SOME APPROACHES TO THE WATER PROJECT

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1. Decomposition

Under decision making concerning Large Scale Systems (LSS) of Water Resources (WR), which is a big river basin with cities, industry and agriculture systems, water reservoirs, hydroelectric power stations, etc., we need some kind of decomposition because of this LSS complexity.

Suppose, with some reason, we divide our LSS into different parts S_i; $i = 1,\ldots,n$, where each system S_i is situated downstream with respect to previous components $S_1,\ldots,S_{i-1}$ (see Fig. 1).

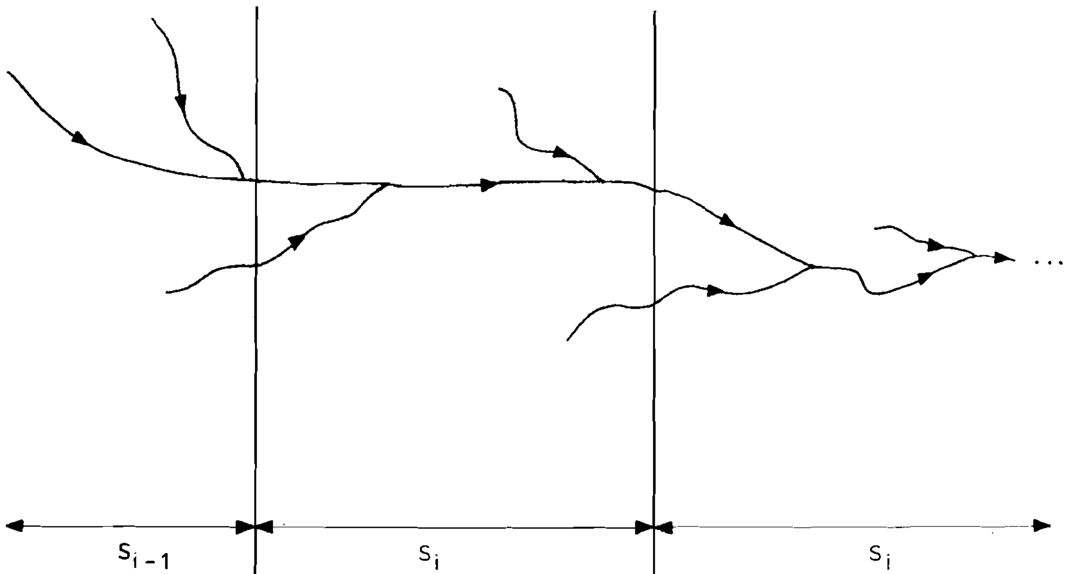

FIGURE 1

The system S_i might be a big complex of various components S_{ik} $(k = 1,\ldots,n_i)$--plants, irrigation systems, etc.--for which it is necessary to supply WR.

It seems reasonable to assume that during a considerable period of time $(t, t + \Delta t)$ for all components S_{ik} we know proper inflows

$$x^*_{ik} = x^*_{ik}(t)$$

under which these components operate in a <u>normal</u> way. Let us assume also that if an actual inflow

$$x_{ik} = x_{ik}(t)$$

is different from the normal inflow $x^*_{ik} = x^*_{ik}(t)$, then we lose (in a proper scale) an amount

$$C_{ik}(t, x_{ik}, x^*_{ik}) \quad .$$

The subproblem for every system S_i $(i = 1,\ldots,n)$ is to minimize the total loss

$$C[t, x_i(t), x^*_i(t)] = \sum_{k=1}^{n_i} C_{ik}(t, x_{ix}, x^*_{ik})$$

which takes place in the case of the total inflow

$$x_i(t) \sum_{k=1}^{n_i} x_{ik}$$

by choosing the optimal inflow distribution

$$(x_{i1},\dots,x_{in_i}) \quad .$$

Say for a water reservoir S_{ik} the value $C_{ik}(t,x_{ik},x_{ik}^*)$ might be an estimate of a proper loss in a future when it will be necessary to supply WR from S_{ik}, for an irrigation system it might be a loss of a corresponding crop, etc.

The decomposition problem is how actually to form the inflows $x_i(t)$ for systems $S_i(t)$, $i = 1,\dots,n$.

Let i_i be a passage-time between S_i and S_{i+1} for a main flow. (Remember that S_{i+1} is situated downstream with respect to S_i.) Roughly speaking, if some part of WR in the main flow is not consumed by S_i at the moment t, then this WR will be available for S_{i+1} at the moment $t + \tau_i$.

Let $w_i(t)$ be a WR innovation which is available for S_i at the moment t and $a_i(t)$, $b_i(t)$ are given low limits for the consumption $x_i(t)$ and the main flow after S_i:

$$a_i(t) \le x_i(t) \le y_{i-1}(t - \tau_{i-1}) + w_i(t) - b_i(t)$$

where $y_{i-1}(t - \tau_{i-1})$ is the main flow after S_{i-1} at the previous moment $t - \tau_{i-1}$ (τ_{i-1} is the passage-time between S_i and previous system S_{i-1}).

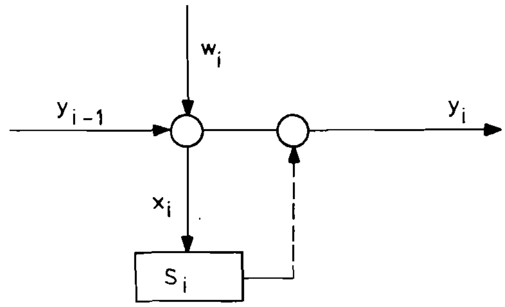

FIGURE 2

Obviously, the inflows for S_i and S_k, $k > i$ are connected only in the corresponding passage-time $\tau_i + \cdots + \tau_{k-1}$, so if we consider

$$x_1 = x_1(t),\ w_1 = w_1(t),\ a_1 = a_1(t),\ b_1 = b_1(t)$$

and

$$x_k = x_k(t + T_k) \quad , \qquad w_k = w_k(t + T_k) \quad ,$$

$$a_k = a_k(t + T_k) \quad , \qquad b_k = b_k(t + T_k) \quad ,$$

$$(T_k = \tau_1 + \cdots + \tau_{k-1}) \quad ; \qquad k = 2,\ldots,n,$$

then the following constraints have to be true:

$$a_1 \leq x_1 \leq w_1 - b_1 \quad ,$$

$$a_k \leq x_k \leq \sum_{i=1}^{k-1} (w_i - x_i) + w_k - b_k \quad , \quad (k = 2,\ldots,n).$$

With the substitution

$$x_1 - a_1 \rightarrow x_1 \quad , \quad w_1 - b_1 - a_1 \rightarrow w_1$$

$$x_k - a_k \rightarrow x_k \quad , \quad w_k - b_k + b_{k+1} \rightarrow w_k \quad , \quad (k = 2,\ldots,n),$$

the constraints for the new variables will be

$$0 \leq x_1 \leq w_1$$

$$0 \leq x_k \leq \sum_{i=1}^{k-1} (w_i - x_i) + W_k \quad ; \quad k = 2,\ldots,n,$$

or (what is the same!)

$$x_k \geq 0$$

$$0 \leq \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} w_i \quad ; \quad k = 1,\ldots,n.$$

Suppose the minimal loss for the system S_i under the inflow x_i is $C_i\ (\cdot,x_i,x_i^*)$. We cannot expect a possibility to find out in an analytical structure of this function. But if we know a few values for different x_i, then we can try to find a proper approximation.

Let us assume that under WR shortage, the inflows x_i have not to exceed the corresponding limits x_i^*:

$$x_i \leq x_i^* \quad ; \qquad i = 1,\dots,n.$$

What kind of approximation for the loss functions $C_i(\cdot,x_i,x_i^*)$ might be used in order to act with respect to the principle of the minimal total loss:

$$\sum_{i=1}^{n} C_i(\cdot,x_i,x_i^*) = \min \quad .$$

Generally such approximation has not to be <u>linear</u> because a small error in such approximation might give us an absolutely wrong result concerning the choice of $x_1,\dots,x_n$. For example, in the case of two systems S_1,S_2 with linear loss functions

$$C_i(\cdot,x_i,x_i^*) = \lambda_i(x_i^* - x_i)$$

the minimization of the total loss

$$\lambda_1(x_i^* - x_1) + \lambda_2(x_2^* - x_2) \quad .$$

With coefficients $\lambda_2 > \lambda_1$ under the constraints (1) makes us supply to the second system S_2 <u>as much as it is possible</u> so the first system S_1 might be without WR at all; that seems obviously non-realistic if λ_1 is only a very little smaller than λ_2.

Thus a linear loss function gives an absolute privilege for one or a few systems that seems non-realistic and we have to take some care about the approximation of actual loss.

We suggest a quadratic approximation which has a good "robustness" property with respect to possible errors in our loss estimation. Namely, we suggest to take

$$C_i(\cdot, x_i, x_i^*) = \lambda_i (x_i^* - x_i)^2$$

with the proper coefficient λ_i, $i = 1,\ldots,n$. Then the problem on optimal choice of the corresponding inflows $x_1,\ldots,x_n$ can be solved on the basis of minimization of the total loss function

$$r(x,x^*) = \sqrt{\sum_{i=1}^{n} \lambda_i (x_i - x_i^*)^2} \quad .$$

The various aspects of this problem, including the various conditions of uncertainty and decision making under risk, were presented recently at an IIASA seminar [1].

Note that under the decision to supply for the system S_i the proper WR, we can divide this total inflow x_i in the corresponding components x_{ik} independently of other systems S_j, $j \neq i$.

2. Inputs of WR LSS

A collection of water data $w\cdot(t) = \{w_1(t),\ldots,w_n(t)\}$ for a water basin has to be considered as a multivariate random process.

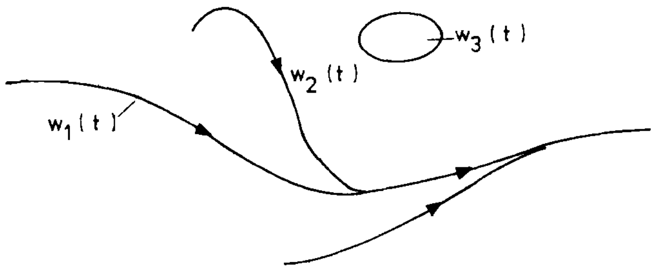

FIGURE 3

Usual tool of its analysis is based on the corresponding mean value vector function.

$$A(t) = Ew(t) = \{A_1(t),\ldots,A_n(t)\}$$

and correlation matrix function

$$B(s,t) = D^{\frac{1}{2}}(s)\ R(s,t)\ D^{\frac{1}{2}}(t)^*$$

where

$$D^{\frac{1}{2}}(t) = \{D_1^{\frac{1}{2}}(t),\ldots,D_n^{\frac{1}{2}}(t)\}$$

is a vector of standard deviations

$$D_k^{\frac{1}{2}}(t) = \left(\mathrm{Var}\ w_k(t)\right)^{\frac{1}{2}} \quad , \quad k = 1,\ldots,n,$$

and

$$R(s,t) = \{R_{kj}(s,t)\}$$

is a matrix of correlation coefficients

$$R_{kj}(s,t) = \frac{E[w_k(s) - A_k(t)]\ [w_j(t) - A_j(t)]}{D_k^{\frac{1}{2}}(s)D_j^{\frac{1}{2}}(t)}$$

Concerning various inputs $w_1(t),\ldots,w_n(t)$ of WR system, which are water streams, levels of water reservoirs, etc. it seems reasonable to assume that all these components are positively correlated:

$$R_{kj}(s,t) \geq 0$$

because its increasing (or decreasing) usually occurs for the same reason--snow melting, rain, drought, etc.--so increasing (or decreasing) of some components occurs with the same phenomena as for other components, and similar connection takes place in time.

We are not going to discuss in detail a structure of functions $A(t)$, $D(t)$, and $R(s,t)$ but note that usually $A(t)$, $D(t)$ are assumed to be seasonal periodic functions and

$$R(s,t) = R(t - s)$$

is assumed to be a correlation function of multivariate stationary Markov type random process (multidimensional auto-regression model).

Concerning probability distributions for $w_1(t),\ldots,w_n(t)$, one usually assumes that each component has a proper gamma distribution.

Now the following problem arises: What type of <u>multi-dimensional distribution</u> for the vector input $w(t) = \{w_1(t),\ldots,w_u(t)\}$ is consistent with all the properties mentioned above, namely with the given positive correlation coefficients $R_{kj}(s,t)$ and marginal gamma distributions?

We suggest considering some kind of multidimensional gamma distribution which is completely determined with the corresponding parameters $A(t)$, $D(t)$ and $R(s,t)$.

We prefer to describe this multivariate distribution in a way which is convenient for actual <u>modelling</u> (for synthetic hydrology).

Let

$$\{\eta_{ik}(t),\ i = 1,\ldots,m_k,\ k = 1,\ldots,n\}$$

be a series of independent standard Gaussian variables. With well known linear methods we can obtain identically distributed Gaussian processes

$$\xi_{ik}(t), \quad i = 1,\ldots,m_k,$$

(independent for different $i = 1,\ldots,m_k$) such that

$$E\xi_{ik}(t) = 0,\ \mathrm{Var}\ \xi_{ik}(t) = \sigma_k(t)^2$$

and

$$\mathrm{Cor}\ \{\xi_{ik}(s),\ \xi_{ij}(t)\} = \rho_{kj}(s,t) \geq 0 \quad .$$

Let us consider

$$w_k(t) = \sum_{i=1}^{m_k} \xi_{ik}^2(t), \qquad k = 1,\dots,n.$$

We have

$$Ew_k(t) = m_k \cdot E\xi_{1k}^2(t) = m_k \cdot 6_k(t)^2$$

$$\text{Var } w_k(t) = m_k \cdot \text{Var } \xi_{1k}^2(t) = 2m_k \cdot 6_k(t)^4$$

and

$$E[w_k(s) - Ew_k(s)][w_j(t) - Ew_j(t)]$$

$$= \min(m_k, m_j)\, E[\xi_k(s)^2 - 6_k(s)^2][\dot{\xi}_j(t)^2 - 6_j(t)^2]$$

$$= \min(m_k, m_j)\, 6_k(s)^2 6_j(t)^2 \rho_{kj}(s,t)^2 \quad ,$$

$$\text{Cor}'\{w_k(s), w_j(t)\} = \frac{\min(m_k, m_j)}{m_k^{\frac{1}{2}}\, m_j^{\frac{1}{2}}}\, \rho_{kj}\ (s,t)^2 \ .$$

Thus, if we set

$$2m_k 6_k(t)^4 = D_k(t) \quad ,$$

$$\frac{\min(m_k, m_j)}{m_k^{\frac{1}{2}}\, m_j^{\frac{1}{2}}}\, \rho_{kj}(s,t) = R_{kj}(s,t) \quad ,$$

then the multivariate random process with components

$$w_k(t) + A_k(t) - m_k 6_k(t)^2 \quad , k = 1,\dots,n$$

has the given parameters

$$A_k(t),\ D_k(t) \text{ and } R_{kj}(s,t), \quad k,j = 1,\ldots,n.$$

All marginal distributions are gamma distributions, namely the probability density of the variable $w_k(th)$ is

$$f(x) = \frac{x^{\beta-1}e^{-x/\alpha}}{\alpha^{\beta}\Gamma(\beta)}, \quad x > 0 \quad ,$$

where

$$\alpha = 2\ 6_{kk}(t) \quad , \qquad \beta = \tfrac{1}{2}\, m_k \quad .$$

More general multidomensional gamma-type distributions wer suggested by D.R. Krishnaiah and H.M. Rao [2]. See also [3].

References

[1] Letov, Alexander M. and Rozanov, Yuri A. "On Optimal Compromise for Multidimensional Resource Distribution." IIASA Research Report RR-74-8, June 1974.

[2] Krishnaiah, D.R. and Rao, H.M. "Remarks on a Multivariate Gamma-distribution," Amer. Math. Monthly, Vol. 68, No. 4 (1961).

[3] Siddiqui, U.U. "Some Properties of Empirical Distribution Function of a Random Process, J. of Research of National Bureau of Standards, Vol. 65B, No. 2 (1961).