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On Misapplications of Diffusion Approximations in Birth and Death Processes of Noisy Evolution

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Abstract

Birth and death processes with a finite number of states are used in modeling different kinds of noisy learning processes in economics. To analyze the long run properties one looks at the corresponding stationary distribution. When the number of states is large, the stationary distribution becomes bulky and difficult to analyze. To simplify the analysis in such a situation and hence to make the long run properties of the learning process more transparent, a diffusion approximation has been suggested. Unfortunately, quite often such approximation is not correctly done. Why this happens and how the situation can be fixed is discussed in this note.

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On Misapplications of Diffusion Approximations in Birth and Death Processes of Noisy Evolution

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Quite a few recent models of noisy evolution exploit the machinery of birth and death processes with a finite number of states. The papers by Orléan (1995), Binmore et al. (1995) and chapters 4 and 5 of the book by Aoki (1996) are among the most recent examples of such use. Often people are interested in the long run outcome of the evolutionary process. So, they look at the stationary distribution of the corresponding birth and death process. When the numbers of states N is large, this distribution is bulky and difficult to analyze. To make the picture more transparent, the set $\{0, 1, \dots, N-1\}$ where the distribution nests, is compressed to $\{0, 1/N, \dots, 1 - 1/N\}$. Now it looks "almost" like a continuous distribution on $[0, 1]$ and it is intuitively plausible to look for a continuous approximation of this distribution as $N \to \infty$. For this purpose a diffusion approximation has been suggested. But, there is a common mistake in many studies exploiting diffusion approximations. Namely, their limiting stochastic differential equation or, equivalently, the corresponding Fokker – Plank equation contains a multiplier $1/N$ in the diffusion coefficient. Such a term would have not been possible, had the passage to the limit as $N \to \infty$ been done correctly. Examples of this mistake can be found in section 7 of Binmore et al. (1995) or in section 2.2.2 of Weidlich and Haag (1983) which is often quoted by studies on noisy evolution. Thus, what is claimed to be a diffusion approximation of stationary distribution does not have much to do with it. Let us look at what *exactly* causes this mistake.

First, let us recall some basic facts concerning birth and death processes. Following Feller (1957 p. 407) a system evolving in continuous time $t \geq 0$ and having 0, 1, $N-1$ as feasible states is called a *birth and death process* if during $[t, t+h)$: (a) it moves from i to $i+1$ with probability $\lambda_i^{(N)} h + o(h)$; (b) the probability of shifting from i to $i-1$ equals $\mu_i^{(N)} h + o(h)$; and (c) any shift for more than one digit occurs with probability $o(h)$. Here $\lambda_i^{(N)} > 0$ for $0 \leq i \leq N-2$ and $\lambda_{N-1}^{(N)} = 0$. Also $\mu_i^{(N)} > 0$ for $1 \leq i \leq N-1$ and $\mu_0^{(N)} = 0$.

Set $p_{j,i}^{(N)}(\tau)$ for the probability that starting at $t = 0$ from j the system is at i for $t = \tau$. Then, regardless of j, $\lim_{\tau \to \infty} p_{j,i}^{(N)}(\tau) = d_i^{(N)}$. The values $d_i^{(N)}$, $0 \leq i \leq N-1$, form the unique *stationary distribution* which is defined as follows (see, for example, Hoel et al. (1972) p. 51)

$$d_i^{(N)} = d_0^{(N)} \prod_{j=1}^{i} p_{j-1}^{(N)} / q_j^{(N)}, \; i = 1, 2, \dots, N-1, \quad d_0^{(N)} = \left[1 + \sum_{i=1}^{N-1} \prod_{j=1}^{i} p_{j-1}^{(N)} / q_j^{(N)} \right]^{-1}.$$

It is proved to be the same as the stationary distribution of a time homogeneous Markov chain ξ_N^t, $t = 0, 1, 2, \dots$. This *birth and death chain* assumes the values $0, 1, \dots, N-1$ and has the following transition probabilities

$$P\{\xi_N^{t+1} = i+1 | \xi_N^t = i\} = p_i^{(N)} = \frac{\lambda_i^{(N)}}{\lambda_i^{(N)} + \mu_i^{(N)}},$$

$$P\{\xi_N^{t+1} = i-1 | \xi_N^t = i\} = q_i^{(N)} = \frac{\mu_i^{(N)}}{\lambda_i^{(N)} + \mu_i^{(N)}},$$

where $0 \le i \le N-1$. Above both the continuous time and the discrete time versions were referred to as a birth and death process. In the following let us deal with a chain.

To derive a diffusion approximation of the stationary distribution shifted to $\{0, 1/N, \ldots, 1-1/N\}$, let us look at the increments of the birth and death chain normalized by $1/N$. Set $\zeta_N^t = \xi_N^t/N$ and $\Delta\zeta_N^t = \zeta_N^{t+1} - \zeta_N^t$. The state space of the Markov chain $\zeta_N^t = \xi_N^t/N$, $t = 0, 1, \ldots$, is $\{0, 1/N, \ldots, 1-1/N\}$. The first two conditional moments $\Delta\zeta_N^t$ are

$$E(\Delta\zeta_N^t | \zeta_N^t = i/N) = \frac{1}{N}[p_i^{(N)} - q_i^{(N)}], \tag{0.1}$$

$$E[(\Delta\zeta_N^t)^2 | \zeta_N^t = i/N] = \frac{1}{N^2}[p_i^{(N)} + q_i^{(N)}]. \tag{0.2}$$

For a positive integer M, define a random process $x_N^M(\cdot)$ on $[0, \infty)$ by setting

$$x_N^M(0) = \zeta_N^M, \quad x_N^M(u) = \zeta_N^{M+i} \text{ for } \frac{i}{N} \le u < \frac{i+1}{N}.$$

At this point it is already clear that a *nontrivial unconstrained* diffusion approximation for $x_N^M(\cdot)$ *is not possible.* Nontrivial here means that the diffusion coefficient is not identically equal to zero. Unconstrained means that there are no barriers.

Indeed, ζ_N^t belongs to $[0, 1)$ and the absolute value of its increment $\Delta\zeta_N^t$ does not exceed $1/N$ with certainty. But a small increment of a nontrivial diffusion without barriers is approximately Gaussian taking with positive probability values from $-\infty$ to ∞. A *constrained nontrivial* diffusion approximation *is not possible either* as the following argument shows.

If there is a Lipschitz function $f(\cdot)$ such that

$$\lim_{N\to\infty} \sup_{0\le i\le N-2} |p_i^{(N)} - q_i^{(N)} - f(i/N)| = 0,$$

one can expect that as $N \to \infty$, $M \to \infty$ the weak limits of $x_N^M(\cdot)$ satisfy the following equation

$$\frac{dx}{dt} = f(x). \tag{0.3}$$

This is a *deterministic equation.* It cannot be stochastic. Indeed, by (1) and (2) the conditional variance of $\Delta\zeta_N^t$ is of the order $1/N^2$. Setting $\Delta t = 1/N$, one sees that if (3) were to be a stochastic differential equation with a nonzero diffusion term, then $E\Big\{\{\Delta x(t) - E[\Delta x(t)|x(t)]\}^2 | x(t)\Big\}$ would have been of the order of $1/N$. Here $\Delta x(t) = x(t+1/N) - x(t)$. Thus, *as long as (2) is in place, no nonzero diffusion term is possible in the limit (3).*

This looks like a puzzle. Indeed a deterministic limit comes out of a stochastic process. But the puzzle resolves if one realizes that the randomness of (3) is in its initial state, $x(0)$. This cannot be revealed by a nontrivial diffusion approximation. For details of what the distribution of $x(0)$ looks like see Kaniovski and Pflug (1997).

Finally, a guess of how the multiplier $1/N$ in the diffusion term comes to exist is as follows. Having recognized the above puzzle, people try to preserve the randomness of the limit. That is why one $1/N$ in (2) is used as Δt in the passage to the limit, while the other $1/N$ is "frozen", turning into the multiplier in the diffusion coefficient.

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