

ON THE OPTIMAL ADAPTIVE PARAMETER ESTIMATION OF
WATER RESOURCES CONTROL SYSTEMS

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Water Resources Control Systems

As water resource systems have grown larger and more complex, the importance of optimum operation of these systems has increased. Several IIASA papers have been published attacking these problems which are in essence, the problems of estimating/controlling the state of WR systems. Casti (1974) gave algorithms for the stochastic inflow-nonlinear objective reservoir control problem; Szöllösi-Nagy (1975) outlined the closed-loop control of linear stochastic water quality systems with quadratic performance measure; and quite recently, Takeuchi (1976) dealt with the problem of typhoon forecasting using stochastic filtering techniques.

There is at least one aspect which is common to these approaches, and that is the way they look at the dynamics of the WR systems. In one way or another, they assume that the system dynamics is linear and given by

$$x(t + 1) = \Phi(t + 1, t)x(t) + \Gamma(t + 1)u(t) + w(t), \quad (1)$$

where

- $x(t)$ is a vector of the system states belonging to a bounded set of state space $X \subset R^n$;
- $\Phi(t + 1, t)$ is the $n \times n$ state transition matrix;
- $u(t)$ is the vector of control variables belonging to the compact set of admissible controls $U \subset R^p$;

$\Gamma(t + 1, t)$ is the $n \times p$ control transition matrix;
 $w(t)$ is a vector white gaussian noise (WGN) sequences called process disturbances with the statistics $w(t) \sim N(0, R_1(t))$;
 $t \in T$ is the discrete time variable,
 $T = \{t : t = 0, 1, 2, \dots\}$.

It is also assumed that the collection of measurement equipments which are attached to the system, modelled by Eq. (1), to monitor its behavior has an output (collection of measured variables) which can be modelled by the relation

$$z(t) = H(t)x(t) + v(t) \quad , \quad (2)$$

where

$z(t)$ is the m vector of measurements (output vector);
 $v(t)$ is a vector of WGN sequences called measurement errors with the statistics $v(t) \sim N(0, R_2(t))$;
 $H(t)$ is the $m \times n$ measurement matrix.

Moreover, it is generally assumed that the uncertainties are independent of each other, i.e.

$$\mathcal{E}\{w(t)v^T(t + \tau)\} = 0, \quad \forall \tau \in T \quad .$$

Clearly, the measurement sequence $z(t)$ generates an increasing σ -algebra

$$\mathcal{Z}_t = [z(1), z(2), \dots, z(t)]$$

with the obvious chain property of

$$\mathcal{X}_t = \left[\mathcal{X}_{t-1}, z(t) \right] .$$

To get an optimal control sequence $u^*(t) \in U$ which minimizes certain expected loss $\mathcal{E}\{J\}$, the famous separation principle (Kalman and Koepcke, 1958) is utilized which states that the optimal stochastic control can be separated into two parts. The first is the estimation/prediction of the state variables and the second is the determination of the closed-loop feedback control $u(\cdot)$ based upon the estimated states $\hat{x}(\cdot)$. It is well known that the estimated values are given by the Kalman filter algorithms (provided that the statistics of the initial state $x(0)$ is known and gaussian, $x(0) \sim N(\hat{x}(0), P(0))$), while the optimal control is obtained through a deterministic dynamic programming performed on the estimated state variables. Due to the separation principle here we concentrate ourselves on the estimation part only, noting that the synthetization of the control strategies is then really straightforward.

Due to the unknown parameters of the matrices in Eq. (1) the separated system dynamics and measurement becomes

$$x(t + 1) = \phi(t + 1, t; \theta) x(t) + w(t) \quad (3)$$

$$z(t) = H(t, \theta) x(t) + v(t), \quad (4)$$

where θ is a q -vector of unknown parameters in ϕ , H , R_1 , R_2 , $x(0)$ belonging to a finite dimensional parameter space θ^q , $\theta \in \theta^q$. It is assumed that θ is time invariant and has an a priori pdf $p(\theta)$. Again, when the parameters of the model are exactly known, the solution is straightforward. In our case, however, the parameters are uncertain and life becomes much more

complicated. As Eykhoff (1974) showed, the joint estimation of the uncertain states/parameters becomes nonlinear even in the case of linear system dynamics. Following Lainiotis (1971), the optimal mean square estimate $\hat{x}(t|t)$ of the state $x(t)$ is then given by

$$\hat{x}(t|t) = \int_{\theta^q} \hat{x}(t|t, \theta) p(\theta | \mathcal{Z}_t) d\theta \quad (5)$$

where

$\hat{x}(t|t, \theta)$ is called θ -conditional (or model conditional) estimation and can be obtained from the Kalman filter applied for a fixed parameter vector $\theta \in \theta^q$;

$p(\theta | \mathcal{Z}_t)$ is called weighting coefficients which assigns different weights to the different parameter vectors.

Since the θ -conditional estimates are given by the Kalman filter algorithms the mean square estimation problem is reduced to finding the weighting coefficient which is in fact a posteriori pdf of the parameter vector θ given the measurement sequence \mathcal{Z}_t . If it is found then the θ -conditional estimates are weighted with respect to the a posteriori pdf and integrated over the q -dimensional parameter space θ^q .

We are going to deal, here, with the discrete case, noting that the continuous case based upon Bucy's representation theorem is treated in Lainiotis (1974). The conditional probability function of the parameter vector θ , when the measurement sequence \mathcal{Z}_t is given, is defined by

$$\begin{aligned}
 p(\theta | \mathcal{Z}_t) &= \frac{p(\theta, \mathcal{Z}_t)}{p(\mathcal{Z}_t)} \\
 &= \frac{p(\theta, \mathcal{Z}_{t-1}, z(t))}{p(\mathcal{Z}_{t-1}, z(t))},
 \end{aligned}$$

or, according to the chain-rule of conditional probabilities, as

$$p(\theta | \mathcal{Z}_t) = \frac{p(z(t) | \theta, \mathcal{Z}_{t-1}) p(\theta | \mathcal{Z}_{t-1}) p(\mathcal{Z}_{t-1})}{p(\mathcal{Z}_{t-1} | z(t)) p(z(t))}. \quad (6)$$

Since

$$p(\mathcal{Z}_{t-1} | z(t)) = \int_{\Theta^q} p(\theta, \mathcal{Z}_{t-1} | z(t)) d\theta$$

and

$$p(\theta, \mathcal{Z}_{t-1} | z(t)) = \frac{p(\theta, \mathcal{Z}_{t-1}, z(t))}{p(z(t))}$$

we get for the marginal distribution

$$p(\mathcal{Z}_{t-1} | z(t)) = \frac{1}{p(z(t))} \int_{\Theta^q} p(\theta, \mathcal{Z}_{t-1}, z(t)) d\theta.$$

Substituting this into the denominator of Eq. (6) and considering that

$$p(\theta, \mathcal{Z}_{t-1}, z(t)) = p(z(t) | \theta, \mathcal{Z}_{t-1}) p(\theta | \mathcal{Z}_{t-1}) p(\mathcal{Z}_{t-1})$$

we have

$$p(\theta | \mathcal{Z}_t) = \frac{p(z(t) | \theta, \mathcal{Z}_{t-1})}{\int_{\Theta^q} p(z(t) | \theta, \mathcal{Z}_{t-1}) p(\theta | \mathcal{Z}_{t-1}) d\theta} p(\theta | \mathcal{Z}_{t-1}) \quad (7)$$

which is in fact a recursive Bayesian algorithm for the calculation of the weighting coefficients with the initial condition $p(\theta | \mathcal{Z}_0) = p(\theta)$.

For the case when the parameter space θ^q is discrete and consists of M elements, the a priori pdf is

$$p(\theta) = \sum_{i=1}^M p(\theta_i) \delta(\theta - \theta_i)$$

and the a posteriori pdf is given by

$$p(\theta | \mathcal{Z}_t) = \sum_{i=1}^M p(\theta_i | \mathcal{Z}_t) \delta(\theta - \theta_i)$$

then the recursive algorithm becomes

$$p(\theta_i | \mathcal{Z}_t) = \frac{\Lambda(t | \theta_i)}{\sum_{j=1}^M \Lambda(t | \theta_j) p(\theta_j | \mathcal{Z}_{t-1})} p(\theta_i | \mathcal{Z}_{t-1})$$

where the likelihood function $\Lambda(t | \theta_i)$ stands for $p(z(t) | \theta_i, \mathcal{Z}_{t-1})$.

Now, let us consider the derivation of the likelihood function $p(z(t) | \theta, \mathcal{Z}_{t-1})$. It is known from the innovation theory (Kailath, 1968) that if we are given a stochastic process $\{z(t) : t \in T\}$ we can define its innovation representation $\{v(t) : t \in T\}$ as a WGN process such that $z(\cdot)$ can be calculated from $v(\cdot)$ by a causal (i.e. nonanticipative) and causally invertible transformation. The point is that $v(\cdot)$ and $z(\cdot)$ contain the same 'statistical information' since we can go back and forth in real-time from one process to the other, but, of course, $v(\cdot)$ will generally be a much simpler process than $z(\cdot)$.

Moreover, since the values of $v(\cdot)$ at different time instants of time are statistically independent of each other, each observation $v(t)$ brings 'new information' only, unlike the observation $z(t)$ which is, in general, statistically related to past values of $z(\cdot)$. (This is the reason why $v(\cdot)$ is called 'new information' or 'innovation' process of $z(\cdot)$). Therefore, the θ -conditional innovation is defined as

$$\begin{aligned} v(t, \theta) &= z(t) - \hat{z}(t|t-1, \theta) \\ &= z(t) - \mathcal{E} \left\{ H(t, \theta)x(t) + v(t) \mid \mathcal{Z}_{t-1} \right\} \\ &= z(t) - H(t, \theta)\hat{x}(t|t-1, \theta) \end{aligned} \quad (8)$$

since it is part of the measured output which contains some information which was not previously available. So, we can replace $z(\cdot)$ by $v(\cdot)$ and according to the theory of derived distributions, we have

$$p(\theta, z(t) \mid \mathcal{Z}_{t-1}) = p(\theta, v(t, \theta) \mid \mathcal{Z}_{t-1}) J \quad ,$$

where due to the linearity, the Jacobian is equal to

$$J = \left| \frac{\partial(z(t) - H(t, \theta)\hat{x}(t|t-1, \theta))}{\partial z(t)} \right| = I \quad ,$$

the identity matrix. Since the innovation process is WGN and is independent of the previous measurements \mathcal{Z}_{t-1} , we have on the one hand

$$p(\theta, z(t) \mid \mathcal{Z}_{t-1}) = p(\theta \mid \mathcal{Z}_{t-1}) p(v(t, \theta))$$

and on the other

$$\begin{aligned} p(\theta, z(t) | \mathcal{Z}_{t-1}) &= p(z(t), \theta | \mathcal{Z}_{t-1}) \\ &= p(z(t) | \theta, \mathcal{Z}_{t-1}) p(\theta | \mathcal{Z}_{t-1}) \quad . \end{aligned}$$

Combining the above two expressions, we get for the likelihood function

$$p(z(t) | \theta, \mathcal{Z}_{t-1}) = p(v(t, \theta)) = p_v(z(t) - H(t, \theta) \hat{x}(t | t - 1, \theta)) \quad .$$

It is easy to see that the innovations form a zero mean WGN process with covariance $P_v(t, \theta)$, $v(t, \theta) \sim N(0, P_v(t, \theta))$. The covariance matrix can readily be determined from an equivalent representation of

$$v(t, \theta) = -H(t, \theta) \tilde{x}(t | t - 1, \theta) + v(t) \quad ,$$

where

$$\tilde{x}(t | t - 1, \theta) = x(t) - \hat{x}(t | t - 1, \theta)$$

is the one-step-ahead prediction error, as

$$\begin{aligned} P_v(t, \theta) &= \text{cov}[v(t, \theta), v(t, \theta)] = \mathcal{E}\{v(t, \theta) v^T(t, \theta)\} \\ &= \mathcal{E}\{[-H(t, \theta) \tilde{x}(t | t - 1, \theta) + v(t)] [\cdot]^T\} \\ &= \mathcal{E}\{H(t, \theta) \tilde{x}(t | t - 1, \theta) \tilde{x}^T(t | t - 1, \theta) H^T(t, \theta)\} \\ &\quad + \mathcal{E}\{v(t) v^T(t)\} \\ &= H(t, \theta) P(t | t - 1, \theta) H^T(t, \theta) + R_2(t, \theta) \quad , \quad (9) \end{aligned}$$

where

$$P(t | t - 1, \theta) = \mathcal{E}\{\tilde{x}(t | t - 1, \theta) \tilde{x}^T(t | t - 1, \theta)\}$$

is the θ -conditional error covariance matrix.

Summing up, we have the following recursive scheme for calculating the weighting coefficients

$$p(\theta | \mathcal{Z}_t) = \frac{|P_v(t, \theta)|^{-\frac{1}{2}} \exp \left[-\frac{1}{2} |v(t, \theta)|^2 P_v^{-1}(t, \theta) \right] p(\theta | \mathcal{Z}_{t-1})}{\int_{\theta^q} |P_v(t, \theta)|^{-\frac{1}{2}} \exp \left[-\frac{1}{2} |v(t, \theta)|^2 P_v^{-1}(t, \theta) \right] p(\theta | \mathcal{Z}_{t-1}) d\theta} \quad (10)$$

where

$$P_v(t, \theta) = H(t, \theta) P(t | t-1, \theta) H^T(t, \theta) + R_2(t, \theta)$$

$$v(t, \theta) = z(t) - H(t, \theta) \hat{x}(t | t-1, \theta)$$

and the initial conditions are

$$p(\theta | \mathcal{Z}_0) = p(\theta) \quad ,$$

$$\Lambda(0 | \theta) = 1 \quad .$$

The conditional error covariance matrix $P(t | t)$, which is useful for the on-line evaluation of the estimator performance, is given by Lainiotis (1974) as

$$P(t | t) = \int_{\theta^q} \left\{ P(t | t, \theta) + [\hat{x}(t | t, \theta) - \hat{x}(t | t)] [\hat{x}(t | t, \theta) - \hat{x}(t | t)]^T \right\} \cdot p(\theta | \mathcal{Z}_t) d\theta \quad .$$

The complete Bayesian recursive algorithm is shown in Table I and the related block diagram is depicted in Figure 1. It should be mentioned that the notion of structure adaption can

also be imbedded into the above algorithms. We should only guess an upper bound \tilde{n} to the system dimensionality and to augment the parameter vector θ up to \tilde{n} . Then the adaptive algorithms automatically give us an estimate n for the system order. Surely, $n \leq \tilde{n}$. In other words, it gives zero elements for the 'superfluous' parameter values.

Since the speed of the evaluation of Eq.(10) highly depends upon the dimension of the parameter space, some tricky numerical integration technique, like the one based upon Monte Carlo simulation, should be used.

Having the estimated values, it is easy to synthesize the optimal control policies. For details, see Aoki (1967).

Table I The Adaptive Sequential Prediction Algorithm

System Model	$x(t+1) = \phi(t+1, t; \theta)x(t) + w(t), w(t) \sim N(0, R_1(t, \theta))$
Measurement Model	$z(t) = H(t, \theta)x(t) + v(t), v(t) \sim N(0, R_2(t, \theta))$
Initial State	$x(0) \sim N(x(0 0, \theta), P(0 0, \theta))$
θ -conditional Estimate	
State Prediction	$\hat{x}(t+1 t, \theta) = \phi(t+1, t; \theta)\hat{x}(t t, \theta)$
Predicted error covariance matrix	$P(t+1 t, \theta) = \phi(t+1, t; \theta)P(t t, \theta)\phi^T(t+1, t; \theta) + R_1(t, \theta)$
Predictor gain	$K(t+1, \theta) = P(t+1 t, \theta)H^T(t, \theta)[H(t, \theta)P(t+1 t, \theta)H^T(t, \theta) + R_2(t, \theta)]^{-1}$
State Estimation	$\hat{x}(t+1 t+1, \theta) = \hat{x}(t+1 t, \theta) + K(t+1, \theta)[z(t+1, \theta) - H(t+1, \theta)\hat{x}(t+1 t, \theta)]$
Error covariance updating	$P(t+1 t+1, \theta) = (I - K(t+1, \theta)H(t+1, \theta))P(t+1 t, \theta)(I - K(t+1, \theta)H(t+1, \theta))^T + K(t+1, \theta)R_2(t+1, \theta)K^T(t+1, \theta)$
Adaptive Estimate	
$\hat{x}(t t) = \int_{\theta^q} \hat{x}(t t, \theta)p(\theta \mathcal{Z}_t) d\theta$	
Weighting coefficient	$p(\theta \mathcal{Z}_t) = \frac{\Lambda(t \theta)}{\int_{\theta^q} \Lambda(t \theta)p(\theta \mathcal{Z}_{t-1}) d\theta}$
Likelihood function	$\Lambda(t \theta) = P_v(t, \theta) ^{-1/2} \exp \{-\frac{1}{2} \ v(t, \theta)\ _{P_v^{-1}(t, \theta)}^2\}$
Innovation sequence	$v(t, \theta) = z(t) - H(t, \theta)\hat{x}(t t-1, \theta)$
Innovation covariance	$P_v(t, \theta) = H(t, \theta)P(t t-1, \theta)H^T(t, \theta) + R_2(t, \theta)$
Initial Conditions	$p(\theta \mathcal{Z}_0) = p(\theta)$ $\Lambda(0 \theta) = 1$

A Posteriori Analysis (nonlinear)

Kalman Filter Algorithms (linear)

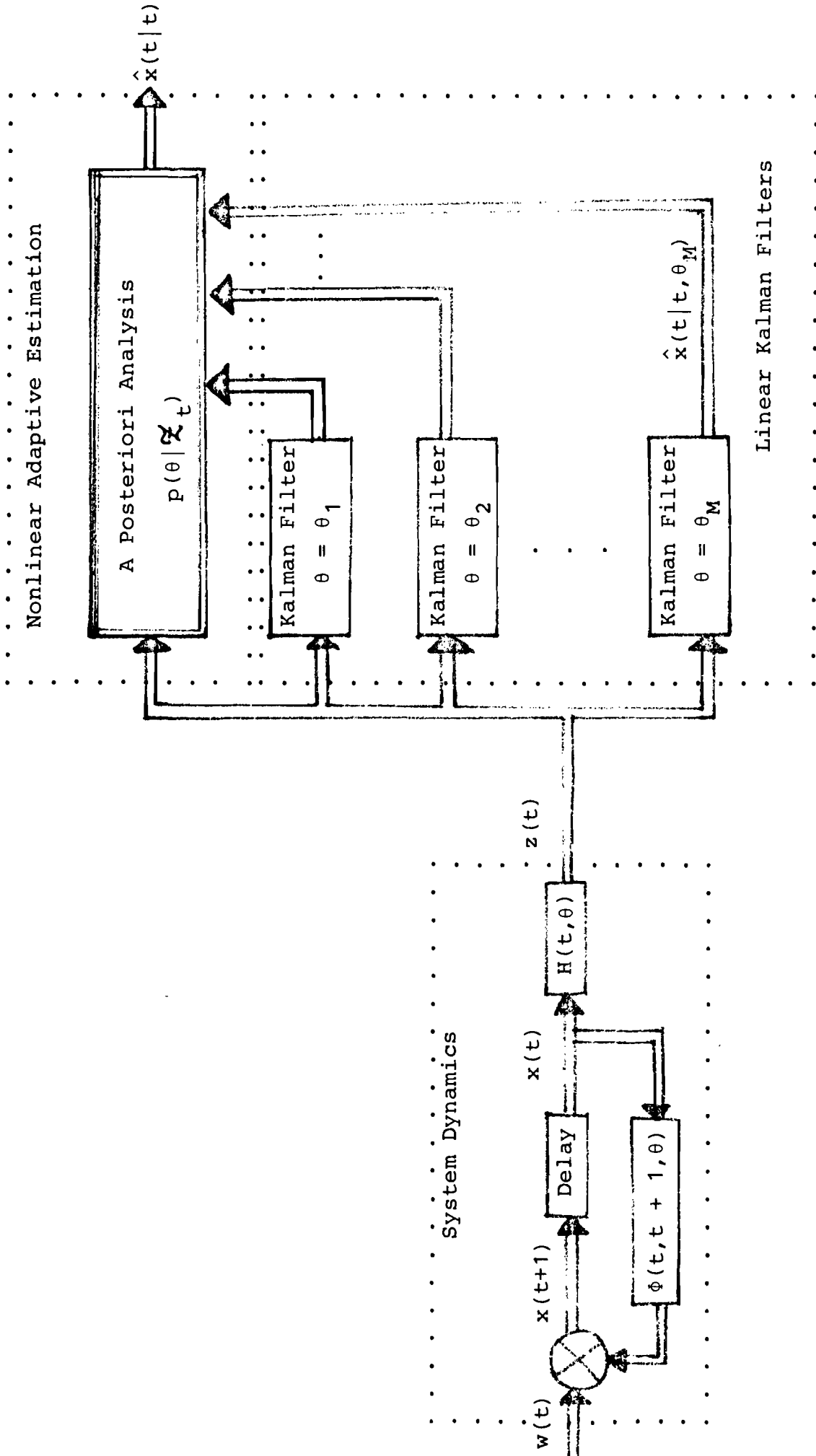


Figure 1: The block diagram of the nonlinear adaptive sequential estimation algorithm (when the range of θ is discrete)

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