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Dependent Risks and Ruin Probabilities in Insurance

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Abstract

Classical risk process models in insurance rely on independency. However, especially when modelling natural events, this assumption is very restrictive. This paper proposes a new approach to introducing dependency structures between events into the model and investigates its effects on a crucial parameter for insurance companies, the probability of ruin. Explicit formulas, numerical simulations and sensitivity results for dependence are established for different dependency models of first-order markovian type indicating that for various scenarios dependency considerably increases the probability of ruin.

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1 Introduction

1.1 The need for dependency models

One of the important issues of insurance industry is how to cope with the effective insurance of natural catastrophies - events of low probability and high impact. It is the size and frequency of these events that considerably influences a crucial parameter for the risk exposure of an insurance company, the probability of ruin.
Its estimation traditionally is done by assuming independence in space and time between these natural events (see Section 2.1). This is mostly due to the fact that up to now there are in general no explicit analytical formulas for the calculation of ruin probabilities for dependent events available in the literature.
Analytical modelling of extremal events can so far only be handled by introducing distributions with heavy tails (i.e. higher probability for large damage occurences) for the claim sizes in the independent model. In such a model the ruin probability turns out to be essentially determined by the distribution of the maximum value of the claim sizes [Embrechts 1997]. But extremal events can be triggered by inherent dependencies and not necessarily have to be the result of an underlying heavy-tail-distribution of the claims. If this fact is neglected, the estimated ruin probability can, as is shown in this paper, in many realistic scenarios be drastically lower than its real value. According to many climate experts, extremal natural events will get more likely in the future due to recent climate changes (cf.[MacDonald 1997]). Thus an improvement of models for dependent catastrophic events is inevitable.

Classical estimations for the ruin probability are furthermore based on the fact that the number of claims up to an arbitrary time t constitutes a Poisson process (cf. [Grandell 1992]), meaning that the time distance back to the last event does not influence the timing for the next event to happen. That essentially would mean that nature has no memory and causal intertwinings concerning its catastrophies do not exist. However, apart from an intuitive objection to that restriction (think for instance of the atmosphere and the ocean which can store

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extremal conditions over years), looking at various data bases of natural catastrophies in the recent decades, this assumption seems very unlikely (see Section 1.2).

There are some generalizations of the Poisson process to so-called Cox-models (for a nice summary see [GRANDELL 1992]), where the intensity of the Poisson-process is not constant anymore, but an M-state Markov process. REINHARD [1984] has worked out an explicit formula for the ruin probability for a 2-state Markov process. However, all these Cox-models need the inter-occurence times to be exponentially distributed and are not capable of considering dependencies between consecutive claim sizes. This paper proposes a different and somewhat more general technique to embed the classical independent models into models including dependency.

Recently numerical tools to optimise portfolio selections and insurance strategies in the presence of dependent catastrophic events have been developed (see [ERMOLIEVA 1997] and [ERMOLIEVA ET AL. 1997]). This approach uses adaptive Monte-Carlo-methods, which essentially require analytical blocks. Therefore it is also from this point of view highly desirable to develop analytical models for dependency structures.
In practice, often the first moments of a certain distribution are estimated and used as the basis of the calculation of ruin probabilities. However, as is shown in this paper, dependent models with first moments identical to those in the independent case can be formulated, ending up with ruin probabilities being higher at several orders of magnitude (see Section 3.2).

Section 2 summarizes the classical approach and gives a theorem for a generalization of an approximation result for the ruin probability. Section 3 discusses general concepts of dependence and thereafter different classes of dependent models of first order markovian type are introduced and compared. Sensitivity results of the ruin probability under dependency are, both analytically and numerically, established and applied to the data of Section 1.2.

1.2 Empirical data

Figure 1 shows the occurence of damages with an insurance industry loss of more than US$ 50,000,000 (normalized to the 1987 market) caused by Tornados in the USA between 1949 and 1986 (data taken from [FRIEDMAN 1987]).
Even for the naked eye correlation and clustering is visible. Calculation of the autocorrelation coefficient ρ for the one-step-autocorrelation model

$$X_{i+1} = \rho X_i + e_i,$$

where e_i is the residuum and X_i is the corresponding variable of interest, yields $\rho = 0.3$ for the correlation between the interoccurence times and $\rho = 0.06$ between the damage heights, clearly indicating that these events are dependent on each other.

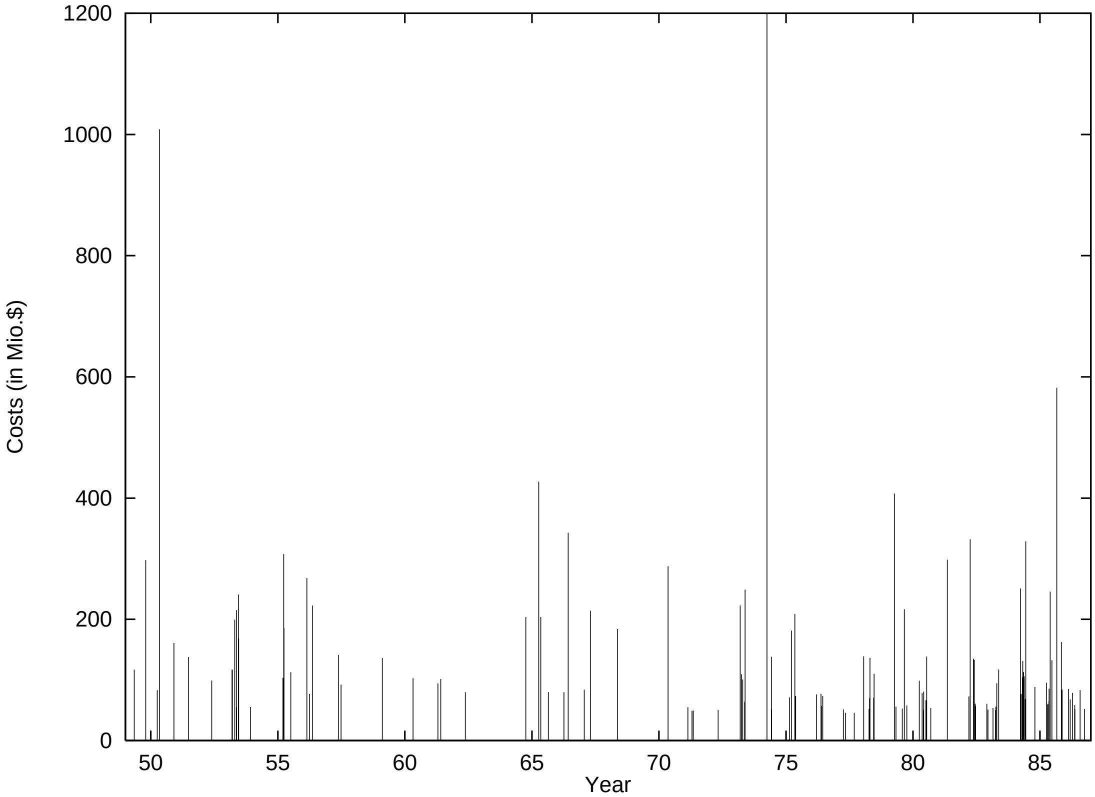

Figure 1: Insurance industry losses caused by Tornados in the USA between 1949 and 1986

2 The Classical Model

A risk process can be modelled as follows: The occurence of the claims is described by a point process and the amounts of money to be paid by the company at each claim by a sequence of random variables. The company receives a certain amount of premium to cover its liability and the difference between the premium income and the (average) cost for the claims is the 'safety loading' r.

In the classical so-called Cramer-Lundberg-model [Grandell 1992] the claim sizes $Y_1, Y_2, \ldots$ are independent identically distributed (iid) positive random variables with distribution function F and finite mean $\mu = E(Y_1)$. The claims Y_k arrive at time $T_k = D_1 + \cdots + D_k$ according to a homogeneous Poission process, where the inter-occurence times $D_1, D_2, \ldots$ are independent exponentially distributed random variables with intensity α. If the corresponding claim number is $N(t) = \sup\{k \geq 1 : T_k \leq t\}$, $t \geq 0$, then the capital at time t is

$$U(t) = u + ct - V(t), \quad t \geq 0,$$

where u is the initial capital, $V(t) = \sum_{n=1}^{N(t)} Y_n$ the total claim amount until time t and $c > 0$ is the premium income rate. The ruin probability in infinite time is then defined by

$$\psi(u) = P\left(\inf_{0 \leq t \leq \infty} U(t) < 0\right).$$

For the case of exponentially distributed claim sizes Y_k there exists an explicit formula for the ruin probability first established by [LUNDBERG 1903]:

$$\psi(u) = \frac{1}{1+r} e^{-\frac{ru}{\mu(1+r)}}, \tag{1}$$

where $r = \frac{c}{\alpha\mu} - 1$ is the safety loading, which is positive for a reasonable insurance policy. Using an empirical value for μ and α, a desired value for the ruin probability can in practice (under the assumption of independence between the claims) be approximated by choosing a suitable risk premium rate c (this procedure relies on the law of large numbers which only holds for independent events).

In the more general case of $Y_1, Y_2, \ldots$ being identically distributed according to an arbitrary (but exponentially bounded) distribution function F, there exists the famous Cramer-Lunberg approximation

$$\lim_{u\to\infty} e^{Ru}\psi(u) = C, \tag{2}$$

where C is a a constant and the so-called *Lundberg exponent* R depends on F and is given by a functional equation (see GRANDELL 1992]).
This asymptotic result (2) is only proved for exponentially distributed D_i. However, the theorem at the end of this section shows, that it holds for a still wider class of distributions.

For further purposes it is convenient to rewrite the process in the form of a random walk

$$S_i = \sum_{j=1}^{i} X_j \tag{3}$$

where $X_i = Y_i - cD_i$, so the process direction is reversed (starting at $U(t) = 0$ and causing ruin, if $U(t) = u$). In the classical model, $X_1, X_2, \ldots$ is an iid sequence of random variables. Note that for a reasonable choice of the premium rate c, $E(X_i) < 0$, so the random walk has negative drift. Using this notation the ruin probability can equivalently be defined by

$$\psi(u) = P\left(\max(0, S_1, S_2, \ldots) \geq u\right). \tag{4}$$

If the distribution function F is normal with mean μ and variance σ^2, then the random walk S_n can be seen as discrete equidistant points taken from its continuous analogon, the well-known Wiener process $W(t)$. But for a Wiener process $W(t)$, it is known that the ruin probability is

$$\psi(u) = P(\max_{t\geq 0}\left[\sigma W(t) - \mu t\right] \geq u) = e^{-\frac{2\mu u}{\sigma^2}},$$

so there exists a Lundberg exponent $R = 2\mu/\sigma^2$ for the continuous case. For our discrete random walk with normal distribution, the following theorem applies and hence shows the existence of a Lundberg exponent R:

Theorem 1 *Let F be the common (compound) distribution function of the iid X_i. If F is such that there exists a constant ν with*

$$\int_{-\infty}^{\infty}(1-F(u))e^{\nu u}du = 1, \tag{5}$$

then the Cramer-Lundberg approximation

$$\lim_{u\to\infty} e^{Ru}\psi(u) = C$$

holds for some constant C.

Proof: The ruin probability can expressed by the recursive equation

$$\psi(u) = \begin{cases} 1 - F(u) + \int_{-\infty}^{u}\psi(u-v)dF(v) & \text{if} \quad u \geq 0 \\ 1 & \text{if} \quad u < 0 \end{cases} \tag{6}$$

or equivalently by

$$\psi(u) = 1 - F(u) + \int_{-\infty}^{u}\psi(u-v)dF(v) + g(u)$$

with

$$g(u) = \left(F(u) - \int_{-\infty}^{u}\psi(u-v)dF(v)\right)1_{\{u<0\}}.$$

We introduce the transformation

$$\hat{\psi}(t) = \int_{-\infty}^{\infty} e^{tu}\psi(u)du$$

for $0 < t \leq \nu$. Let $h(u) = 1 - F(u) + g(u)$. Using the transformation rule for the convolution this yields

$$\hat{\psi}(t) = \hat{h}(t) + \hat{\psi}(t)\hat{F}(t)$$

or

$$\hat{\psi}(t) = \frac{\hat{h}(t)}{1-\hat{F}(t)}.$$

Now $1 - \widehat{F}(u)$ is by the precondition for $t \leq \nu$ finite and $\hat{g}(t)$ is finite, because $g(u) = 0$ for $u \geq 0$, so $\hat{h}(t)$ is finite. For $t = \nu$ the transorm $\hat{\psi}$ has a singularity and therefore the behavior of ψ in the original domain for $u \to \infty$ corresponds to $t \to \nu$ in the transformed domain. To reveal the behavior of $\hat{\psi}(t)$ near $t = \nu$, we linearize and obtain

$$\hat{\psi}(t) = \frac{\hat{h}(\nu)}{c_1(\nu - t)}$$

in the vicinity of $t = \nu$ with $c_1 = \hat{F}'(\nu) = \int_{-\infty}^{\infty} ue^{\nu u}dF(u)$ being a positive constant. But this is exactly the transform of a function of the form

$$\psi(u) = Ce^{-\nu u}. \qquad //$$

3 The Dependent Model

3.1 Concepts of dependence

We now want to introduce dependency between consecutive random variables in a sequence into the model (throughout this paper we will consider first order Markovian processes only (i.e. a random variable X_{i+1} depends on the past $X_1, \ldots, X_i$ only through its direct 'precessor' X_i)). Let us define what is meant by dependency: For simplicity we will restrict our considerations to positive dependence or 'association'. Heuristically spoken a pair (X, Y) is positively dependent, if large values of Y tend to be associated with large values of X and small values of Y tend to be associated with small values of X.

Following [Lehmann 1966] there are three successively stronger definitions of positive dependence:
For a first definition, one can compare the probability of any quadrant $X \leq x, Y \leq y$ with the corresponding probability in the case of independence. A pair (X, Y) is then *positively quadrant dependent*, if

$$P(X \leq x, Y \leq y) \geq P(X \leq x)P(Y \leq y) \qquad \text{for all}\, x, y. \tag{7}$$

In that case, as can be shown,

$$E(XY) \geq E(X)E(Y)$$

if the expectations exist, with equality holding if X and Y are independent. This essentially means that if a pair (X, Y) is positively quadrant dependent, then the covariance (one of the well-known measures of association), when existing, is $Cov(X, Y) \geq 0$.
Definition (7) expresses the fact that knowledge of X being small increases the probability of Y being small. A stronger condition is the following: Y is *positively regression dependent* on X, if

$$P(Y \leq y \,|\, X = x) \text{ is non-increasing in}\, x, \tag{8}$$

Condition (8) requires the conditional variable Y given x to be stochastically increasing. An even stronger condition is obtained by requiring the conditional density of Y given x to have monotone likelihood ratio, i.e.

$$f(x, y')f(x', y) \leq f(x, y)f(x', y') \qquad \text{for all } x < x', \quad y < y'. \tag{9}$$

If (9) holds, then (X, Y) are said to be *positive likelihood ratio dependent.*

3.2 The normal copula model

One typical way of realizing dependence between two random variables in a mathematical model is to combine them through a copula, which is defined on the unit square and combines two uniformly distributed variables according to some dependence structure. There are various copula functions available in the literature (cf.

[JOE 1993]). However, for our purpose of consecutive variables in a sequence we need to express the dependent variable by its 'precessor' and an independent variable in a compact form. This is only possible for the so-called 'normal copula'

$$C(u, v; \rho) = \Phi_{\Sigma(\rho)}(\Phi^{-1}(u), \Phi^{-1}(v)),$$

where Φ_Σ is the bivariate normal distribution function with mean vector zero and covariance matrix

$$\Sigma(\rho) = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \; -1 \leq \rho \leq 1$$

and u and v are uniformly distributed.

Let now $X_1, Z_2, Z_3, \ldots$ be iid with distribution function F (where the notation of (3) is used). By the fact that a distribution function of a random variable is itself a uniformly distributed variable, the copula can be rewritten to

$$F(X_{i+1}) = \Phi\left(\rho\Phi^{-1}(F(X_i)) + \sqrt{1-\rho^2}\Phi^{-1}(F(Z_{i+1}))\right), \tag{10}$$

with Φ denoting the standard normal distribution. This formula can now be used to simulate a sequence X_i, with adjacent variables in the sequence being connected through the normal copula. It should be noted, that the marginal distributions of the X_i stay unchanged, which is highly desirable for a dependence model, as changes in the marginal distribution ('the independent variables') would have additional impact on the value of the ruin probability and this effect could not be separated from dependency reasons.

In (10) we have $X_{i+1} = X_i$ (complete dependence) for $\rho = 1$ and X_{i+1} is independent from X_i for $\rho = 0$. It can easily be verified that X_{i+1} is positive regression dependent on X_i for $\rho \geq 0$.

We now establish a sensitivity result for the trajectory on the boundary between the independent and dependent case. From

$$X_{i+1} = F^{-1}\,\Phi\left(\rho\Phi^{-1}(F(X_i)) + \sqrt{1-\rho^2}\Phi^{-1}(F(Z_{i+1}))\right)$$

we get

$$\left.\frac{\partial X_{i+1}}{\partial \rho}\right|_{\rho=0} = \frac{\varphi(\Phi^{-1}(F(X_{i+1})))\Phi^{-1}(F(X_i))}{f(F^{-1}(Z_{i+1}))},$$

with φ and f denoting the densities of Φ and F, respectively. We eventually derive

$$\left.\frac{\partial X_{i+1}}{\partial \rho}\right|_{\rho=0} = \frac{e^{\frac{-\Phi^{-1}(F(X_{i+1}))^2}{2}}\Phi^{-1}(F(X_i))}{2\pi f(X_{i+1})}. \tag{11}$$

This formula depends on F and is a sensitivity result only for the single X_i, not for the sum; however it shows that extreme events increase this derivative for later events considerably and thus inherit dependency can increase the probability of ruin. For the simple case when F is a normal distribution $N(-a, 1),\ a > 0$, formula (11) simplifies and by summation one gets

$$\frac{\partial S_{i+1}}{\partial \rho} = S_i + ia,$$

a structure that for the autoregressive model in the next section will occur for a larger class of distributions F and will be interpreted there.
It is not possible to draw general conclusions from (11), but for certain distributions F the above result certainly means an increase of ruin probability for increasing ρ; the following simulation gives for instance some insight into the case, where F is a convolution of an exponential distribution for the inter-occurence times and an exponential distribution for the claim sizes:

3.2.1 Simulation technique

The risk process can be simulated on a computer by randomly drawing sample paths (starting at initial capital u) according to a given distribution and measure the percentage of trajectories leading to ruin until some finite time T, which is then for a suitable large T an estimation for the ruin probability for infinite time:

$$\hat{\psi}(u) = \frac{1}{N} \sum_{Z} 1_A(z_t),$$

where Z is the set of all N trajectories z_t and $A \subset Z$ is the set of trajectories leading to ruin. As this Monte-Carlo-Method would need a huge number N of runs to keep the variance (proportional to $1/\sqrt{N}$) of the result below some appropriate threshold, an additional reduction of variance, a method called *importance sampling* was applied:
If f is the density of the single steps X_i of a trajectory, and g is a density chosen in a way that it leads considerably more trajectories to ruin than f, then

$$\begin{aligned} P(z_t \in A) &= E_f(1_A(z_t)) = \int 1_A(z_t) f(x) dx = \int 1_A(z_t) \frac{f(x)}{g(x)} g(x) dx \\ &= E_g \left(1_A(z_t) \frac{f(z_t)}{g(z_t)} \right). \end{aligned}$$

So the simulation uses deviates from a distribution according to density g and corrects the values in each step obtaining

$$\hat{\psi}(u) = \frac{1}{N} \sum_{Z} 1_A(z_t) \frac{\prod_i f(z_{t_i})}{\prod_i g(z_{t_i})}.$$

This method is most effective, if approximately half of the trajectories due to g lead to ruin (see e.g. [Pflug 1996]).
The random numbers used in the simulations are generated by the well-known linear-congruence-method and the random variables are then derived using the inversion method [Devroye 1986].

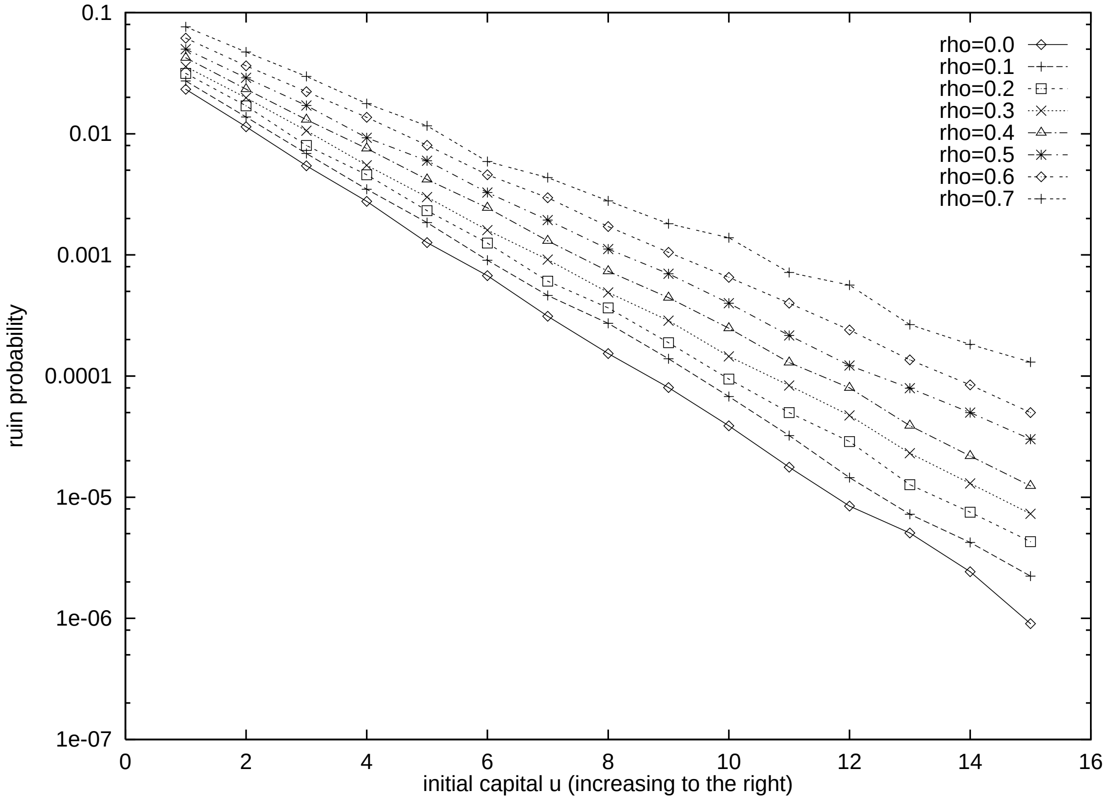

Figure 2: Simulation of ruin probabilities for normal-Copula dependent variables ($N = 10000$)

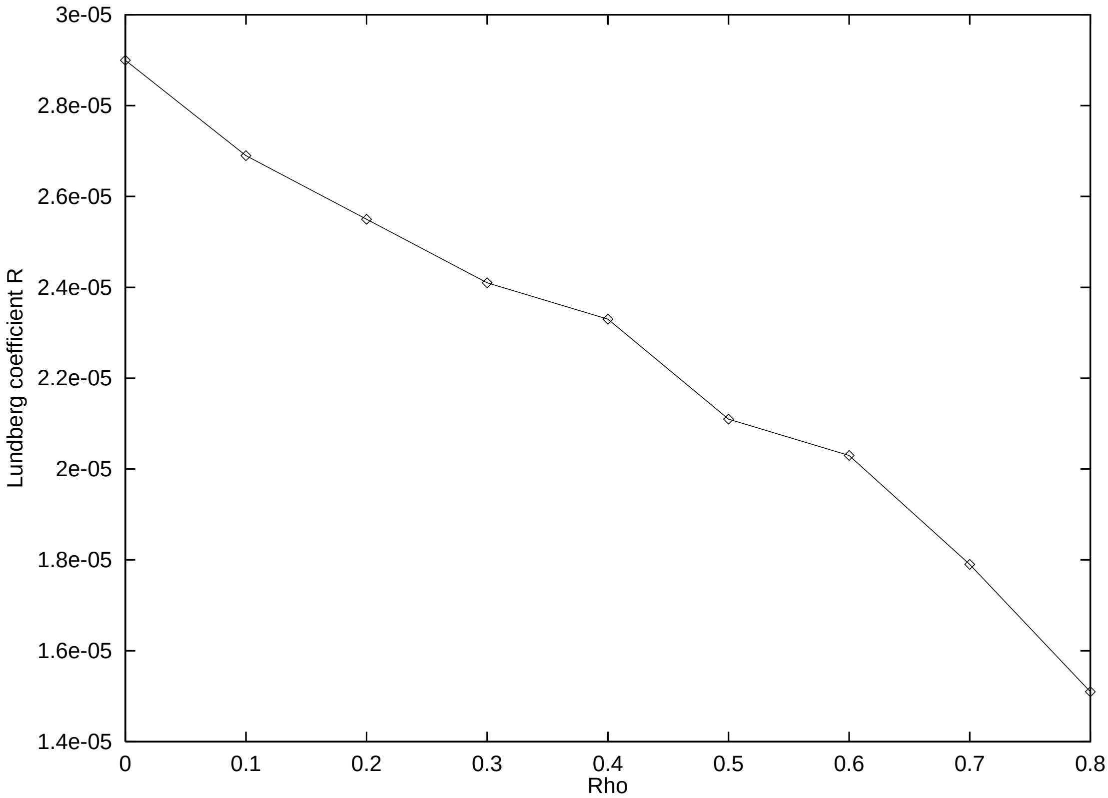

Figure 3: Relationship between R and Rho

3.2.2 Results

Figure 2 shows a simulation for the ruin probabilities for a range of realistic initial capitals u. Y_i are exponentially distributed with intensity α and the D_i are exponentially distributed with mean μ; the values of α and μ chosen according to the data of the example of Section 1.2. The premium rate c is constant throughout the simulation and its value is chosen such that for the independent case $\rho = 0$ we get a value for the ruin probability that would be desireable for insurance companies. Note that the line $\rho = 0$ represents the analytical Lundberg-formula (1).
It is clearly visible that an increase of ρ increases the ruin probability by orders of magnitude and as the diagram depicts the behavior on a logarithmic scale, one can conclude that there exists a Lundberg-exponent R also for the dependent cases, with its value decreasing for increasing ρ. Figure 3 illustrates the relationship between ρ and R implying a lower rate for the decrease of the ruin probability when the initial capital u is increased (the absolute numerical values of R depend on the units of all the parameters involved in the model, therefore it is the qualitative behavior that is of interest in this diagram). Ignoring possible dependence of the normal-copula kind might thus lead to wrong decisions for the choice of u.

The probable existence of R and its derivation for dependent cases is discussed from the viewpoint of the autoregressive model at the end of the following section.

3.3 Autoregressive models

Following the notation of (3) let $X_1, Z_2, Z_3, \ldots$ be independently distributed with (compound) distribution function F. Let furthermore $E(X_1) = E(Z_i) = E(Y_1) - cE(D_1) = -a, \quad a \geq 0$. Then we define the autoregressive scheme

$$X_{i+1} = -a + \rho(X_i + a) + \sqrt{1-\rho^2}(Z_{i+1} + a). \tag{12}$$

Obviously X_{i+1} is independent of X_i if $\rho = 0$, and they are identical, if $\rho = 1$. It is easy to see that $E(X_{i+1}) = -a$ and $Var(X_{i+1}) = Var(X_1)$, so the first two moments of X_{i+1} are the same as in the independent case for every ρ. However, if we calculate the distribution function for X_{i+1}, we get

$$\begin{aligned} P(X_{i+1} \leq w) &= \int P\left(X_{i+1} \leq w | X_i = v\right) dF(v) = \\ &= \int F\left(\frac{w - a(\rho - 1 + \sqrt{1-\rho^2}) - \rho v}{\sqrt{1-\rho^2}}\right) dF(v). \end{aligned} \tag{13}$$

Clearly, X_{i+1} is regression dependent on X_i.
For $\rho \neq 0$ the distribution function has changed its shape compared to the independent case (opposed to the Normal-copula model of the last section), but as we now want to obtain results for the vicinity of the independent case, we see that the distribution function changes very slowly with increasing ρ, as its derivative equals zero:

$$\left.\frac{\partial F(w)}{\partial \rho}\right|_{\rho=0} = \int f(w)(-a - v) dF(v) = 0.$$

Note that for $F = N(-a,1)$ the autoregressive model and the Copula model of the last section are the same.
From (12) we see that

$$\left.\frac{\partial X_{i+1}}{\partial\rho}\right|_{\rho=0} = X_i + a$$

and therefore

$$\left.\frac{\partial S_{i+1}}{\partial\rho}\right|_{\rho=0} = S_i + (i-1)a \tag{14}$$

3.3.1 Results

We can now establish a result on the qualitative behavior of the ruin probability, if dependency of autoregressive kind comes into the model:

Theorem 2 *Let $I = \inf\{i : S_i > u\}$. Then for the autogressive model (12)*

$$\left.\frac{\partial\psi(u)}{\partial\rho}\right|_{\rho=0} = E\left([S_{I-1} + (I-1)a]\, f(u - S_{I-1})\right) > 0. \tag{15}$$

Proof: $\psi(u) = \sum_{i=1}^{\infty} P(S_i^\rho \geq u, S_{i-1}^\rho < u, \ldots, S_1^\rho < u)$, where the upper index ρ indicates the dependence on ρ. Now S_i^ρ can (by virtue of (14) be linearized (w.l.o.g. as will be seen in the sequel) to $S_i^\rho = S_i^0 + \rho(S_{i-1} + (i-1)a)$ and hence

$$\begin{aligned}
\frac{\partial\psi(u)}{\partial\rho} &= \sum_{i=2}^{\infty}\frac{\partial}{\partial\rho}P(S_i^\rho \geq u, S_{i-1}^0 < u, \ldots, S_1^0 < u) \\
&= \sum_{i=2}^{\infty}\frac{\partial}{\partial\rho}P(S_i + \rho(S_{i-1} + (i-1)a) \geq u, S_{i-1}^0 < u, \ldots, S_1^0 < u)
\end{aligned}$$

$$= \sum_{i=2}^{\infty}\frac{\partial}{\partial\rho} \underset{\substack{S_i+\rho(S_{i-1}+(i-1)a)\geq u \\ S_{i-1}<u,\ldots,S_1<u}}{\int\cdots\int} f(S_i - S_{i-1})f(S_{i-1} - S_{i-2})\cdots f(S_1)dS_i\cdots dS_1$$

and by the transformation $S_i - u = t$

$$= \sum_{i=2}^{\infty}\lim_{\rho\to 0}\frac{1}{\rho} \underset{S_{i-1}<u,..,S_1<u}{\int\cdots\int} \int_{-\rho(S_{i-1}+(i-1)a)}^{0} f(u{+}t{-}S_{i-1}))dt\, f(S_{i-1}{-}S_{i-2})\cdots f(S_1)dS_{i-1}\cdots dS_1$$

$$= \sum_{i=2}^{\infty} \underset{S_{i-1}<u,..,S_1<u,S_i>u}{\int\cdots\int} [S_{i-1}{+}(i{-}1)a]f(u{-}S_{i-1})f(S_{i-1}{-}S_{i-2})\cdots f(S_1)dS_{i-1}\cdots dS_1.$$

Looking at the disjoint sets $A_{i+1} = \{S_1 < u, \ldots, S_i < u, S_{i+1} \geq u\}$, we see that the integration domain of the above equation is just the disjoint union $\cup_{i=2}^{\infty}A_i$ and this yields the expected value (15) on all the ruin trajectories. //

This theorem essentially expresses the remarkable fact that only the last value S_{I-1} before ruin of a ruin trajectory is relevant for the dependence of the ruin probability on increasing ρ. ASMUSSEN [1982] has shown, that under condition (5) a sample path leading to ruin has locally a linear drift with slope $\kappa'(\nu)$ just before ruin happens, where $\kappa(s) = E(e^{sX})$ and ν is the exponent of (5). But the derivative $\kappa'(\nu)$ is positive, because $\kappa'(0) = E(Z)$ is negative and $\kappa(s)$ is a convex function. This local behavior is in contrast to the global picture (where the drift of S_n is negative) and is closely related to the fact that the distribution function of a sample path leading to ruin can be approximated by the 'associated' Esscher-transformed distribution function

$$H_\nu(x) = \int_{-\infty}^{x} e^{\nu y} dF(y), \quad x \in \mathcal{R},$$

which does not need the knowledge about whether or not the trajectory will lead to ruin [EMBRECHTS 1997].
For interpreting Theorem 2 this now means (qualitatively spoken) that the value of S_{I-1} is close to u and most probably positive. Therefore the value of the derivative $\frac{\partial\psi(u)}{\partial\rho}\Big|_{\rho=0}$ is certainly positive and very probably high, indicating a drastic increase of the ruin probability compared to the independent case, if dependency according to the autoregressive scheme (12) is in the model. Hence neglecting dependency structures of this kind leads to severe underestimations of the ruin probability.

3.3.2 An advanced model

A refinement of model (12) is that different correlation coefficients for inter-occurence times and damage heights are allowed (which might be the case in most of the applications). So the variable $X_i = Y_i - cD_i$ is now composed by

$$\begin{cases} Y_i = \mu + \rho_1(Y_{i-1} - \mu) + \sqrt{1-\rho_1^2}(Y_i^0 - \mu) \\ D_i = \frac{1}{\alpha} + \rho_2(D_{i-1} - \frac{1}{\alpha}) + \sqrt{1-\rho_2^2}(D_i^0 - \frac{1}{\alpha}), \end{cases}$$

where Y^0 and D^0 are independent variables of the corresponding distributions. Note that for $\rho_1 = \rho_2$ we have (12). (In the Tornado damage case of Section 1.2 we had $\rho_1 = 0.06$ and $\rho_2 = 0.3$). It can easily be shown that again the X_i have the same mean value and standard deviation as in the independent case for all ρ. If we now for instance fix ρ_1 we get

$$\frac{\partial S_i}{\partial \rho_2}\Big|_{\rho_2=0} = c\left(\frac{i-1}{\alpha} - \sum_{j=1}^{i-1} D_j\right)$$

and applying the technique of the proof of Theorem 2 we obtain

$$\frac{\partial \psi}{\partial \rho_2}\Big|_{\rho_2=0} = E\left(c\left[\frac{I-1}{\alpha} - \sum_{j=1}^{I-1} D_j\right] f(u - S_{I-1})\right) \tag{16}$$

with I again being the index for which ruin occurs. This expected value is clearly positive (as the inter-occurence times of a ruin trajectory have to be below their expectation) and therefore the ruin probability increases considerably, if ρ increases. We also see that the premium rate c directly influences the sensitivity of ψ for dependency of the variables (which was also the case for model (12), where $a = cE(D_i) - E(Y_i)$).

3.3.3 An integral equation for complete solution

The simulation results for the copula model in Section 3.2 seem to indicate the existence of a Lundberg-coefficent (that is an expoenential rate R for the decrease of the ruin probability with u) also in the dependent model. Motivated to establish a similar result for the autoregressive model, it looks appealing to generalize the integral equation (6) of Section 2 to the dependent case. Let $\psi(u, v)$ denote the probability of ruin for a capital u, if the last claim size was v. Then, with (12), we get

$$\psi(u,v) = \begin{cases} 1 - F(u - \rho(v+a)) + \int\limits_{-\infty}^{u} \psi(u-w, w) dF(w - \rho(v+a)) & \text{if } u \geq 0 \ \forall v \\ 1 & \text{if } u < 0 \ \forall v \end{cases} \tag{17}$$

and finally the solution of

$$\psi(u) = 1 - F(u) + \int_{-\infty}^{u} \psi(u, v) dF(v)$$

is the desired ruin probability for the dependent autoregressive model. The analytic solution of equation (17) will reveal the exact behavior of ψ concerning changes in ρ and answer the question about the existence of a Lundberg exponent R. However, so far it remains an open problem to solve (17).

3.4 Further models

There are plenty of possibilities to formulate reasonable dependency structures. We want to show simulation results for two more models:

Model (a) gives some insight into the impact of long memory: Let us assume the inter-occurence times D_i to be exponentially distributed with intensity α and the claim sizes be exponentially distributed with intensity λ_0, if the previous event happened 'sufficiently long' ago ($D_{i+1} \geq t^*$), where the threshold t^* is a parameter for dependence and is expressed in multiples of the expected value of the inter-occurence time. If the event X_{i+1} happens for $D_{i+1} < t^*$, then its claim size Y_{i+1} is simulated by a normal distribution with mean Y_i and the variance linearly decreasing for deacreasing time D_{i+1} between Y_i and Y_{i+1} is. So if the threshold is set at $t^* = 0$, our model is just the independent double-exponential classical model of Section 2.1 and its ruin probability can be calculated by (1). The bigger the threshold is, the longer is the 'memory' about the last event (so the claim sizes are dependent, whereas the events happen independently from each other). It is

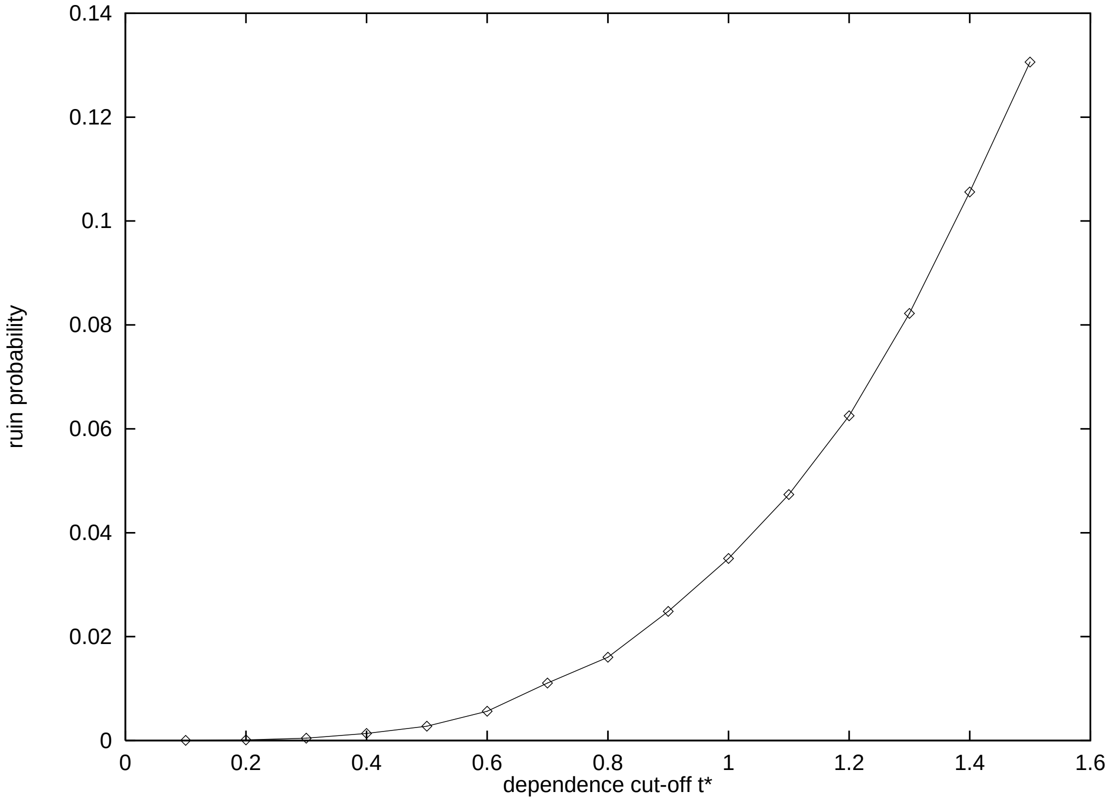

Figure 4: Ruin probabilities for Model (a) (N=50000)

important to note that this model leaves the expected value μ for the claim sizes unchanged (compared to the independent model), so the results really represent dependence impacts and are not due to other biases. Figure 4 shows a typical simulation for the t^*-range $[0.1\beta, \ldots, 1.5\beta]$, where $\beta = 1/\alpha$ is the expected value of D_{i+1}. It is clearly visible that longer memory (in the above definition) increases the ruin probability by several orders of magnitude.

Model (b) again assumes exponentially distributed inter-occurence times D_i (intensity α) and the claim sizes Y_i to be exponentially distributed, but with the intensity depending on the time D_{i+1} since the previous event linearly:

$$\lambda(D_{i+1}) = \lambda_0 + (\rho(D_{i+1} - 1/\alpha)).$$

So the expected value of a claim size increases with decreasing D_{i+1} (a model that might be realistic for some natural catastrophy scenarios, where events blow up because of their frequency). It can be calculated, that the expected value μ for the claim sizes is again (this time approximately, however) the same as for the independent case. Note that ρ is the slope of the intensity line and for $\rho = 0$ we have again the independent double-exponential model with known solution (1). The simulation depicted in Figure 5 shows an increase of the ruin probability for increasing ρ. In addition to that it tells something about the influence of the *Coefficient of Variation* of a random variable X: It is defined as

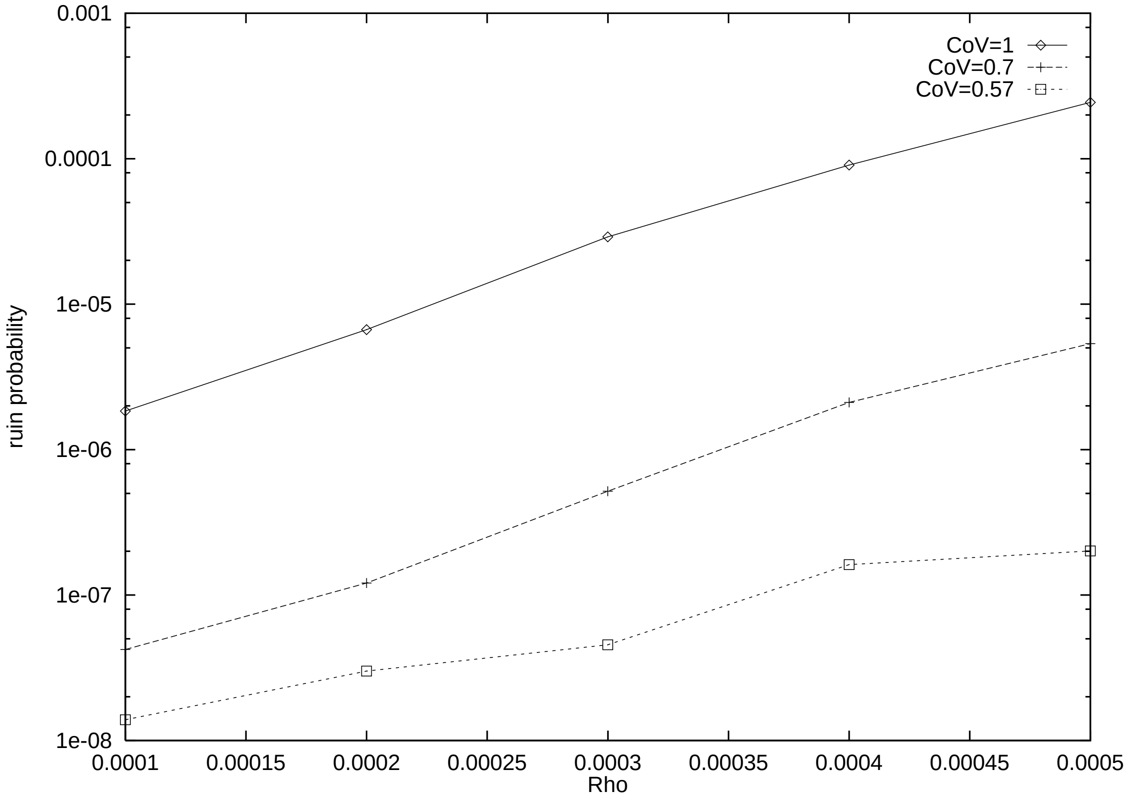

Figure 5: Model (b) for an exponential and two gamma distributions (N=50000)

$$\mathrm{CoV}(X) = \frac{\sqrt{\mathrm{Var}(X)}}{E(X)}$$

and is a measure of how regularly the claims occur. For the exponential distribution we have $\mathrm{CoV} = 1$ and for equidistant deterministic intervals $\mathrm{CoV} = 0$. So if we model the inter-occurence times to be gamma-distributed, we have lower values for the CoV and, as the simulation shows, this considerably decreases the probability of ruin and furthermore decreases the rate, at which ruin becomes more likely when dependency comes into the model (the expected values of the installed gamma-distributions have been normalized to the exponential case to avoid additional biases).

4 Conclusion

It is shown that for various scenarios with dependent damage occurences the ruin probability is considerably higher than for independent events. Therefore independence models may be dramatically misleading; this effect is particularly strong, if extreme events are involved. However, there are a lot of open questions; the introduced models might be generalized to Markov processes of higher order. Moreover a model, where clustering of consecutive events only exists above a certain threshold (which could be realized by using upper tail dependent copula functions), looks promising for an improvement of the estimation of risk exposure of insurance companies. Facing the fact that possible climate changes may increase the likelihood of extremal events, it seems inevitable to look more intensively into the intrinsic behavior and causal structure of natural catastrophies and its implications for the insurance industry.

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