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Mathematical simulation of an autonomous network of retail outlets at a local market

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Abstract

The paper is devoted to economic and mathematical simulation of an autonomous network of retail outlets at a local market. The question of simulation of redistribution processes in the network is discussed. Two interconnected and complemented economic-mathematical models are suggested. These models describe the evolution processes of the sale conditions in the network. They are constructed on the basis of the price demand and the system state functions. The mathematical models are described by a dynamic system of partial differential equations. The main attention is given to mathematical simulation of the redistribution process of consumer demand through retail outlets in the network under fluctuations of prices. The numerical realization of the models is discussed. Computer simulations show that the proposed models can adequately reflect some real processes of the sales redistribution in the network.

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Mathematical simulation of an autonomous network of retail outlets at a local market

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Introduction

The issues of economic and mathematical simulation of processes in autonomous networks of retail outlets at a local market are discussed in this paper. The simulation of the redistribution processes in the network is in the focus. The list of the redistributed characteristics, first of all, includes the total number of sales and the total volume of goods realized on the market.

The initiative to study such network belongs to a large company selling gasoline in one of regional markets of Russia through its network of petrol stations (PS). Main interests of this company are connected with the study of the qualitative factors, which influence on the state of business. The list of these interests includes the investigation of quantitative interrelations between the market structures. Forecast of the market state under changing the dominant factors is the main task for managers. The interests of the company in this problem are caused by intensification of competition and, hence, by the necessity in the development of the price strategy that ensures competitive advantage of the company over other market participants.

It is necessary to stress that many processes connected with operation of a single PS or several PSs united in the network have a probabilistic nature. For example, intensity of entering flow of cars, queue length, duration of servicing a client can be considered as random quantities when problems are studied at the local level. The methods of the theory of traffic flows [1], methods of the queuing theory [2,3] can help in analysis of the problem of pricing discussed in this paper (see [4]).

It is obvious that a client creates many casual influences in economic processes. The existence of subjective factors (among the others) in social-economic areas makes these areas difficult to formalize. The subject area, the set of affecting factors, and the time interval must be clearly determined to get an acceptable forecast. In this case it is necessary to emphasize that the simulation of processes in PS network is based on large-scale marketing researches of a local petrol market. The results of the conducted marketing researches created an informational data base for economic and mathematical simulations. They allowed to reveal a system of the network preferences in retail outlets and to make segmentation of the market. The well known methods in marketing, discriminant analysis and cluster analysis [5], have been used for these purposes.

The variables and factors of the model reflect the main qualitative and quantitative interrelations of the network. These parameters were found as a result of the conducted marketing research. The list of these factors consists of the following positions.

- The local market is considered as a one-commodity model.
- The local market has no interrelations with other local markets or influence of other markets is negligibly small (the autonomous property of the network).
- The price is the main factor of redistribution of goods in the network.
- The total amount of goods in the network is not changed under retail price fluctuations (the total demand for goods is constant).
- The considered time interval is characterized by the constancy of values of main global parameters of the network.
- Changes of selling conditions in the network don't cause crucial changes of the client preferences in retail sales outlets.
- The total quantity of clients of the network on the considered time interval is a constant value. The fluctuations of realization volumes at each retail outlet of the network happen only because clients change their preferred outlets.
- Each retail outlet has its own specific reaction on price changes.

Two interconnected and complemented one-commodity economic-mathematical models are suggested in this paper. These models describe the processes in the autonomous network of a local market of retail sales under changes of the sale conditions in the network.

The main attention is given to mathematical simulation of the redistribution process of consumer demand through retail outlets in the network under fluctuations of prices. The models are developed in the framework of one real market of retail sales of petrol. However the models are not restricted by specific features of petrol markets. They can be practically applied for investigation of other autonomous local markets that satisfy to the above requirements. For examples, food markets can be studied in the framework of the proposed models.

Main terms and notations

The considered network will be often referred as a system and each its retail outlet will be called a node. Suppose that the network consists of m nodes and it is considered on the time interval $\Omega = [t_0, t_0 + T]$. As it was noted above the time interval Ω is characterized by the constancy of values of the main system parameters.

Let $x_i(t)$, $i = 1,2,3,...,m$ denote the value of sales at the i-th node of the system at time $t \in \Omega$. This value is called the node state.

Let $p_i(t)$ denote the price at the i-th node of the system at time $t \in \Omega$.

The state vector (or simply – a state) is denoted by the symbol $X(t) = (x_1(t), \quad x_2(t), \quad \dots \quad x_m(t))^T$.

The price vector is denoted by the symbol $P(t) = (p_1(t), \quad p_2(t), \quad \dots, \quad p_m(t))^T$. We assume that the price vector is the main factor defining the system state $X(t) = X(P(t)), \quad t \in \Omega$. Sometimes the vector $P(t)$ will be called by a control vector.

Let us introduce the following notation $X_\Sigma(t) = \sum_{i=1}^{m} x_i(t), \quad t \in \Omega$. This value defines the total characteristic of the system at the time moment $t \in \Omega$. It is supposed that the total characteristic of the system is a constant within the considered time interval Ω, $X_\Sigma(t) \overset{t\in\Omega}{\equiv} S > 0, \quad S = const$.

Define the value $y_i(t) = \dfrac{x_i(t)}{S} \geq 0 \quad (i = 1,2,...,m)$. This value defines a portion (share) of each concrete node in the total characteristic of the system under the price vector $P(t) = (p_1(t), \quad p_2(t), \quad \ldots \quad p_m(t))^T$. Obviously, the value $y_i(t)$ satisfies to the equality $\sum_{i=1}^{m} y_i(t) = 1 \quad \forall t \in \Omega$. Notice that $y_i(t)$ depends on distribution of price through nodes of the network $y_i(t) = y_i(p_1(t), \ldots, p_m(t))$.

We denote by symbol $Y(t) = (y_1(t), \quad y_2(t), \quad \ldots, \quad y_m(t))^T$ the vector of node shares.

Chapter 1. Simulation on the basis of demand function

1. Analysis at the microlevel

Consider an arbitrary node of the network. The symbol $x = x(p)$ denotes demand at this node when the price is p. It is assumed that the demand monotonously decreases when the price grows according to the law

$$x(p) = \left(\frac{p}{p^{(0)}} \right)^{\varepsilon}. \tag{1.1}$$

Here $p^{(0)}$ is the price at the time of beginning of observation, ε is price demand elasticity, $\varepsilon < 0$. Demand is normalized with respect to the initial price $p = p^{(0)}$

$$x(p^{(0)}) = 1$$

In this case elasticity shows the portion (in percent) of decrease in demand under the price growth on one percent. The formal definition of elasticity is given by formula

$$\varepsilon = \frac{dx}{dp} \cdot \frac{p}{x(p)}$$

Often in economic literature a discrete variant of the elasticity definition is used

$$\varepsilon = \frac{\Delta x}{\Delta p} \cdot \frac{p}{x(p)}. \tag{1.2}$$

Functions (1.1) belong to the class of Cobb-Douglas functions. Such functions are used in the theory of household as utility functions. In the theory of firms functions are used as production functions where they define dependency of production output on volumes of expenses.

In practice elasticity can be found on the base of the marketing research results. Here the algorithm of the elasticity reconstruction over measurements of the system characteristics by the method of the least squares is suggested. The hypothetical elasticity is defined as the demand elasticity connected with a priori intentions of clients of the network. The discrete definition of elasticity (1.2) leads to equation

$$\Delta x = \varepsilon \frac{\Delta p}{p} x(p)$$

Then under the initial price $p = p^{(0)}$ we have the equality

$$\Delta x = \varepsilon \frac{\Delta p}{p^{(0)}}$$

The value Δx shows reduction (or increment) of volume of realized goods when price changes from the $p^{(0)}$ on Δp. The portion Δx of the lost volume of realized goods has sign "minus" and the portion of the volume of additionally realized goods has sign "plus" under fluctuation Δp of the price.

Let us suppose that L measurements of client reactions on hypothetical change of the price are executed during the marketing research. Here $\Delta x^{(l)}$ is a change of a volume of realized goods under l-th measurement when a hypothetical price fluctuation is equal to $\Delta \tilde{p}^{(l)}$. Function

$$\Phi(\varepsilon) = \sum_{l=1}^{L} (\varepsilon \frac{\Delta \tilde{p}^{(l)}}{p^{(0)}} - \Delta x^{(l)})^2$$

estimates the difference between the measured values and the model values (discrepancy over measurements).

Define the elasticity by minimizing this function

$$\Phi(\varepsilon) \xrightarrow[\varepsilon]{} \min. \qquad (1.3)$$

The solution of the optimization problem (1.3) is given by formula

$$\varepsilon = p^{(0)} \frac{\sum_{l=1}^{L} \Delta \tilde{p}^{(l)} \Delta x^{(l)}}{\sum_{l=1}^{L} (\Delta \tilde{p}^{(l)})^2}.$$

2. Analysis at the macrolevel

In what follows the price demand elasticity is not considered as a network characteristic. A sufficiently common situation is studied in this chapter. The client reaction on a price fluctuation is specific at each retail outlet. The price demand elasticity is considered as the numeric value characterizing peculiarities of demand at each concrete node of the network. Here prices are considered as functions of time $p_i = p_i(t), \quad i = 1, \ldots, m$.

Let us introduce demand functions

$$x_1(p_1(t)) = \left(\frac{p_1(t)}{p_1(t_0)}\right)^{\varepsilon_1}, \ldots, x_m(p_m(t)) = \left(\frac{p_m(t)}{p_m(t_0)}\right)^{\varepsilon_m}$$

at corresponding nodes. Here $p_1(t), \ldots, p_m(t)$ define prices at time $t \geq t_0$. Where $t = t_0$ is the initial time of the system observation, symbols $\varepsilon_1, \ldots, \varepsilon_m$ denote the price demand elasticities.

3. Elasticities of the network interrelations

For investigation of nodes interrelations one can use formalism of the introduced demand functions. Let us consider two different nodes with indexes i and j ($i \neq j$). Function

$$x_{ij}(p_j) = (x_j(p_j))^{a_{ij}} = \left(\frac{p_j}{p_j^{(0)}}\right)^{a_{ij}\varepsilon_j} \qquad (3.1)$$

is called by the function of interrelation of *i*-th node with *j*-th node. The function (3.1) belongs to the class of Cobb-Douglas functions. The value

$$a_{ij} = \frac{dx_{ij}}{dx_j} \times \frac{x_j}{x_{ij}} \qquad (3.2)$$

can be interpreted as the elasticity of this interrelation. Really, the definition (3.2) leads to equality

$$a_{ij} = \frac{dx_{ij}}{dx_j} \times \frac{x_j}{x_{ij}}\bigg|_{p_j = p_j^{(0)}} = \frac{dx_{ij}}{dx_j} \approx \frac{\Delta x_{ij}}{\Delta x_j}.$$

Hence

$$\Delta x_{ij} \approx a_{ij} \Delta x_j.$$

If Δx_j is a part of the lost clients (sales) at the *j*-th node due to price increase at this node then the value a_{ij} defines a share of Δx_j which pass to *i*-th network node. In other words, coefficients a_{ij} characterize redistribution of the clients (sales) through nodes of the network under price fluctuation at the *i*-th node when $i \neq j$. It is assumed that such numerical characteristics of the node as the volume of the realized goods and the amount of the clients are interdependent and mutually deducible. The increase of one of these parameters leads to growth of other parameter. Therefore, the value a_{ij} can be treated as a portion of the lost clients (sales) which leave the *j*-th node for the *i*-th node.

If the functions of interrelation of all nodes with the considered retail outlet

$$x_{1j}(p_j) = (x_j(p_j))^{a_{1j}}, \ \ldots, \ x_{mj}(p_j) = (x_j(p_j))^{a_{mj}}$$

are known (function $x_{jj}(p_j) \equiv 0$ in this list) then the balance equality

$$\sum_{\substack{i=1\\ i\neq j}}^{m} \Delta x_{ij} = \Delta x_j \qquad (3.3)$$

is valid with the necessity.

The equality (3.3) is equivalent to the following equalities

$$\sum_{\substack{i=1\\ i\neq j}}^{m} a_{ij} = 1, a_{ij} \geq 0 \text{ for all } i \neq j .$$

Interrelation elasticities can be united in a matrix of elasticities

$$A = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mm} \end{pmatrix} = \begin{pmatrix} 0 & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & 0 \end{pmatrix}. \qquad (3.4)$$

Matrix A contains the information on the graph of the client's motion. Hence this matrix represents the information on the redistribution of the volumes of realized goods under price fluctuations at nodes of the network. Each vector-column of matrix A characterizes the redistribution of clients of the corresponding node through all nodes of the network under price fluctuation at this node.

The method of the least squares is used for calculation of the matrix of elasticities (3.4). Construction of matrix A can be realized by columns. Fix a network node with index j. Find interrelation elasticities

$$a_{ij} = \frac{\Delta x_{ij}}{\Delta x_j},\ i \neq j$$

of this node with all other nodes of the network.

Let us suppose that L measurements of clients reaction on hypothetical change of the price are executed during marketing research. The symbol $\Delta x_j^{(l)}$ denotes a share (portion) of clients which left the j-th node under the l-th measurement. The symbol $\Delta x_{ij}^{(l)}$ denotes a share of $\Delta x_j^{(l)}$ which pass to the i-th node when a hypothetical price fluctuation at the j-th node is equal to $\Delta \tilde{p}_j^{(l)}$.

Let us solve the mathematical programming problem

$$\sum_{l=1}^{L}\left(a_{1j} - \frac{\Delta x_{1j}^{(l)}}{\Delta x_j^{(l)}}\right)^2 + \dots + \sum_{l=1}^{L}\left(a_{mj} - \frac{\Delta x_{mj}^{(l)}}{\Delta x_j^{(l)}}\right)^2 \to \min \qquad (3.5)$$

$$a_{1j} + \dots + a_{mj} = 1, a_{1j} \geq 0, \dots, a_{mj} \geq 0,$$

of minimization of the sum of squares of discrepancies over measurements. This problem has *(m-1)* variables (remind that $a_{jj} = 0$).

The solution of the optimization problem (3.5) is given by formulas

$$a_{1j} = \frac{1}{L}\sum_{l=1}^{L}\frac{\Delta x_{1j}^{(l)}}{\Delta x_j^{(l)}} + \frac{1}{m-1}\left(1 - \frac{1}{L}\left[\sum_{l=1}^{L}\frac{\Delta x_{1j}^{(l)}}{\Delta x_j^{(l)}} + \dots + \sum_{l=1}^{L}\frac{\Delta x_{mj}^{(l)}}{\Delta x_j^{(l)}}\right]\right)$$

$$\ldots$$

$$a_{mj} = \frac{1}{L}\sum_{l=1}^{L}\frac{\Delta x_{mj}^{(l)}}{\Delta x_j^{(l)}} + \frac{1}{m-1}\left(1 - \frac{1}{L}\left[\sum_{l=1}^{L}\frac{\Delta x_{1j}^{(l)}}{\Delta x_j^{(l)}} + \ldots + \sum_{l=1}^{L}\frac{\Delta x_{mj}^{(l)}}{\Delta x_j^{(l)}}\right]\right)$$

4. Potential of a network

Let us simulate a particular situation. Assume that the price changes at the j-th node when the price remains previous at others nodes of the network. Then this node will lose (or will get – depending on sign of Δp_j) the share of the clients

$$\Delta x_j = \varepsilon_j \frac{\Delta p_j}{p_j^{(0)}}.$$

If to take into account the "weight" of this node in the sharing of the market at the initial time of the system observation then the share of loss of the clients is given by equality

$$y_j(t_0)\Delta x_j = y_j(t_0)\varepsilon_j \frac{\Delta p_j}{p_j^{(0)}}.$$

For definiteness one can suppose that the price increases ($\Delta p_j > 0$) at the j-th node. It leads to outflow of clientele from the considered node. The share Δx_j of the clients (these clients left the j-th node) redistributes through other nodes with the corresponding weights (interrelation elasticities) $a_{ij} \geq 0, \sum_{\substack{i=1 \\ i \neq j}}^{m} a_{ij} = 1$:

$$\Delta x_j = a_{1j}\Delta x_j + \ldots + 0 + \ldots + a_{mj}\Delta x_j. \tag{4.1}$$

In equality (4.1) the first summand shows that a part of clients which leave the *j*-th node and pass to the *i*-th node of the network. The *j*-th summand shows that part of clients of *j*-th node which pass to this node. Hence, this summand equals to zero (since the growth of the price does not lead to increase of the clients inflow).

Knowing the initial "weight" $y_j(t_0)$ of the *j*-th node in sharing of the market, one can derive from (4.1) the following equations

$$y_j(t_0)\Delta x_j = y_j(t_0)\varepsilon_j \frac{\Delta p_j}{p_j^{(0)}} = a_{1j} y_j(t_0)\varepsilon_j \frac{\Delta p_j}{p_j^{(0)}} + \ldots + 0 + \ldots + a_{mj} y_j(t_0)\varepsilon_j \frac{\Delta p_j}{p_j^{(0)}}.$$

In the considered situation the share of each retail outlet to the next moment of time will define by equations

$$\begin{cases} y_1(t_1) = y_1(t_0) - a_{1j}\varepsilon_j \dfrac{\Delta p_j}{p_j^{(0)}} y_j(t_0) \\ \vdots \\ y_j(t_1) = y_j(t_0) + \varepsilon_j \dfrac{\Delta p_j}{p_j^{(0)}} y_j(t_0) \\ \vdots \\ y_m(t_1) = y_m(t_0) - a_{mj}\varepsilon_j \dfrac{\Delta p_j}{p_j^{(0)}} y_j(t_0) \end{cases} . \qquad (4.2)$$

Here all weights (except of the *j*-th weight) do not decrease due to sign of elasticity. Only for the *j*-th node we have the opposite inequality $y_j(t_1) \le y_j(t_0)$. It means that the share of clients of *j*-th node does not increase under the price growth at this node. It is clear that the sum of the left parts is equal to the sum of the right parts when $\sum_{i=1}^{m} a_{ij} = 1$.

Thus, we have the equality

$$\sum_{j=1}^{m} y_j(t_1) = \sum_{j=1}^{m} y(t_0) = 1 . \qquad (4.3)$$

The equality (4.3) expresses invariance of goods demand under the price fluctuation at the *j*-th retail outlet.

Let us rewrite equations (4.2) in more details

$$\begin{cases} y_1(t_1) - y_1(t_0) = y_1(p_1^{(0)},\ldots,p_{j-1}^{(0)},p_j^{(0)} + \Delta p_j, p_{j+1}^{(0)},\ldots,p_m^{(0)}) - y_1(p^{(0)}) = -a_{1j}\varepsilon_j \dfrac{\Delta p_j}{p_j^{(0)}} y_j(t_0) \\ \vdots \\ y_j(t_1) - y_j(t_0) = y_j(p_1^{(0)},\ldots,p_{j-1}^{(0)},p_j^{(0)} + \Delta p_j, p_{j+1}^{(0)},\ldots,p_m^{(0)}) - y_j(p^{(0)}) = \varepsilon_j \dfrac{\Delta p_j}{p_j^{(0)}} y_j(t_0) \\ \vdots \\ y_m(t_1) - y_m(t_0) = y_m(p_1^{(0)},\ldots,p_{j-1}^{(0)},p_j^{(0)} + \Delta p_j, p_{j+1}^{(0)},\ldots,p_m^{(0)}) - y_m(p^{(0)}) = -a_{mj}\varepsilon_j \dfrac{\Delta p_j}{p_j^{(0)}} y_j(t_0) \end{cases}$$

Passing to the limit in the price increments Δp_j one can obtain equations

$$\begin{cases} \dfrac{\partial y_1}{\partial p_j} = -a_{1j}\varepsilon_j \dfrac{1}{p_j^{(0)}} y_j \\ \vdots \\ \dfrac{\partial y_j}{\partial p_j} = \varepsilon_j \dfrac{1}{p_j^{(0)}} y_j \\ \vdots \\ \dfrac{\partial y_m}{\partial p_j} = -a_{mj}\varepsilon_j \dfrac{1}{p_j^{(0)}} y_j \end{cases}$$

If to suppose the possibility of price fluctuations at all nodes of the network, one can construct the mapping

$$P \xrightarrow{F} y . \tag{4.4}$$

This mapping specifies the market shares distribution through nodes of the network depending on the prices. For brevity the mapping (4.4) will be called by potential of the network. The Jacobi matrix of potential is given by formula

$$\frac{\partial y}{\partial P} = \begin{pmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} & \cdots & \frac{\partial y_1}{\partial p_m} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} & & \frac{\partial y_2}{\partial p_m} \\ \vdots & & \ddots & \vdots \\ \frac{\partial y_m}{\partial p_1} & \frac{\partial y_m}{\partial p_2} & \cdots & \frac{\partial y_m}{\partial p_m} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\varepsilon_1}{p_1^{(0)}} y_1 & -a_{12} \frac{\varepsilon_2}{p_2^{(0)}} y_2 & \cdots & -a_{1m} \frac{\varepsilon_m}{p_m^{(0)}} y_m \\ -a_{21} \frac{\varepsilon_1}{p_1^{(0)}} y_1 & \frac{\varepsilon_2}{p_2^{(0)}} y_2 & & -a_{2m} \frac{\varepsilon_m}{p_m^{(0)}} y_m \\ \vdots & & \ddots & \vdots \\ -a_{m1} \frac{\varepsilon_1}{p_1^{(0)}} y_1 & -a_{m2} \frac{\varepsilon_2}{p_2^{(0)}} y_2 & \cdots & \frac{\varepsilon_m}{p_m^{(0)}} y_m \end{pmatrix}$$

The Jacobi matrix of potential can be presented by the matrix equality

$$\frac{\partial y}{\partial P} = (E - A) B(\varepsilon, P^{(0)}) y . \tag{4.5}$$

Here E denotes the identity matrix,

$$A = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mm} \end{pmatrix} = \begin{pmatrix} 0 & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & 0 \end{pmatrix}$$ is the matrix of interrelations,

$$B(\varepsilon, p^{(0)}) = \begin{pmatrix} \frac{\varepsilon_1}{p_1^{(0)}} & 0 & \cdots & 0 \\ 0 & \frac{\varepsilon_2}{p_2^{(0)}} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\varepsilon_m}{p_m^{(0)}} \end{pmatrix}$$ is the elasticity matrix,

and $y = y(P) = \begin{pmatrix} y_1(P) \\ \vdots \\ y(P) \end{pmatrix}$ is the vector of the shares distribution in sharing of the market.

In the equality (4.5) the first matrix multiplicand *E-A* reflects the structure of interrelations in the network. This matrix is a result of the analysis at the macrolevel and it should be related to the network parameter of the model. The elements of the second matrix multiplicand $B(\varepsilon, P^{(0)})$ are the numerical characteristics of the nodes. These multipliers reflect the analysis at the microlevel. The diagonal feature of this matrix expresses the accepted earlier assumption that at each retail outlet the demand function depends on the price at this node only (and the reaction of the clients of this outlet to the price fluctuations at other nodes is not taken into account). It is possible to agree with this assumption under insignificant fluctuations of the price. More general situation is considered in the second chapter.

Let us note that the sum of elements at each column of the Jacobi matrix is equal to zero, i.e.

$$\frac{\partial(y_1 + \ldots + y_m)}{\partial p_j} = 0$$

for all prices p_j. Hence the dynamics of the network is described by the system of partial differential equations

$$\begin{cases} \dfrac{\partial(y_1 + \ldots + y_m)}{\partial p_1} = 0 \\ \quad\vdots \\ \dfrac{\partial(y_1 + \ldots + y_m)}{\partial p_m} = 0 \end{cases} \tag{4.6}$$

with the initial condition $y_1 + \ldots + y_m\big|_{p=(p_1^{(0)},\ldots,p_m^{(0)})} = 1$.

The autonomy condition for the considered system is expressed in the equation (4.6). The closed network does not lose the clients under changes of prices. There is only their redistribution in the network. The system of the differential equations (4.6) is called the *system of the equations of the network dynamics*.

It is not possible to determine an analytical expression for potential of the network. Therefore, we linearize system (4.4). The linearization of mapping (4.4) is defined by equations

$$y_1(P) \approx y_1^{(0)} + \frac{\varepsilon_1}{p_1^{(0)}} y_1^{(0)}(p_1 - p_1^{(0)}) - a_{12}\frac{\varepsilon_2}{p_2^{(0)}} y_2^{(0)}(p_2 - p_2^{(0)}) - \ldots - a_{1m}\frac{\varepsilon_m}{p_m^{(0)}} y_m^{(0)}(p_m - p_m^{(0)})$$

$$y_2(P) \approx y_2^{(0)} - a_{21}\frac{\varepsilon_1}{p_1^{(0)}} y_1^{(0)}(p_1 - p_1^{(0)}) + \frac{\varepsilon_2}{p_2^{(0)}} y_2^{(0)}(p_2 - p_2^{(0)}) - \ldots - a_{2m}\frac{\varepsilon_m}{p_m^{(0)}} y_m^{(0)}(p_m - p_m^{(0)})$$

$$y_m(P) \approx y_m^{(0)} - a_{m1}\frac{\varepsilon_1}{p_1^{(0)}} y_1^{(0)}(p_1 - p_1^{(0)}) - a_{m2}\frac{\varepsilon_2}{p_2^{(0)}} y_2^{(0)}(p_2 - p_2^{(0)}) - \ldots + \frac{\varepsilon_m}{p_m^{(0)}} y_m^{(0)}(p_m - p_m^{(0)})$$

Here symbol $P = (p_1, \ldots, p_m)$ denotes the vector of prices at all nodes of the network, symbol $y_j^{(0)} = y_j(p_1^{(0)}, \ldots, p_m^{(0)})$ denotes the share (portion) of the *j*-th node in sharing of the market under the initial prices distribution.

It is necessary to note that the obtained equalities have the approximation character. Namely, the linear approximation of the network potential is constructed. This approximation defines values of the shares in sharing of the market under small price fluctuations in the network.

Chapter 2. Simulation on the basis of state function and potential of interrelation

1. Function of the node state

In the first chapter the problem of mathematical simulation of the processes in a local market is investigated on the basis of demand function. Demand function is a local characteristic of the node. This function defines the specific reaction of the node state in change of the price at the node. Besides this approach, one can use the node state function for the model construction. The node sate function can be defined as the function of m variables

$$x_i(t) = x_i(p_1(t), p_2(t), \ldots, p_m(t)), \quad t \in \Omega, \quad i = 1,2,3,\ldots,m. \tag{1.1}$$

This definition is quite natural and can be explained by the following reasons. The volume of realized goods at each concrete node of the network at any moment of time depends not only on the price at this node, but also on prices in some "neighbourhood" of the node. Thus, function (1.1) defines more sophisticated dependence of the node state on the network parameters than interconnections determined by the demand function. It takes into account not only the price at the given node but also prices established at other nodes of the network.

It is necessary to note that the function of the node state (1.1) should satisfy to the following condition

$$x_i(p_1(\tilde{t}), p_2(\tilde{t}), \ldots, p_m(\tilde{t})) = \left(\frac{p_i(\tilde{t})}{p_i(t_0)}\right)^{\varepsilon_i}, \quad \forall \tilde{t} \in \Omega, \quad i = 1,2,3,\ldots,m.$$

Here equality

$$x_i(p_i(t)) = \left(\frac{p_i(\tilde{t})}{p_i(t_0)}\right)^{\varepsilon_i}, \quad \forall t \in \Omega, \quad i = 1,2,3,\ldots,m. \tag{1.2}$$

defines the state of the i-th node at time $t \in \Omega$ through the demand function (see (1.1) in chapter 1).

Let us note that the equalities (1.1) and (1.2) describe the same characteristic of a node. The numerical values of this characteristic can be calculated using different information.

2. Potential of interrelation of system nodes

Let us consider the function which describes interrelations in the network

$$G_{ij}(t,\Delta t)=G\big(p_i(t),\,p_i(t+\Delta t),\,p_j(t+\Delta t)\big)=\left(\frac{p_i(t+\Delta t)-p_j(t+\Delta t)}{p_i(t)}\right)^{2\alpha_{ij}+1}.$$

$$t\in\Omega,\ \Delta t\geq 0: t+\Delta t\in\Omega$$

This function is called by potential of interrelation of the i-th node with the j-th node of the network. Coefficient α_{ij} is called by elasticity of interrelation of the i-th node with the j-th node.

An interpretation of potential of interrelation can be the following one. At each time $t\in\Omega$ the value $\delta p_{ij}(t)=p_i(t)-p_j(t)$ defines the difference (price discrepancy) between the prices at the i-th and the j-th nodes. However, condition of the system equilibrium $\delta p_{ij}(t)=0$ not necessarily is valid. This condition means that "appreciable" redistribution of clients (or sales) through nodes of the network is absent. The value $\Delta p_{ij}(t,\Delta t)=\dfrac{\delta p_{ij}(t+\Delta t)}{p_i(t)}$ defines the relative difference of prices at the i-th and the j-th nodes at time $t+\Delta t\in\Omega$ with respect to the price at the i-th node at the moment of time $t\in\Omega$. Redistribution of sales through nodes of the network takes place due to fluctuation of prices in the network, and, hence, due to fluctuation of the relative differences of prices between each node and other nodes "connected" with it. The quantity of sales (clients) passing from the i-th node to the j-th node or arriving to the i-th node from the j-th node during the time period $\Delta t>0$ depends on the value of fluctuation $\Delta p_{ij}(t,\Delta t)-\Delta p_{ij}(t,0)$. This expression defines the relative change of the difference of prices between the i-th and the j-th nodes on a time interval $[t,t+\Delta t]\subseteq\Omega$ with respect to the price at the i-th node at time t. The interrelation elasticity α_{ij} allows to take into account the fact of existence of interrelation between the i-th and the j-th retail outlets of the network. Namely, if $\alpha_{ij}>0$ then interrelation exists, and if $\alpha_{ij}=-\dfrac{1}{2}$ then interrelation does not exist. This parameter can be used to define the share of sales (clients) which passed from the j-th node to the i-th node in the difference between the quantity of sales (clients) at the i-th node at time $t+\Delta t$ and the quantity of sales at the same node at time t.

Thus, the variation of potential of interrelation of the i-th node with the j-th node is defined by formula

$$\delta G_{ij}(t,\Delta t)=G_{ij}(t,\Delta t)-G_{ij}(t,0). \qquad (2.1)$$

This variation can be used for calculation of either the quantity of clients (sales) arriving to the i-th retail outlet from the j-th retail outlet during the time interval $[t,t+\Delta t]\subseteq\Omega$ in the case of sales increase at this retail outlet, or the quantity of clients (sales) arriving to the j-th retail outlet from the i-th retail outlet during the time interval $[t,t+\Delta t]\subseteq\Omega$ in the case of sales reduction at the i-th retail outlet.

Dividing equality (2.1) by $\Delta t>0$ and passing to the limit by $\Delta t\to 0$ one can derive the function

$$g_{ij}(t)=\frac{d}{d\Delta t}(G(t,0)) \quad t\in\Omega .$$

Explicitly this function is defined by formulas

$$g_{ij}(t)=\frac{2\alpha_{ij}+1}{p_i(t)}\left(\frac{p_i(t)-p_j(t)}{p_i(t)}\right)^{2\alpha_{ij}}\left(\frac{dp_i}{dt}(t)-\frac{dp_j}{dt}(t)\right)$$

$$g_{ij}(t)=K_{ij}(t)\left(\frac{dp_i}{dt}(t)-\frac{dp_j}{dt}(t)\right), \quad K_{ij}(t)=\frac{2\alpha_{ij}+1}{p_i(t)}\left(\frac{p_i(t)-p_j(t)}{p_i(t)}\right)^{2\alpha_{ij}} . \qquad (2.2)$$

$$t\in\Omega, \quad \forall i, j=1,2,...,m$$

It is necessary to notice that formulas (2.2) define the analytical expression for the interrelation elasticities a_{ij} of the network nodes. These parameters have been considered in the first chapter (see item 3 chapter 1). The main differences of the suggested here approach to definition of such parameters from the approach discussed in the first chapter consist in the following circumstances. First, equalities (2.2) take into account the price not only at one node, but also at two nodes connected with each other. Second, the rates of change of these prices are also taken into account in (2.2).

These arguments lead to the following relation

$$\frac{\partial}{\partial p_j}x_i(p_1(t),p_2(t),...,p_m(t))=g_{ij}(t)\frac{\partial}{\partial p_j}x_j(p_1(t),p_2(t),...,p_m(t)), \quad .$$

$$\forall t\in\Omega \quad \forall i, j=1,2,...,m$$

Substitution of the state function in this equality by the demand function (1.2) leads to equations

$$\frac{\partial}{\partial p_j}x_i(p_1(t),p_2(t),...,p_m(t))=g_{ij}(t)\frac{d}{dp_j}x_j(p_j(t)), \quad . \qquad (2.3)$$

$$\forall t\in\Omega \quad \forall i, j=1,2,...,m$$

Calculating the derivative of the demand function in the right hand side of the equality (2.3) one can obtain the following formula

$$\frac{\partial}{\partial p_j}x_i(p_1(t),p_2(t),...,p_m(t))=\varepsilon_j\frac{g_{ij}(t)}{p_j(t)}x_j(p_j(t)),$$

$$\forall t\in\Omega \quad \forall i, j=1,2,...,m$$

In turn, substitution of the state function (1.1) instead of the demand function leads to the final relations

$$\frac{\partial}{\partial p_j}x_i(p_1(t),p_2(t),...,p_m(t))=\varepsilon_j\frac{g_{ij}(t)}{p_j(t)}x_j(p_1(t),p_2(t),...,p_m(t)) \quad . \qquad (2.4)$$

$$\forall t\in\Omega \quad \forall i, j=1,2,...,m$$

Let us consider two nodes of the network (the i-th and the j-th nodes) and analyse some practically important special situations using equations (2.3) and (2.4).

The first situation.
Suppose that the price at the i-th retail outlet is constant and the price alters at the j-th retail outlet. Then equation (2.2) implies the relation

$$g_{ij}(t) = -K_{ij}(t)\frac{dp_j}{dt}(t), \quad K_{ij}(t) \geq 0, \quad \forall t \in \Omega .$$

Two cases are possible in this situation: A) the price increases at the j-th retail outlet ($\frac{dp_j}{dt}(t) > 0$), B) the price decreases at the j-th retail outlet ($\frac{dp_j}{dt}(t) < 0$). Let us consider each case separately.

Case (A). $\frac{dp_j}{dt}(t) > 0$. In this case under natural reaction of clients guided by the price change it is necessary to expect the inflow of the clients at the i-th retail outlet including clients from the j-th retail outlet. It means that the inequalities $\frac{d}{dp_j} x_j(p_j(t)) \leq 0$ and $\frac{\partial}{\partial p_j} x_i(p_1(t), p_2(t),..., p_m(t)) \geq 0$ should take be valid. Really, the first inequality follows directly from the definition of demand function (1.2). The second inequality follows from (2.3) since multipliers in its right hand side satisfy to the following conditions

$$\frac{d}{dp_j} x_j(p_j(t)) \leq 0, \quad g_{ij}(t) < 0 \quad \Rightarrow \quad \frac{\partial x_i}{\partial p_j} = g_{ij}\frac{dx_j}{dp_j} \geq 0 .$$

Case (B). $\frac{dp_j}{dt}(t) < 0$. In this case it is natural to expect outflow of clients (sales) from the i-th retail outlet to other retail outlets including the j-th retail outlet of the network. It means that the inequalities $\frac{d}{dp_j} x_j(p_j(t)) \geq 0$ and $\frac{\partial}{\partial p_j} x_i(p_1(t), p_2(t),..., p_m(t)) \leq 0$ should take place. Similarly to case (A), the first inequality follows from (1.2). The second inequality follows from (2.3) since multipliers in the right hand side in this case satisfy to conditions

$$\frac{d}{dp_j} x_j(p_j(t)) \geq 0, \quad g_{ij}(t) \leq 0 \quad \Rightarrow \quad \frac{\partial x_i}{\partial p_j} = g_{ij}\frac{dx_j}{dp_j} \leq 0 .$$

The second situation.
Let us suppose that the price alters at the i-th retail outlet and the price is constant at the j-th retail outlet. Then from (2.2) we get

$$g_{ij}(t) = K_{ij}(t)\frac{dp_i}{dt}(t), \quad K_{ij}(t) \geq 0, \quad \forall t \in \Omega .$$

In this situation two cases are possible as in the situation considered above: A) the price increases at the i-th retail outlet ($\frac{dp_i}{dt}(t) > 0$), B) the price decreases at the i-th retail outlet ($\frac{dp_i}{dt}(t) < 0$). The same arguments show that in both cases (under "natural" reaction of clients on the price change) the corresponding inequalities for $\frac{\partial x_i}{\partial p_j}(t)$ are validt. That is, in case (A) $\frac{\partial x_i}{\partial p_j}(t) \le 0$ (as $g_{ij}(t) \ge 0$, $\frac{dx_j}{dp_j} = 0$), and in case (B) $\frac{\partial x_i}{\partial p_j}(t) \ge 0$ (as $g_{ij}(t) \le 0$, $\frac{dx_j}{dp_j} = 0$).

The common situation for two considered retail outlets (the i-th and j-th nodes) corresponds to the simultaneous change of prices at these retail outlets. The similar analysis of change of the clients quantity (the quantity of sales) in these retail outlets can be conducted. In this situation it is necessary to investigate the greater number of variants of ratios in the trends of price changes (increase and decrease) at the i-th and the j-th nodes, and to analyse the greater number of trends in rates of the price changes (the values of derivatives $\frac{dp_i}{dt}(t)$ and $\frac{dp_j}{dt}(t)$). The conducted analysis confirms, that, from the qualitative point of view, the equations (2.3), (2.4) adequately reflect "natural" reaction of clients to the prices fluctuations at two retail outlets. For calculation of the quantitative characteristics of such reaction it is necessary to define the values of interrelation elasticity α_{ij}.

In conclusion of analysis of equations (2.2) - (2.4) let us note the following important observation. Let us return to the first particular situation analysed in case (A). Let us suppose that the price increases at the j-th retail outlet while the price is constant at the i-th retail outlet. In this case one can to notice that increase of the clients quantity (the quantity of sales) at the i-th retail outlet (due to the clients outflow from the j-th retail outlet) is less under condition $p_i(t) > p_j(t)$ than under condition $p_i(t) < p_j(t)$. This effect happens due to the multiplier

$$R_{ij}(t) = \left(\frac{p_i(t) - p_j(t)}{p_i(t)} \right)^{2\alpha_{ij}}$$

in the right hand side of the equality (2.2). In this situation the value $R_{ij}(t)$ decreases if $p_i(t) > p_j(t)$ and the value $R_{ij}(t)$ increases if $p_i(t) < p_j(t)$. The non-negative value of the derivative $\frac{\partial x_i}{\partial p_j}(t)$ (the increment of the client quantity at the i-th retail outlet) decreases or increases accordingly. Such behaviour of this derivative adequately reflects the real reaction of the market under various price fluctuations at the retail outlets if the client reaction on the prices fluctuation is "natural".

3. Equation of the system state

Let us construct equations which connect values of the state functions at each time $t \in \Omega$ with values of these functions at preceding moment of time. Let us differentiate the state function (1.1) with respect to time

$$\frac{dx_i}{dt}(t) = \sum_{j=1}^{m} \frac{\partial x_i(p_1, p_2, ..., p_m)}{\partial p_j} \frac{dp_j}{dt}(t), \qquad t \in \Omega, \quad i = 1,2,...,m.$$

The substitution of the right hand side of the equality (2.4) to the right hand side of the last equality instead of the partial derivative $\frac{\partial x_i(p_1, p_2, ..., p_m)}{\partial p_j}$ leads to the system of differential equations

$$\frac{dx_i}{dt}(t) = \sum_{j=1}^{m} v_{ij}(t) x_j(p_1(t), p_2(t), ..., p_m(t)),$$

$$v_{ij}(t) = \frac{\varepsilon_j g_{ij}(t)}{p_j(t)} \frac{dp_j}{dt}(t), \qquad . \tag{3.1}$$

$t \in \Omega, \quad i = 1,2,...,m$

Equations (3.1) are called by equations of the system state. Rewriting equations (3.1) in the matrix form one can obtain the following relation

$$\frac{dX}{dt}(t) = B(\Lambda, P(t)) A(\Sigma, P(t)) X(t), \qquad t \in \Omega. \tag{3.2}$$

Here

$$\Sigma = (\varepsilon_1 \quad \varepsilon_2 \quad \dots \quad \varepsilon_m)^T, \qquad \Lambda = \{\alpha_{ij}\}_{i,j=1}^{m}$$

$$A(\Sigma, P(t)) = \{a_{ij}(\Sigma, P(t))\}_{i,j=1}^{m}, \quad a_{ij}(\Sigma, P(t)) = \begin{cases} \frac{\varepsilon_i}{p_i(t)} \frac{dp_i}{dt}(t), & i = j \\ 0, & i \neq j \end{cases} \tag{3.3}$$

$$B(\Lambda, P(t)) = \{g_{ij}(t)\}_{i,j=1}^{m}. \tag{3.4}$$

One can see that the equalities (3.2) - (3.4) reflect the fact that dynamics of the system state depends both on the local nodes characteristics (matrix $A(\Sigma, P(t))$) and on the characteristics of the nodes interrelations (matrix $A(\Sigma, P(t))$).

The values of the diagonal elements of the matrix $A(\Sigma, P(t))$ are defined by the demand elasticity ε_i at the i-th node and by the relative rate of the price change $\frac{1}{p_i(t)} \frac{dp_i}{dt}(t)$ at this node at time $t \in \Omega$ only. All other elements of the matrix $A(\Sigma, P(t))$ are equal to zero. If the price value at some node does not vary then from the equalities (3.2), (3.3) it follows that such node does not influence directly on the state of the nodes connected with it.

Unlike elements of the matrix $A(\Sigma, P(t))$ the values of elements of the matrix $B(\Lambda, P(t))$ depend on parameters of interrelations between nodes of the network. Thus, this matrix defines redistribution of clients (sales) through the network nodes under fluctuation of the prices at the market.

Within the framework of the suggested mathematical model the numerical simulation of processes at a local market can be reduced to the numerical solution of the Cauchy problem for the equation of the system state (3.2) with the initial condition $X(t_0) = X^0$. Here symbol X^0 denotes the known system state at the initial time of the system observation. The mathematical statement of this problem can be given by the equations

$$\begin{cases} \dfrac{dX}{dt}(t) = B(\Lambda, P(t))A(\Sigma, P(t))X(t), \qquad t \in \Omega, \\ X(t_0) = X^0 \end{cases}. \tag{3.5}$$

In terms of the vector $Y(t)$ the problem (3.5) can be formulated with the equations

$$\begin{cases} \dfrac{dY}{dt}(t) = B(\Lambda, P(t))A(\Sigma, P(t))Y(t), \qquad t \in \Omega, \\ \sum_{i=1}^{m} y_i(t) = 1, \quad \forall t \in \Omega \\ Y(t_0) = Y^0 \end{cases}. \tag{3.6}$$

The problems (3.5) and (3.6) are equivalent since their solutions are connected by the equality $X(t) = S \times Y(t), \forall t \in \Omega$. In the problem (3.6) it is possible to abstract from the absolute values of the nodes states (the sales quantity, the volumes of realized goods etc.) and to operate with the shares of the values at each node state.

4. Numerical simulation of the system state

In this section some approaches to the numerical solution of the problem (3.6) are suggested. The variants of finite difference approximation admitting the computer realization for the problem solution are considered. Here the finite difference approximation of the problem means transition from continuous time $t \in \Omega$ to a discrete set of moments of the system observation.

Let $\{t_0, t_1, t_2, t_3, ..., t_N\} \subseteq \Omega$ be a finite set of moments of the system observation. Without loss of generality, one can assume that

$$t_{k+1} = t_k + \Delta t, \quad k = 0,1,2,..., N-1 \qquad \Delta t = \frac{T}{N}.$$

Here t_0 is the initial moment of time of the system observation.

Denote by symbols X^k, Y^k the values of the vectors $X(t), \quad Y(t)$ respectively at time $t_k, \quad k = 0,1,2,..., N$. One of discrete analogues (see [7]) of the problem (3.6) is given by formulas

$$\begin{cases} Y^{k+1} = W(\Delta t, \Lambda, \Sigma, P^k)Y^k, \qquad k = 0,1,2,..., N-1, \\ \sum_{i=1}^{m} y_i^k = 1, \quad \forall k = 0,1,2,..., N-1 \\ Y^0 = Y(t_0) \end{cases}, \tag{4.1}$$

$$P^k = P(t_k), \quad Y^k = \begin{pmatrix} y_1^k & y_2^k & \dots & y_m^k \end{pmatrix}^T, \quad y_i^k = y_i(t_k), \quad i = 1,2,...,m, \quad k = 0,1,2,...,N.$$

Here matrices $W(\Delta t, \Lambda, \Sigma, P^k) \quad (k = 0,1,2,...,N-1)$ are defined by the equality

$$W(\Delta t, \Lambda, \Sigma, P^k) = E + \Delta t B(\Lambda, P^k) A(\Sigma, P^k), \quad k = 0,1,2,...,N-1. \tag{4.2}$$

The symbol E denotes the identity matrix.

It is clear that in order to use the equalities (4.1) and (4.2) for the numerical solution of the problem (3.6) it is necessary to define the values of all parameters $\varepsilon_i \quad (i = 1,2,...,m)$ (the price demand elasticities - local characteristics of the nodes) and $\alpha_{ij} \quad (i, j = 1,2,...,m)$ (the elasticities of interrelations between nodes - network characteristics of the system). The numerical algorithm for finding the price demand elasticities is described in the first chapter in detail. Let us consider the problem of calculation of the interrelation elasticities $\alpha_{ij} \quad (i, j = 1,2,...,m)$.

Suppose that L measurements are realized during the system observation at moments of time $\tilde{t}_i \in \Omega \quad (i = 1,2,...,L)$. Denote by symbols $\tilde{Y}^k = Y(\tilde{t}_k) \quad (k = 1,2,...,L)$ the results of the system state measurements at corresponding moments of time.

The values of interrelation elasticities can be determined as the solution of the problem

$$\Phi(\Lambda) = \sum_{k=1}^{L} (\tilde{Y}^k - Y^k)^2 \longrightarrow \min_{\Lambda}. \tag{4.3}$$

Here $Y^k \quad (k = 1,2,...,L)$ are the model system states calculated by formulas

$$\begin{cases} Y^{k+1} = W(\tilde{t}_{k+1} - \tilde{t}_k, \Lambda, \Sigma, P^k)\tilde{Y}^k, \qquad k = 0,1,2,...,L-1, \\ \sum_{i=1}^{m} y_i^{k+1} = 1, \quad \forall k = 0,1,2,...,L-1 \\ Y^0 = Y(t_0) \end{cases}. \tag{4.4}$$

Note that the one-step procedure (4.4) is used for calculation of the model system states. This procedure generates the model system state Y^{k+1} at the $k+1$-th step on the basis of the real system state $\tilde{Y}^k$ at the previous (k-th) step.

Let us introduce the notations

$$y_i^k = y_i(\tilde{t}_k), \quad p_i^k = p_i(\tilde{t}_k), \quad \Delta t_k = \tilde{t}_{k+1} - \tilde{t}_k, \quad k = 0,1,2,...,L-1, \quad i = 1,2,...,m.$$

The difference expressions $\frac{dp_i}{dt}(t_k) \approx \frac{p_i^{k+1} - p_i^k}{\Delta t_k}$ of the first order accuracy [9] will be used for the difference approximation of the first derivatives of control parameters (of the prices at nodes of the network) in the matrices $A(\Sigma, P(t))$ and $B(\Lambda, P(t))$. Then the system (4.4) can be presented in the explicit form

$$\begin{cases} y_i^{k+1} = \tilde{y}_i^k + \frac{1}{\Delta t_k} \sum_{j=1}^{m} (2\alpha_{ij} + 1) \left(\frac{p_i^k - p_j^k}{p_i^k} \right)^{2\alpha_{ij}} \frac{(p_j^{k+1} - p_j^k)(p_i^{k+1} - p_i^k) - (p_j^{k+1} - p_j^k)^2}{p_j^k p_i^k} \varepsilon_j \tilde{y}_j^k, \\ \sum_{j=1}^{m} y_j^k = 1, \quad i = 1,2,...,m, \quad k = 0,1,2,...,L-1 \end{cases}.$$

One can rewrite this equation in a compact form

$$\begin{cases} y_i^{k+1} = \tilde{y}_i^k + \tau_k \sum_{j=1}^{m} (2\alpha_{ij} + 1)(\pi_{ij}^k)^{2\alpha_{ij}} \left(\rho_i^k - \dfrac{p_j^k}{p_i^k} \rho_j^k \right) \varepsilon_j \rho_j^k \tilde{y}_j^k , \\ \sum_{j=1}^{m} y_j^k = 1, \quad i = 1,2,...,m, \quad k = 0,1,2,...,L-1 \end{cases} . \qquad (4.5)$$

Here $\tau_k = \dfrac{1}{\Delta t_k}, \quad \pi_{ij}^k = \dfrac{p_i^k - p_j^k}{p_i^k}, \quad \rho_i^k = \dfrac{p_i^{k+1} - p_i^k}{p_i^k}, \quad i = 1,2,...,m, \quad k = 0,1,2,...,L-1$.

Conditions $\sum_{j=1}^{m} y_j^k = 1, \quad k = 0,1,2,...,L-1$ in (4.5) define the additional constraints on the interrelation elasticities α_{ij} of the network nodes. Really, the sum of the equalities (4.5) over i from 1 to m leads to the following relation

$$\sum_{i=1}^{m} y_i^{k+1} = \sum_{i=1}^{m} \tilde{y}_i^k + \tau_k \sum_{i=1}^{m} \sum_{j=1}^{m} (2\alpha_{ij} + 1)(\pi_{ij}^k)^{2\alpha_{ij}} \left(\rho_i^k - \frac{p_j^k}{p_i^k} \rho_j^k \right) \varepsilon_j \rho_j^k \tilde{y}_j^k , .$$
$$k = 0,1,2,...,L-1$$

Taking into account that $\sum_{j=1}^{m} y_j^k = \sum_{j=1}^{m} \tilde{y}_j^k = 1, \quad k = 0,1,2,...,L-1$, one can obtain the equality

$$\sum_{i=1}^{m} \sum_{j=1}^{m} (2\alpha_{ij} + 1)(\pi_{ij}^k)^{2\alpha_{ij}} \left(\rho_i^k - \frac{p_j^k}{p_i^k} \rho_j^k \right) \varepsilon_j \rho_j^k \tilde{y}_j^k = 0, \quad k = 0,1,2,...,L-1 . \qquad (4.6)$$

Note that $\rho_i^k - \dfrac{p_j^k}{p_i^k} \rho_j^k = \rho_i^k + (\pi_{ij}^k - 1)\rho_j^k$. Finally, taking into account equalities (4.6), one can derive equations for the model system state

$$\begin{cases} y_i^{k+1} = \tilde{y}_i^k + \tau_k \sum_{j=1}^{m} (2\alpha_{ij} + 1)(\pi_{ij}^k)^{2\alpha_{ij}} \left(\rho_i^k + (\pi_{ij}^k - 1)\rho_j^k \right) \varepsilon_j \rho_j^k \tilde{y}_j^k , \\ \qquad i = 1,2,...,m \\ \sum_{i=1}^{m} \sum_{j=1}^{m} (2\alpha_{ij} + 1)(\pi_{ij}^k)^{2\alpha_{ij}} \left(\rho_i^k + (\pi_{ij}^k - 1)\rho_j^k \right) \varepsilon_j \rho_j^k \tilde{y}_j^k = 0, \\ \qquad k = 0,1,2,...,L-1 \end{cases} . \qquad (4.7)$$

Let us represent the functional $\Phi(\Lambda)$ in the explicit form

$$\Phi(\Lambda) = \sum_{k=1}^{L} (\tilde{Y}^k - Y^k)^2 = \sum_{k=0}^{L-1} (\tilde{Y}^{k+1} - Y^{k+1})^2 =$$
$$= \sum_{k=0}^{L-1} \sum_{i=1}^{m} \left[(\tilde{y}_i^{k+1} - \tilde{y}_i^k) - \tau_k \sum_{j=1}^{m} (2\alpha_{ij} + 1)(\pi_{ij}^k)^{2\alpha_{ij}} \left(\rho_i^k + (\pi_{ij}^k - 1)\rho_j^k \right) \varepsilon_j \rho_j^k \tilde{y}_j^k \right]^2 .$$

Finally, the problem (4.3) can be formulated as the classical problem of conditional extremum [7]

$$\Phi(\Lambda)=\sum_{k=0}^{L-1}\sum_{i=1}^{m}\begin{bmatrix}\left(\tilde{y}_i^{k+1}-\tilde{y}_i^k\right)- \\ \tau_k\sum_{j=1}^{m}\left(2\alpha_{ij}+1\right)\left(\pi_{ij}^k\right)^{2\alpha_{ij}}\left(\rho_i^k+\left(\pi_{ij}^k-1\right)\rho_j^k\right)\varepsilon_j\rho_j^k\tilde{y}_j^k\end{bmatrix}^2\to\min_{\alpha_{ij}} \qquad (4.8)$$

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\left(2\alpha_{ij}+1\right)\left(\pi_{ij}^k\right)^{2\alpha_{ij}}\left(\rho_i^k+\left(\pi_{ij}^k-1\right)\rho_j^k\right)\varepsilon_j\rho_j^k\tilde{y}_j^k=0,\quad k=0,1,\dots,L-1 \qquad (4.9)$$

The problem (4.8)-(4.9) can be solved with the help of the standard methods. In this case (due to nonlinearity of conditions (4.9)) it is better to use the method of Lagrange multipliers

$$\Psi(\Lambda,Z)=\Phi(\Lambda)+\sum_{k=0}^{L-1}\left(z_k\sum_{i=1}^{m}U_{ik}(\Lambda)\right)\to\min_{\Lambda,Z}$$

$$U_{ik}(\Lambda)=\sum_{j=1}^{m}\left(2\alpha_{ij}+1\right)\left(\pi_{ij}^k\right)^{2\alpha_{ij}}\left(\rho_i^k+\left(\pi_{ij}^k-1\right)\rho_j^k\right)\varepsilon_j\rho_j^k\tilde{y}_j^k, \qquad (4.10)$$
$$i=1,2,\dots,m,\quad k=0,1,\dots,L-1$$

The problem (4.10) can be solved either on the basis of the necessary and sufficient conditions of extremum for the functional $\Psi(\Lambda,Z)$, or with the help of the approximation methods for solving extremum problems. The first approach is connected with the numerical solution of the system of nonlinear equations with high dimension. The second approach can be realized with the help of gradient methods [7]. In the both cases, if the values of parameters m (the quantity of nodes) and L (the quantity of measurements of the system states) are large, the quantity of variables in the problem (4.10) is large. However, there exists a possibility for essential reduction of the quantity of variables. For this purpose one can use the data of the marketing research. This information allows to find existence of interrelations between the system nodes and describe them analytically.

Conclusion

The models considered in the paper are developed on the common principles but they differ from each other in description of interaction of nodes. In the first variant the quantity of sale at a node depends mainly on the price change at this node. This assumption allows to construct a compact numerical model of the network with a constructive computer realization. An attempt to take into account the price factors of interaction of the network nodes generates technical difficulties in numerical realization of the second variant of the model. The numerical experiments have been conducted on the basis of the proposed models using real data obtained as a result of the marketing research of a petrol market. The developed mathematical models reflect adequately the interaction of sales at petrol stations under the price fluctuations at a market. The proposed mathematical models can be used for solving optimization problems and development of the optimal price strategy of a firm at a retail market.

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