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Interim Report IR-01-036

Oscillations In Optional Public Good Games

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IIASA STUDIES IN ADAPTIVE DYNAMICS NO. 54

The Adaptive Dynamics Network at IIASA fosters the development of new mathematical and conceptual techniques for understanding the evolution of complex adaptive systems.

Focusing on these long-term implications of adaptive processes in systems of limited growth, the Adaptive Dynamics Network brings together scientists and institutions from around the world with IIASA acting as the central node.

Scientific progress within the network is reported in the IIASA Studies in Adaptive Dynamics series.

THE ADAPTIVE DYNAMICS NETWORK

The pivotal role of evolutionary theory in life sciences derives from its capability to provide causal explanations for phenomena that are highly improbable in the physicochemical sense. Yet, until recently, many facts in biology could not be accounted for in the light of evolution. Just as physicists for a long time ignored the presence of chaos, these phenomena were basically not perceived by biologists.

Two examples illustrate this assertion. Although Darwin's publication of "The Origin of Species" sparked off the whole evolutionary revolution, oddly enough, the population genetic framework underlying the modern synthesis holds no clues to speciation events. A second illustration is the more recently appreciated issue of jump increases in biological complexity that result from the aggregation of individuals into mutualistic wholes.

These and many more problems possess a common source: the interactions of individuals are bound to change the environments these individuals live in. By closing the feedback loop in the evolutionary explanation, a new mathematical theory of the evolution of complex adaptive systems arises. It is this general theoretical option that lies at the core of the emerging field of adaptive dynamics. In consequence a major promise of adaptive dynamics studies is to elucidate the long-term effects of the interactions between ecological and evolutionary processes.

A commitment to interfacing the theory with empirical applications is necessary both for validation and for management problems. For example, empirical evidence indicates that to control pests and diseases or to achieve sustainable harvesting of renewable resources evolutionary deliberation is already crucial on the time scale of two decades.

The Adaptive Dynamics Network has as its primary objective the development of mathematical tools for the analysis of adaptive systems inside and outside the biological realm.

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Contents

Abstract

We present a new mechanism promoting cooperative behavior among selfish individuals in the public goods game. This game represents a straightforward generalization of the prisoner's dilemma to an arbitrary number of players. In contrast to the compulsory public goods game, optional participation provides a natural way to avoid deadlocks in the state of mutual defection. The three resulting strategies – collaboration or defection in the public goods game, as well as not joining at all – are studied by means of a replicator dynamics, which can be completely analysed in spite of the fact that some payoff terms are nonlinear. If cooperation is valuable enough, the dynamics exhibits a rock-scissors-paper type of cycling between the three strategies, leading to sizeable average levels of cooperation in the population. Thus, voluntary participation makes cooperation possible. But for each strategy, the average payoff value remains equal to the earnings of those not participating in the public goods game.

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1 Introduction

Most theories on the emergence of cooperation among selfish individuals are based on kin selection (Hamilton, 1963), group selection (Wilson & Sober, 1994) and reciprocal altruism (Trivers, 1971). In all three models, cooperative behavior is attained through basic mechanisms of discrimination enabling individuals to target their altruistic acts towards certain partners only.

In this article, we present another mechanism to achieve sizeable levels of cooperation in a population. The following investigation is based on the public goods game (see Fehr & Gächter, 1999; Kagel & Roth, 1995), which represents a natural extension of the prisoner's dilemma to an arbitrary number of players (see Boyd & Richerson, 1988; Dawes, 1980; Hauert & Schuster, 1997).

In a typical public goods game, an experimenter gives 10 dollars to each of six players. The players may contribute part or all of their money to some common pool. The experimenter then triples this amount and divides it equally among the six players, irrespective of the amount of their individual contribution. If all six players contribute maximally, they will end up with 30 dollars each. But each individual is faced with the temptation to exploit, as a free rider, the contributions of the co-players. Since investing one dollar yields only a return of 50 cents to the investor, the dominating strategy is to invest nothing at all. If all players do this, they will not increase their initial capital. In this sense, the 'rational' equilibrium solution prescribed to 'homo oeconomicus' leads to economic stalemate. In actual experiments, players tend to invest a lot, however (see Fehr & Gächter, 1999).

Public goods games are abundant in human and animal societies, and can be seen as basic examples of economic interactions (see e.g. Binmore, 1994; Dugatkin, 1997).

Their essence was already brought to light by Jean Jacques Rousseau (1755) in his 'Discourse on Inequality,' when he described a dilemma experienced by the participants in a stag hunt. The success of a stag hunt depends on the cooperation of a group of hunters. Each hunter can improve his lot by defecting, and joining only for dinner. However, if all hunters defect, there will be nothing for dinner.

Actually Rousseau's example is a bit more complex, and this in a highly relevant way. In Rousseau's example, individuals have the option either to join a group of stag hunters, or else to hunt for a hare on their own. If we include this asocial 'fallback' solution, we shall see that a certain amount of cooperation indeed emerges.

We mention that a 'stag hunt game' motivated by Rousseau's example plays an important role in game theory, and in particular in equilibrium selection (see for instance Samuelson, 1997) or Young (1998)), who however does not use the term, but this is not related to the following model.

2 The Model

We consider a large population of players. From time to time, N such players are chosen randomly – the 'stag hunt party.' Within such a group, players can either contribute some fixed amount c or nothing at all. The return of the public good, i.e. the payoff to the players in the group, depends on the frequency of cooperators. If n_c denotes their number among the public goods players, the net payoff for cooperators P_c and defectors P_d will be given by:

$$P_c = -c + rc\frac{n_c}{N}$$

$$P_d = rc\frac{n_c}{N},$$

where r denotes the interest rate on the common pool. For a public goods game deserving its name, we must have:

$$1 < r < N. \tag{1}$$

The first inequality states that if all do the same, they are better off cooperating than defecting; the second inequality states that each individual is better off defecting than cooperating. Selfish players will therefore always avoid the cost of cooperation c, so that a collective of selfish players will not cooperate. Defection is the dominating strategy. Hence both classical and evolutionary game theory predict that all players will defect, and obtain payoff 0.

We now extend the public goods game. In this optional public goods game, players can decide whether to participate in the public goods game or not. (For a similar approach in the prisoner's dilemma, see Batali & Kitcher (1995); Orbell & Dawes (1993).) Individuals unwilling to join the public goods game are termed loners. These players prefer to rely on a small but fixed payoff $P_l = \sigma c$ with

$$0 < \sigma < r - 1, \tag{2}$$

such that the members in a group where all cooperate are better off than loners, but loners are better off than members in a group of defectors.

In the stag hunt example, players unwilling to join the stag hunt can hunt hares, an activity for which a collective effort is not necessary. Players joining a stag hunt or participating in a public goods game are effectively speculating that it will contain few free riders.

For the optional public goods game, there are thus three behavioural types in the population: (a) the loners unwilling to join the public goods game, (b) the cooperators ready to join the group and to contribute their effort, and (c) the defectors who join, but do not contribute. Assuming that groups form randomly, the payoffs for the different strategies P_c, P_d and P_l are then determined by the relative frequencies x,y and z of the three strategies.

3 The Equations of Motion

Evolutionary game theory assumes that a strategy's payoff determines the growth rate of its frequency within the population. More precisely, following Weibull (1995); Schlag (1998) and Hofbauer & Sigmund (1998), we postulate in our model that players using strategies $i = 1, ..., n$ occasionally compare their payoff with that of a randomly chosen 'model' member of the population, and adopt the strategy of their model with a probability proportional to the difference between the model's payoff and their own, if this is positive (and with probability 0 otherwise). In the continuous time model, the evolution of the frequencies x_i of the strategies i is given by

$$\dot{x_i} = \sum_j x_i x_j (P_i - P_j) \tag{3}$$

with $1 \leq i, j \leq n$, which reduces to the replicator equation

$$\dot{x_i} = x_i \sum (P_i - \bar{P}) \tag{4}$$

where $\bar{P} = \sum x_j P_j$ is the average payoff in the population.

For simplicity and without loss of generality, we set the cost c of cooperation equal to 1. The payoff for loners is then given by the constant

$$P_l = \sigma.$$

In order to compute the payoff values for cooperators and defectors, we first derive the probability that S of the N sampled individuals are actually willing to join the public goods game. In the case $S = 1$ (no co-player shows up) we assume that the player has no other option than to play as a loner, and obtains payoff σ. This happens with probability z^{N-1}. For a given player willing to join the public goods game, the probability of finding, among the $N-1$ other players in the sample, $S-1$ co-players joining the group ($S > 1$), is

$$\binom{N-1}{S-1} (1-z)^{S-1} z^{N-S}.$$

The probability that m of these players are cooperators, and $S-1-m$ defectors, is

$$\left(\frac{x}{x+y}\right)^m \left(\frac{y}{x+y}\right)^{S-1-m} \binom{S-1}{m}.$$

In that case, the payoff for defectors is $r \cdot m/S$. Hence the expected payoff for a defector in a group of S players ($S = 2, ..., N$) is

$$\frac{r}{S}\sum_{m=0}^{S-1} m\left(\frac{x}{x+y}\right)^m\left(\frac{y}{x+y}\right)^{S-1-m}\binom{S-1}{m} = \frac{r}{S}(S-1)\frac{x}{x+y}.$$

Thus,

$$\begin{aligned}P_d &= \sigma z^{N-1} + r\frac{x}{1-z}\sum_{S=1}^{N}\binom{N-1}{S-1}(1-z)^{S-1}z^{N-S}\left(1-\frac{1}{S}\right)\\ &= \sigma z^{N-1} + r\frac{x}{1-z}\left[1-\sum_{S=1}^{N}\binom{N-1}{S-1}(1-z)^{S-1}z^{N-S}\frac{1}{S}\right]\end{aligned}$$

and using $\binom{N-1}{S-1} = \binom{N}{S}\frac{S}{N}$ leads to

$$P_d = \sigma z^{N-1} + r\frac{x}{1-z}\left(1-\frac{1-z^N}{N(1-z)}\right). \tag{5}$$

In a group with $S-1$ co-players playing the public goods game, switching from cooperation to defection yields $1 - r/S$. Hence,

$$P_d - P_c = \sum_{S=2}^{N}\left(1-\frac{r}{S}\right)\binom{N-1}{S-1}(1-z)^{S-1}z^{N-S}.$$

Using the same arguments as before, we obtain

$$P_d - P_c = 1 + (r-1)z^{N-1} - \frac{r}{N}\frac{1-z^N}{1-z} =: F(z). \tag{6}$$

The advantage of defectors over cooperators depends only on the fraction of individuals actually willing to play i.e. on the fraction of loners z. At the same time, it is independent of the loner's payoff σ.

The sign of $P_d - P_c$ in fact determines whether it pays to switch from cooperation to defection or vice versa, $F(z) = 0$ being the equilibrium condition. We claim that for $r \leq 2$, F has no root, and for $r > 2$ exactly one root $\hat{z}$ in the interval $(0, 1)$. In order to show this, we consider the function $G(z) = F(z)(1-z)$ which has the same roots as $F(z)$ in $(0,1)$ and note that (a) $G(0) = 1 - r/N > 0$, (b) $G(1) = 0$, (c) $G(z) \asymp (2-r)(N-1)(1-z)^2$ for $z \to 1$, such that in a neighborhood of $z = 1$ $G(z)$ is negative for $r > 2$, and (d) $G''(z) = z^{N-3}(N-1)((N-2)(r-1) - z(Nr - N - r))$ changes sign at most once in $(0,1)$. Thus, for $r > 2$ (which by equation (1) implies $N > 2$) there exists a threshold value of the loners frequency $\hat{z}$ above which cooperators fare better than defectors (see figure 1).

The average population payoff $\bar{P}$ can now be rewritten using the condition $y = 1 - x - z$:

$$\begin{aligned}\bar{P} &= xP_c + yP_d + zP_l = x(P_c - P_d) + z(\sigma - P_d) + P_d\\ &= -x(P_d - P_c) + (1-z)(P_d - \sigma) + \sigma.\end{aligned}$$

Substituting equations (5) and (6) then yields

$$\bar{P} = \sigma - \left[(1-z)\sigma - (r-1)x\right](1 - z^{N-1}). \tag{7}$$

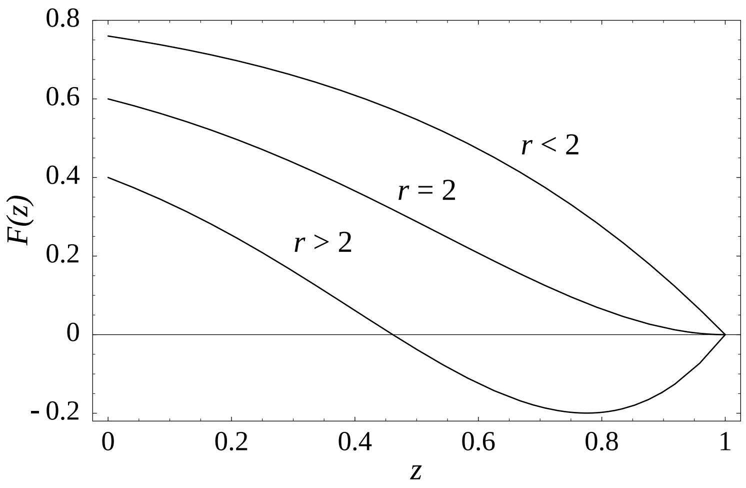

Figure 1: The difference between the payoff of cooperators P_c and defectors P_d is a function of the fraction of loners z: $F(z) = P_d - P_c$. If almost everybody is participating in the public goods game ($z \to 0$) then $F(z) > 0$ holds and it pays to defect. However, for interest rates $r > 2$, if the proportion z of loners increases, it eventually pays to cooperate ($F(z) < 0$) and the social dilemma disappears – at least for a while. $F(z)$ has either no or a unique root in the interval $(0, 1)$.

4 The Dynamics

Let us now analyse the replicator dynamics. The corners of the simplex $S_3 = \{(x, y, z) : x, y, z \geq 0, x + y + z = 1\}$, i.e. the vectors $\mathbf{e}_i$ of the standard basis ($i = c, d, l$ in a straightforward notation), are obviously fixed points. There are no other fixed points on the boundary of S_3. In fact, the edge $\mathbf{e}_c\mathbf{e}_d$ consists of an orbit leading from $\mathbf{e}_c$ (cooperators only) to $\mathbf{e}_d$ (defectors only), the edge $\mathbf{e}_d\mathbf{e}_l$ is an orbit leading to the state consisting of loners only, and the orbit $\mathbf{e}_l\mathbf{e}_c$ closes this heteroclinic cycle of rock-scissors-paper type on the boundary.

In order to analyse the dynamics in the interior, it is useful to show that the replicator equation, defined on the simplex S_3, can be rewritten in the form of a Hamiltonian system, and thus admits an invariant of motion. Indeed, defining as a new variable $f = x/(x + y)$, i.e. the fraction of cooperators among the individuals actually participating in the public goods game, we obtain

$$\dot{f} = \frac{y\dot{x} - x\dot{y}}{(x + y)^2} = \frac{xy}{(x + y)^2}(P_c - P_d).$$

This, as well as substituting equation (7) into the replicator equation $\dot{z} = z(\sigma - \bar{P})$, yields

$$\dot{f} = -f(1 - f)F(z) \tag{8}$$

$$\dot{z} = [\sigma - f(r - 1)]\, z(1 - z)(1 - z^{N-1}) \tag{9}$$

with (f, z) on the unit square $(0, 1)^2$. Dividing the right hand side by the function $f(1 - f)z(1 - z)(1 - z^{N-1})$, which is positive on the unit square, corresponds to a change in velocity which does not affect the orbits. This yields

$$\dot{f} = \frac{-F(z)}{z(1-z)(1-z^{N-1})} =: -g(z)$$
$$\dot{z} = \frac{\sigma - f(r-1)}{f(1-f)} =: l(f).$$

Introducing $H := G + L$, where $G(z)$ and $L(f)$ are primitives of $g(z)$ and $l(f)$:

$$G(z) = (1 - \frac{r}{N})\log z + (\frac{r}{2} - 1)\log(1-z) + R(z) \tag{10}$$
$$L(f) = \sigma \log f + (r - 1 - \sigma)\log(1-f) \tag{11}$$

with $R(z)$ bounded on $[0,1]$, we obtain the Hamiltonian system

$$\dot{f} = -\frac{\partial H}{\partial z}$$
$$\dot{z} = \frac{\partial H}{\partial f}.$$

The actual dynamics of the system depends on whether the condition $P_d = P_c$ can be satisfied in the interior S_3, and hence on the interest rate r. For $r \leq 2$ there are no fixed points except the corners and all trajectories in $intS_3$ are homoclinic orbits of $\mathbf{e}_l$. Thus, the system will display intermittently brief bursts of cooperation, but always ends up with no one willing to participate in the public goods game, as shown in figure 2.

For $r > 2$, equation (2) implies that there exists a unique fixed point $\mathbf{Q} = (\hat{x}, \hat{y}, \hat{z})$ in $intS_3$ such that $F(\hat{z}) = 0$ and:

$$\hat{x} = \frac{\sigma}{r-1}(1 - \hat{z}) \tag{12}$$

as well as

$$\hat{y} = (1 - \frac{\sigma}{r-1})(1 - \hat{z}) \tag{13}$$

which follows from $P_d = P_l$. Due to the fact that the system is conservative, and the Hamiltonian H attains a strict (global) maximum at $(\frac{\sigma}{r-1}, \hat{z})$, the interior equilibrium $\mathbf{Q}$ is a center, i.e., it is neutrally stable and surrounded by closed orbits (see figure 3).

Actually *all* interior orbits are closed: equation (10) shows that $G(z) \to -\infty$ for $z \to 0, 1$ if $2 < r < N$, and equation (11) implies that $L(f) \to -\infty$ as $f \to 0, 1$ if $\sigma < r - 1$. Therefore $H \to -\infty$ uniformly near the boundary of $[0,1]^2$ and hence all level sets of H are closed curves. In particular, no interior orbit converges to the nonhyperbolic equilibrium $\mathbf{e}_l$.

Variations of the three parameters N, r, σ allow to position $\mathbf{Q}$ anywhere in the interior of the simplex (see figure 4). Note that in general all three parameters must be adjusted to place $\mathbf{Q}$ in a particular location. According to equations (12) and (13), the fixed point $\mathbf{Q}$ lies on the line

$$x = \frac{\sigma}{r - 1 - \sigma} y \tag{14}$$

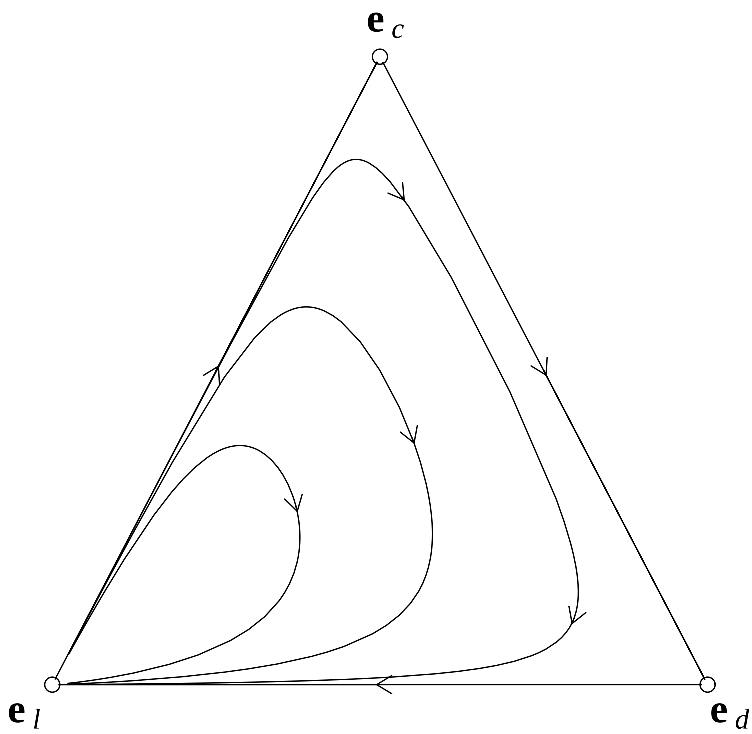

Figure 2: The three corners $\mathbf{e}_c, \mathbf{e}_d, \mathbf{e}_l$ of S_3 are saddle points (but $\mathbf{e}_l$ is not hyperbolic) and the boundary bdS_3 represents a rock-scissors-paper type heteroclinic cycle. For small interest rates, $r < 2$, no fixed point exists in $intS_3$ and all orbits converge to $\mathbf{e}_l$. But $\mathbf{e}_l$ is not Lyapunov stable. Parameters: $N = 5, r = 1.8, \sigma = 0.5$.

independent of the group size N. For increasing N, $\mathbf{Q}$ moves towards the corner $\mathbf{e}_l$ and in the limit $N \to \infty$ homoclinic orbits issuing from and leading to $\mathbf{e}_l$ are obtained.

For the limiting cases $r = N$, $\sigma = r - 1$ and $\sigma = 0$, $\mathbf{Q}$ approaches the edges $\mathbf{e}_c\mathbf{e}_d$, $\mathbf{e}_l\mathbf{e}_c$ or $\mathbf{e}_l\mathbf{e}_d$, respectively. In particular, for $r = N$, cooperation becomes stable in the sense that, while the state can fluctuate along the edge $z = 0$ by random drift, any small fluctuation introducing the missing loners will be offset in such a way that the loners vanish again and the number of cooperators is larger than previously (see figure 5).

Although the time averages of the state variables over an orbit of period T, defined as $\bar{v} = \frac{1}{T}\int_0^T v\,dt$, depend on the initial conditions, the following relations hold for every orbit. First, the average fraction of cooperators among playing individuals corresponds to its value at the equilibrium point $\mathbf{Q}$:

$$\frac{\bar{x}}{\bar{x} + \bar{y}} = \frac{\sigma}{r - 1}. \tag{15}$$

This means that the time average lies on the solid line in figure 4 which connects $\mathbf{Q}$ and $\mathbf{e}_l$. Second, the average of the fraction of cooperators among participants in public goods games $\bar{f}$ corresponds to the fraction of the averages:

$$\bar{f} = \frac{\sigma}{r - 1}. \tag{16}$$

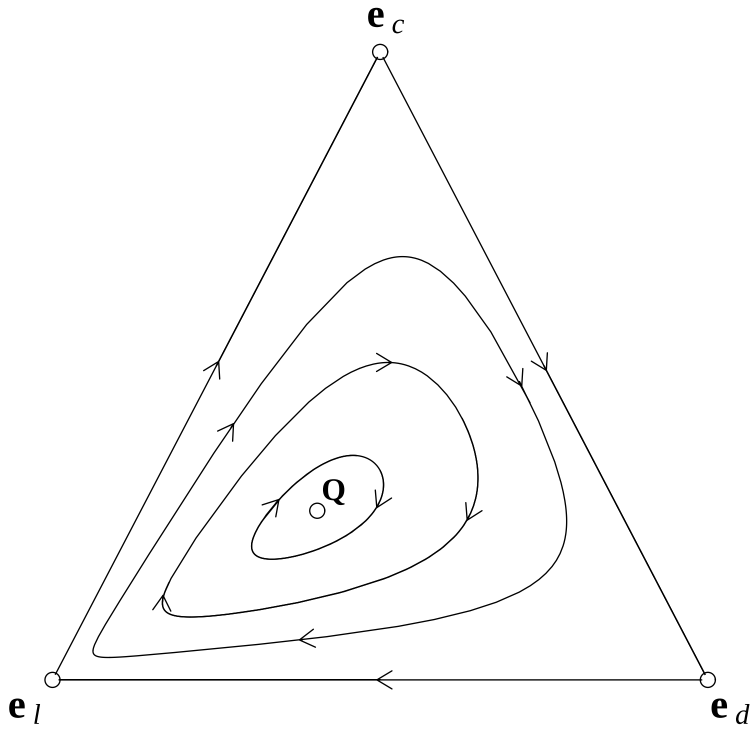

Figure 3: For $r > 2$, the three corners $\mathbf{e}_c$, $\mathbf{e}_d$, $\mathbf{e}_l$ are again saddle points and bdS_3 represents a heteroclinic cycle. In $intS_3$ a single fixed point $\mathbf{Q}$ appears. It is a center surrounded by closed orbits (see text). Parameters: $N = 5, r = 3, \sigma = 1$.

Surprisingly, perhaps, increasing r always favours defection, i.e. it decreases the fraction f of cooperators among those actually engaging in the public goods game.

According to numerical calculations, the time average lies on the line segment $\mathbf{Q}\mathbf{e}_l$ and converges to $\mathbf{e}_l$ as the closed orbit approaches the boundary of S_3. We can offer only a heuristic explanation of this observation: The closer the periodic orbit is to the boundary the more time it will spend near the degenerate equilibrium $\mathbf{e}_l$ (both eigenvalues zero) where motion is much slower than close to the hyperbolic equilibria $\mathbf{e}_c$ and $\mathbf{e}_d$.

Let us show how equation (15) is deduced by integrating equation (9). Remembering that, by definition, $x = f(1 - z)$, and dividing both sides of equation (9) by $z\,(1 - z^{N-1})$, we get:

$$\int_0^T \left[\sigma(1-z) - (r-1)\,x\right]\,dt = \int_0^T \frac{\dot{z}dt}{z\,(1-z^{N-1})} = p(z)\Big|_{z(0)}^{z(T)}$$

$p(z)$ being a primitive of $[z(1 - z^{N-1})]^{-1}$. Since the orbits are closed, the last term vanishes and the proportionality between $\bar{x}$ and $1 - \bar{z}$, i.e. $\bar{x} + \bar{y}$ follows. The time average (16) follows in the same way after dividing equation (9) by $z(1-z)(1-z^{N-1})$.

Due to the properties of the replicator equation, the time averages of the payoffs for the three different strategies are equal and reduce to the payoff of loners σ:

$$\bar{P}_c = \bar{P}_d = \bar{P}_l = \sigma.$$

Thus, in the long run, no one does better or worse than the loners.

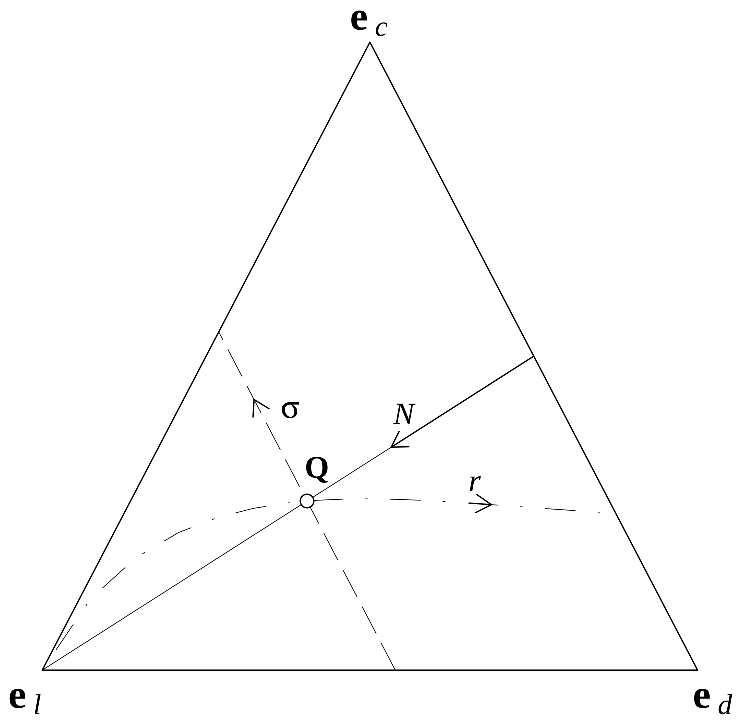

Figure 4: The position of the center $\mathbf{Q}$ in S_3 depends on the values of the parameters N, r and σ. The intersection of the three lines corresponds to $N = 5, r = 3, \sigma = 1$. Each line indicates the displacement of the center when varying a single parameter. Increasing the number of potential participants N shifts the center along the solid line in the direction indicated by the arrow, i.e. towards the corner $\mathbf{e}_l$. Similarly, increasing σ shifts the center upwards on the dashed line $z = \hat{z}$ and increasing r moves the center to the right, along the dash-dotted line. For $r \to 2$, the center approaches the corner $\mathbf{e}_l$.

5 Discussion

The oscillations, and thus the recurrent increase in cooperation, are due to the fact that a public goods game needs not always be a social dilemma. In a public goods game, those players who are defecting are always better off than those players who are cooperating. Nevertheless, if the group size S of participating players is less than the interest rate r, it pays the individual player to switch from defection to cooperation. If players have the option of an asocial 'fallback solution,' they can refuse to join the public goods game. If enough players refuse to join, the group becomes so small that the game is no longer a social dilemma. But then, the higher payoff obtained by the cooperators in the public goods game causes more players to join, and larger groups of public goods players create the temptation to defect, i.e. the social dilemma. This requires $r > 2$, a condition which is similar to the condition that in the Prisoner's dilemma game, the benefit exceeds twice the cost: this condition is essential for the stability of the Pavlov strategy (Nowak & Sigmund, 1995). It may be argued that this condition can also be found in Hamilton's rule for kinship selection. Here, the cost-to-benefit ratio should exceed the degree of relatedness between donor and recipient, but under 'normal' conditions

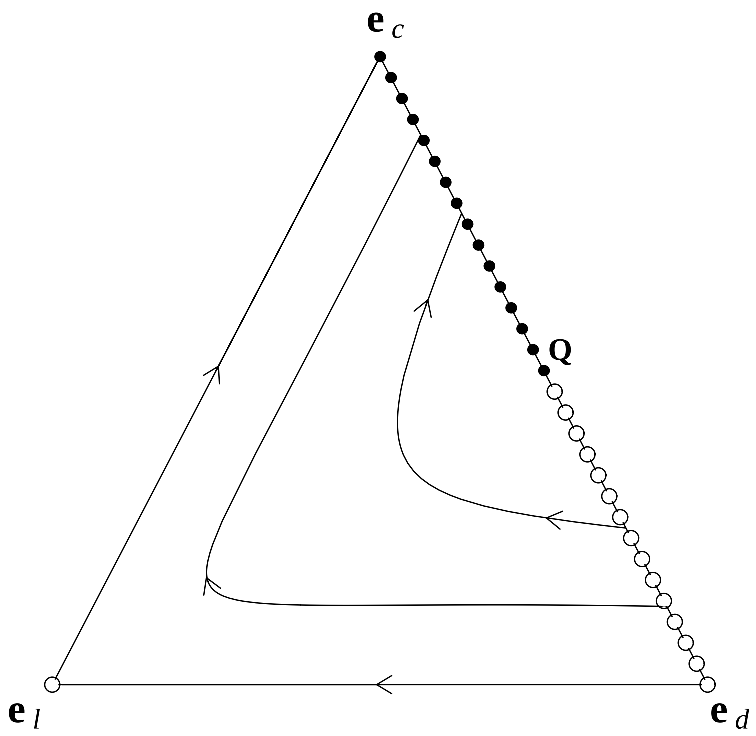

Figure 5: In the limiting case $r = N$, the edge $\mathbf{e}_c\mathbf{e}_d$ is a line of fixed points, stable on $\mathbf{e}_c\mathbf{Q}$ (closed circles) and unstable on $\mathbf{Q}\mathbf{e}_d$ (open circles). Random drift and occasional appearances of the missing loner strategy will eventually drive the system close to the corner $\mathbf{e}_c$ with almost everybody cooperating. Parameters: $N = 3, r = 3, \sigma = 1$.

(no inbreeding, etc) this relatedness is at most 1/2.

The proposed model for an optional public goods game represents one of the rare cases where a highly non-linear system of replicator equations can be fully analyzed by purely analytical means. For small interest rates, $r \leq 2$, homoclinic orbits are observed starting in and returning to $\mathbf{e}_l$, i.e. the state where no one participates in the public goods game. For $r > 2$ a fixed point occurs in the interior of the simplex S_3. By reducing the replicator equations to a hamiltonian system, we could see that $\mathbf{Q}$ is actually a center and that in $intS_3$ only closed orbits appear. From this follow various conditions on the time averages of the frequencies and payoffs of the three strategies. For example, the average ratio of cooperators and defectors corresponds to the ratio of the averages and is independent of the initial configuration and the group size N. It turns out to be impossible to increase cooperation by increasing the interest rate r – on the contrary, it favours defection and lowers $\bar{x}/\bar{y}$. In order to promote cooperation, one should rather increase the loner's payoff σ or reduce the group size N. Note that in the latter case $\bar{x}/\bar{y}$ still increases even when keeping the profits for each invested dollar constant ($r/N = const$). The fact that cooperation is favoured in smaller groups agrees with other theoretical as well as experimental results (Bonacich *et al.*, 1976; Boyd & Richerson, 1988; Milinski *et al.*, 1990; Hauert & Schuster, 1998).

We stress that the dynamics obtained in this simple and, we believe, natural model is highly degenerate: it has a center, an invariant of motion, a heteroclinic cycle, a nonhyperbolic fixed point, and an even number of Nash equilibria. All these properties are nongeneric under the usual assumptions.

The option to drop out from a public goods game, i.e. a social and economic enterprise, avoids deadlocks in states of mutual defection and economic stalemate. As a prerequisite, the possible gain – i.e. the 'interest' r – has to be quite large. The enterprise must offer a considerable advantage. In simple societies, such situations may occur in big game hunting or in war. Small groups of volunteers are known to be efficient for these tasks. Success attracts larger groups of participants, but growth may inherently spell decline. This mechanism leads to oscillations in the composition of the population. However, the average effect on the individual's payoff is just the same as if this possibility did not exist and all members of the population were loners.

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