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On Mutual Insurance

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Abstract

Owners of stochastic assets can pool their endowments to smoothen and insure individual payoffs across outcomes and time. We explore, in such a setting, how contingent shadow prices on aggregate resources can be used for *three* purposes: *First*, to design mutual contracts for risk averse agents; *second*, to quantify the malfunctioning of such contracts when there are risk lovers (or scale economies); and *third*, to estimate reasonable premiums for insurance offered by outside agents.

Key words: risk, insurance, mutuals, cooperative games, core, contingent prices, stochastic Lagrange multipliers, duality gap, modulus of nonconvexity, randomization.

Contents

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1 Introduction

This paper considers several extensions of Borch's classical study of a reinsurance market [5]. Novelties include state-dependent payoffs, multi-dimensional risks, stochastic dependence, dynamic allocations, and above all: computable core solutions. The setting is broadly as follows. Suppose individual i, when operating alone, could obtain expected payoff $\pi_i(e_i)$ from a stochastic commodity bundle e_i fully owned, delivered, produced, or handled by him. Examples are manifold. For instance,

$*$ e_i might be the randomly varying water endowments of agricultural region (or hydro-electric power station) i;

$*$ e_i could stand for nation i's state-dependent quotas in producing diverse pollutants (or in catching various fish species);

$*$ e_i could account for uncertain quantities of different goods that transportation firm i must bring from various origins to specified destinations;

$*$ e_i can be the financial risk, in the form of unknown monetary claims, against insurance company i.

Present several such individuals $i \in I$, we consider pooling and exchange of their private holdings e_i as a mean of protection against inconvenient outcomes. More precisely, we shall deal with exchange aimed at providing the concerned parties with mutual insurance. Focus will be on three problems:

• If all agents are risk averse, can they write an efficient, socially stable, and computable contract?

• Otherwise, when some agents love risk, or if there are economies of scale, what properties will such contracts have?

• In any case, if the mutual company at hand exploits internal insurance optimally, how should it evaluate supplementary insurance offered by outside agents? What would be reasonable premia?

To come to grips with these problems we use as benchmark the instance where all parties have concave objectives. Then, as explained in Section 2, an efficient and socially stable contract can be fully written in terms of a contingent shadow price vector on the aggregate endowment $e_I := \sum_{i \in I} e_i$. That contract is, in principle, readily computable and easy to implement. It resembles a competitive equilibrium. As argued in Sections 2 and 3, it decomposes across events and time into a family of equilibria akin to spot markets.

However, and not surprisingly, when preferences are *nonconcave*, that sort of insurance contract, designed in terms of the said shadow prices, cannot be viable. In fact, any such contract will cause overspending and insolvency. Section 4 explains why. To mitigate these severe deficiencies, Section 5 briefly explores how randomization may help to restore concavity - whence enhance the prospects for good insurance.

This is a companion paper to [8]. As such it elaborates on *four* novelties. *First*, we make publicity for contracts which, under concavity assumptions, are not only efficient in the aggregate, but also robust against defection. More precisely, emphasis is here on modes of risk sharing that belong to *core* of a transferable-utility cooperative game. This perspective is surprisingly uncommon and far from fully explored. *Second*, present nonconcave preferences, we explain and quantify some basic deficiencies of insurance contracts based on Lagrangian duality and shadow prices. *Third*, we indicate how randomization may facilitate the writing of good contracts. *Fourth* and last, we estimate the willingness of a mutual company to pay for supplementary insurance offered by outside agents.

The paper should interest a mixed audience, comprising actuaries, economists, game theorists, operations researchers, and statisticians. All assumptions and results are stated formally and precisely. Detailed proofs are given elsewhere [8]. Here we only outline demonstrations of two key results - and rather discuss some ramifications and seek to explain the economic significance of the main propositions.

2 Mutual insurance under concave preferences

We accommodate henceforth a finite fixed set I of risk exposed agents (or industries, regions, sectors etc.) Each individual $i \in I$ owns a private, random endowment e_i in some Euclidean space $\mathbb{E}$. More precisely, if the state (or scenario) $s \in S$ comes up, then i is fully entitled to the commodity vector $e_i(s) \in \mathbb{E}$ (say, $\mathbb{R}^m$ if m commodities are at stake).[1] One may construe these vectors as accounting for state-dependent outputs of various enterprises or natural resources.

Since part of our motivation is computational, we do not hesitate in assuming S finite. Moreover, all agents use the same S as an exhaustive set of mutually exclusive states. In addition, the agents agree on the probability distribution $p(s) > 0$ of the still unknown future state s. This means that uncertainty is objective and external. It also means that the environment is unaffected by actions considered below, - and there are no informational asymmetries. Note that we can allow all sorts of dependencies and associations between the stochastic vectors $s \mapsto e_i(s), i \in I$. For simplicity we shall start by considering instances where uncertainty about s is fully resolved in *one* step. The case when information comes gradually, over several steps, will be discussed in Section 3.

If individual i were to contend with his random endowment e_i in splendid isola-

[1] Generation of relevant scenarios is demanding since their number easily gets out of hand. This problem, prominent in stochastic programming, will not be explored here.

tion, he would a priori look forward to expected payoff

$$\pi_i(e_i) := E\Pi_i(e_i) := \sum_{s\in S} p(s)\Pi_i(s, e_i(s)) \tag{1}$$

where $\Pi_i(s,\cdot)$ is his payoff function in state s. Often i can do better by collaborating with others, such collaboration implying that the holdings of the participants be pooled. Specifically, the members of some coalition $C \subseteq I$ could, in principle, join forces and compute their presumably finite, ex ante, aggregate, *stand-alone payoff*

$$\pi_C(e_C) := \sup\left\{\sum_{i\in C}\pi_i(x_i)\left|\sum_{i\in C}x_i(s) = \sum_{i\in C}e_i(s) =: e_C(s) \textit{for all } s\in S\right.\right\}, \tag{2}$$

the aim being to distribute prospective gains among themselves.[2] Since $\pi_C(e_C)$ is finite by assumption, so is the corresponding state-dependent, ex post value

$$\Pi_C(s, e_C(s)) := \sup\left\{\sum_{i\in C}\Pi_i(s, x_i(s))\left|\sum_{i\in C}x_i(s) = e_C(s)\right.\right\} \tag{3}$$

for each $s\in S$. We ask: *Can the grand coalition I form (be it ex ante, ex post, or both)? And if so, how might payoffs then be shared?*

These questions point to a family of cooperative games, all with player set I, but with various characteristic functions $I \supseteq C \mapsto v_C \in \mathbb{R}$, and each game allowing side payments. In other words, main objects here are so-called *production games*, featuring transferable (maybe stochastic) payoffs. Given such a game, codified by the characteristic function $v = [v_C]$, a *payoff allocation* $u = (u_i) \in \mathbb{R}^I$ is said to belong to its *core* iff u entails

$$\left.\begin{array}{lrcl} \textit{efficiency:} & \sum_{i\in I}u_i & = & v_I, \\ \textit{and social stability:} & \sum_{i\in C}u_i & \geq & v_C \quad \textit{for all coalitions } C\subset I \end{array}\right\} \tag{4}$$

Social stability means that no coalition $C \subset I$ could improve its members' outcome by splitting away from the others. Note that mere stability is easy to achieve: Simply let the numbers (the utilities) u_i be so large that $\sum_{i\in C}u_i \geq v_C, \forall C\subseteq I$. Thus, not very surprising, the essential difficulty resides in the requirement that total payoff be efficient and not distributed excessively.

Remarks:

* Cooperative games do not figure prominently in the insurance literature, a notable exception being [2]. Borch [4], [5] also deals with cooperative games, but he never uses the core solution concept.

* Creation of a mutual is, of course, only one of many ways to reduce risk exposure. Self-insurance, treaties with outside providers of insurance, and specific

[2] *Sup* is short notation for *supremum value*. We could posit that all these extremal values are attained, i.e., that they be maxima. This issue will not be elaborated though because it detracts attention from the main issues.

investments to enhance safety (or reliability) are other measures. We regard these alternatives as exogenous supplements to the actions taken here. In many settings, particularly those involving fairly homogeneous communities with well spread risks, a mutual contract will be part of the overall arrangement, and worth exploring.

* The insurance literature mostly considers state-independent payoffs. That optic appears reasonable for low-consequence, conventional risks such as damage on cars. It does, however, not fit major events like severe illness or catastrophes.

* Insurance theory often assumes independent or weakly associated risks. *No* such assumption is made here. Consequently, we cannot - and shall not - rely on any law of large numbers or central limit theorem. In fact, the subsequent analysis is applicable for major events, say catastrophes, inflicting severe and highly correlated losses.

* We stress that endowments e_i can be multi-dimensional. While the insurance literature, and insurance circles, often consider merely only one good, namely money, we can accommodate several "securities", be it financial papers or real assets, these giving various sorts of dividends. Multi-dimensionality might also stem from some components of e_i referring to goods available only in specified combinations of state, location, and time.

* Note that $\Pi_i(s, \cdot)$ is *net payoff* or *net utility*, obtainable after adverse affects of the "hazard" s have been mitigated.

* Admittedly, the use of expected payoffs $E\Pi_i(s, \cdot)$is best justified under repeated interaction, allowing probabilities to be estimated from observed data. Nothing precludes, however, that a mutual will be set up to protect against specific rare events the "statistics" of which represents expert judgements.

* Payoff $\Pi_i(s, x_i(s)) = -\infty$ is far from excluded. In fact, the value $-\infty$ represents infinite loss (or total dissatisfaction) and accounts for violation of implicit constraints, not spelled out at this aggregate level. This abstract way of incorporating constraints is analytically very convenient (albeit not useful in computation). It helps to keep focus on some key issues. For example, we could have

$$\Pi_i(s, x_i(s)) := \sup \left\{ \hat{\Pi}_i(s, y_i) : L_i y = x_i(s) \right\}$$

with the understanding that $\Pi_i(s, x_i(s)) = -\infty$ whenever the equation $L_i y = x_i(s)$ has no solution. Explicit representation of constraints is illustrated in Section 3.

* It is tacitly assumed, when it comes to any program (2) or (3), that no concerned agent misrepresents his preferences. We posit that all functions Π_i are common knowledge, or these objects can be readily synthesized, or they are reported honestly. Admittedly, this assumption is quite stringent. In the same vein, it may take some faith - or good will - to presume that all agents have the same perception of uncertainty. Coincidence of probability assessments is rare and difficult when it comes to exceptionally important states that occur with very low frequencies. We believe, however, that iterative experiments or computations may be set up which require neither common knowledge nor equal risk perceptions, but which nonetheless converge to core solutions.

* Constructions (2) and (3) are like the ones given by Shapley and Shubik [14] in their classical analysis of so-called market games.

* Note the complete absence of direct externalities in the individual objectives (1). This feature is crucial - in fact, indispensable for the subsequent analysis. It invites decomposition and decentralized decision making. To wit, if all payoff functions $\Pi_i(s,\cdot)$ are concave, then - as stated in Theorem 1 - decomposition can be supported by prices associated with contingent relaxation of the balance requirement $\sum_{i\in I} x_i(\cdot) = e_I(\cdot)$. Facing appropriate prices each (presumable risk averse) agent is - as we shall see - free to make a best choice.

* Equation (2) models pooling and exchange of perfectly divisible goods, freely transferable among members of C. The advantage of doing so is evident and twofold: First, aggregation offers increased leeway and better substitution possibilities; second, it makes possible transfers of goods across time and contingencies. Less evident is the fact that, granted *concave* payoffs, that is, given risk averse agents, then cooperative incentives become so strong and well distributed that the grand coalition can safely form. Its formation means that payoff can be shared in ways not blocked by any subgroup. This is stated in the following

Proposition 1 *(Nonempty cores ex ante and ex post)*

* *Suppose all payoff functions $e_i \mapsto \pi_i(e_i)$, defined in (1), are concave. Then the ex ante payoff-sharing game $[\pi_C(e_C)]$ becomes* ***totally balanced****. This means that the game itself and all its subgames (comprising fewer players) have nonempty cores.*

* *Suppose state s has already happened, and that all payoff functions $\Pi_i(s,\cdot)$ are concave. Then, the ex post payoff-sharing game $[\Pi_C(s, e_C(s)]$ also becomes totally balanced.*

* *If the allocations $u(s) = [u_i(s)]$ belongs to the core of the ex post game (3) for every s, then $[Eu_i(s)]$ belongs to the core of the ex ante game (2).* ■

To make these insights useful we must give some computational advice concerning how to find core elements in the various settings. Denote by

$$L_C(s, x(s), \lambda(s)) := \sum_{i\in C} [\Pi_i(s, x_i(s)) + \lambda(s)(e_i(s) - x_i(s))]$$

the ex post, state s, Lagrangian of coalition C, naturally associated to problem (3). Here and elsewhere we write simply ab for the usual inner product $a \cdot b$ in $\mathbb{E}$. It is notationally convenient though to use the alternative inner product $E(\lambda x_i) := \sum_{s\in S} p(s)\lambda(s)x_i(s)$ on $\mathbb{E}^S$. Thus, via (1), we get that

$$L_C(x, \lambda) := EL_C(s, x(s), \lambda(s)) = \sum_{i\in C} [\pi_i(x_i) + E\lambda(e_i - x_i)]$$

is the standard Lagrangian associated to problem (2). Any $\lambda_I \in \mathbb{E}^S$ such that $\sup_x L_I(x, \lambda_I) \le \pi_I(e_I)$ will be named a *Lagrange multiplier.* Similarly, for given state s, any $\lambda_I(s) \in \mathbb{E}$ such that $\sup_{x(s)} L_I(s, x(s), \lambda_I(s)) \le \Pi_I(s, e_I(s))$ will be called a *contingent Lagrange multiplier.* A little more notation is needed now. For any pairing $\langle\cdot,\cdot\rangle$ (or inner product) between a Euclidean space and its dual let

$$f_*(\lambda) := \sup_x \{f(x) - \langle\lambda, x\rangle\} \tag{5}$$

denote the *conjugate* of the extended real-valued function f. Note that f_* is defined on the dual space. In terms of the customary Fenchel conjugate $f^*(\lambda) := \sup_x \{\langle \lambda, x\rangle - f(x)\}$ we have $f_*(\lambda) = (-f)^*(-\lambda)$. Definition (5) fits well to a perfectly competitive setting. Namely, if some agent buys production factors (input bundles) x at fixed linear cost $\langle \lambda, x\rangle$ to achieve revenue $f(x)$, then, at most, he gets profit $f_*(\lambda)$.

Theorem 1. (Lagrange multipliers yield core solutions)
* *For any Lagrange multiplier* λ_I *the ex ante, deterministic payoff allocation* $u_i := E(\lambda_I e_i) + \pi_{i_*}(\lambda_I), i \in I$, *belongs to the core of game (2).*
* *Similarly, for any state* s *and associated Lagrange multiplier* $\lambda_I(s)$ *the contingent, ex post, payoff allocation* $u_i(s) := \lambda_I(s)e_i(s) + \Pi_{i_*}(s, \lambda_I(s)), i \in I$, *belongs to the state-dependent core of game (3).*
* λ_I *is an overall Lagrange multiplier iff* $\lambda_I(s)$ *is a contingent multiplier for each* s. ■

Proof. Social stability obtains in the ex ante game because any coalition C receives

$$\sum_{i\in C} u_i = \sup_x L_C(x, \lambda_I) \geq \inf_\lambda \sup_x L_C(x, \lambda) \geq \sup_x \inf_\lambda L_C(x, \lambda) = \pi_C(e_C).$$

The very last inequality is often referred to as *weak duality.* The hypothesis concerning λ_I ensures *strong duality:*

$$\pi_I(e_I) \geq \sup_x L_I(x, \lambda_I) \geq \inf_\lambda \sup_x L_I(x, \lambda) \geq \sup_\lambda \inf_x L_I(x, \lambda) = \pi_I(e_I)$$

so that Pareto efficiency prevails: $\pi_I(e_F) = \sup_x L_I(x, \lambda_I) = \sum_{i\in I} u_i$. The ex post games are in the same manner. ■

Remarks:
* A special version of Theorem 1 was first proven by Owen [11] who dealt with linear programs, plagued by no uncertainty. For extensions see [8], [13], and references therein.
* Theorem 1 has a nice interpretation. Suppose contingent commodity bundles $e \in \mathbb{E}^S$ were traded at a constant price vector $\lambda \in \mathbb{E}^S$. Then, if individual i were a price-taker, he could - at best - envisage expected profit $\pi_{i*}(\lambda) := \sup_{x_i} \{\pi_i(x_i) - E(\lambda x_i)\}$. For arbitrary contingent price regime λ, given already his endowment e_i, potential profit always dominates the fait accompli, i.e., $\pi_{i*}(\lambda) \geq \pi_i(e_i) - E(\lambda e_i)$. Now, the particular nature of any Lagrange multiplier λ_I is that allocation - and profit considerations - can be decentralized as follows: In state s each individual i chooses a vector $x_i(s)$ such that $\Pi_i(s, x_i(s)) - \lambda(s)x_i(s)$ becomes maximal.
* The risk averse parties $i \in I$ are under no compulsion to reach the agreement designed in terms of the multiplier. The corresponding treaty will be incentive-compatible in that no individual or group can do better alone.

* We tacitly assume that for $C = I$ the suprema in both problems (2) and (3) are attained. Sufficient conditions for attainment are that

$$\left\{x : \sum_{i\in I} \pi_i(x_i) \le r, \sum_{i\in I} x_i = e_I\right\} \text{ and } \left\{x(s) : \sum_{i\in I} \Pi_i(s, x_i(s)) \le r, \sum_{i\in I} x_i(s) = e_I(s)\right\}$$

be compact for every real r. Then the side-payments, embodied in the core allocation, can be supported by a treaty saying how the aggregate endowment e_I should be split in various circumstances. Often there is no need to write that treaty though. To wit, if all objectives $\Pi_i(s, \cdot)$ are strictly concave, then two desirable things occur: First, the optimal distribution of the aggregate endowment (both ex ante and ex post) will be unique; second, the said unique choices will be made by the agents themselves. Nobody would need persuasion or coercion. In fact, the modified objective $\Pi_i(s, x_i(s)) - \lambda(s)x_i(s)$ of i calls forth, by itself, his seemingly agreed upon, best choice.

* Theorem 1 should be seen as a generalization of Borch's seminal study [5]. It hinges upon Lagrangian duality, exploiting weak and strong versions to generate core solutions by means of prices. Those prices are stochastic and constitute a contingent price system, as brought out in [6]. Any agent will be paid $E(\lambda_I e_i)$ for his endowment plus the amount $\pi_{i_*}(\lambda_I)$ for his profit contribution (if any), both entities being computed in terms of the said price λ_I. The result thus resembles competitive equilibrium, giving emphasis to non-strategic, price-taking behavior and decentralized actions.

* Theorem 1 begs questions whether multipliers do exist. At this juncture enters the concavity of preferences:

Proposition 2 *(Existence of multipliers) Suppose here that all functions* $\Pi_i(s, \cdot)$ *are concave and that*

$$e_I(s) \text{ belongs to the interior of } \sum_{i\in I} \{x_i(s) : \Pi_i(s, x_i(s)) > -\infty\} \text{ for all } s. \quad (6)$$

Then there exist multipliers λ_I *and* $\lambda_I(s)$ *for each* s. ■

It is informative, and helpful for economic interpretation, to relate Lagrange multipliers to marginal payoffs or shadow prices. The following result essentially derives from Danskin's envelope theorem.[3] Denote by ∂ the subdifferential of convex analysis [12].

Proposition 3 *(Shadow prices yield core solutions) Suppose here that all functions* $\Pi_i(s, \cdot)$ *are concave. Then:*

* λ_I *is a Lagrange multiplier iff* $\lambda_I \in \partial\pi_I(e_I)$*. Similarly,* $\lambda_I(s)$ *is a contingent Lagrange multiplier iff* $\lambda_I(s) \in \partial\Pi_I(s, e_I(s))$*.*

* *Given* $\lambda_I \in \partial\pi_I(e_I)$*, then for any optimal* x *we have* $\lambda_I \in \partial\pi_i(x_i)$ *for all* i*. Similarly, given* $\lambda_I(s) \in \partial\Pi_I(s, e_I(s))$*, then for any optimal* $x(s)$ *we have* $\lambda_I(s) \in \partial\Pi_i(s, x_i(s))$ *for all* i. ■

[3] It also follows from the analysis of so-called *inf-convolutions* in [10].

3 Cooperation over time

It is fitting to elaborate briefly on dynamic problems. For simplicity let there be only two time periods $t = 0, 1$, representing now and "tomorrow". (Extensions to more periods is, in principle, easy. It requires explicit modeling of the information flow though, and it comes with more cumbersome notations.) Decompose the ambient space $\mathbb{E} = \mathbb{E}_0 \times \mathbb{E}_1$ as well as endowments $e_i(s) = (e_{i0}, e_{i1})(s)$ into corresponding stage-relevant parts. Most important, since s is unknown at time 0, we require that the time 0 component $e_{io}(s)$ be constant as a function of s. This restriction is commonly referred to as *non-anticipativity*: Future knowledge cannot be exploited before it comes about.

We now define $i's$ problem as follows. If he uses an information-adapted strategy $s \longmapsto y_i(s) = (y_{io}, y_{i1}(s))$, he enjoys state-dependent, time-separable payoff

$$f_i(s, y_i) := f_{i0}(y_{io}) + f_{i1}(s, y_{i1}(s))$$

provided the constraints

$$g_i(s, y_i) := [g_{io}(y_{io}), g_{i1}(y_{io}, y_{i1}(s))] \leq e_i(s) := [e_{io}, e_{i1}(s)] \quad \text{for all} \quad s \in S.$$

are satisfied. Otherwise his payoff is $-\infty$. (As usual, inequality between vectors is meant to hold coordinatewise.) Define $\pi_i(e_i)$ to be the maximum expected value of this program. One may easily argue that $\pi_C(e_C)$, as defined in (2), assumes the alternative form

$$\pi_C(e_C) = \sup_y \left\{ \sum_{i \in C} E f_i(s, y_i) : \sum_{i \in C} g_i(s, y_i) \leq e_C(s) \text{ for all } s \right\}.$$

Let here $L_C(y, \lambda) := E \sum_{i \in C} [f_i(s, y_i) + \lambda(s) \{e_i(s) - g_i(s, y_i)\}]$.

Proposition 4 *Suppose $\pi_I(e_I) \geq \sup_y L_I(y, \lambda_I)$ for some Lagrange multiplier rule $s \mapsto \lambda_I(s) = (\lambda_{I0}, \lambda_{I1}(s)) \geq 0$. Then, paying each i the amount*

$$u_i := \sup_{y_i} E [f_i(s, y_i) + \lambda_I(s) \{e_i(s) - g_i(s, y_i)\}],$$

yields a core allocation ex ante. Ex post, at time $t = 1$, in state s, with already sunk optimal decisions y_o, the remaining game with conditional payoffs

$$\Pi_C(s, e_{C1}(s)) := \sup_{y_1} \left\{ \sum_{i \in C} f_{i1}(s, y_{i1}) : \sum_{i \in C} g_{i1}(s, y_i) \leq e_{C1}(s) \right\}$$

admits an ex post core allocation

$$u_i(s, y_{i0}) = \sup_{y_{i1}} [f_{i1}(s, y_{i1}) + \lambda_{I1}(s) \{e_{i1}(s) - g_{i1}(s, y_i)\}]. \quad \blacksquare$$

As above, concavity - and some constraint qualification - suffices to guarantee existence of multipliers. In particular, it would be enough for (6) to have all functions $f_i(s, \cdot)$ and (components of) $-g_i(s, \cdot)$ concave finite-valued, and that each strict inequality $g_i(s, \cdot) < e_i(s)$ be solvable. This is the so-called Slater condition. Numerical techniques for solving such stochastic programs are presented in [7].

4 Insuring risk lovers

Insurance has positive value to risk averters. They are willing to pay for greater security. Mathematically, this phenomenon manifests itself here as an equality between two extremal quantities. These are, on one side the presumably finite, efficient payoff $P := \pi_I(e_I)$, and on the other side, the optimal value $D := \inf_\lambda \sup_x L_I(x, \lambda)$ of the associated dual program. Theory tells that under (6) the equality $P = D$, so conducive for computation and decomposition, does indeed obtain provided the (presumably continuous) aggregate objective $\sum_{i \in I} \pi_i(x_i)$ is concave. In other words, *aggregate risk aversion* suffices for the design of the contracts described above. *Absent* aggregate risk aversion (i.e., *absent* aggregate concavity) that design no longer works well. The Lagrangian $L_I(x, \lambda)$ could then have no saddle value:

$$P := \pi_I(e_I) = \sup_x \inf_\lambda L_I(x, \lambda) < \inf_\lambda \sup_x L_I(x, \lambda) = \sup_\lambda \sum_{i \in I} [E\lambda e_i + \pi_{i*}(\lambda)] =: D$$

and there would be a positive so-called *duality gap* $d := D - P$. That gap determines how well solutions to (4) can be approximated:

Theorem 2. *(Approximate core allocations) Suppose (6) holds. Then there exists some $\lambda_I \in E^S$ which maximizes $\sum_{i \in I} [E\lambda e_i + \pi_{i*}(\lambda)]$. Any such λ_I defines an allocation $u_i := E\lambda_I e_i + \pi_{i*}(\lambda_I), i \in I$, which is socially stable in so far as*

$$\sum_{i \in C} u_i \geq \pi_C(e_C) \text{ for all coalitions } C \subseteq I.$$

Moreover, the grand coalition will overspend with excess d:

$$\sum_{i \in I} u_i = \pi_I(e_I) + d.$$

Proof. The "primal value" $P = \pi_I(e_I)$ is finite by assumption. Let λ_I be any optimal solution to the dual problem $\inf_\lambda \sum_{i \in I} [\lambda e_i + \pi_{i*}(\lambda)]$. Such a solution is known to exist under (6). Then evidently, $\sum_{i \in I} u_i = \sup_\lambda \sum_{i \in I} [\lambda e_i + \pi_{i*}(\lambda)] = \pi_I(e_I) + d$, and

$$\sum_{i \in C} u_i = \sum_{i \in C} [\lambda_I e_i + \pi_{i*}(\lambda_I)] = \sup_x L_C(x, \lambda_I) \geq$$

$$\inf_\lambda \sup_x L_C(x, \lambda) \geq \sup_x \inf_\lambda L_C(x, \lambda) = \pi_C(e_C)$$

for all coalitions $C \subseteq I$. ■

Theorem 2 says that if some outside benefactor would contribute d on the condition that coalition I forms, then cooperation could indeed come about. If that transfer does not come, the mutual company faces bankruptcy. It cannot, in the average, honor the contracts. Some bills will be left unpaid, and contingent plans may be hard to implement.

This result may - at first glance - inspire some sadness or frustration. In mathematical terms is shows the limitation of convex analysis. In economic terms it stresses that insurance becomes quite difficult under economies of scale. That feature is certainly not surprising. It merely expresses that risk lovers require compensation for getting rid of uncertainty.

On second thoughts Theorem 2 provides some useful insights. We speculate briefly about these next. First, one may argue that when $d > 0$, the core is likely to be empty. Risk loving members of I will shy away from insurance. They fit the company badly and should, if possible, be excluded. Alternatively, Theorem 1 can be seen as making a case for self-insurance of some parties: sectors or individuals enjoying increasing returns should live without some sorts of insurance. They probably will not join the mutual; they are foreign parties to the community I. In particular, this could apply to public utilities and to holders of options which are subject to price uncertainty. Theorem 2 could also be stretched as an argument for collective rescue operations: some rare "disasters" call upon society at large to mitigate the post-event consequences. It might also happen that insurance, as designed here, improves so much on efficiency, that a tax d can justifiably be levied on the members of I (and on others maybe).

Anyway, to quantify the disheartening deficiency d, that number must be related to data. To simplify this task we suppose that all domains of functions are convex.[4] Then, following Aubin and Ekeland [1], given any function f from a real vector space into $\mathbb{R}\cup\{\pm\infty\}$ with convex effective domain $domf := f^{-1}(\mathbb{R})$, we measure that function's lack of convexity by the number

$$\rho(f) := \sup\left\{ f(\sum_{k\in K}\alpha_k x_k) - \sum_{k\in K}\alpha_k f(x_k)\right\},$$

the supremum being taken over all finite families $\alpha_k \in [0,1]$, $x_k \in domf$, $\sum_{k\in K}\alpha_k = 1$. Clearly, $\rho(f) \geq 0$, $\rho(f+g) \leq \rho(f)+\rho(g)$, $\rho(f) = 0 \iff f$ is convex, and the largest convex function $convf \leq f$ must satisfy $f - \rho(f) \leq convf$. Suppose henceforth that all payoff functions $\pi_i(\cdot)$ are upper semicontinuous (usc for short). The following result derives from [1]:

Proposition 5 *(Estimating the duality gap) With $\pi_I(e_I)$ finite suppose (6) holds and that $dom\pi_i := \{x_i : \pi_i(x_i) > -\infty\}$ is nonempty convex for every $i \in I$. Then $d \leq \sum_{i\in I}\rho(-\pi_i)$.*

Let henceforth $\mathbb{E} = \mathbb{R}^m$. Using the Shapley-Folkman Lemma Aubin and Ekeland [1] proved

Proposition 6 *(A tighter estimate on the duality gap) With $\pi_I(e_I)$ finite and (6) in vigor, suppose $\mathbb{E} = \mathbb{R}^m$ and that $dom\pi_i$ is convex for all $i \in I$. Then $d \leq (m+1)\max_i \rho(-\pi_i)$.* ■

[4] For discrete domains see [3].

Remark: The core incorporates much stability and facilitates the study of many games with side-payments. It can be either empty though or so large as to loose predictive power. But insurance (production) games are somewhat different and at advantage: Granted concave preferences the core contains price-supported elements. And absent concavity, but present many players, the core comprises computable and good approximations. For large games it is widely known that cooperation and competition approximate each other well; they can nearly be reconciled [9], [15], [16], [17]. This can also be brought out here by letting I be a nonatomic measure space and invoking the Lyapunov convexity theorem.

5 Randomization

To mitigate the deficiency d under nonconcave preferences one might proceed as follows. Suppose agent i is restricted to apply a finite set $\mathcal{F}_i \subset \mathbb{E}^S$ of strategies. Presumably $e_i \in \mathcal{F}_i$. Let X_i be the set of probability distributions (the simplex) over $\mathcal{F}_i$ and define (with slight abuse of notation)

$$\Pi_i(s, x_i(s)) := \sum_{f_i \in \mathcal{F}_i} x_i(f_i)\Pi_i(s, f_i(s))$$

and, as before,

$$\pi_i(x_i) := \sum_{s \in S} p(s)\Pi_i(s, x_i(s))$$

when $x_i \in X_i$, $-\infty$ elsewhere. Evidently, concavity (in fact, linearity) of objectives now obtains. A constraint of the sort $\sum_{i \in C} x_i(s) = e_C(s)$ now means that

$$\sum_{i \in C} \sum_{f_i \in \mathcal{F}_i} x_i(f_i) f_i(s) = e_C(s).$$

So, one would use the convention $x_i(s) := \sum_{f_i \in \mathcal{F}_i} x_i(f_i) f_i(s)$ and, modulo this rule, the analysis of Section 2 applies.

Admittedly, implementation of randomized devices is not straightforward, and to find a practical form appears much of a challenge. In some cases implementation may amount to maintenance of activities that yield inferior return, but offer good recourse options when such are needed. In other cases, randomization can take the form of part-time utilization of alternative production lines, technologies, strategies, or modes of behavior.

6 Paying for supplementary insurance

Suppose some outside agent offers to add the some random vector $\hat{e}$ to the existing aggregate e_I. This means that in state s he would transfer the resource bundle $\hat{e}(s)$ to the mutual company. How would that company price such an offer? Clearly, a reasonable price would equal the total value added, namely $\pi_I(e_I + \hat{e}) - \pi_I(e_I)$. Let

$\hat{\lambda}_I$ denote a Lagrange multiplier, if any, under the new aggregate endowment $e_I + \hat{e}$. An ex ante core allocation would then be to pay i the amount

$$u_i = E(\hat{\lambda}_I e_i) + \pi_{i*}(\hat{\lambda}_I)$$

and let the outside insurer have $E\hat{\lambda}_I\hat{e}$. If $\hat{e}$ is only a minor addition to the aggregate endowment, then, under qualification (6), $\pi_I(\cdot)$ is likely to be Lipschitz continuous near e_I. Any dual optimal solution λ_I will then be a generalized gradient of $\pi_I(\cdot)$, and $E(\lambda_I e_I)$ becomes a reasonable estimate of $\pi_I(e_I + \hat{e}) - \pi_I(e_I)$.

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