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ECONOMY PHASE PORTRAITS

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Preface

In the past few years, at IIASA and other places, people have become interested in the phase portrait of systems, i.e. in a qualitative description of their global structure and long-time behavior. These terms also provide a natural language for talking about resilience. To illustrate this concept, the Energy Program has set up a series of small, schematic models of an economy and its energy sector. Their phase portraits are described in this paper.

Abstract

Topological terms like attractors and basins are expected to play an ever increasing role in the qualitative description of systems described by differential equations. This paper illustrates those concepts through a detailed study of the phase portraits of various "economic models" treated in the Energy Program of IIASA.

ECONOMY PHASE PORTRAITS

1. Introduction

During the last year the Energy Program has set up a series of models of the development of an abstract economy [1,2,3]. These models treat in a phenomenological way the embedding of the energy system and its structural changes into the economy ("economy in transition"). They have been studied with respect to the structure of the dynamics in their state space (the phase portrait), and this Research Memorandum is intended as a collection of the results obtained.

The aim of this work should not be misunderstood. The models, of course, make no claim to numerical predictions, nor are they based on detailed understanding of the underlying dynamics. On the contrary, we wanted to emphasize the structural-topological approach pioneered by Holling [5] in his work on the resilience concept and studied by one of us (H.R.G) [6,7]. In this spirit, our omission of all numerical values for the parameter is deliberate. In studying these models, we are more interested in the overall picture of the state space than in individual trajectories. Moreover, the dynamic assumptions contained in the model should not be taken too seriously; the qualitative results should be regarded as "topological scenario-writing." In short, the models present "myths" as defined by Holling [8]. On the other hand, they are useful despite their shortcomings; in training our group in a methodology, as well as in showing trends - the latter, because of the structural stability of the model. In this way the qualitative predictions can make sense even if some of the dynamical details do not.

2. Explanation of Terms.

We want to maintain the flow and character of our writing: not a precise mathematical theory but a verbal description, and an indication of what we were thinking when discussing resilience. We will therefore only illustrate here what the reader should understand by the expressions used. For precise definitions of those terms the interested reader is referred to [7]; a more technical introduction to the methodology utilized here can be found in [9] and [10].

What you will see in the figures are mainly phase portraits. By this we mean a qualitative description of the solution curves of a given system of differential equations. In general, one cannot trace from a phase portrait a particular trajectory starting from a given initial condition; but one can see trends, the number of basins (explained below), and relationships among basins.

The terms attractor, basin, and separatrix are closely related. An attractor is a generalization of the concept of a stable fixed point (a sink). The term takes into account the fact that we need a common notion of a place that attracts trajectories; that place can be of dimension zero (for example a stable fixed point), or be a manifold of higher dimension (e.g. a stable closed orbit), or be a more complicated set. To each attractor belongs a basin. This is simply the subset of phase space containing those points whose trajectories tend to that attractor.

A basic set is related to an attractor as a general fixed point is to a stable one. Its stable manifold consists of the points tending to that basic set in the future. (Thus, the stable manifold of an attractor is identical with its basin.) If a stable manifold has one dimension less than the state space itself (i.e., is a hypersurface) it is called a separatrix, since points on one side of it have a markedly different future behavior than points on the other. The basin boundaries are to be found among the separatrices; but see Model C (chapter 5) for an example of a separatrix that is not a border between two different basins.

A generic property is one that holds for almost all (in a topological sense) elements of a given set.

3. Model A.

This model was presented by Häfele in [1] to demonstrate the meaning and usefulness of Holling's work on resilience to the Energy project. The model is two-dimensional and has as its state variables population P and per capita energy consumption e. The myths contained in the model are the following:

1. The birth rate will decrease with increased standard of living (expressed by the per capita energy consumption e).
2. Risk acceptance by society will also decrease with increased e; thus the cost of safety measures will rise.
3. Total energy consumption will rise proportionally to "effective" GNP, given by GNP in the usual sense minus safety expenditures.
4. A Cobb-Douglas production function for GNP is assumed.

The following additional variables are introduced:

$E = e \cdot P$: total energy consumption r* : risk acceptance

K : safety expenditures

G : "effective" GNP.

The equations of the model are

$$1.\quad \frac{dP}{dt} = \sigma P - \kappa e \qquad (3.1)$$

$$2.\quad \frac{K}{Ko} \frac{Eo}{E} = \frac{ro}{r} = (\frac{e}{eo})^{\lambda} \qquad (3.2)$$

$$3.\quad \frac{dE}{dt} = \mu G \qquad (3.3)$$

$$4.\quad G = AE^{\frac{1}{2}} P^{\frac{1}{2}} - K, \qquad (3.4)$$

where σ, κ, λ, μ, and A are parameters, and K_o, E_o, r_o are initial conditions. The number in front of each equation refers to the corresponding "myth".

The phase portrait (in e and P) is shown in Fig. 3.1. The interesting part of it ($e > O$ and $P > O$) contains one fixed point--a saddle point, whose stable manifold (the separatrix) divides the phase plane into basins B_1 and B_2. These basins differ by their asymptotic tendencies**, shown by the respective parts of the unstable manifold, W_1^u for basin B_1 and W_2^u for basin B_2.

All trajectories of basin B_1 tend to zero population and infinite per capita energy consumption; in contrast, the trajectories of basin B_2 tend to a finite per capita energy consumption but infinite population.

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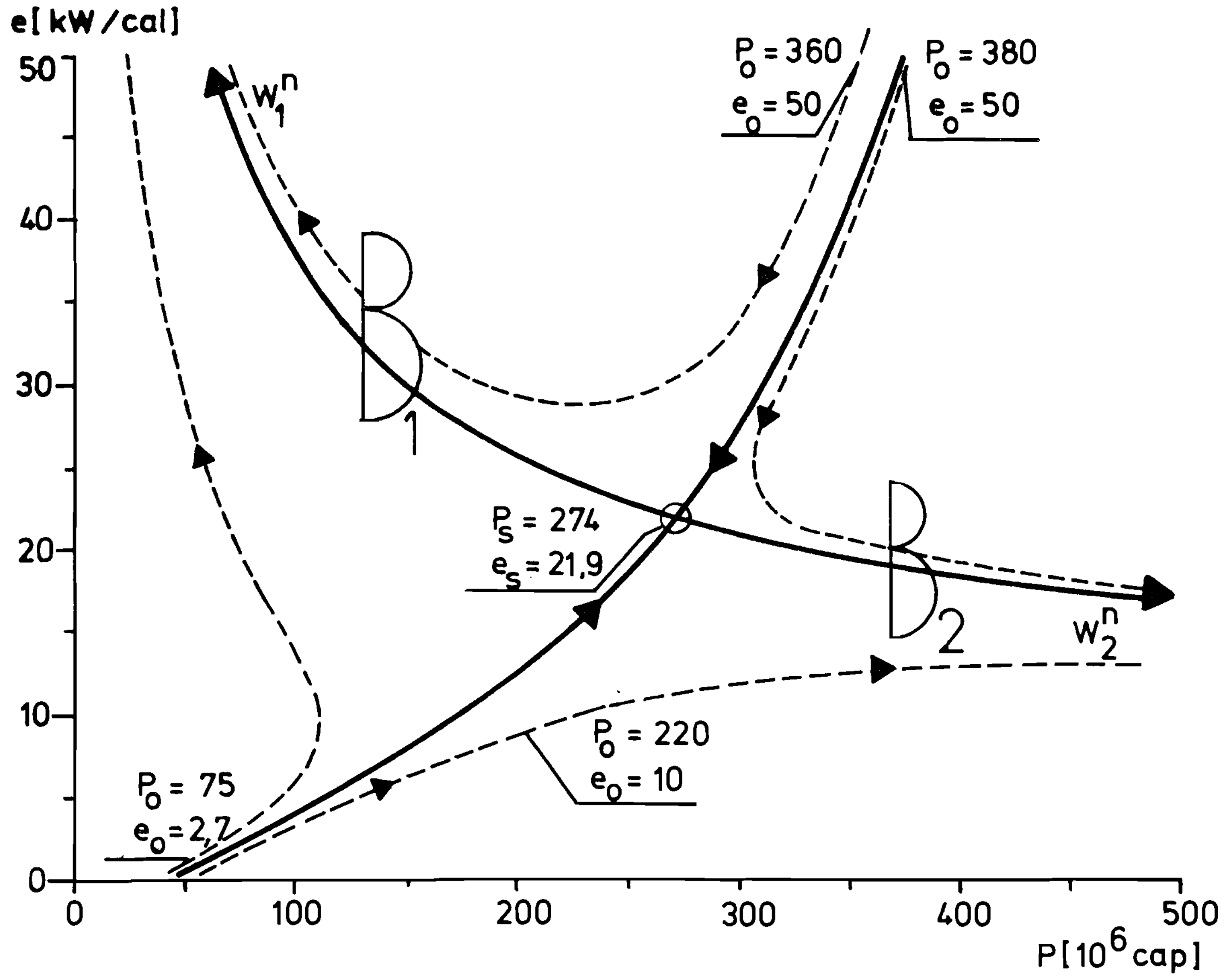
*It seems difficult to quantify risk acceptance. However, it will be seen that in the equations r represents only an intermediate stage in the relationship between standard of living and safety expenditures.

**We cannot quite call them attractors: the trajectories go off to infinity without approaching a fixed subset of the phase space (a familiar problem for non-compact phase spaces). One could "cut off" the model at utterly unrealistic values of the state variables (e.g. $P < 1$); thus W_1^u and W_2^u would end in two new stable fixed points which would serve as honest attractors for the basins.



e[kW/cal]
50
40
30
20
10
0
0
100
200
300
400
500
P[10^6 cap]
w_1^n
w_2^n
P_o = 360
e_o = 50
P_o = 380
e_o = 50
P_s = 274
e_s = 21,9
P_o = 220
e_o = 10
P_o = 75
e_o = 2,7
1
2

Figure 3.1.

4. Model B.

This model [2] was intended as the first step into a higher-dimensional state space. At the same time the pure phenomenology of Model A was brought closer to established theories of economics. Model B splits the economy into two aggregated parts: the energy production sector and all the rest. Thus the state variables are:

P : total population

g : GNP/capita

E : total energy production

Auxiliary* variables are introduced:

G : total GNP

M : total capital stock invested outside the energy sector

C : total consumption

c : per capita consumption

r : risk level (see Model A)

K : specific energy production costs

i : specific energy investment costs

I_0, i_0, r_0 : initial values.

Myths 1 (on population dynamics) and 2 (on the dependence of energy production and investment costs on standard of living via risk level) are taken over from Model A, the former with the modification of a logistic growth of population. In addition, the following assumptions are introduced:

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*"Auxiliary" in the mathematical sense; the variables are economic quantities in their own right. But through the equations of the model, they can be expressed in terms of three state variables (chosen in a slightly arbitrary way).

3. A logistic growth of GNP per capita is enforced.
4. A Cobb-Douglas production function with three production factors (E, P, and K) is assumed.
5. $(K-K_0)$ E has to be paid out of GNP as unproductive safety expenditures.
6. Total consumption is a fixed fraction of GNP (the "trade union myth").

Therefore, the equations of the model are

1. $$\frac{dP}{dt} = a_p \cdot P(1-{}^{P}/P_\infty) - a_c \cdot C \qquad (4.1)$$

2. $$\frac{r}{r_0} = \left(\frac{c}{c_0}\right)^\lambda ; \quad \frac{K}{K_0} = \frac{r_0}{r} ; \quad \frac{i}{i_0} = \frac{r_0}{r} \qquad (4.2\text{-}4.4)$$

3. $$\frac{dg}{dt} = \mu g\left(1-\frac{g}{g_\infty}\right) \qquad (4.5)$$

4. $$G = AP^\alpha E^\beta M^\gamma \qquad (4.6)$$

5. $$D = a_v \cdot G \qquad (4.7)$$

6. $$G = C + i\frac{dE}{dt} + \frac{dM}{dt} + \kappa(E+M) + (K-K_0)E ; \qquad (4.8)$$

a_p, a_c, a_v, λ and μ are phenomenological parameters. P_∞ and g_∞ are limiting levels and κ denotes the depreciation rate (assumed) uniform for the energy and non-energy sectors).

According to eq.(4.5) the interesting part of state space is defined by $g \leq g_\infty$. We immediately note the following features:

- Fixed points
  - α) Fixed lines at $g = 0$, $P = P_\infty$, or $P = 0$ and arbitrary E. These lines are in any case artificial and repelling
  - β) Two fixed points F_1 and F_2 at $g = g_\infty$, $P = P_{fix} = P_\infty\left(1-\frac{a_c a_v g_\infty}{a_p}\right)$ and $E = E_{fix,1}$ or

$$E = E_{fix,2}\ {}^{*}\ (E_{fix,1} > E_{fix,2}) \qquad (4.9)$$

- A divergence surface at

$$iE\beta = M\gamma \qquad (4.10)$$

where M and i should be expressed by g, P and E via (4.6), (4.2), and (4.4). At this surface, the equation for $\frac{dE}{dt}$ becomes singular. One notices that (4.10) is the condition for the optimal ratio of the production factors E and M. At such points, $\frac{dg}{dt}$ is already determined from eq. (4.6), and (4.8) and can no longer be constrained as by (4.5). We define E_{div} by

$$E_{div} = \left[\left(\frac{\gamma}{i_{fix}\beta}\right)^{\gamma} g_{\infty} P_{fix}{}^{1-\alpha}/A\right]^{1/\beta+\gamma} \qquad (4.10)$$

$$i_{fix} = \left(\frac{g_{\infty}}{g_0}\right)^{\lambda} i_0 \quad ,$$

the value of E on the divergence surface for $g = g_{\infty}$ and $P = P_{fix}$. The corresponding point is denoted by D. For the mathematical treatment, one has to multiply the whole system of equations with the denominator of $\dot{E}$, in order to

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*For $\beta=\gamma$ these values are given by:

$$E_{fix,1,2} = \frac{G = (1-a_v)}{2(\kappa+K_0(1-\frac{g_{\infty}}{g_0})^2)} \pm \sqrt{\frac{G^2(1-a_v)^2}{4(\kappa+K_0(1-\frac{g_{\infty}}{g_0})^2)^2} - \kappa N}$$

with $N = (g_{\infty}P^{1-\alpha}/A)^{1/\beta}$

thus for $\kappa \rightarrow 0$, $E_{fix,1} \rightarrow \frac{G(1-a_v)}{K_0(1-\frac{g_{\infty}}{g_0})^2}$ and $E_{fix,2} \rightarrow 0$.

get rid of this singularity. This procedure is not purely formal; in fact, a line of "spurious fixed points" appears when both numerator and denominator of $\dot{E}$ vanish. Although these fixed points are not true stationary states of the model, they nevertheless influence the phase portrait.

As soon as the solution reaches the divergence surface, we can only state that the assumptions of the model have become incompatible (except for one case described below).

We find two distinct parameter regions corresponding to three types of phase portraits. In sketching the phase portraits, we note that equations (4.1) and (4.3) are independent of E. Thus all trajectories lie on cylinders whose bases are given by the solution of these two equations in the (g,P) plane. Figure 4.1 sketches these curves:

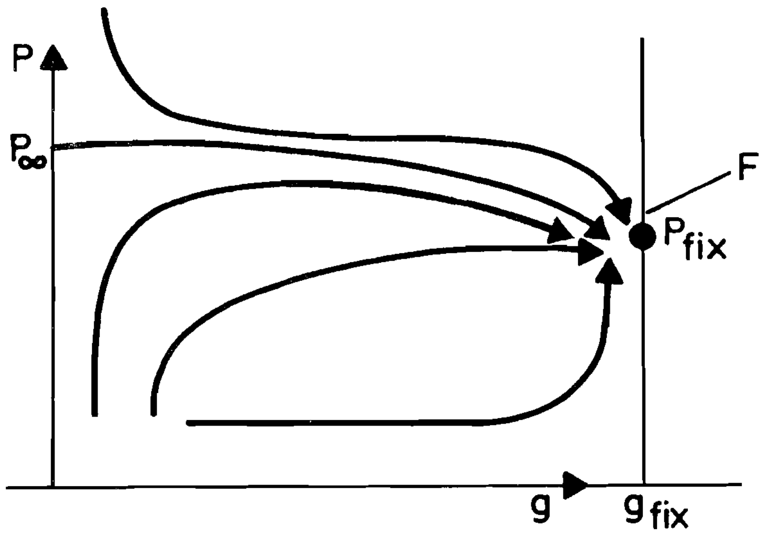

Fig.4.1.

The fixed point F is globally stable. In the following sketches we show a typical cylinder.

Region I: $E_{fix,1} < E_{div}$. F_1 is unstable, F_2 stable.

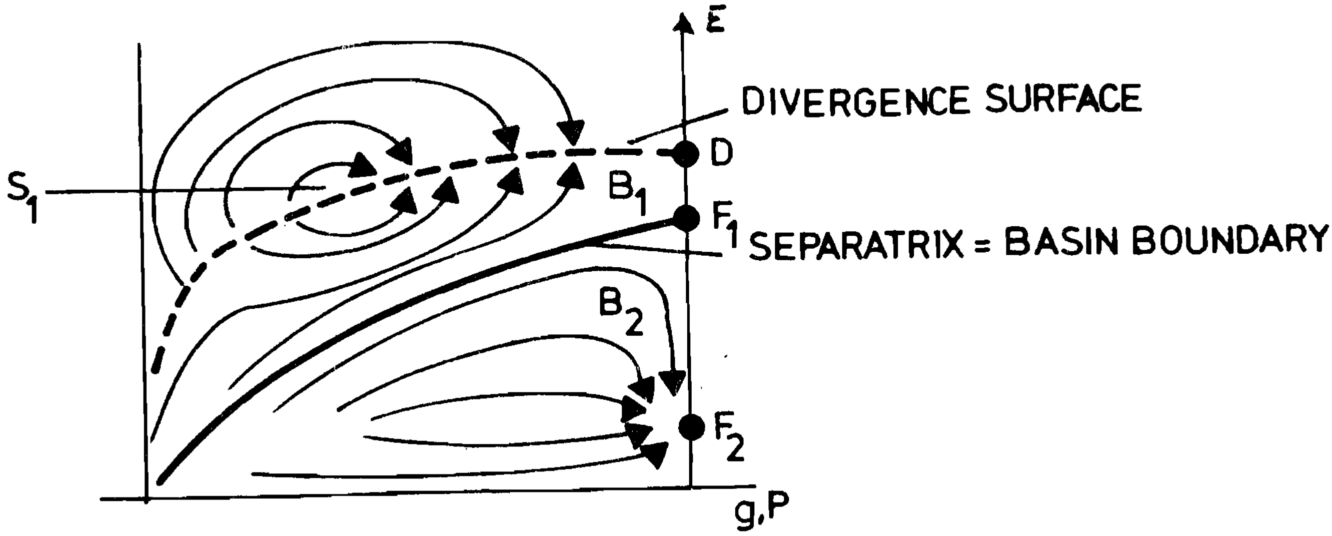

Fig.4.2.

We distinguish two basins: in the lower one, all trajectories tend to the fixed point F_2 (low E, high M); in the upper one, all trajectories sooner or later reach the divergence surface where evelution according to the model has to stop.* Above the divergence surface, the curves look as if time had been reversed; we will meet this situation again in Model D.

*A "spurious fixed point" at S acts as formal attractor for this basin. See [2] for details.

Region II: $E_{fix} > E_{div}$. Both fixed points are now stable. In this case the solution cylinder will look different, depending on the number of spurious fixed points introduced (the fixed line can intersect the cylinder in 0 or 2 points).

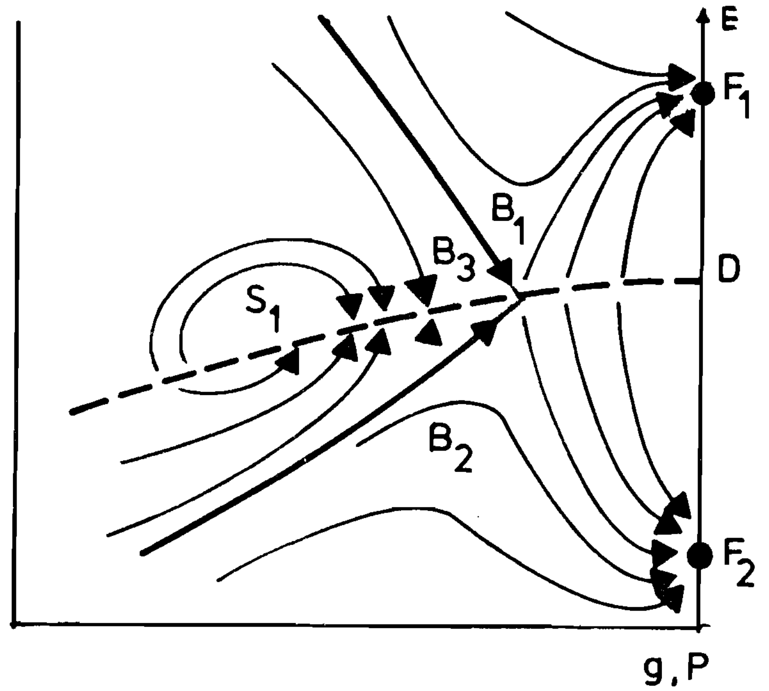

Fig.4.3.

Three basins exist: B_1 and B_2 are the basins of the fixed points F_1 and F_2, while in B_3 (the basin of the "spurious" S_1) all trajectories limit the divergence surface. Between B_1 and B_2 the divergence surface acts as a boundary.

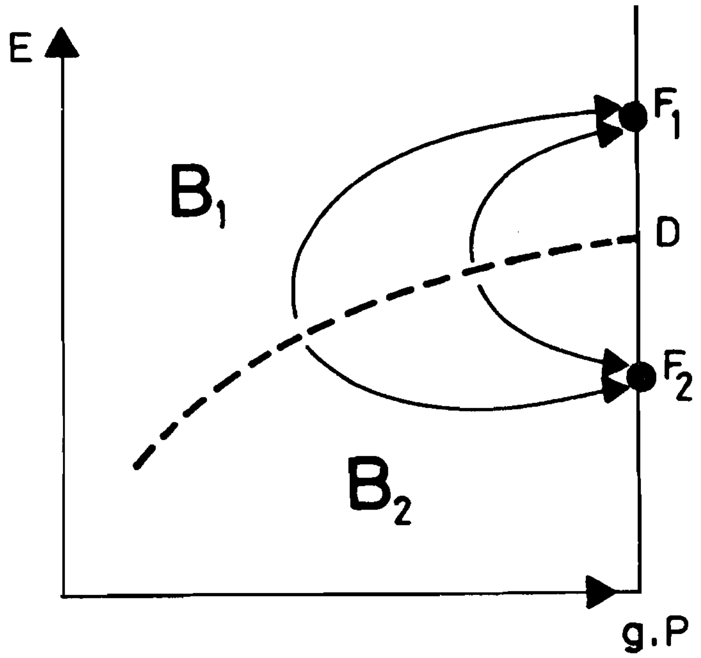

Fig.4.4

Here, the situation is very simple: the divergence surface separates two basins with fixed points as attractors.

5. Model C.

This model followed model B and is derived from the model described in [3]. It additionally takes labor into account but does not anymore model the risk acceptance and thereby also the explicit modelling of the additional safety expenditures is omitted. Besides of the inclusion of labor the objective for this model was to avoid the "divergence surface" of Model B. However, the equation for population dynamics is almost exactly the same as in Model B. The myths of the model are:

1. The distribution of GNP G is into the following sectors: Population P, energy E, capital K, labor L and the net increases of energy, capital and labor $\dot{E}$, $\dot{K}$, $\dot{L}$.
2. A Cobb-Douglas production function containing energy, capital and labor as production factors.
3. These two myths together with the above mentioned equation for the population growth are not sufficient to close the system. Therefore, to close the system, a driving myth has to be employed. In the case of this model it is therefore assumed that investment is made in that sector which yields a maximum growth of GNP. Thus, we have the following equations:

1. $G = cP + e_o^E E + e_o^K K + e_o^L L + e_1^E \dot{E} + e_1^K \dot{K} + e_1^L \dot{L}$ (5.1)

2. $G = AE^{\alpha} L^{\beta} K^{\gamma}$ (5.2)

3. $\dot{P} = a_p P\,(1-P/P_{\infty}) - a_c G$ (5.3)

4. $\max \dot{G}$ (5.4)

with the parameters:

c	per capita consumption
e_o^E, e_o^K, e_o^L	specific current plus replacement costs for energy, capital and labor.
e_1^E, e_1^K, e_1^L	specific costs for (net) investment into energy, capital and labor.
A, α, β, γ	are the usual parameters of a Cobb-Douglas function
a_p, P_∞	growth and limiting parameters for the logistic function
a_c	correction factor, like in Model B.

The way this model was implemented was the following: It was numerically integrated, the increases in $\dot{E}$, $\dot{K}$ and $\dot{L}$ have been determined by a linear program in these three variables which is described by

$$\text{a)} \quad \dot{E} \geq 0 \quad \dot{K} \geq 0 \quad \dot{L} \geq 0 \tag{5.5}$$

$$\text{b)} \quad e_1^E\dot{E} + e_1^K\dot{K} + e_1^L\dot{L} \leq G - cP - e_o^E E - e_o^K K - e_o^L L \tag{5.6}$$

$$\text{c)} \quad \max \frac{\dot{G}}{G} = \alpha\frac{\dot{E}}{E} + \beta\frac{\dot{L}}{L} + \gamma\frac{\dot{K}}{K} \tag{5.7}$$

An additional operational constraint was

$$\text{d)} \quad L \leq \delta P \tag{5.8}$$

This is to say that the step size for numerical integration had to be chosen in a way so as to make sure that L did not exceed δP.

In order to study the long term, qualitative behavior of the model, it was transformed into a closed system of differential equations with the help of the following observations: Firstly, the most effective production factor for investment (according to our assumption about the parameters) is labor and constraint (5.8) becomes an equality almost immediately. The next feature is that either E or K is increased until indifference (which means a degenerated solution of the LP) between investing into E or K is reached. From that time on it is optimal to keep the E/K ratio fixed. This optimal ratio depends, among others on e_o^E and e_o^K. This is well in accordance with economic theory which says that there is an optimal mix of production factors given a production function and a production level [11].

Following these observations, the maximization in (5.4) has been replaced by

$$\kappa E = K \tag{5.9}$$

and $$L = \delta P \tag{5.10}$$

and the system is closed in a way that we end up with two dynamic equations for $\dot{E}$ and $\dot{P}$.

The phase portrait of what we called the base case (meaning that we used a reference set of parameters) is quite simple:

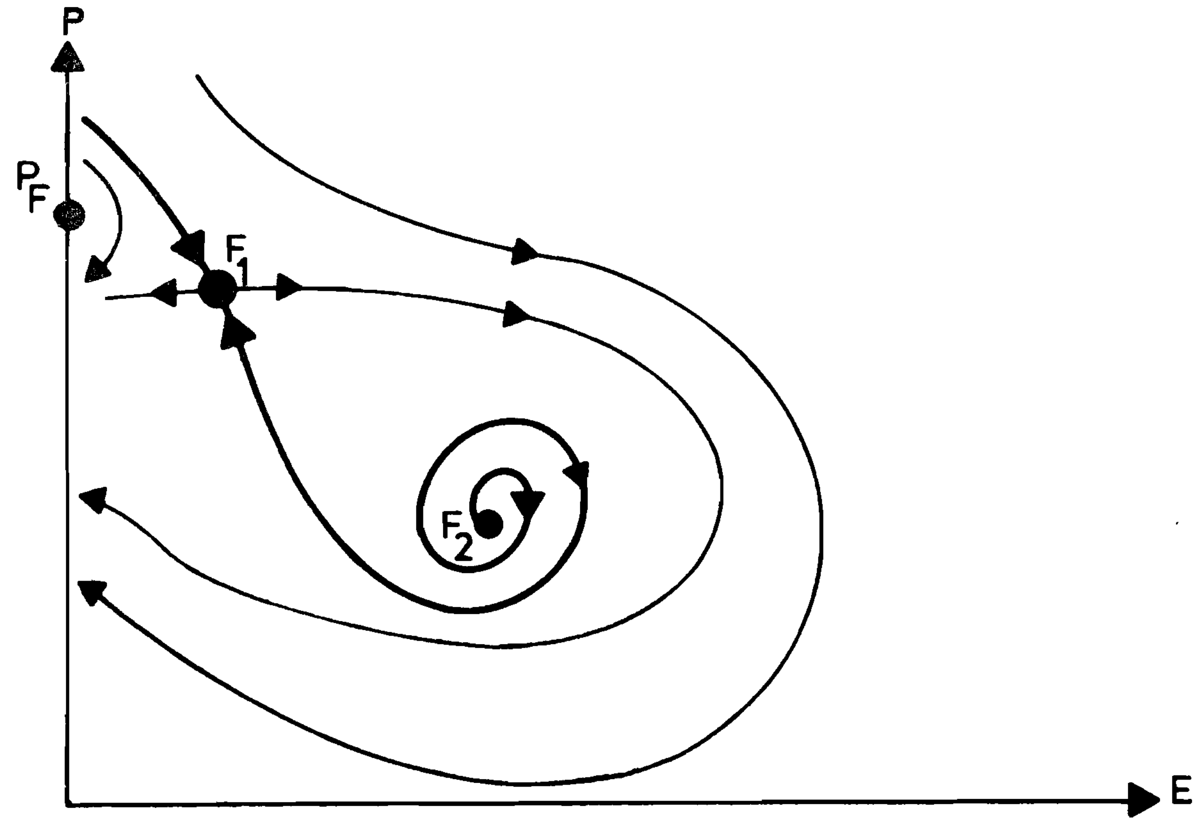

Fig.5.1.

The phase space considered (E>O, P>O) contains two fixed points: One saddle point (F_1) and one source (F_2).

Here, the stable manifold of the saddle point is a separatrix but not a basin boundary, since the entire phase space consists of only one basin which is attracted by the straight line between the points (E=O, P=O) and (E=O, $P=P_f$) where P_f is defined by $\dot{E}(O, P_f) = O$. The different future behavior, as mentioned in chapter 2 consists in the fact that trajectories on the one side of the separatrix go once more around the unstable fixed point than trajectories on the other side. Finally, they all end up on the axis E=O where the model ceases to be meaningful. In a purely formal extrapolation, the attractor is a stable fixed point at P=O and E<O.

In addition to the just described base case some sensitivity analysis has been performed. In conclusion, we will present a part of it showing a dependence of the phase portrait on the parameter a_c ($a_c \cdot G$ was the correction factor in the population equation) that has been varied in an interval that seemed reasonable*. It turned out that by decreasing a_c the former unstable fixed point F_2 changes its character and becomes stable. The according phase portrait is shown in Fig. 5.2. The phase space now consists of two basins and the separatrix becomes the basin boundary.

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*Since there are no analytic expressions for the coordinates of the fixed points, the different cases (in dependence of the parameters) cannot be described as exactly as for Model B. The results reported here are based on numerical evaluations of the equations.

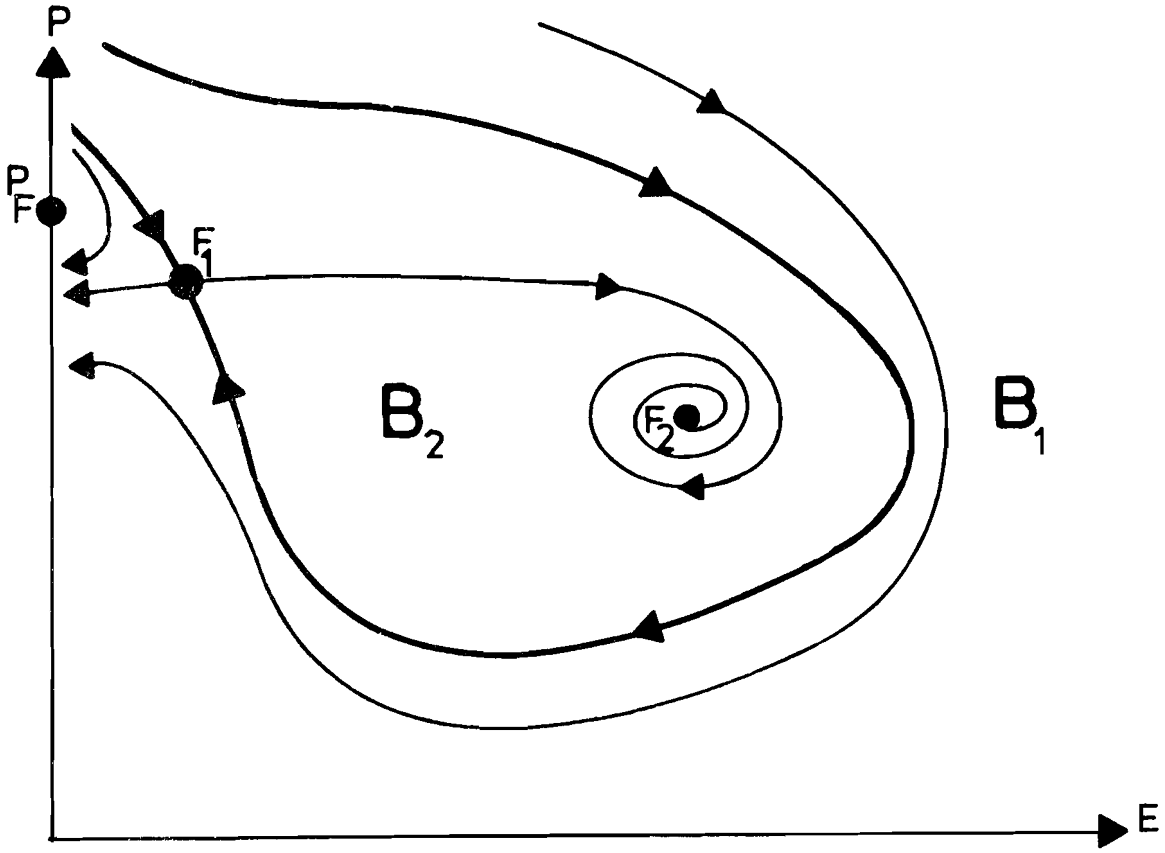

Fig.5.2.

The question arises now how the transition between these cases takes place. Studying this question one realizes that two things happened during the transition from the phase portrait in Fig. 5.1 to the one in Fig. 5.2. First, the fixed point changed its character and secondly, the one part of the unstable manifold does no longer go "under" the stable manifold but stays "above". Depending on which one of these two changes happens first, there are different possibilities of transition.*

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*These two transitions correspond to two well-known bifurcations of a vector field in two dimensions: coincidence of one-dimensional stable and unstable manifolds and non-hyperbolicity (=change in stability) of a fixed point. See [10].

In our case, the first change is in the character of the fixed point. Therefore, the now stable fixed point has to be "screened" by an unstable closed orbit (in Fig. 5.3) which becomes a separatrix and a basin boundary. Not because of its relevance (it is not generic) but just for completion of the story the special case between Figures 5.2 and 5.3 is shown in Fig. 5.4. Transition from Fig. 5.3 to 5.1 is just that the closed orbit becomes smaller and smaller until it collapses to a single point.

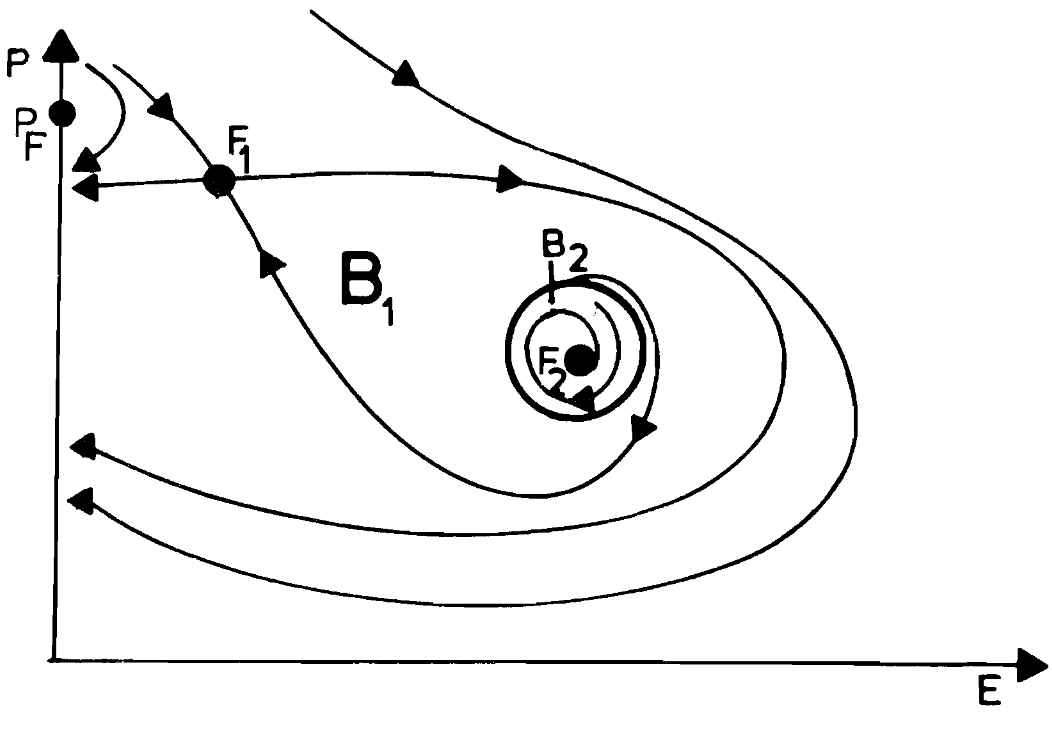

Fig.5.3.

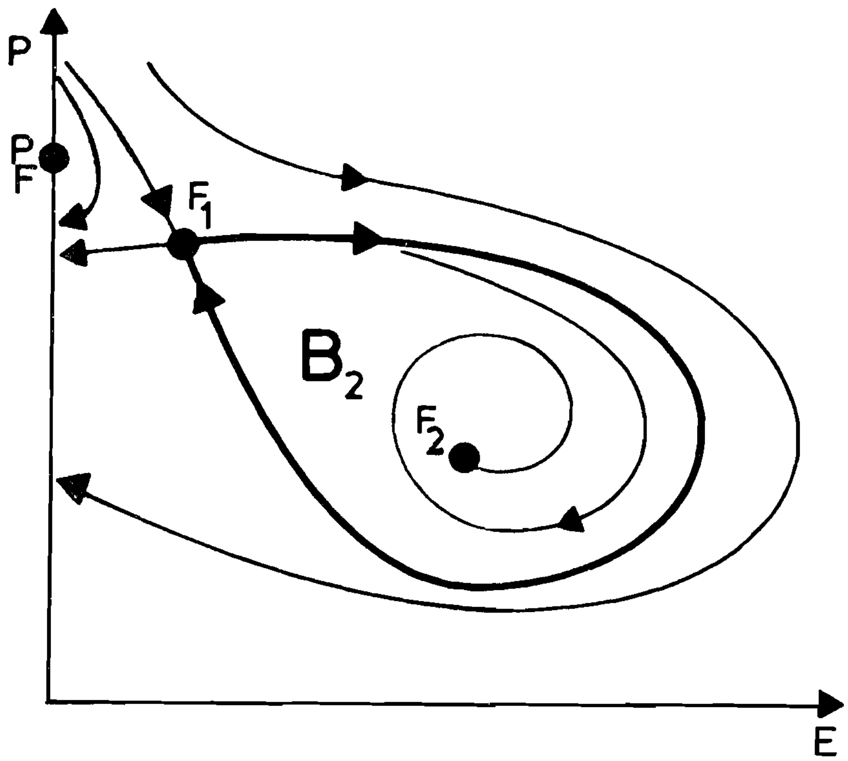

Fig.5.4.

There is one further phase portrait for this model which we do not show because it does not occur for reasonable values of the parameters. It corresponds to an inverted sequence of the transitions mentioned above.

6. Model D.

The model we are going to describe now has its origin in [4]. It is not exactly the same as described by Häfele for two reasons: Firstly, equations have been changed as the discussion evolved and secondly in order to proceed step-by-step simplifying assumptions have been made for this first step. Therefore we again end up with a two-dimensional model with the variables E (total energy consumption) and K (capital stock). Its myths could be described as follows:

1. Dynamics of energy and capital flow can be described by input/output type relations.
2. The GNP is divided into two parts: one part accounts for satisfying the current demand, the rest remaining for investment.
3. A Cobb-Douglas production function for GNP.

There are the following auxiliary variables:

L : labor

G : GNP

α : fraction of GNP for consumption

P : population

Now we have the following equations:

$$\dot{E} = (1-\alpha)\ \{a_{11}E + a_{12}K + a_{13}L\} \tag{6.1}$$

$$\dot{K} = (1-\alpha)\ \{a_{21}E + a_{22}K + a_{23}L\} \tag{6.2}$$

$$\alpha G = cP + e_o^E E + e_o^K K + e_o^L L \tag{6.3}$$

$$G = AE^{\hat{\alpha}}\ K^{\hat{\beta}}\ K^{\hat{\gamma}} \tag{6.4}$$

Parameters:

a_{ij} : input/output coefficients

c : per capita consumption

e_o^E, e_o^K, e_o^K : specific current plus replacement costs for energy, capital and labor

e_1^E, e_1^K, e_1^L : specific costs for (net) investment into energy, capital and labor

A, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$: usual parameters of a Cobb-Douglas function

Labor L and population P are kept constant or, in our language one could say that they are determined by an autonomous system that has reached (a stable) equilibrium as in Model B.

Inspection of equations (6.1) and (6.2) shows that except for the common factor $(1-\alpha)$ we deal with a linear system. This means that the given system can by the parameter transformation

$$\frac{d\tau}{dt} = 1-\alpha$$

be transformed into a linear system in τ, or in other words the solution of the given system is obtained by solving the linear system and changing the "speed" accordingly. The solution of the linear system is straightforward. With our assumptions, it turns out that its only fixed point, F_o, is a saddle point close to the origin in the first quadrant of the (E,K) plane. The solution of the linear system near the fixed point is shown in Fig. 6.1. The two basins of this phase portrait are obvious.

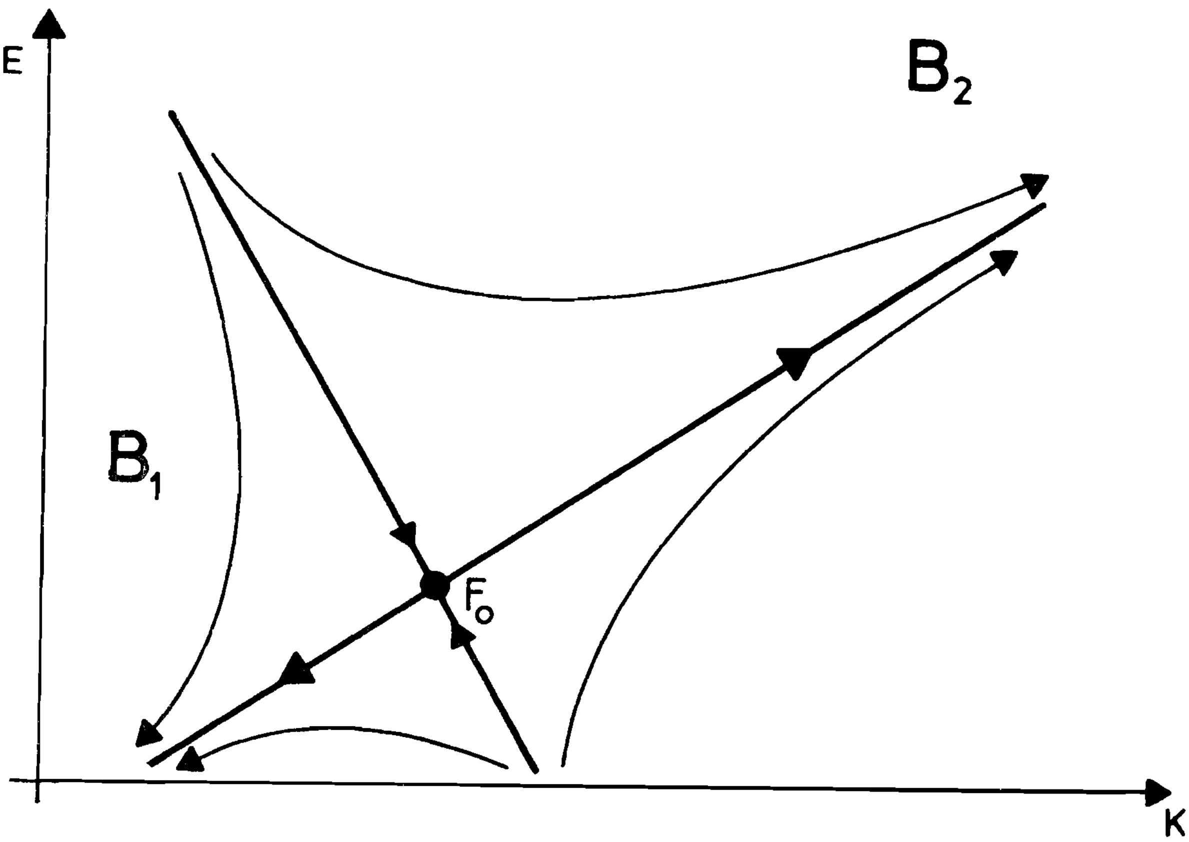

Fig.6.1.

The next step in obtaining the phase portrait of the given system is to investigate the "speed-factor" 1-α. If this factor is positive (negative), the sense of movement is kept (**reversed**) as compared to the linear system. Therefore, it is of particular interest to know the solution of the equation $1-\alpha=0$ which consists of fixed points of our given system since the "speed-factor" vanishes. Since we assumed

$$\hat{\alpha} + \hat{\beta} + \hat{\gamma} = 1 \text{ and } \hat{\alpha},\ \hat{\beta},\ \hat{\gamma} > 0$$

it follows that $\hat{\alpha} + \hat{\beta} < 1$ and therefore the solution for $1-\alpha=0$ is a closed curve of fixed points thus called the "fixed curve". An impression of the fixed curve is given in Fig. 6.2. The inside of this curve consists of points with $(1-\alpha)>0$ and the sense of motion is not changed as compared to the linear system. The contrary is true for the outside: $1-\alpha$ is less than zero and the sense of motion is reversed.

In order to complete the description of the phase portrait the nature of the fixed points has to be discussed: The fixed point of the linear system remains, of course, only the stable and the unstable manifold interchange their nature because the fixed point lies in the outside of the fixed curve. Now, let us look for the properties of the fixed points that lie on the fixed curve. The fixed curve consists of four segments--separated by four points--with fixed points of the same nature. Two of these segments consist of stable fixed points and the other two consist of unstable ones whereas the four separating points show neutral stability. The consequences for the phase portrait are given in Figures 6.2 and 6.3. The four above mentioned special (or extremal) points are denoted by P_1 through P_4 and in these points the tangents for both the fixed curve and the trajectory coincide. To either of the two stable segments belongs an according basin of attraction. The larger one of the two, B_2,

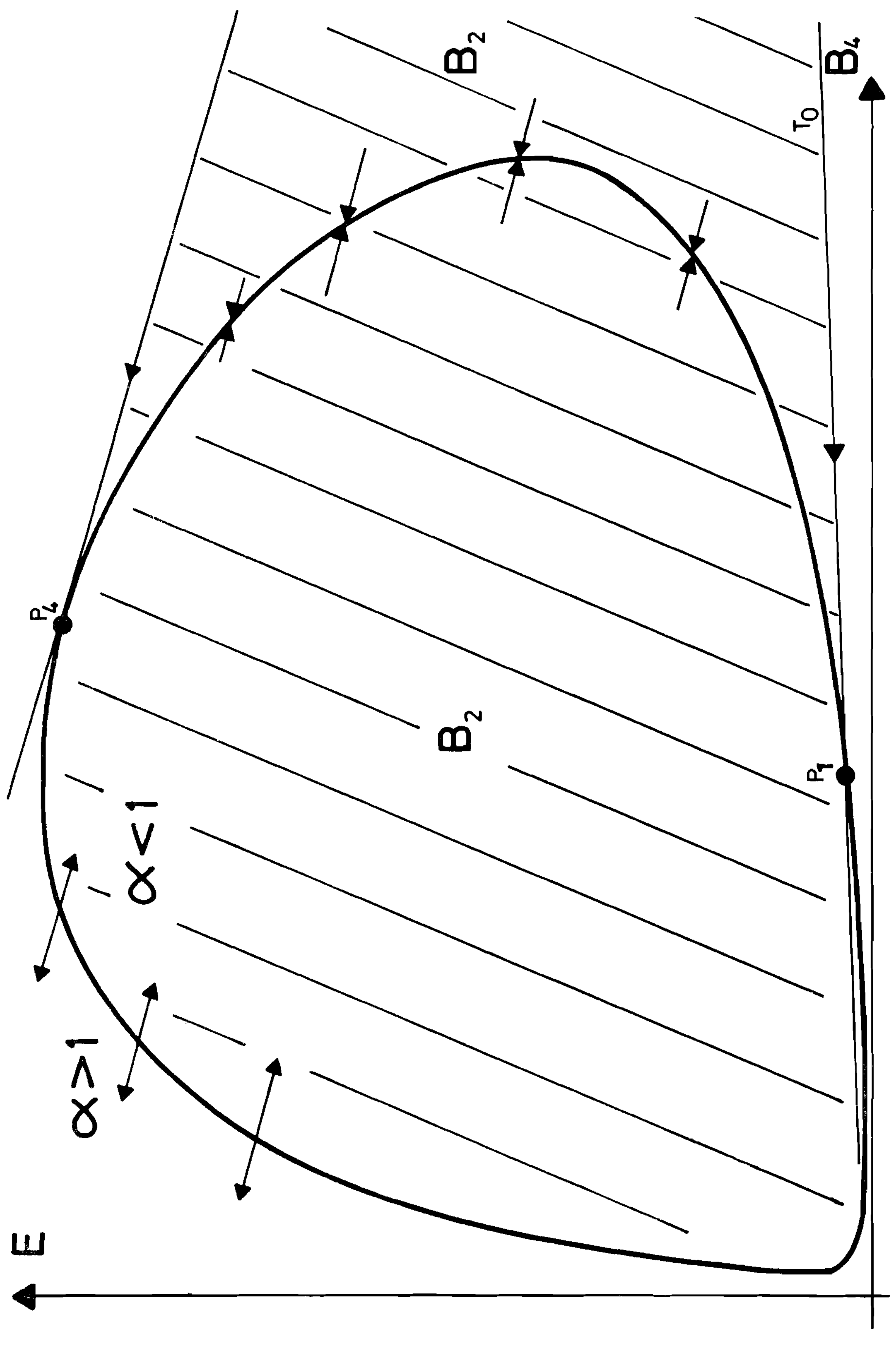

Fig.6.2.

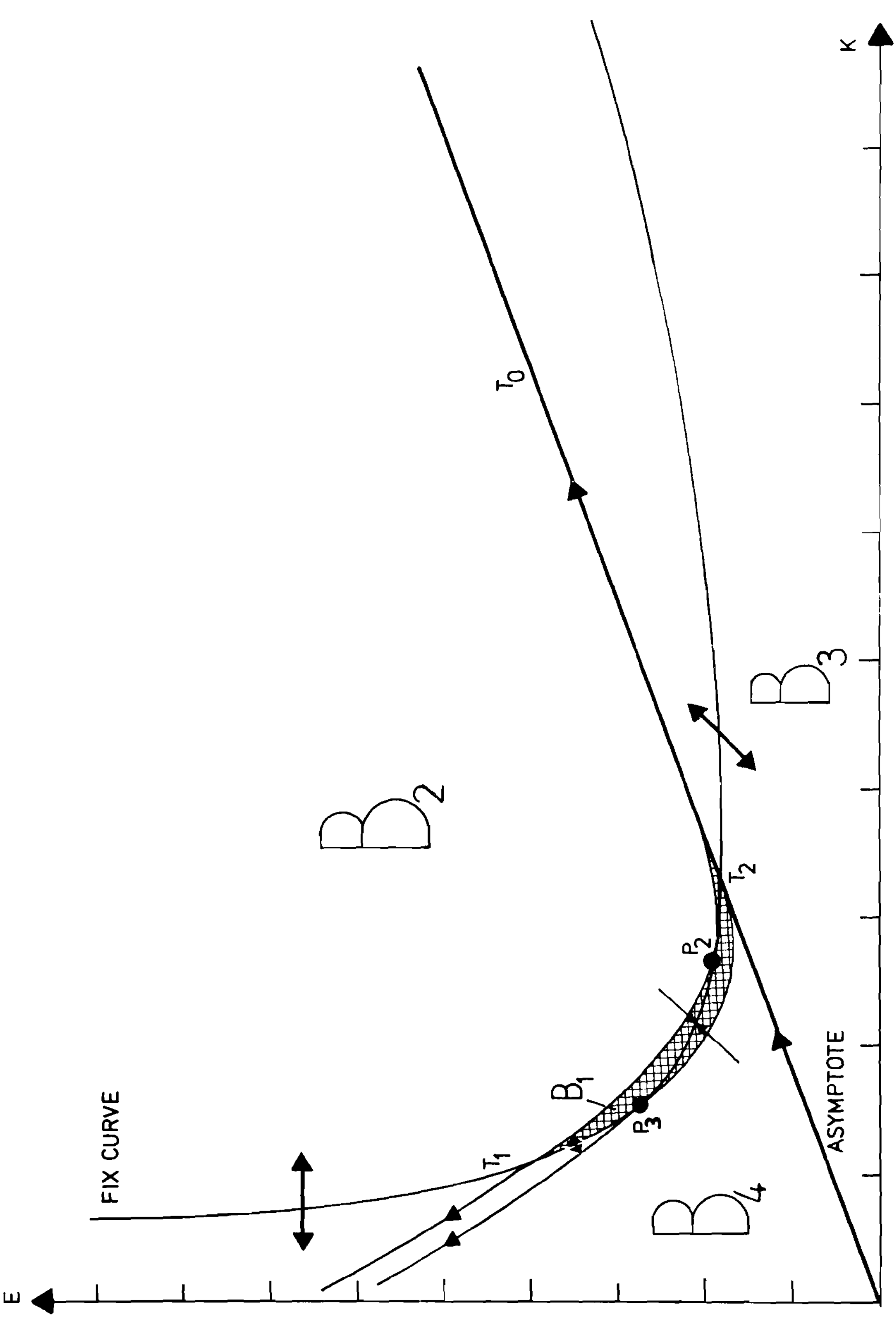

Fig.6.3.

which belongs to the stable segment between P_1 and P_4 is sketched in Fig. 6.2. This figure is not quite correct in so far as it does not show basin B_1, but this is due to the big difference in sizes of those two basins. However, the details concerning basin B_1 which belongs to the stable segment P_2-P_3 can be seen on Fig. 6.3.

The rest of the phase space consists of what is left of the two basins of the linear system (of course not the basins of Fig. 6.1 because of the reversed sense of motion outside the fixed curve). In Figs. 6.2 and 6.3 those are denoted by B_3 and B_4.

To analyze further possibilities for the phase portrait, we perform a change of coordinates and thereby transform the trajectories of the linear system mentioned above into straight lines.* This, of course, deforms the fixed curve as well. The classification of all possible phase portraits can now be performed through classifying the following situation (Fig.6.4):

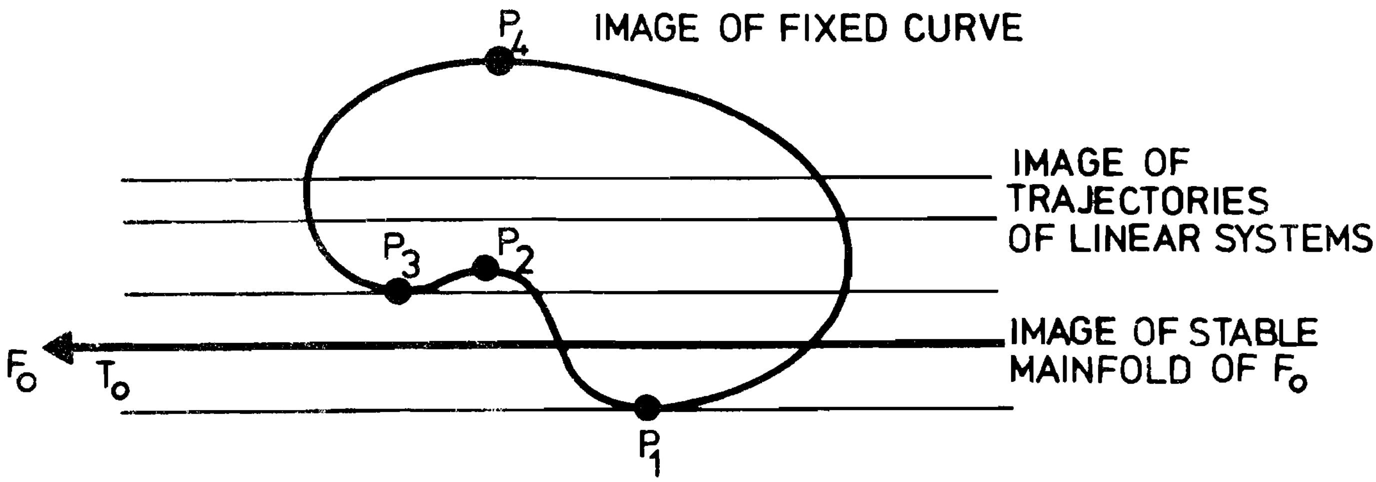

Fig.6.4.

*Of course, this can only be done in the "right-half plane" delimited by the unstable manifold of F_0.

Essential in this picture are only the relative locations of the minima and maxima (P_1 to P_4) with respect to each other and to the "distinguished trajectory" T_o. There are several constraints to delimit the possible structures; e.g., T_o intersect the image of the fixed curve at most twice.

The following phase portraits can occur (for clarity we show them in the new coordinate system, as in Fig.6.4). Other possible phase portraits do not occur for reasonable values of the parameters.

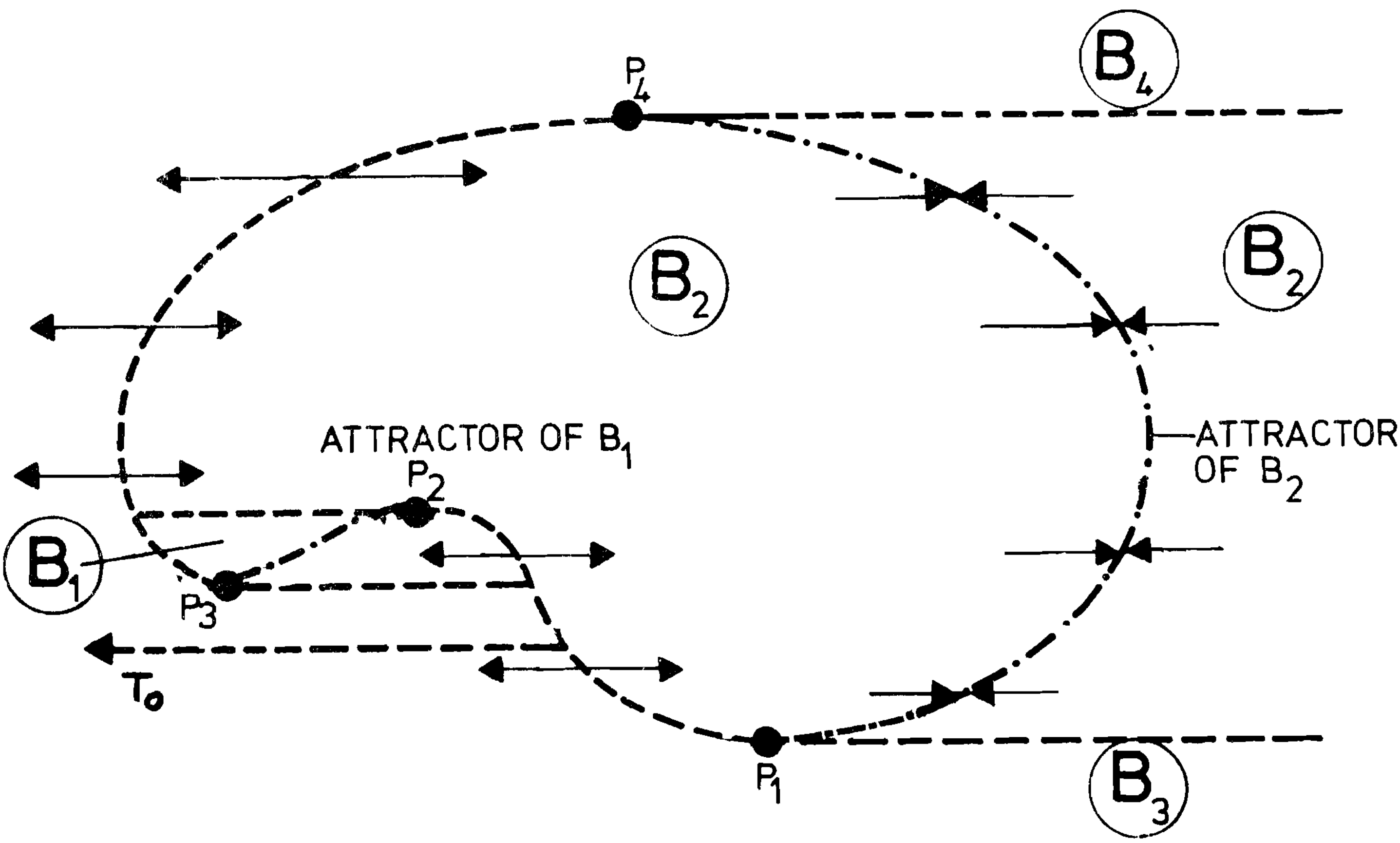

Fig.6.5. 4 extrema, To intersects fixed curve
4 basins

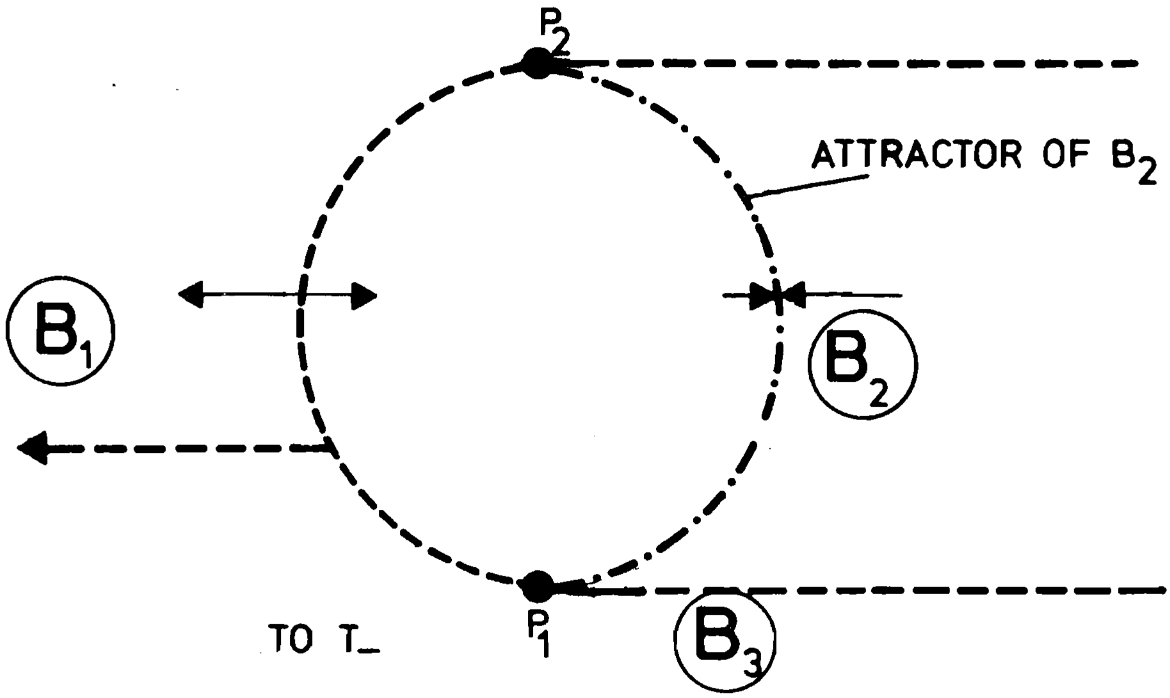

Fig.6.6. 2 extrema, To intersects fixed curve
3 basins

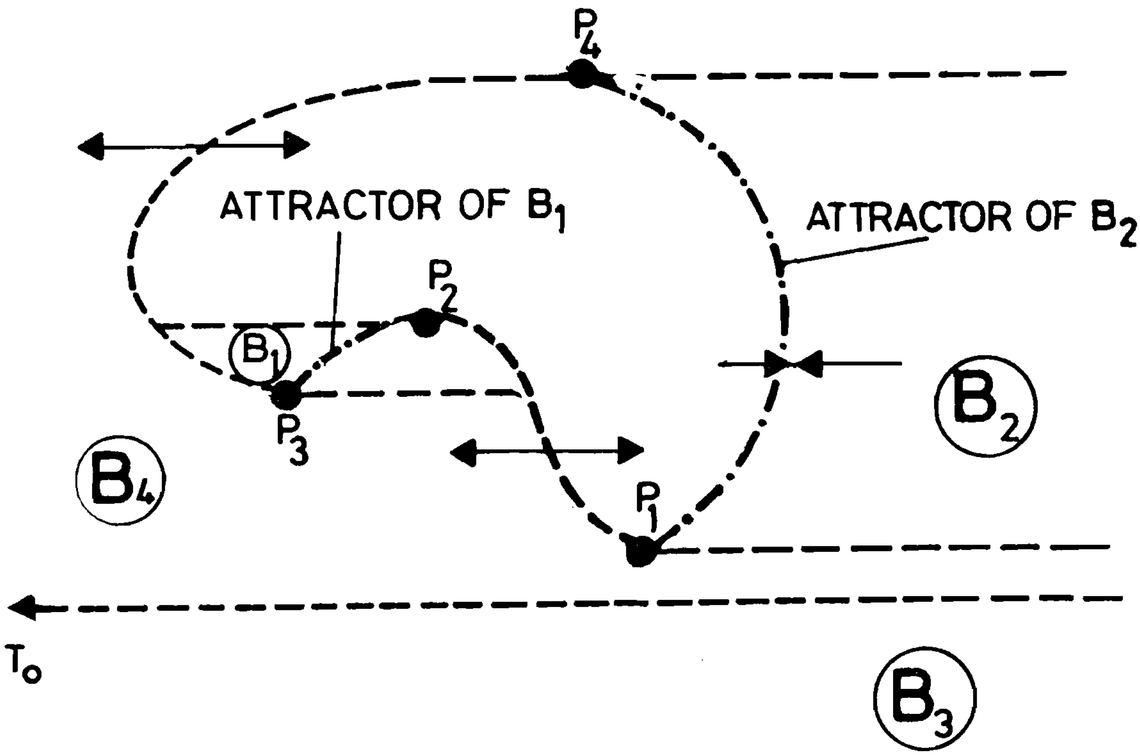

Fig.6.7. 4 extrema, To does not intersect
4 basins

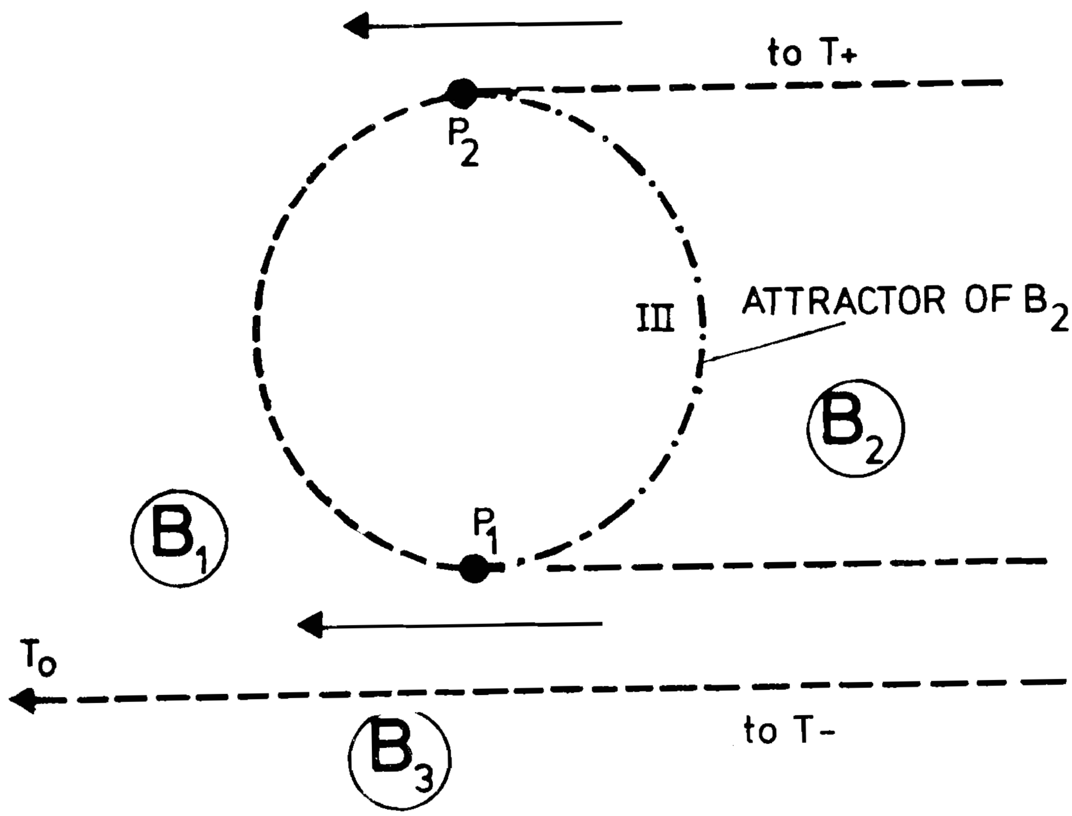

Fig.6.8. 2 extrema, To does not intersect
3 basins

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