensional models, the basins will generally have rather complicated structures, such that they could contain a large volume while the boundary is still close to each point in the basin.

Of course for the computation of these numbers in a concrete problem, the exact locations of the separatrices have to be determined. Two kinds of approaches seem most suitable for this: either a method proving that some region lies wholly within the domain of attraction (Lyapunov's method as used in [10], or Zubov's method as described in [11]), or a direct determination of the separatrices as stable manifolds of co-dimension one. The latter method was used in [7].

6. MATHEMATICAL DEFINITION: R_2

Resilience of the second kind as conceived in this paper will obviously be a concept related to the structural stability idea of Smale [12]. It is well known that any notion of qualitative equivalence between systems gives rise to a corresponding notion of structural stability in the general sense: a system will have a certain structural stability property if its equivalence class under the given equivalence notion is open in some C^r-topology. Structural stability in the technical sense is connected in this way to topological conjugacy, i.e., the existence of a homeomorphism transforming the systems into each other. While structural stability is too strong a concept, since it is concerned with the whole phase portrait and not with positive time asymptotics alone, Ω-stability is too weak since it implies nothing about structural change of the basins. In [4], it was illustrated that basins could jump under a small perturbation of an Ω-stable system (hyperbolic fixed points and closed orbits), a non-resilient situation as explained in Section 3.

As a further distinction from the usual mathematical concepts of structural stability, Ω-stability, etc., we have to note that we will not vary the system over a whole neighborhood in Diff(M) or X^1(M) (the space of all discrete or continuous systems on M). We therefore assume a sub-manifold P of Diff(M) or X^1(M) to be given such that the original $\phi \in P$. P can be thought of as described by a finite set of parameters contained in ϕ that we want to vary; then P will be finite-dimensional. But P could very well be infinite-dimensional if some equations of the model are assumptions and implicitly inexact while others are exact identities. Then one might study resilience of the model to arbitrary small variations of the first set of equations, in the spirit of Thom's insistence on structurally stable models [13]. Many of the mathematical distinctions in what follows will become trivial in the case of stable equilibria and are included partly for completeness; however, there are strong suggestions that non-trivial attractors will appear even in simple models. A particularly nasty example is given by the Lorenz attractor (see [18]).

We again use the Hausdorff distance to formulate the condition that basins and attractors do not vary much.

Definition (6.1.a) (R_2, discrete case):

Given a system ϕ on M and a manifold P as above such that $\phi \in P \subseteq$ Diff (M), ϕ is called resilient in the second sense if:

i) There exists a neighborhood U of ϕ in the C^1-topology on Diff (M) such that all systems $\phi' \in U \cap P$ have the same finite number of attractors (and, therefore, the same number of basins!);

ii) The maps $\phi' \mapsto A_i(\phi')$ (i-th attractor of ϕ') and $\phi' \mapsto \bar{B}_i(\phi')$ (closure of i-th basin of ϕ') are continuous with the C^1-topology on $U \cap P$ and the Hausdorff metric on the A_i and $\bar{B}_i$.

Definition (6.1.b) (R_2, continuous case):

Assume the continuous system $\{\phi_t\}$ given by a differential equation $\dot{x} = F(x)$ as in Section 2, and P a sub-manifold of $X^1(M)$, $F \in P$. Then F is called resilient in the second sense if:

i) There exists a neighborhood U of F in the C^1-topology such that all systems $\{\phi'_t\}$ defined by the $F' \in U \cap P$ have the same finite number of attractors;

ii) The maps $F \mapsto A_i(F)$, $F \mapsto \overline{B_i}(F)$ are continuous with the C^1-topology on $U \cap P$ and the Hausdorff metric on the A_i and $\overline{B_i}$.

If the manifold P is finite-dimensional, given by variations of some parameters in the functional form of ϕ (or F in the continuous case), the continuity conditions (2) simply mean continuity in those parameters.

Comparing this definition with the well-known stability concepts, we see that we have put a very weak condition on the attractors. Usual formulations require topological conjugation on the attractor,[6] or at least that nearby systems give homeomorphic attractors. Thus, the Lorenz attractor [18], known to be topologically unstable, could very well be R_2. This weakening of the definition is due to the intended applications of the resilience concept; one does not want to be bothered with "sub-shifts of finite type" in an ecological model, for example. For the first steps we are interested only in the location of attractors. However, a more refined analysis should include the invariant--under $\{\phi_t\}$--measures μ_i on the attractors describing time averages in the basins [8]. Some climatologists have expressed great interest in those averages, which can be used for defining mean resilience in Section 5. Thus, for a

[6]This means that the orbits of $\{\phi'_t\}$ on $A_i(\phi')$ can be carried into the orbits of $\{\phi_t\}$ on $A_i(\phi)$ by a homeomorphism of the attractor sets.

stable fixed point, R_m would simply be the minimum distance to the basin boundary. Then the conditions (ii) on the attractors should be replaced by (ii'):

The maps $\phi' \mapsto \mu_i(\phi')$ are continuous with the weak topology on the space of measures.[7] This implies (ii) since A_i = the support of μ_i.

On the other hand, a possible generalization might take into account that "splitting" one attractor under variation into several--still close together--does not essentially change the asymptotic structure of the system. For each attractor A_i of ϕ, and each perturbed system ϕ', we then have a finite set $a_i(\phi')$ of attractors of ϕ' (close together), and we replace (ii) by (ii"):

the maps $\phi' \mapsto \bigcup_{A \in a_i(\phi')} A(\phi')$

and $\phi \mapsto \bigcup_{B_i \in \mathscr{B}} \overline{B_i}(\phi')$

[where $\mathscr{B}$ is the set of all basins belonging to attractors in $a_i(\phi')$] are continuous.

The system as defined by

$$\dot{X} = \mu X - \varepsilon X^3 \tag{6.3}$$

would still be resilient in this sense to variations of μ around $\mu = 0$. Although the stable fixed point for $\mu < 0$ splits at $\mu = 0$ into one unstable and two stable points, the attracting points are still close together.

[7]This means that time averages of continuous functions on the state space will vary continuously under variations of ϕ.

If we look for a numerical expression connected with R_2 we first might try:

Definition (6.2) (minimal resilience):

Given a metric $\bar{d}$ on the "parameter manifold" P, and denoting by S_P (parameter separatrix) the set $S_P = \{\phi \in P | \phi'$ does not satisfy $R_2\}$, we define

$$\bar{R}(\phi)_{min} = \bar{d}(\phi, S_P)$$

a normative number such as R_{min} (Section 5).[8] This number, of course, tells us a range of parameter variations which do not induce qualitative changes in the behavior of the system. By choosing the parameter manifold P in different ways, one could then study resilience with respect to various combinations of parameters.

Another possible definition of a resilience number more in line with standard sensitivity analysis is suggested here. Given the continuous dependence of attractors and basins required by Definition (6.1), their "speed" under parameter variation may be interesting. Although the phase portrait does not change--the systems might even be structurally stable, i.e., more than just R_2--a very sensitive dependence of basin boundaries does not correspond to the intuitive concept of a resilient system. A large reduction in size of a particular basin is considered almost as catastrophic as its complete disappearance. We therefore propose

Definition (6.3) (speed resilience):

Under the assumptions of Definition (6.2), and denoting by B_h a ball of radius h around the system ϕ in the parameter manifold P,

$$\bar{R}_{SP} = 1/\lim_{h \to 0} \frac{1}{h} \sup_{\phi' \in B_h} \sup_i [d(A_i, A_i'), d(\bar{B}_i, \bar{B}_i')] \quad ,$$

[8] We denote resilience values connected to R_2 by a bar.

with d the Hausdorff metric of closed sets in state space and $A_i, B_i, \ldots,$ as in (6.1);

Definition (6.4) (volume sensitivity resilience):

$$\bar{R}_V = \lim_{h \to 0} \frac{1}{h} \sup_{\phi', B_h} \, \sup_{P} \, |v(B_1) - v(B_1')| \quad ,$$

with B_1 the "desired" basin and notations as above. But any volume-type measures must be regarded with some reservation, as explained in Section 5.

$\bar{R}_{SP}$ may well be 0 even if the system is R_2, if the location of attractors or basins depends non-differentiably on the parameters; see, for an example, equation (6.3). This system is resilient only in the generalized sense (stable fixed points at $\pm\sqrt{\frac{\mu}{\varepsilon}}$) mentioned there, however.

For slow changes of the parameters in the sense of Section 3, the conditions about the variations of basins in Definitions (6.1) and (6.4) should be left out. Under these conditions, a basin boundary can never overtake the system on its course towards the attractor. In Definition (6.1), only the disappearance of the attractor should be counted as non-resilient behavior, as in catastrophe theory. Approaching such parameter values will involve a shrinking of the basin as observed in several ecological examples.

7. CONCLUSION AND DIRECTION OF FURTHER RESEARCH

The theory of differentiable dynamical systems gives a satisfactory language for describing the many different facets that make up the resilience concept. At the same time, we have been able to distill from the applications an interesting mathematical concept that has not been studied yet in ordinary stability theory. In relation to catastrophe theory, the resilience concept is a two-fold extension: it takes more complicated attractors than fixed points into account right from the start, and it also involves properties of the basins.

The following directions of further research--some of them already under discussion--are suggested:[9]

- The mathematical theory of resilience should be investigated: general necessary of sufficient criteria for resilience of the state space would be extremely interesting. Some starting points for this are contained in Section 6.
- In relation to the "kit concept"--constructing a system having a predetermined structure of attractors and basins--two tasks are important: starting a list of attractors relevant for applications, and studying consistency questions along the lines of [15].
- Studying the change of the phase portrait as we cross a parameter separatrix: the problem of bifurcation. This has been suggested as a mechanism for generating turbulent and erratic behavior of systems [17]. For the resilience concept, its study gives us an understanding of "what goes wrong."
- Numerical methods for the calculation of boundary basins should be developed and tried in simple models; a list of candidates is given at the end of Section 5.
- The connections between the choice of resilient measure (distance, volume in state space, etc.) and our knowledge or our implicit assumptions about the structure of perturbations of the system should be described in detail.

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[9]Most of them, of course, are also relevant for the general structural study of dynamical systems, apart from the question of resilience. As a forthcoming paper [16] hopes to show, this theory can extend our understanding of the dynamics of a system "beyond numerical integration."

APPENDIX

A short introduction to the mathematical foundations of the theory can be found in P. Walters.[10] Only a brief list of informal definitions will therefore be given here. We shall deal with the discrete case ($\phi_n = f^n$; D) and the continuous case (C) simultaneously.

A *fixed point* of ϕ is an $x_0 \in M$ with $\phi_t(x_0) = x_0$ for all t.

A *periodic point* (D)/*closed orbit* (C) is an $x_0 \in M$ with $\phi_t(x_0) = x_0$ for some t.

The *non-wandering set* Ω is defined as the set of $x \in M$ which do not wander in the following sense: x wanders if there is a neighborhood U(x) with $\phi_t U(x) \cap U(x) = \emptyset$ for all t large enough. Non-wandering is the weakest recurrent-like property; of course, all periodic points/closed orbits, but in general also more complicated things, lie in Ω.

A fixed point x_0 is *hyperbolic* if the Jacobian of f (D)/ the matrix $\frac{\partial F_i}{\partial x_i}$ (C)/at x_0 has no eigenvalues on the unit circle (D)/on the imaginary axis (C). In this case, the *stable* and *unstable* manifolds $W^s(x_0)$ and $W^u(x_0)$ can be defined as points tending exponentially to x_0 as $x \to +\infty$, $x \to -\infty$, respectively. They are smooth sub-manifolds of M, invariant under ϕ_t. A hyperbolic fixed point cannot disappear under a small perturbation of ϕ.

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[10] P. Walters, "An Outline of Structural Stability Theory," in H.-R. Grümm, ed., Analysis and Computation of Equilibria and Regions of Stability, CP-75-8, International Institute for Applied Systems Analysis, Laxenburg, Austria, 1975.

A *continuous hyperbolic splitting* over a closed subset $N \subsetneq M$ is given by defining in a continuous way a splitting of the tangent space $T_x(M) = E_x^s \oplus E_x^u$ (for these concepts see Walters[10]) into directions where ϕ is exponentially contracting or expanding, respectively (D). In the continuous case, $T_x(M) = E_x^s(M) = E_x^s \oplus E_x^u \oplus F_x$, where F is a one-dimensional subspace along the direction of the vector field at x.

ϕ satisfies *Axiom A* if

1) The periodic points (D)/closed orbits (C) are dense in Ω;

2) There exists a continuous hyperbolic splitting over Ω. (Axiom A thus deals only with the behavior of ϕ on the non-wandering set.)

In this case, Ω can be partitioned into *basic sets* Λ_i; they are the proper generalizations of fixed points. Every Λ_i has its *stable* and *unstable* manifold $W^s(\Lambda_i)$ and $W^u(\Lambda_i)$ with properties analogous to those of fixed points.

An *attractor* is a closed minimal invariant subset $A \subset N$ with an open neighborhood U contracting to A in the future ($A = \bigcap_{t>0} \phi_t U$). This is the proper generalization of a stable fixed point. The *basin* B of an attractor A is the set of $x \in M$ tending to A as $t \to \infty$; it is open. In the case of Axiom A, a basic set Λ_i is an attractor if $W^s(\Lambda_i)$ is open; then $W^s(\Lambda_i)$ is the basin of Λ_i.

An attractor is called *strange* if it is not a smooth submanifold of M, e.g., the Lorenz attractor. Warning: most attractors are strange!

A *separatrix* (in the terminology of [4]) is a stable manifold of co-dimension one. In the case of Axiom A, the basin boundaries are to be found among the separatrices.

For an Axiom A attractor Λ, there exists an *invariant measure* μ on Λ for which the following is true:

$$\text{(D)} \quad \lim_{T\to\infty} \frac{1}{T} \sum_{n=1}^{T} f(\phi_i x)$$

$$= \int_{\Lambda} f(x)\,d\mu(x)$$

$$\text{(C)} \quad \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} f(\phi_t x)\,dt$$

for almost all x in the basin of Λ and all continuous functions f on M (Bowen-Ruelle theorem). So μ allows us to calculate time averages.

The C^r*-topology* on the space of all dynamical systems on M[11] is defined in the following way: two systems ϕ and ϕ' are C^r-close if, together with their derivatives up to order r, they are uniformly close as maps ϕ_t' and ϕ_t from M to M for a fixed t.

A system ϕ is C^r*-structurally-stable* if for all ϕ', C^r-close enough to ϕ, there exists a homeomorphism[12] h of M transforming orbits of ϕ into orbits of ϕ'. Thus, up to a topological deformation, ϕ' looks exactly like ϕ.

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[11] Here, M has to be compact, or awkward technical problems will occur.

[12] There are good reasons for insisting only on bicontinuity of h, instead of on differentiability.

A system ϕ is *C^r-Ω-stable* if the above holds at least on the non-wandering sets Ω of ϕ and Ω of ϕ' (so that h transforms Ω into Ω'). Here, we are interested only in non-transient behavior: on Ω. If ϕ is Ω-stable, it must satisfy Axiom A.

A system ϕ is *topologically stable* if all C^r-close ϕ' have non-wandering sets homeomorphic to the one of ϕ. The Lorenz attractor is not even topologically stable.

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