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General equilibrium and welfare modeling in spatial continuum: a practical framework for land use planning

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Abstract

The application of continuous distributions from statistics in spatial modelling makes it possible to represent discrete choices in a spatial continuum, and to obtain efficiency results and competitive equilibrium prices where aggregate or discretized models fail. Along these lines, and combining principles established by Aumann and Hildenbrand in the sixties with recent results from stochastic optimization, the paper develops a practical modeling framework for land use planning and presents the associated stochastic algorithms for numerical implementation. We consider groups of consumers and producers whose activities are distributed over space, and who have to make decisions, say, about where to live, which marketplace to visit, and which infrastructure facilities to invest in. After presenting a general equilibrium model in which all consumers meet their own budget with given transfers, we focus on the case in which transfers among consumer groups adjust to support the maximization of a given social welfare criterion. It appears that this optimization problem becomes more tractable if it is treated as the minimization of a dual welfare function, that solely depends on prices but is evaluated after integration over space. Next, we apply the dual welfare function to represent (non-rival) demand that simultaneously benefits several agents, reflecting a general informational infrastructure as well as investments with uncertain outcomes. This leads to a minimax problem, in which the dual welfare function is to be minimized with respect to prices and maximized with respect to non-rival demand. Finally, we endogenize welfare weights jointly with prices to model, for example, a land consolidation process whereby none of the participants should lose relative to the initial situation, and the gains could be shared according to agreed principles. This gives rise to a bargaining problem whose solution can be found by jointly minimizing the dual welfare function over prices and welfare weights, subject to constraints.

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1. Introduction

At the interface of geography and economics, the practical relevance of applied policy models has often been limited by their lack of empirical detail in representing the distribution of spatial and social characteristics of the economy under study. Elaborate household surveys have been conducted and detailed geographic information systems were set up, but the databases are rarely used in regional or national models, due to the relatively high level of social and spatial aggregation that is required to keep the analysis tractable at that level. The situation is even less satisfactory when it comes to dealing with spatial distributions of uncertain events, which are either neglected altogether or dealt with through a small number of alternative states of nature. Continuous joint densities would seem to be the natural setting for representation of spatial characteristics. They often significantly simplify the analysis, and also have the advantage that they permit to address current issues in land use planning, that naturally involve discrete choices and other non-convexities, and are very difficult to handle within general equilibrium theory and welfare modeling if space is represented by a classification in separate categories.

First, the selection of an optimal location for a facility is a classical problem that often involves discrete choice. This problem usually amounts to finding the geographical location of an industrial facility that minimizes the cost of transporting goods to that location from a surrounding region or vice versa. The early location models are classical transportation models and only select the best out of a finite number of alternatives and treat the region as a finite number of fields identified by their barycentre. Subsequent location/pricing models treat the site as a continuous choice variable and simultaneously calculate the consumer price at every point in the region on the basis of the distance from the facility (see Hansen et al., 1987, and Drezner, 1995, for surveys). Second, land use itself has become an important field of application especially in relation to environmental issues. Here the vocation of the land (nature-agriculture-industry, or by crop) is the discrete choice variable to be determined so as to maximize, say, the social welfare in the region, possibly subject to zoning constraints. And for urban uses, indivisibilities of infrastructure and agglomeration effects must be accounted for. Third, the analysis of transportation and migration is also faced with discrete choices as to the destination and the mode of transportation. (see e.g. Fujita et al., 1999).

This paper proposes a framework and a solution algorithm for a spatial general equilibrium model that can incorporate the full distribution of such characteristics, and can accommodate discrete choice. It assumes that individuals in society are located on a joint density over physical space, social characteristics and random events. Thus, we interpret "spatial" in a broad sense, and the geographical coordinates may only be two among the many coordinates of the space. Representation in a continuum is often used in mathematical modeling, to bypass non-essential effects associated with the discontinuous nature of real world processes. For example, in control theory (see e.g. Alekseev, Tihomirov, Fomin, 1979), the assumption of continuous time convexifies the attainable sets, and leads to Pontryagin's maximum principle. Kantorovich (1942) studied classical transportation problems in a spatial continuum. In economics Aumann (1964, 1966) and Hildenbrand (1970, 1973) were first to study a continuum of agents within general equilibrium theory. Aumann (1966) and Hildenbrand (1970) assume continuity of consumer preferences and hence continuity of the corresponding utility functions but relax the usual concavity requirements on utilities. Consequently, consumer demands may be discontinuous. Hildenbrand (1970) includes a finite number of profit maximizing producers with a convex aggregate production set. Treating consumers as atomless and distributed according to a smooth density enables both authors to prove existence of competitive equilibria, – i.e. the existence of endogenously generated prices at which aggregate demand does not exceed supply, while consumers and producers take prices as given and maximize utility and profits, respectively, according to optimization problems that may exhibit nonconvexities – essentially because all individuals are assumed to be sufficiently different to ensure that agents whose demand or supply exhibits a discontinuity at the prevailing prices have measure zero and can be disregarded.

However, this assumption is non-constructive, in the sense that it is introduced after derivation of individual behavior from preferences and technology. This makes it difficult actually to build a model that meets the requirement, and may be one reason for the class of models not to have found numerical application so far, another being that the common computational approaches require discretization of infinite dimensional models that could destroy continuity of the aggregate excess demand.

In this paper, building on the non-convex spatial problems for producers considered in Keyzer and Ermoliev (1999), we develop a formulation that also considers consumers and endogenizes the market prices, with the aim to model spatial decisions and to construct geographic maps of consumption, production, income and poverty, under alternative policy regimes. Despite the non-convexities considered, we establish sufficient conditions for the continuity of the aggregate excess demand function, and hence for the existence and efficiency of the resulting market equilibrium.

Regarding the specification as a general equilibrium model, we restrict the class considered in several respects. First, we deal with discrete choice, as a particular source of non-convexity. But since virtually every non-convexity can be represented in the way, this hardly imposes a limitation. Second, we allow for full satiation of consumers. When consumers are atomless, their demand could remain infinite at equilibrium, provided it has zero measure. This seems unrealistic and creates unnecessary complications. It is avoided by allowing for satiation. Third, the critical step in ensuring that non-convexities may be bypassed is to guarantee that all agents making the same discrete choice are sufficiently different. For this, it suffices to require nonstationarity with respect to a single characteristic. Fourth, our convexity requirements are strict, ensuring that individual net demands are almost everywhere single- rather than multivalued. Finally, after introducing a model that treats transfers between consumer groups (regions) as given, we soon focus on the special case where transfers adjust to maximize a given social welfare objective. It appears that the problem becomes more tractable if it is treated as the minimization of a dual welfare function, that solely depends on prices, rather than as a maximization in terms of quantities, since these are unknown functions of characteristics.

We also introduce two extensions. First, we allow for (nonrival) demand that simultaneously benefits several agents, in order to represent general informational infrastructure as well as investments with uncertain outcomes. This leads to a minimax problem, with a dual welfare function to be minimized with respect to prices and maximized with respect to nonrival demand. Second, we endogenize welfare weights jointly with prices to obtain a model that could represent, for example, a land consolidation process whereby none of the participants should lose relative to the initial situation, and the gains could be shared according to agreed principles. This gives rise to a bargaining problem whose solution is found by joint minimization of the dual welfare function over prices and welfare weights, subject to constraints.

Regarding computation, the main step is that we treat the distributions of characteristics as probability distributions. For solving the general equilibrium model with exogenous transfers, this enables us to rely on the stochastic procedures given in Ermoliev et al. (2000). However, we focus on the special case in which aggregate excess demand satisfies the Weak Axiom of Revealed Preference, specifically on the situation in which transfers adjust to maximize a given social welfare criterion. We show that, after integration of individual excess demands over space, the aggregate excess demand for such models is the negative of the gradient of the dual welfare function and hence satisfies the Weak Axiom. This enables us to develop variants of stochastic tâtonnement for such models, which are in effect stochastic equivalents of the classical Walrasian tâtonnement. But whereas the classical tâtonnement at each step adjusts the market price for every commodity on the basis of its aggregate excess demand, the stochastic variant only activates a random sample of agents from an infinite set. The tâtonnement indicates a direction of price change, purely on the basis of the net demand within this sample. Though the case satisfying the Weak Axiom is admittedly restrictive, it permits to apply a faster algorithm that has well understood convergence properties and that can naturally be extended into a minimax procedure which adjusts quantities and welfare weights in parallel with prices, to represent nonrivalry and bargaining.

The paper proceeds as follows. Section 2 introduces the distribution of spatial and social characteristics. Producer and consumer behavior under discrete choice are described in sections 3 and 4, respectively. Existence of a competitive equilibrium and of a solution to the spatial welfare model is established in section 5 and 6, respectively. Section 7 discusses the stochastic tâtonnement procedure and its application to both. Next, section 8 incorporates non-rival demand and formulates the associated minimax procedure. Finally, section 9 shows how bargain can be included and leads to joint minimization over weights and prices.

2. The continuum of agents: distribution of characteristics

We seek to develop an operational modeling framework that can combine geographical information with household survey data while avoiding aggregation to a discrete number of income groups and subregions. Each answer in the survey questionnaire defines one characteristic, while the frequency of answers specifies the distribution of these characteristics in the sample. If a characteristic relates to an exogenous variable of the analysis (e.g. previous occupation of the respondent, or geographical location), it can be treated as part of a vector $x \in X \subset R^m$. Assuming that the survey was well designed, and representative, it is possible to infer from this sample the distribution $G(x)$ at the level of the population.

In our computational procedure we view $G(x)$ as a probability measure on an appropriate probability space for which we use the following formal general concepts.

Let $L_q(X)$ denote the Banach space of integrable in power q functions on X for some $q, 1 \le q < \infty$. The multifunction $A: x \to A(x) \subset R^n$ is called (Borel)-measurable if it has a Borel graph in $X \otimes R^n$, (see Aumann, 1965, Hildenbrand, 1974, and Castaing and Valadier, 1977, for the concept of measurable multifunctions).

Assumption 2.1 *Let* x *be an m-dimensional real vector of characteristics and* $x \in X$, *where* X *is a compact in* R^m. *The distribution* $G(x)$ *defines a measure on* X, *and this measure* $G = G_1 \otimes G_y$ *on* $X = X_1 \otimes Y$ *is a product of the absolutely continuous (with respect to Lebesgue measure) measure* G_1 *on* $X_1 \subseteq R$ *and the σ-additive and complete measure* G_y *on* Y.

We note that continuity is only required for a single characteristic, say, element x_1. This is important because survey data often comprise a large number of discrete characteristics, such as farm/non-farm or male/female. It is always possible to introduce an artificial, continuous variable, say, x_{m+1}, that creates a "pseudo"-continuum and only serves to eliminate discontinuities that might arise from non-convexities. In this paper, all characteristics are taken to describe the spatial or social diversity of agents. This makes it safe to treat X as a compact set, and also ensures that for a continuous, Borel measurable function $u(c,x)$ and distribution $G(x)$, the integrability over X is assured and the function $U(c) = \int_X u(c,x) dG(x)$ is continuous. By Lebesgue's dominance convergence theorem $U(c)$ is continuous if $u(c,x)$ is continuous in c and majorated by some integrable function, for example, if $u(c,x) \le \bar{u}$ for some given $\bar{u}$. However, some components of x could also be taken to represent uncertain events for which the compactness of X is no longer guaranteed.

Before formulating the producer model with setup costs and discrete decisions in general terms, we present a simple example that introduces the approach to eliminate discontinuities. This is essentially based on the following lemmas that ensure the non-stationarity w.r.t. x_1 of the value function $F(x)$ of the decision problems, i.e. when $F(x^1) \ne F(x^2)$ for $x^1 \ne x^2$.

Lemma 1. *Assume that (i) the problem* $F(x) = \sup_z \{ f(z,x) \mid h(z,x) \le 0 \}$ *has a solution for any* $x \in R^1$; *(ii) the function* $f(z,x)$ *is strictly increasing in x; and (iii)* $h(z,x)$ *is nonincreasing in x. Then, the function F(x) is strictly increasing in x.*

Proof. Choose $x^1 < x^2, F(x^1) = f(z(x^1), x^1)$, and $h(z(x^1), x^1) \le 0$. Since $f(z(x^1), x^1) < f(z(x^1), x^2)$ while $h(z(x^1), x^2) \le 0$, it follows that $F(x^1) < F(x^2)$. □

Lemma 2. *Assume that (i) the problem* $F(x) = \sup_z \{ f(z,x) \mid h(z,x) \le 0 \}$ *has a solution for any* $x \in R^1$; *(ii) the function* $f(z,x)$ *is non-decreasing in x; (iii)* $h(z,x)$ *is strictly decreasing in x; and (iv) for any (z, x) there exists an arbitrary small* Δz *such that* $f(z + \Delta z, x) > f(z,x)$ *(for example,* $\Delta z = \varepsilon \nabla_z f(z,x) \ne 0$, $\varepsilon > 0$*). Then, the function F(x) is strictly increasing in x.*

Proof. Choose $x^1 < x^2$ and $F(x^1) = f(z(x^1),x^1)$, and $h(z(x^1),x^1) \le 0$. Now $z(x^1)$ is an internal point of the set $\{z \mid h(z,x^2) \le 0\}$, since by (ii) $h(z(x^1),x^2) < h(z(x^1),x^1) \le 0$. Furthermore, by (iii) and assumption (iv), there exists a value Δz such that $h(z(x^1)+\Delta z,x^2) \le 0$ and $F(x^2) \ge f(z(x^1)+\Delta z,x^2) \ge f(z(x^1)+\Delta z,x^1) > f(z(x^1),x^1)$ $= F(x^1)$. □

Finally, the next lemma gives a sufficient condition for the level set of a partially nonstationary function to have zero measure. This is the main regularity property that makes it possible to neglect the discontinuities in response functions after integration.

Lemma 3. *Assume that (i) function* $f(x,y): X \otimes Y \to R^1$ *is measurable in* (x,y) *on a product of measurable sets* $X \subseteq R^1, Y \subseteq R^m$; *(ii)* $f(x,y)$ *is nonstationary in variable* x*, i.e.* $f(x^1,y) \ne f(x^2,y)$ *for any* $x^1 \ne x^2 \in X$ *and* $y \in Y$; *(iii) measure* $G = G_x \otimes G_y$ *on* $X \otimes Y$ *is a product of a σ-additive and complete measure* G_y *on Y and (iv) absolutely continuous (with respect to Lebesgue measure) measure* G_x *on X. Then* $G\{(x,y) \mid f(x,y) = 0\} = 0$.

Proof. For any $y \in Y$ by (ii) the set $\{x \mid f(x,y) = 0\}$ consists of no more than one point. By (iv), $G_x\{x \mid f(x,y) = 0\}$. And by the Fubini theorem (e.g. Kolmogorov and Fomin, 1981):

$$G\{(x,y) \mid f(x,y) = 0\} = \int_Y G_x\{x \mid f(x,y) = 0\} dG(y) = 0 . \square$$

In subsequent sections we often use the following important fact. Let

$$V(p) = \int_X max_{d \in D}\, v(p,d,x) dG(x),$$

where $D \in R^n$ is a compact set, $v(p,d,x)$ is convex and continuous in p, continuous in d and integrable in x, its subdifferential $\partial_p v(p,d,x)$ is bounded for all $d \in D$ by an integrable in x function. Then, by well known results on subdifferentiation of integral functions and the differentiation of a maximum function (see e.g. Clarke, 1983, Levin, 1985), and for $co\{\cdot\}$ denoting the convex hull, the following result holds.

Lemma 4. *The subdifferential* $\partial V(p)$ *of* $V(p)$ *is expressed as follows:*

$$\partial V(p) = \int_X \partial_p\, max_{d \in D}\, v(p,d,x) dG(x) = \int_X co\{\partial_p v(p,d,x) \mid d \in d(p,x)\} dG(x),$$

where $d(p,x) = arg\,max_{d \in D}\, v(p,d,x)$. *In addition, if* $v(p,d,x)$ *is continuously differentiable in* p *with gradient* $v_p(p,d,x)$*, then* $d(p,x)$ *is single valued for any* p *and* $V(p)$ *is continuously differentiable with gradient*

$$V_p(p) = \int_X v_p(p,d,x)\Big|_{d=d(p,x)} dG(x).$$

Proof. See Clarke (1983), Levin (1985). □

Example 1: Single output and input commodity

Suppose that "producers" are distinguished by a characteristic x, say, geographic location and distributed over an area according to the smooth distribution function $G(x)$. The firm at spot x produces a single output commodity, using a single input commodity, according to a strictly concave production function $f(v,x)$, with setup costs $g_0(x)$, where v denotes input use. The firm maximizes the discontinuous profit function:

$$\pi(p,x) = max_{v\geq 0}\left[p_1 f(v,x) - g(v,x) - p_2 v\right], \tag{2.1}$$

where $g(v,x) = 0$ if $v = 0$ and $g(v,x) = g_0(x)$ if $v > 0$, while p_1 and p_2 are the given prices of the output and input, respectively, and $p = (p_1, p_2)$. This defines a discontinuous input demand $v(p,x)$ such that $v(p,x) = 0$ if $\pi(p,x) = 0$, and $v(p,x) > 0$ if $\pi(p,x) > 0$. Since $f(0,x) = 0$, this discontinuous problem can also be rewritten as the mixed-integer program:

$$\pi(p,x) = max_{v\geq 0,\delta=0,1}\,\delta\left[p_1 f(v,x) - g(v,x)\right] - p_2 v\,. \tag{2.2}$$

Furthermore,

$$\Pi(p) = \int \pi(p,x)\,dG(x) = \int\left[max(\overline{\pi}(p,x),0)\right]dG(x), \tag{2.3}$$

for $\overline{\pi}(p,x) = max_{v\geq 0}\left(p_1 f(v,x) - g_0(x) - p_2 v\right)$, assuming that this function is integrable. Now if $\overline{\pi}(p,x)$ is non-stationary with respect to x_1, then $G\{x : \overline{\pi}(p,x) = 0\} = 0$, implying that the points at which a switch takes place can be neglected in the integration. Consequently, the following properties hold. First, aggregate profit $\Pi(p)$ is continuously differentiable and convex in p. Second, the aggregate output and input coincide with the one obtained after integration of input demand in the original problem (2.1), and, by Hotelling's lemma (Varian, 1992), are equal to the negative of the derivative of the profit function (see also Lemma 4).

To our knowledge, this approach has not found practical application so far, presumably because of the difficulties in dealing with maximization problems generally involving, multi-dimensional integrals, as in (2.3). Stochastic quasigradient procedures − to be discussed in section 7 − enable us to deal with the maximization of multidimensional integrals without having to approximate or evaluate them explicitly.

3. Producer behavior

We consider a set of marketplaces indexed λ, located at x^λ, with $\lambda = 1,\ldots,L$. Commodities are traded at these marketplaces and fetch a price p^λ. Let $p \in R^n_+$ denote the vector of stacked prices of all marketplaces partitioned into $(p^1,\ldots,p^\lambda,\ldots,p^L)$.

Next, we introduce the production model with discrete characteristics, representing H technology types indexed h, J firm types, indexed j, K commodities, indexed k. At x, every firm of type j maximizes profits, at given prices p solving:

$$\pi_j(p,x) = max_{y_j^h,\delta_j^h} \sum_h \delta_j^h \left(py_j^h\right)$$

$$s.t. \quad \sum_h \delta_j^h H_j^h(y_j^h, x) \le 0, \tag{3.1}$$

$$\sum_h \delta_j^h = 1, \delta_j^h \in \{0,1\},$$

where y_j^h denotes net supply of firm j at x using technology h; $\pi_j(p,x)$ is the optimal profit; and x has a distribution $G(x)$ satisfying assumptions 2.1, 2.2. By definition, this profit is equal to the sum of the value of net supplies at the different locations: $\pi_j(p,x) = \sum_\lambda p^\lambda y_j^{h,\lambda}$. Hence, the firm can in principle buy and sell at every market place λ, taking charge of the transportation costs of outputs to, and of inputs from this market. Clearly, the technology index h might also be associated to a particular configuration of marketplaces at which the producer trades. For notational convenience we do not in the sequel refer explicitly to the marketplace.

Every producer chooses one technology, represented by a transformation function $H_j^h(\cdot)$. The transformation function may have a positive value at $y_j^h = 0$, so as to reflect that setup costs must be incurred before any production can take place, but we also assume that it is feasible to close down the factory, i.e. that there is a technology h for which the transformation function is non-positive at $y_j^h = 0$. In Example 1, the associated transformation function can be defined as:

$$H_j^1(y_{1j}^1, y_{2j}^1, x) = y_{1j}^1 - f(-y_{2j}^1, x) \text{ and } H_j^2(y_{1j}^2, y_{2j}^2, x) = 0 .$$

We note that model (3.1) has discrete decision variables. This reflects an indivisibility and hence a non-convexity in production. Alternatively, this indivisibility can be expressed in the space of products but on nonconvex and, in general, disconnected sets, as follows:

$$Y_j^h(x) = \{y \in R^n \mid H_j^h(y,x) \le 0\}, \qquad Y_j(x) = \cup_h Y_j^h(x) .$$

Hence, model (3.1) can also be written as the maximization of the profit function py on the generally nonconvex and possibly disconnected set $Y_j(x)$ but we maintain a representation with discrete choice because this permits to eliminate the discontinuity at aggregate level in a constructive way.

Assumption 3.1. *(Transformation) Every firm x with technology j has transformation functions* $H_j^h : R^n \times X \to R$, $H_j^h(y_j^h, x)$, *and every such function satisfies the following properties: (i)* *it is continuous and strictly quasiconvex in* y_j^h*, measurable in* x*; (ii) for each j it has possibility of inaction* $H_j^h(0,x) \le 0$ *for some* h *; (iii)* $\sup_{y_j^h}\{\|y_j^h\| : H_j^h(y_j^h,x) \le 0\} \le \bar{\gamma}_j^h(x) \in L_2(X)$ all *j,h.* ◊

Measurability in (i) is a far weaker requirement than continuity and permits to accommodate abrupt changes in technological conditions over the space over characteristics. Condition (iii) generates a scalar $\bar{\gamma}_j^h(x)$, which is the upper bound on feasible output. Now we can re-define the profit functions in the following way:

$$\pi_j(p,x) = max_h\, \pi_j^h(p,x) \tag{3.2}$$

for

$$\pi_j^h(p,x) = max_{y_j^h}\{ py_j^h \mid H_j^h(y_j^h,x) \le 0\}. \tag{3.3}$$

We remark that if we replace the technology constraint $H_j^h(y_j^h,x) \le 0$ by the full set production constraints and balances, it becomes possible to calculate the price $p_j^h(x)$ at location *x*, as a shadow price to the program. The following assumption is the key step to ensure that the aggregate net supply is G-a.s. a continuously differentiable function.

Assumption 3.2.*(Regularity) For any positive p and fixed h≠h′:* $G(x \mid \pi_j^h(p,x) = \pi_j^{h'}(p,x) = \pi_j(p,x)\} = 0$. ◊

For a two-dimensional vector *x*, this assumption means that the boundaries between regions choosing different technologies are lines of zero surface. This illustrates how the optimization model can be used to generate a zoning map $x) = \arg\max_h \quad (p,x)$, defined so as to maximize $\pi_j^h(p,x)$, the value of land. Clearly, it is possible to impose legal restrictions on this zoning, expressed as the index set , say, to keep land under natural vegetation. The model to be presented can be used to analyze both the direct effect of such restrictions, and the indirect effect via the adjustment of prices.

Assumption 3.2 is satisfied if for all pairs h,h', the difference between the profit functions are nonstationary with respect to one characteristic, say, x_1 whenever h and h' are both maximal in (3.3) (see Lemma 1). It is mild since it only requires that two competing best technologies should not lead to profits that coincide within any sub-region, while the underlying best supplies do not. The requirement can be considered constructive as it can always be enforced by including in the transformation function of Assumption 3.1, an additional perturbation that differentiates between h and h'. For this, we can define a nonnegative perturbation function $\varepsilon_j^h(x_1)$ that is measurable and nonstationary in x_1 and enters as: $H_j^h(y_j^h - \varepsilon_j^h(x_1), x) \le 0$.

Proposition 1 establishes continuous differentiability of the aggregate profit function, and hence single-valuedness and continuity of aggregate net supply.

Proposition 1 *(Aggregate net supply): Let the distribution of characteristics and the transformation function satisfy assumptions 2.1, 2.2 and 3.1, 3.2. Then, the aggregate profit* $\Pi_j(p) = \int_X \pi_j(p,x)\, dG(x)$, *where* $\pi_j(p,x) = \max_h \pi_j^h(p,x)$, *of firms in group j is continuously differentiable, convex, nonnegative and homogeneous of degree one in p and the aggregate net supply mapping* $Y_j(p) = \partial \Pi_j(p)/\partial p$ *is continuous and homogeneous of degree zero in p;* $Y_j(p) = \int_X \sum_h \delta_j^h(p,x) y_j^h(p,x) dG(x)$, *where* $\delta_j^h(p,x), y_j^h(p,x)$ *solve problem (3.1).*

Proof. The proof proceeds in three steps.

(1) The profit function $\pi_j^h(p,x)$ *is continuously differentiable and convex in p.* This follows from the Maximum Theorem (see for instance, Varian, 1992) and assumption 3.1(i)); $\partial \pi_j^h(p,x) \big/ \partial p = y_j^h(p,x)$, where $y_j^h(p,x)$ is a solution of (3.3), that is measurable due to assumption 3.1(i). Function $\pi_j^h(p,x)$ is measurable in x as the optimal value of the optimization problem whose feasible is set measurable in x (assumption 3.1(i), see Castaing and Valadier(1977)).

(2) The profit $\pi_j(p,x) = max_h \pi_j^h(p,x)$ *of firms in group j is, almost everywhere w.r.t. G(x), continuously differentiable, and convex in p, it is nonnegative and homogeneous of degree one;* by assumption 3.1(i) and 3.2, the function is continuously differentiable in *p* almost everywhere in *x;* convexity in *p* follows from (1)*;* homogeneity follows from the definition of problem (3.1); non-negativity from the possibility of inaction 3.1(ii); the "almost everywhere" property follows from the regularity assumption (see lemmas 1-3). By the rule of subdifferentiation of maximum function (see for instance, Rockafellar, 1973):

$$\partial \pi_j(p,x) = co\{\partial \pi_j^h(p,x) \big/ \partial p \mid h \in arg\,max_h\, \pi_j^h(p,x)\}$$
$$= co\{ y_j^h(p,x) \mid h \in arg\,max_h\, \pi_j^h(p,x),$$

where $co\{\cdot\}$ denotes the convex hull. Since all $\pi_j^h(p,x)$ are measurable, $\pi_j(p,x)$ inherits this property, and multifunction $\partial \pi_j(p,x)$ is also measurable (see Castaing and Valadier(1977)).

(3) By assumption 3.1(iii), profit function $\pi_j(p,x)$ and subdifferential $\partial \pi_j(p,x)$ are bounded by integrable functions, so $\Pi_j(p) = \int_X \pi_j(p,x)\, dG(x)$ is well defined and by Lemma 4

$$\partial \Pi_j(p) = \int_X \partial \pi_j(p,x)\, dG(x) = \int_X co\{ y_j^h(p,x) \mid h \in arg\,max_h\, \pi_j^h(p,x)\} dG(x) .$$

The subdifferential $\partial\pi_j(p,x)$ is G-a.s. single valued, hence subdifferential $\partial\Pi_j(p)$ is single valued and continuous, and thus $\Pi_j(p)$ is a convex, continuously differentiable function. Choose a measurable function $\bar{h}(p,x) \in \arg\max_h \pi_j^h(p,x)$ and define

$$\delta_j^h(p,x) = \begin{cases} 1, & h = h(p,x), \\ 0, & h \neq h(p,x), \end{cases} \text{ all } h, \qquad Y_j(p) = \int_X \sum_h \delta_j^h(p,x) y_j^h(p,x) dG(x).$$

Remark that since $\sum_h \delta_j^h(p,x) y_j^h(p,x) \in co\{ y_j^h(p,x) \mid h \in \arg\max_h \pi_j^h(p,x)\}$, it follows that $Y_j(p) \subseteq \partial\Pi_j(p)$, and because of single valuedness of $\partial\Pi_j(p)$, we finally obtain $Y_j(p) = \partial\Pi_j(p) = \partial\Pi_j(p)/\partial p$. □

4. Consumer behavior

As indicated in the introduction, the representation of consumers by means of a continuum offers two major advantages. It allows to include detailed empirical distributions of consumer characteristics, as obtained through geo-referenced household surveys and censuses, and it permits to deal with discrete choices, which seems important since consumers in general buy goods in discrete quantities, such as one car, two pairs of shoes, and they face discrete personal choices as to which town, province or country they want to live in, the job they will apply to, and so on.

To describe consumer behavior, we distinguish r consumer groups, indexed i, coinciding with one of the discrete characteristics, say, $i = x_m$ of the households in the survey. To represent discrete choices, we introduce the option for the consumer to migrate to alternative destinations, indexed s, each with a specific utility function and income. The destination might be a physical location, a specific career or a lifestyle. Migration might be highly temporary and only reflect a shopping visit to the city, or permanent. Like for producers, we assume that all consumer purchases take place at the marketplace and that transportation appear as a separate commodity demand.

Assumption 4.1 *(Endowments): Each consumer x of group i owns fixed endowments* $e_i^s(x) \in R^n$ *after choosing destination s, such that (i)* $e_i^s(x) \in L_2(X)$; *(ii)* $e_I^s(x) \geq e_I(x)$ *and* $\int_X e_I(x) dG(x) > 0$; *(iii) for every x there exists a destination s(x) such that* $e_i^{s(x)}(x) \geq 0$, *with at least one strict inequality.* ◊

Note that this specification can also be used to describe purchases of indivisible commodities and that setup costs of migration could be treated in this way.

Assumption 4.2 *(Utility): Each consumer x of group i has, for every s, a utility function* $u_i^s : R^n \times X \to R$, *such that it is (i) Borel in (c,x), for some* $\bar{c}_i^s(x) \in L_2(X)$ $u_i^s(\bar{c}_i^s(x), x) \in L_1(X)$, *(ii) continuously differentiable with respect to consumption vector* $c \in R^n_{++}$ *G-a.s. in x, (iii) strictly concave in* $c \in R^n_{++}$ *G-a.s. in x; (iv)* $u_i^s(0,x) = 0$; *and (v) G-a.s. in x,* $\partial u_i^s(c,x)/\partial c_k \geq 0$ *for* $c \leq \bar{c}_i^s(x)$ *with at least one strict inequality and*

$\partial u_i^s(c,x)/\partial c_k < 0$ *whenever* $c_k > \bar{c}_{ik}^s(x)$ *; (vii) for* $i = 1$, $u_i^s(c,x) = \tilde{u}_i(c)$ *is increasing in c* with $\partial u_1(c)/\partial c_k \to +\infty$ for $c_k \downarrow 0$; *(vii) for* $i = 1$, $\bar{c}_{1,k} \geq \int_X \left(\bar{y}(x) + \sum_i max_s e_{i,k}^s(x) \right) dG(x)$.
◊

The Borel measurability requirement in (i) is weaker than a continuity requirement in (c,x). We do not impose continuity with respect to *x*, in order to maintain all flexibility with respect to possibly abrupt changes in consumer properties in the space of characteristics.
Condition (v) defines an individual satiation level $\bar{c}_i^s(x) \in L_2(X)$. Utility is non-satiated as long as all consumption falls below this level, but it is non-increasing in any commodity for which consumption exceeds it. This guarantees boundedness (even out of equilibrium) and hence integrability of the demand by any member in state *s*. Assumption (vi) expresses that there is one (possibly very small) consumer group whose utility function is increasing everywhere but does not vary with either the state *s* or the location *x*. The requirement on the derivative guarantees positive consumption of all commodities. Condition (vii) indicates that for this consumer group the satiation level is so high that it exceeds maximal potential supply (see also assumption 3.1(iii)). The integrability of its demand is not an issue because all members are identical. Imposing these relatively tight requirements on group 1 enables us to maintain weak assumptions for all other groups.

Consumer *x* of group *i* owns endowments $e_i^s(x)$ and receives a fixed share $\theta_{ij}(x)$ of the profits of firms in group *j*; hence the identity $\sum_i \int_X \theta_{ij}(x)\, dG(x) = 1$ must hold. Thus, consumer income $r_i^s(p,x)$ consists before transfers of the value of commodity endowments $pe_i^s(x)$ plus profits:

$$r_i^s(p,x) = pe_i^s(x) + \sum_j \theta_{ij}(x) \Pi_j(p) \ . \tag{4.1}$$

Assumption 4.3 *(transfers): Each consumer x of group i receives transfers* $t_i(p,x) \in R$ *, such that (i)* $t_i(p,x) \in L_1(X)$ *; (ii)* $t_i(p,x)$ *is continuous and homogeneous of degree one in p (iii)* $max_s \left(r_i^s(p,x) \right) + t_i(p,x)$ *is positive for all p; and (iv)* $\sum_i \int_X t_i(p,x) dG(x) = 0$ *.* ◊

Now the model of consumer x in group *i* reads:

$$u_i^*(p,x) = max_{c^s \geq 0, \kappa^s \in \{0,1\}} \sum_s \kappa^s u_i^s(c^s, x)$$

subject to

$$\sum_s \kappa^s pc^s \leq \sum_s \kappa^s r_i^s(p,x) + t_i(p,x) \tag{4.2}$$

$$\sum_s \kappa^s = 1,$$

for given $t_i(p,x)$.Observe that in view of the satiation assumption 4.2(v), program (4.2) has a bounded solution even for $p_k = 0$, that determines an optimal destination $\kappa_i^{*s}(p,x)$ as well as a (nonnegative) optimal consumption $c_i^{*s}(p,x) \le \bar{c}_i^s(x)$, that is defined as that the solution of (4.2) under the additional restriction: $\kappa_s = 1.$ Because of strict concavity of utility, $c_i^{*s}(p,x)$ is single valued and continuous for all $p \ge 0$.

As in the case of producer problem we can reformulate the consumer problem (4.2) in terms of continuous variables only, but with, in general, piecewise continuous utility functions and non-convex consumption sets, by defining consumption sets

$$C_i^s(p,x) = \{c \in R_+^n \mid pc \le r_i^s(p,x) + t_i(p,x)\}, \quad C_i(p,x) = \cup_s C_i^s(p,x),$$

as well as the index sets $S_i(c,p,x) = \{s \mid c \in C_i^s(p,x)$, and the piecewise continuous utility functions

$$U_i(c,p,x) = max_{s \in S_i(c,p,x)} u_i^s(c,x).$$

Now problem (4.2) is equivalent to the maximization of $U_i(c,p,x)$ over $c \in C_i(p,x)$.

Next, we reformulate problem (4.2) in a more convenient form. For given destination *s*, consumption can be determined from:

$$\begin{aligned} u_i^{*s}(p,x) &= max_{c_i^s \ge 0}\, u_i^s(c_i^s,x) \\ &\quad subject\ to \\ &\quad\quad pc_i^s \le r_i^s(p,x) + t_i(p,x). \end{aligned} \tag{4.3}$$

Note that by nonsatiation assumption (4.2.v), for $s \ne 1$, utility and consumption will at all non-negative prices, be bounded for every *s* and all *x*. and for satiation level $\bar{c}$ high enough, the budget constraint will hold with equality. Now the functions $u_i^*(p,x)$ can also be determined as

$$u_i^*(p,x) = u_i^{*s_i}(p,x) = max_s\, u_i^{*s}(p,x), \tag{4.4}$$

while $c_i^{*s}(p,x) = c_i^{s_i}(p,x)$, $\kappa_i^{*s}(p,x) = 1$ for $s = s_i$ and $c_i^{*s}(p,x) = 0$, $\kappa_i^{*s}(p,x) = 0$ for $s \ne s_i$. As for production, the maximization can be used to generate a zoning map $s(p,x)$ describing the assignments for every location *x*. Likewise, it is possible to impose restrictions $S(x)$ on land use, and to analyze their direct effect as well as their price induced effect. The following assumption, similar to (3.2), ensures that the aggregate consumption is G-a.s. a continuously differentiable function.

Assumption 4.4 *(Regularity) For* $s \ne 1$, $s \ne t$, *and any positive p:*

(i) $G(x \mid u_i^{*s}(p,x) = u_i^{*t}(p,x) = u_i^{*}(p,x)\} = 0.$

Like for assumption (3.2) on profits, this assumption is satisfied if for all pairs that correspond to maximal utility, the difference between both value functions is nonstationary with respect to x_1. The property is easily constructed via a perturbation function $\varepsilon_i^s(x_1)$ that is measurable and nonstationary in x_1 and enters utility as: $u_i^s(c_i^s, x) = \tilde{u}_i^s(c_i^s, x) + \varepsilon_i^s(x_1)$.

Proposition 2 *(Aggregate consumption): Let the distribution of characteristics and the utility, endowment and transfer functions satisfy assumptions 2.1, 2.2. Then, the aggregate consumer demand and the aggregate endowment supply are continuous and homogeneous of degree zero in p.*

***Proof*.** Define $S_i(p,x) = \{s : u_i^{*s}(p,x) = u_i^{*}(p,x)\}$ and

$$C_i^*(p,x) = \{c_i^{*s}(p,x) \mid s \in S_i(p,x)\},\ E_i^*(p,x) = \{e_i^s(x) \mid s \in S_i(p,x)\}.$$

Multivalued mappings $S_i(p,x)$, $C_i^*(p,x)$, $E_i^*(p,x)$ are:

(i) measurable in x for all $p \geq 0$;

(ii) closed valued and even single valued for almost all x;

(iii) homogenous of degree zero in p for almost all x;

(iv) upper semicontinuous in $p \geq 0$ for almost all x;

(v) bounded by a function that is integrable in x.

In conditions (i), (iv), (v) upper semicontinuity is preserved after integration over x. Hence, mappings $C_i^*(p) = \int_X C_i^*(p,x)dG(x)$, $E_i^*(p) = \int_X E_i^*(p,x)dG(x)$ are upper semicontinuous in $p \geq 0$, single-valued by (ii) and thus continuous. Since $S_i(p,x) = S_i(\lambda p,x)$ for any $\lambda > 0$, $C_i^*(p,x) = C_i^*(\lambda p,x)$, $E_i^*(p,x) = E_i^*(\lambda p,x)$ hence $C_i^*(p) = C_i^*(\lambda p)$, $E_i^*(p) = E_i^*(\lambda p)$, mappings $C_i^*(p)$, $E_i^*(p)$ are homogenuous in $p \geq 0$ of degree zero.□

5. Existence of a competitive Equilibrium

The aggregate net supply and demand of consumers and producers in the general equilibrium model (3.1), (4.1) and (4.2) with given transfers are obtained as integral values:

$$C_i^*(p) = \int_X \left(\sum_s \kappa_i^{*s}(p,x)\, c_i^{*s}(p,x)\right) dG(x) \tag{5.1}$$

$$E_i^*(p) = \int_X \left(\sum_s \kappa_i^{*s}(p,x) e_i^s(x)\right) dG(x). \tag{5.2}$$

$$Y_j^*(p) = \int_X \left(\sum_h \delta_j^{*h}(p,x) y_j^{*h}(p,x)\right) dG(x) \tag{5.3}$$

where $\kappa_i^{*s}(p,x), c_i^{*s}(p,x)$ solve consumer problem (4.2) and $\delta_j^{*h}(p,x), y_j^{*h}(p,x)$ solve producer problem (3.1). A competitive equilibrium is characterized by a price vector such that

$$Z^*(p) = 0, \tag{5.4}$$

for

$$Z^*(p) = \sum_i C_i^*(p) - \sum_i E_i^*(p) - \sum_j Y_j^*(p),$$

and

$$p \in P,$$

where P denotes the price simplex $P = \{ p \geq 0, \sum_k p_k = 1 \}$.

Proposition 3 *(Competitive equilibrium): Let the distribution of characteristics and the utility, endowment and transfer functions satisfy assumptions 2.1, 2.2 and 4.1-4.4, and the transformation functions satisfy assumptions 3.1, 3.2, then model (5.1)-(5.4) has an equilibrium, with positive prices.*

Proof. By propositions 1 and 2 for all $p \in P$, excess demand $Z^*(p)$ is continuous and homogeneous of degree zero in prices. And by assumption 4.3, equation (4.1) and nonnegativity of profit in proposition 1, it satisfies Walras Law ($pZ^*(p) = 0$ for all $p \in P$). Furthermore, by assumptions 4.1(ii) and 4.2(v), consumers $i = 1$ have positive income at all prices and demand more than can be supplied if any price drops to zero. Then, by standard arguments (see, e.g. Arrow and Hahn, 1971, chapter 1) there exists an equilibrium, and price must be positive since, by assumption 4.2 (vii) excess demand could not be nonnegative otherwise.□

6. Spatial welfare optimum: a dual approach

While the spatial competitive equilibrium determines prices for a specified transfer function, through the spatial welfare optimum to be considered in this section the transfers are determined on the basis of a welfare program with given positive weights $\alpha_i(x)$ on the various consumers. This welfare program maximizes the weighted sum over groups *i* of the integral over *x* of individual utilities multiplied by the destination factor κ_i^s and summed over *s*. The resulting welfare program is hard to handle numerically in a straightforward manner, because it is defined in a functional space. Therefore, we propose to formulate the equivalent dual welfare program, that essentially replaces the budget constraint from the model of the previous section by a fixed welfare weight, from which the transfers and the solution of the original program follow.

Thus, for given positive marginal utility of expenditure $\mu_i(x) = 1/\alpha_i(x)$, i.e. equal to the inverse welfare weight we can maximize the surplus of consumer (i, x):

$$w_i^{o}(p,x) = \max_{c^s \geq 0, \kappa^s \in \{0,1\}} \sum_s \kappa^s [u_i^s(c^s, x) - \mu_i(x)(pc^s - r_i^s(p,x))] \tag{6.1}$$

$$\text{subject to}$$

$$\sum_s \kappa^s = 1,$$

with optimal surplus $w_i^{o}(p,x)$, consumption $c_i^{os}(p,x)$ and switches $\kappa_i^{os}(p,x)$. By construction, $\kappa_i^{os}(p,x)=0$ for all s except some s_i, $\kappa_i^{os_i}(p,x)=1$. By assumptions (4.1)-(4.3), this problem has a bounded solution. Program (6.1) defines the (i,x)-specific subproblem:

$$w_i^{os}(p,x) = max_{c\geq 0}\{u_i^s(c,x) - \mu_i(x)(pc - r_i^s(p,x))\} \ . \tag{6.2}$$

By assumptions 4.2(v),(vi), $w_i^{os}(p,x)$, are well defined for all $p \geq 0$, since satiation ensures that consumption will not exceed $\bar{c}_i^s(x)$, this program attains its optimum. The same applies to function

$$w_i^{o}(p,x) = max_s \ w_i^{os}(p,x) = w_i^{os_i}(p,x),$$

that, by (6.2), is equal to the sum of the consumer surplus $(u_i^{s_i}(c_i^{os_i}(p,x),x) - \mu_i(x)pc_i^{os_i}(p,x))$ and the producer surplus $r_i^{s_i}(p,x)$, multiplied by $\mu_i(x)$, where $s_i = s_i(p,x)$ is the optimal state for member x of group i and $c_i^{os_i}(p,x)$ denotes the optimal consumption in this state. This value can be interpreted as the "self-earned" utility since for $\mu_i(x)$ such that the revenue balance with expenditure, it coincides with individual utility. The associated regularity assumption is:

Assumption 6.1 *(Regularity): For* $i \neq 1$ *and any positive* p*:*

$$G\{x \mid w_i^{os}(p,x) = w_i^{ot}(p,x) = w_i^{o}(p,x); s \neq t\} = 0.$$

The assumption on consumer surplus can be enforced constructively in the same way as for assumption 4.4 for utility itself. For every *i*, we can now define associated income transfers:

$$t_i^{o}(p,x) = \textstyle\sum_s \kappa_i^{os}(p,x)(pc_i^{os}(p,x) - r_i^s(p,x)). \tag{6.3}$$

The following proposition establishes that the Second Theorem of welfare economics also applies in our case. For $\mu_i(x) = 1/\alpha_i(x)$, the dual social welfare function can be defined as

$$W(p) = \int_X W^{o}(p,x)dG(x) \tag{6.4}$$

for

$$W^{o}(p,x) = \textstyle\sum_i \alpha_i(x) w_i^{o}(p,x), \tag{6.5}$$

where $\alpha_i(x)$ satisfies:

Assumption 6.2 *(Welfare weight normalization and nonnegligibility of consumer 1)*

(i) $\alpha_i(x)$ are nonnegative integrable functions and $\int_X \sum_i \alpha_i(x)dG(x) = 1$;
(ii) $\alpha_1(x) \equiv \alpha_1 > 0$.◊

By assumptions 4.1(i), 4.2(v),(vi), and 6.2, functions $\alpha_i(x)w_i^o(p,x)$ are integrable and hence function $W(p)$ is well defined for $p \geq 0$.

Proposition 4 *(Equilibrium with transfers): Let the distribution of characteristics and the utility, endowment and transfer functions satisfy assumptions 2.1, 2.2 and 4.1-4.4, 6.1, and the transformation functions satisfy assumptions 3.1, 3.2, while welfare weights satisfy assumption 6.2, then the solution of*

$$min_{p\geq 0} W(p) \tag{6.6}$$

defined as in (6.4)-(6.5) supports a competitive equilibrium (5.4) with transfers (6.3), with unique and positive optimal prices.

Proof. Function $W(p)$ is convex because $w_i^{os}(p,x)$ is convex in *p* (see e.g. Avriel, 1976, Theorem 5.1). Since by assumption 4.2(iv), we have $u_1^s(0,x) = 0$, it follows that $w_1^{os}(p,x) \geq pe_1^s(x)$ and $\alpha_1 w_1^o(p,x) \geq p\alpha_1 \sum_s \kappa_1^{os}(p,x)e_1^s(x)$. By assumption 4.1(ii)

$$\int_X \alpha_1 w_1^o(p,x)dG(x) \geq p\alpha_1 \int_X e_1(x)dG(x) \to +\infty \quad \text{if } p \to +\infty. \tag{a}$$

By assumptions 3.1(ii), 4.2(iv) all $w_i^o(p,x) \geq 0$, hence since by assumption 6.6 consumer 1 is nonnegligible, it follows from (6.4) and (a) that the convex function satisfies $W(p) \geq 0$ and

$$W(p) \to +\infty \quad \text{if} \quad p \to \infty.$$

Hence, $W(p)$ achieves its minimum. Thus, we have the following representation:

$$\begin{aligned}
W(p) &= \sum_i \int_X max_s [\alpha_i(x)u_i^s(c_i^{os}(p,x),x) - (pc_i^{os}(p,x) - r_i^{os}(p,x))]dG(x) \\
&= \sum_i \int_X max_s [\alpha_i(x)u_i^s(c_i^{os}(p,x),x) - (pc_i^{os}(p,x) - pe_i^s(x) - \sum_j \theta_{ij}(x)\Pi_j(p)]dG(x) \\
&= \sum_i \int_X max_s [\alpha_i(x)u_i^s(c_i^{os}(p,x),x) - p(c_i^{os}(p,x) - e_i^s(x))]dG(x) + \sum_j \Pi_j(p) \\
&= \sum_i \int_X max_s [max_{c\geq 0}(\alpha_i(x)u_i^s(c,x) - p(c - e_i^s(x))]dG(x) + \sum_j \Pi_j(p).
\end{aligned}$$

Now by regularity assumption 6.1, function $W(p)$ is differentiable for $p \geq 0$, and by Lemma 4 and Proposition 1:

$$\partial W/\partial p = -Z^o(p) = -(\sum_i C_i^o(p) - \sum_i E_i^o(p) - \sum_j Y_j(p)).$$

A stationary point p^* of the convex function $W(p)$, clears the market, i.e. corresponds to nonnegative excess demand:

$$\partial W/\partial p = -Z^{o}(p^*) \geq 0 \ , \qquad p^* Z^{o}(p^*) = 0 \, . \tag{b}$$

Now, by assumption 4.2(vii), consumer $i = 1$ has demand strictly below satiation level and by 4.2(vi), positive consumption. By (6.2) $\partial u_1 / \partial c_1 = \mu_1 p^*$, and hence $p^* > 0$ and $Z^{o}(p^*) = 0$. Moreover, because of strict concavity of utility, differentiability and the boundary property 4.2(vi) of the derivative, $w_1^{os}(p,x)$ is strictly convex (it is a Legendre transformation, see Avriel, 1976, p. 109), and since by assumption 6.6 consumer $i = 1$ has positive measure, and $w_i^{os}(p,x)$ is convex for $i \neq 1$, this property carries over to $W(p)$. Hence, p^* is unique.

Part 2. Equilibrium with transfers. We show that for $\mu_i(x) > 0$ a stationary point of $W(p)$ is equivalent to a competitive equilibrium (4.2) with transfers (6.3) , where $\mu_i(x)$ is the value of the Lagrange multiplier associated to the budget constraint of the individual consumer problem. Let $\{\kappa_i^{os}(p,x), c_i^{os}(p,x)\}$ be a solution of (6.1), i.e. for all $c^s \geq 0$, $\kappa^s \in \{0,1\}$, $\sum_s \kappa^s = 1$, we have

$$\sum_s \kappa^s (u_i^s(c^s,x) - \mu_i(x)(pc^s - r_i^s(p,x)) \leq$$
$$\leq \sum_s \kappa_i^{os}(p,x)(u_i^s(c_i^{os}(p,x),x) - \mu_i(x)(pc_i^{os}(p,x) - r_i^s(p,x)) , \tag{c}$$

Then,

$$\sum_s \kappa^s u_i^s(c^s,x) \leq \sum_s \kappa_i^{os}(p,x) u_i^s(c_i^{os}(p,x),x)$$

for all $c^s \geq 0$, $\kappa^s \in \{0,1\}$, $\sum_s \kappa^s = 1$, and such that

$$\sum_s \kappa^s pc^s \leq \sum_s \kappa^s r_i^s(p,x) + t_i(p,x) , \tag{d}$$

where $t_i(p,x)$ are defined by (6.3). Since $\{\kappa_i^{os}(p,x), c_i^{os}(p,x)\}$ satisfies (d). Hence, it also provides a solution to (4.2) with transfers (6.3) (implicitly dependent on $\mu_i(x)$). Conversely, for given $\mu_i(x)$, solution $\{\kappa_i^{os}(p,x), c_i^{os}(p,x)\}$ and transfers (6.3) inequality (c) can be rewritten in the form

$$\sum_s \kappa^s (u_i^s(c^s,x) - \mu_i(x) \sum_s \kappa^s (pc^s - r_i^s(p,x) - t_i(p,x)) \leq \sum_s \kappa_i^{os}(p,x) u_i^s(c_i^{os}(p,x),x) ,$$

i.e. $\mu_i(x)$ is a Lagrange multiplier for a budget constraint in (4.2). The same applies to producer decisions. Obviously, transfers sum to zero. □

This proposition shows that the minimum of *W(p)* uniquely defines a competitive equilibrium with transfers. Such a competitive equilibrium is known to be Pareto efficient in terms of the aggregate utility of every group *i,* and more generally of any group of consumers with positive measure: no group could achieve higher utility without any group being worse off. Yet, Pareto efficiency cannot be established, and in fact becomes a meaningless concept for the atomless consumer, since it always is possible to improve utility for any group that has zero measure.

We observe that in conditions of proposition 4, since commodity balances hold and solutions of (6.1) are specialized in the optimum, despite of nonconvexities, (6.2)-(6.5) imply that dual welfare is equal to primal (social) welfare:

$$W(p^*) = \sum_i \int_X \alpha_i(x) u_i^{s_i(x)}(c_i^{s_i(x)}(p^*,x)dG(x) =$$
$$= max_{c_i^s(\cdot),k_i^s(\cdot),y_j^h(\cdot),\delta_j^h(\cdot)} \sum_{i,s} \int_X \alpha_i(x) k_i^s(x) u_i^s(c_i^s(x),x)dG(x), \qquad (6.7)$$

where the maximum is taken over measurable functions $c_i^s(x) \geq 0, k_i^s(x) \in \{0,1\}$, all i,s, and $y_j^h(x), \delta_j^h(x) \in \{0,1\}$, all j,h, subject to constraints

$$\sum_{i,s} k_i^s(x) c_i^s(x) \leq \sum_{i,s} k_i^s(x) e_i^s(x) + \sum_{i,s} \delta_j^h(x) y_j^h(x);$$
$$\sum_h \delta_j^h(x) H_j^h(y_j^h(x),x) \leq 0, \text{ all } j;$$
$$\sum_s k_i^s(x) = 1, \text{ all } i; \qquad \sum_h \delta_j^h(x) = 1, \text{ all } j.$$

We also note that in this model the welfare weights define price normalization, so that there is no scope for further normalization on the simplex, and $Z^{o}(p)$ is not homogeneous of degree zero in prices, unlike $Z^*(p)$.

Clearly, use of the welfare program obviously has the disadvantage that it does not impose restrictions on transfers. Yet, we note that trade balances at fixed prices can be incorporated within the technology set, as a technique to transform imports into exports.

Finally, the minimization of $W(p)$ as it is seen from (6.4), (6.5) belongs to the class of so-called stochastic minimax problems (see Ermoliev, 1988). In the following section we shall use this fact to develop a stochastic tâtonnement procedure for searching equilibrium.

7. Deterministic versus stochastic welfare tatonnement

If it was easy to evaluate excess demand, a deterministic price adjustment procedure could be used to compute equilibrium prices. Specifically, Arrow and Hurwicz (1958) have proved that if the excess demand $Z(p)$ satisfies the Weak Axiom of Revealed Preference (WARP)

$$p^* Z(p) > 0 \text{ for all } p^* \in P \text{ and } p \in P \text{ such that } Z(p^*) = 0,\ Z(p) \neq 0, \qquad (7.1)$$

then Walrasian tatonnement can be used. For excess demand as defined by the general equilibrium model of proposition 3, and prices on the simplex $P = \{p \geq 0 \mid \sum_k p_k = 1\}$, the property can only be proved to hold in very special cases. But any excess demand $Z^{o}(p)$ in (6.7) associated to a welfare optimum satisfies this condition, with price on a compact set

$P = \{0 \le p \le \overline{p}\}$. There is in this case no scope for normalization on the simplex, since price normalization already follows from the welfare weights. Starting from given $p(1) = p_1$, one could specify the algorithm:

$$p(t+1) = Proj_P[\,p(t) + \rho_t Z^{o}(p(t))],\ t = 1,2,\ldots \tag{7.2}$$

where $Proj_P$ is the projection operator on P and step-size multipliers ρ_t are sufficiently small.[1] Yet the difficulty in applying this tâtonnement rule to our model is that, due to the integrals, computation of excess demand becomes very hard and necessarily inaccurate. In fact, the procedure presupposes that there is a central planner who is able to compute aggregate excess demand without error, and hence has to possess all information about all points x. Suppose on the contrary that we possess at every iteration t, a statistical estimate of $Z^{o}(p(t))$, then one might expect that, if this estimate is asymptotically unbiased, the iteration process will eventually converge to an equilibrium. The proposed stochastic Walrasian tâtonnement process builds on this idea. The key observation is that in (6.1) aggregate excess demand is the expected value of the total net demand $z^{o}(p,x)$ of all consumers with characteristic *x*, if we treat *G(x)* as distribution of random events. The stochastic tâtonnement process uses a sequence of independent random drawings *x(t)* from the distribution *G(x)*,[2] and starting from a given $p(1) = p_1 \in P$ adjusts *p(t)* according to:

$$p(t+1) = Proj_P[\,p(t) + \rho_t z^{o}(p(t),x(t))]\ ,\ t = 1,2,\ \ldots \tag{7.3}$$

for

$$z^{o}(p(t),x) = \sum_{i,s}\kappa_i^{os}(p,x)\,c_i^{os}(p,x) - \sum_{i,s}\kappa_i^{os}(p,x)\,e_i^{s}(x) - \sum_{j,h}\delta_j^{h}(p,x)\,y_j^{h}(p,x).$$

We remark that the evaluation of excess demand and the associated price adjustment are only required for commodities and marketplaces where the agents located at *x* are active. Indeed, this process converges.

Proposition 5. *(Convergence of stochastic tâtonnement to a welfare equilibrium)*: Let the assumptions of proposition 3 hold. Then for step-sizes ρ_t such that:

$$\rho_t \ge 0, \sum_t \rho_t = \infty, \sum_t \rho_t^2 < \infty, \tag{7.4}$$

process (6.3) converges, with probability 1, to an equilibrium price.

Proof. see Ermoliev et al. (2000), taking into account that $Z^{o}(p)$ satisfies WARP as a subgradient of the convex function $W(p)$.□

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[1] Recently, Brown and Shannon (2000) have formulated sufficient conditions for deterministic tatonnement to converge locally.

[2] See Rubinstein (1981) and Kalos and Whitlock (1986) for random sampling from the given distribution *G(x)*.

In fact, the rule $\rho_t = const./t$ satisfies requirement (7.4). As argued in Ermoliev et al. (2000), in case WARP does not hold, process (7.4) requires additional shocks for convergence. The stochastic Walrasian tâtonnement process adds to the intuitive appeal of the classical tâtonnement the property of full decentralization. In the classical process (7.2) there is an auctioneer who adjusts prices in proportion to the prevailing excess demand whose calculation requires all agents to communicate their net trades. In the stochastic version, at any given point during the iteration process, only a random collection of consumers have to communicate their intentions.

However, this "purely" dual approach has the limitation that, to avoid solving problems (6.7) in functional space, it requires explicit demand $c_i^{\alpha s}(p,x)$ and net supply $y_j^h(p,x)$. In practice only the primal functions will be available for calculations. In other words, $c_i^{\alpha s}(p,x)$, $y_j^h(p,x)$ are solutions of internal problems that require internal iterations and cannot be obtained without errors. Hence, in (7.3) the estimates of the gradient of $W(p)$ in (6.6) are subject to errors, say, error $\varepsilon(t)$ at iteration t. Consequently, at every iteration the Weak Axiom only holds up to a certain accuracy. Fortunately, convergence of (7.3) is ensured, nonetheless, provided $\varepsilon(t) \to 0$, which is a relatively mild requirement, since the change in p tends to zero by construction, making it easier to achieve accuracy. This convergence property is based on the fact that the approximation of $Z^{o}(p(t))$ calculated in this case is the so-called $\varepsilon(t)$-subgradient of $W(p)$.

Furthermore, we note that the strict quasiconvexity requirement of assumption 3.1(i) on the transformation function and the strict concavity requirement of assumption 4.2(iii) on utility could be relaxed into continuous and quasiconvex, and quasiconcave, respectively. This would result in set-valued (uppersemicontinuous, convex valued) correspondences for excess demand, and non-differentiability of the welfare function $W(p)$. Yet, for the welfare programs and the bargaining procedure, the SQG-procedure applies as before as gradients are to be replaced by subdifferentials, from which the procedure can estimate an element. For the competitive equilibrium the same principles apply but the derivation of the excess demand correspondence is more involved.

Finally, we mention some properties of the rate of convergence. The asymptotic rate of convergence of the sequence $\{p(t)\}$ is usually of order $1/t^{\beta}$, $0 < \beta \le 1$ (see, e.g. Polyak (1983)). There is also the following interesting non-asymptotic result by Nemirovski and Yudin (1978), that illustrates the importance of introducing additional averaging to speed up convergence.

Proposition 6. *(rate of convergence). Construct an averaged (Cesàro) sequence of approximations* $(\underline{p}(1) = p(1) \in P)$:

$$\underline{p}(t+1) = (1-\sigma_{t+1})\underline{p}(t) + \sigma_{t+1} p(t+1), \quad t = 1,2,... \tag{7.5}$$

with $\sigma_t = \rho_t / \sum_{\tau=1}^{t} \rho_\tau$ *, then for any optimal* $p^* \in P$ *and for any sequence of steps* $\rho_t \ge 0$ *, the following estimate holds true:*

$$E\left[W(\underline{p}(t))\right]-W(p^*)\le\left(E\left\|p(1)-p^*\right\|^2+C\sum_{\tau=1}^{t}\rho_\tau^2\right)\Big/\left(2\sum_{\tau=1}^{t}\rho_\tau\right)$$

where expectation E is taken over all sequences {p(t)}, generated by (7.5), and $C\ge sup_{p\in P}\int_X\left\|z(p,x)\right\|^2 dG(x)$.

Proof. See Nemirowski and Yudin (1978).□

This proposition shows that the Cesàro sequence $\{\underline{p}(t)\}$ may converge to the equilibrium p^* in the mean (and hence in probability) under weaker assumptions than (7.4) with, for example, $\rho_t = const/\sqrt{t}.$

8. Nonrival demand and infrastructure development

Suppose that there is an input, denoted by the vector q_j^h that can benefit all producers of firm *j* using technology *h*, and also a demand d_i^s that brings utility to all consumers of group i who choose destination *s*. These inputs are used to maintain general facilities like road signs, and more generally information networks, for which users do not compete, unlike the rival goods of the previous sections, such as cars. Recall that prices as well as demand and supply are differentiated by marketplace, and that producers and consumers can choose where they want to buy or sell. Hence, it may be of interest to allow for adjustment in the capacity of the marketing and transportation infrastructure.

Infrastructure facilities usually have both a rival and a non-rival aspect. Information is typically non-rival and a marketplace where it is amply available will tend to be more attractive. As non-rivalry does not create any increasing returns, there is no immediate agglomeration effect around marketplaces. Rather because of decreasing returns, there will be a tendency to spread the non-rival inputs. Yet, the possibility to save on transportation costs may endogenously create a pattern of spatial concentration and agglomeration. In short, various patterns may emerge and economies of scope may be present but the present assumptions do not produce any specific pattern by necessity.

Alternatively, the vector *x* might refer to uncertain, possibly geo-referenced events, with the non-rival demand relating to beginning-of-period investments, before uncertainty is revealed, and the discrete decisions are part of the coping mechanism once *x* is known. As clearing prices are independent of *x,* all uncertainty is realized in the proportions dictated by the distribution *G(x)*, and may thus be interpreted as idiosyncratic. Conversely, incorporating nonrival demand becomes inescapable whenever we intend to include this type of uncertainty.

Finally, there may be additional (separate) variables q_j, d_i common for all h, s and variables q, d common to all j, i, h, s but we disregard them here.

Assume that the nonrival demand enter the utility and endowment functions via separate terms, and similarly for production. Utility and endowment functions are then written $u_i^s(c_i^s, d_i^s, x)$ and $e_i^s(c_i^s, d_i^s, x)$, for $i\ne 1$, but independent of *s* and *x* for $i=1$, while transformations functions read $H_j^h(y_j^h, q_j^h, x)$. The non-rival consumer and input demand d_i^s and q_j^h are independent of *x*. We remark that welfare problem (6.6) remains unchanged if it is modified to hold for given values of non-rival demand *(d,q)*. The main issue is now to avoid non-concavity when prices and nonrival demand are determined simultaneously

$$w_i^{\mathrm{o}}(p,d,q,x) = max_s \; w_i^{\mathrm{o}s}(p,d,q,x) \tag{8.1}$$

and

$$w_i^{\mathrm{o}s}(p,d,q,x) = max_{c_i^s \geq 0} \; u_i^s(c_i^s, d_i^s, x) - \lambda_i(x)(pc_i^s - r_i^s(p,d,q,x)) \tag{8.2}$$

measures the sum of the consumer and the producer surplus attributable to agent *i* in position *s-x.* For this, we define the "aggregate" variables $D_i^s = \kappa_i^s d_i^s$ and $Q_j^h = \delta_j^h q_j^h$ and make the following limit assumption on the transformation functions, and the utility and endowment functions.

Assumption 8.1 (limit property) $lim_{\delta \downarrow 0} \, \delta H_j^h(Y_j^h / \delta, x) = 0$, *G-a.s., for all j,h and* $lim_{\kappa \downarrow 0} \, \kappa u_i^s(C_i^s / \kappa, D_i^s / \kappa, x) = 0$, *and* $lim_{\kappa \downarrow 0} \, \kappa(e_i^s(C_i^s / \kappa, D_i^s / \kappa, x) - e_i^s(0,0,x)) = 0$, $e_i^s(c,d)$ *concave, and* $e_i^s(0,0) \geq 0$, *for some s, G-a.s., for all i,s.*

Assumption 8.1 ensures that the extended functions $\delta_j^h H_j^h(Y_j^h / \delta_j^h, Q_j^h / \delta_j^h, x)$ and $\kappa_i^s u_i^s(C_i^s / \kappa_i^s, D_i^s / \kappa_i^s, x)$, $\kappa_i^s e_i^s(C_i^s / \kappa_i^s, D_i^s / \kappa_i^s, x)$ are jointly concave in $(\delta_j^h, Y_j^h, Q_j^h)$ and $(\kappa_i^s, C_i^s, D_i^s)$, respectively (see Ginsburgh and Keyzer (1997), Theorem A.1.5). We can now verify that the profit

$$\Pi(p,q) = \max_{\delta_j^h \geq 0, Q_j^h \geq 0, Y_j^h} \; p \textstyle\sum_h Y_j^h$$

subject to

$$\textstyle\sum_{\mathrm{h}} \delta_j^h H_j^h(Y_j^h, -Q_j^h, x) \leq 0 \tag{8.3}$$

$$\mathrm{Q}_{\mathrm{j}}^{\mathrm{h}} \leq q_{\mathrm{j}}^{\mathrm{h}}$$

$$\textstyle\sum_{\mathrm{h}} \delta_{\mathrm{j}}^{\mathrm{h}} = 1$$

is continuous, convex in *p* and concave in *q*. Consequently, for the consumer the value function

$$w_i^{o}(p,d,q,x) = \max_{C_i^s \geq 0, D_i^s \geq 0, \kappa_i^s \geq 0} \sum_s \kappa_i^s u_i^s(C_i^s / \kappa_i^s, D_i^s / \kappa_i^s, x) - \lambda_i(x)(pC_i^s - \kappa_i^s pe_i^s(C_i^s / \kappa_i^s, D_i^s / \kappa_i^s, x) + \sum_j \theta_{ij}(x)\Pi(p,q))$$
$$\text{subject to} \quad D_i^s \leq d_i^s, \quad \sum_s \kappa_i^s = 1, \tag{8.4}$$

is continuous and concave in (d,q). Furthermore, if $q_j^h = q_j$ for all j, or $d_i^s = d_i$ for all i, then programs (8.3) and (8.4) will under the regularity assumption G-a.s. yield δ_j^h and κ_i^s equal to either zero or unity (full specialization) and by monotonicity in assumption (4.2) the inequality constraint can be taken to hold with equality in both programs. Yet, in the general case, δ_j^h and κ_i^s can lie inside the unit interval and measure the fraction from j that moves to h and from i that moves to s. Therefore, we can define the welfare function

$$W(p,d,q) = \sum_i \int_X \alpha_i(x) w_i^{o}(p,d,q,x) dG(x) - pT_d d - pT_q q \tag{8.5}$$

where T_d and T_q map d and q to demand vectors of the same dimension as prices p.

***Proposition** 7 (nonrival demand): The saddlepoint of the minimax problem:*

$$W^* = \min_{p \geq 0} \max_{d \geq 0, q \geq 0} W(p,d,q), \tag{8.6}$$

supports a competitive equilibrium with transfers at unique and positive equilibrium prices p^* and unique optimal collective decisions (d^*, q^*).

Proof. Maximum function $W(p) = \max_{d \geq 0, q \geq 0} W(p,d,q) = W(p, d^*(p), q^*(p))$ is convex, hence it has subdifferential

$$\partial W(p) = \int_X [\sum_{i,s}(C_i^{os}(p,d^*(p),q^*(p),x) - E_i^{os}(p,d^*(p),q^*(p),x)) - \sum_{j,h} Y_j^{oh}(p,q^*(p))] dG(x) - T_d d - T_q q \tag{8.7}$$

which is the aggregated excess demand at given prices p and social decisions $d^*(p), q^*(p)$. The remainder of the proof is as for proposition 4. □

Computation

This enables us to develop a version of the stochastic tâtonement with sequential adjustment of prices and decisions p,d,q. This is a stochastic version of the Arrow-Hurwicz algorithm for finding the saddle point (see Arrow et al. (1958), Goldstein (1972) and Bertsekas (1982)). For $b=(d,q)$,

$$p(t+1)=Proj_P[p(t)-\rho_t w_p(p(t),b(t),x(t))], \qquad (8.8)$$
$$b(t+1)=Proj_B[b(t)+\sigma_t w_b(p(t),b(t),x(t)),$$

and its Cesàro averaging:

$$(\bar{p}(t),\bar{b}(t))=\sum_{\tau=1}^{t}\rho_\tau\sigma_\tau(p(\tau),b(\tau))\Big/\sum_{\tau=1}^{t}\rho_\tau\sigma_\tau, \qquad (8.9)$$

where *x(t)* are independently drawn from the distribution *G(x)* and $w_p(p(t),b(t),x(t))$ and $w_b(p(t),b(t),x(t))$ are estimates of the sub-differentials $\partial_p W(p,b)$, $\partial_b W(p,b)$ at $p=p(t)$, $b=b(t)$. The first process is a stochastic version of saddlepoint processes. The convergence of Cesàro averaged process was studied by Nemirowski and Yudin (1978) and Uryas'ev (1990).

Proposition 8 *(convergence):* Let $P\subset R^n$ and $B\subset R^m$ be convex compact sets, $W(p,b)$ be a function convex in p and concave in *b*, and $W^*=W(p^*,b^*)$ its value at a saddle point. Denote by F_t an σ-algebra generated by random variables $\{p(1),...,p(t),b(1),...,b(t)\}$. Assume that almost surely

$$E\{w_p(t)\,|\,F_t\}\in\partial_p W(p(t),b(t)), \qquad E\{w_b(t)\,|\,F_t\}\in\partial_b W(p(t),b(t)),$$
$$E\|w_p(t)\|^2+E\|w_b(t)\|^2\le Const<+\infty,$$

random variables ρ_t, σ_t are F_t-measurable and a.s.

$$\rho_t>0,\sigma_t>0,\ E\rho_t^2\sigma_t^2<+\infty,$$
$$lim_t\frac{\rho_t-\rho_{t-1}}{\rho_t\sigma_t}=0,\ lim_t\frac{\sigma_t-\sigma_{t-1}}{\rho_t\sigma_t}=0,$$
$$\lim_t\rho_t=0,\quad lim_t\sigma_t=0,\ \sum_{t=1}^{+\infty}\rho_t\sigma_t=+\infty.$$

Then, for almost all trajectories $\{p(t),b(t)\}$:

$$\lim_t[\max_{b\in B}W(\bar{p}(t),b)-W^*]=0, \qquad lim_t[W^*-\min_{p\in P}W(p,\bar{b}(t))]=0.$$

Proof. See Uryas'ev (1990). □

This proposition indicates that convergence to an optimum with respect to prices can be established with probability one, if we average in parallel over nonrival decisions b for convergence of prices, and conversely over prices p for convergence of nonrival decisions.

9. Adjusting welfare weights in a bargaining procedure

In the general equilibrium model with given transfers it possible to distribute income according to entitlements. In the welfare maximizing model transfers follow from given welfare weights. Yet it is possible to go a step further and endogenize welfare weight so as to meet distribution rules and to respect initial claims. In this section, we address this issue along the lines of the bargaining literature and construct a model in which a fair distribution of expenditures is determined through bargaining, i.e. by sharing the surplus over given reservation utilities $\underline{u}_i$ according to a specified constraint. In fact this constraint appears to be the dual representation of an underlying social welfare criterion. Such a problem naturally arises in the context of land consolidation when households agree to pool their land resources and consolidate the holdings while insisting that the gains from consolidation should be shared fairly (see e.g. Keyzer and Ermoliev, 1999). At a higher scale, neighboring districts of a country may agree to redesign their boundaries.

Since the welfare weight become decision variables, we assume across x constant marginal utility of income: $\mu_i(x) = 1/\alpha_i$. This means that individuals x in group i have the same welfare weight α_i, implying that they share a common budget constraint even though the income distribution may be uneven within the group. The economic justification for such an assumption could be that x refers to uncertain random events. In this case the budget constraint would equalize expected revenue with expected expenditure under perfect insurability.

We write $W(\alpha, p), W^{o}(\alpha, p, x)$ instead of $W(p), W^{o}(p, x)$ in (6.4), (6.5) to reflect the dependence on welfare weights, and note that these functions are jointly convex in (α, p), nondecreasing, homogeneous of degree one in the first argument and nonincreasing in the second.

To endogenize the welfare weights in a bargaining context, there are basically two options. The first postulates a social welfare function $V(u)$ and maximizes social welfare, subject to all the utility and technological constraints. However, as mentioned earlier, the associated welfare program is hard to handle numerically, because it is defined in functional space. Hence, we pursue the alternative option that uses a dual representation in which bargaining minimizes the undistributed surplus over the reservation utilities, subject to a social constraint that incorporates the specific bargaining concept. For example, under the principles of Kalai-Smorodinski (1975) the constraint is linear and the dual program reads:

$$\min_{\alpha\geq 0,\, p\geq 0}(W(\alpha,p)-\sum_i \alpha_i \underline{u}_i)$$
subject to (9.1)
$$\sum_i \alpha_i \gamma_i \geq 1$$

where $\underline{u}_i$ are given values. Differentiating under the integral (see Lemma 4) and applying the envelope theorem, while keeping $\mu_i(x)=1/\alpha_i(x)$ in (6.2), we obtain:

$$\partial W(\alpha,p)\big/\partial\alpha_i = u_i^{o}(\alpha,p)=\int_X u_i^{o}(\alpha,p,x)\,dG(x), \tag{9.2}$$

which evaluates the utility $u_i^{o}(\alpha,p,x)=u_i^{s_i}(c_i^{os_i}(\alpha,p,x),x))$ of every group *i*. The first-order conditions of (9.1) imply, for multiplier $\lambda\geq 0$:

$$u_i^{o}(\alpha,p)-\underline{u}_i \geq \lambda\gamma_i, \text{ with equality whenever } \alpha_i \text{ is positive,} \tag{9.3}$$

and
$$W(\alpha,p)-\sum_i \alpha_i(\underline{u}_i+\lambda\gamma_i)=0. \tag{9.4}$$

This form of Kalai-Smorodinski bargaining corresponds exactly to a social welfare criterion without any substitutability across groups, i.e. with $V(u)=min_i((u_i-\underline{u}_i)/\gamma_i)$. Clearly, several generalizations are possible to accommodate alternative rules for sharing the surplus, such as Nash (1950), where $V(u)=\prod_i max(u_i-\underline{u}_i,0)$. The dual formulation can actually accommodate any concave linear homogenous $F(\alpha)\geq 1$ but it is not possible to include (variable) prices in this constraint since this would undermine the market clearing at the optimum.

Stochastic bargaining tatonnement

Alternatively, minimization problem (9.1) can be written as the following stochastic minimax problem:

$$min_{\alpha\in L,p\geq 0}\{[\sum_i\int_X max_s\, max_{c\geq 0}(\alpha_i u_i^s(c,x)-pc+pe_i^s(x))dG(x)+$$
$$+\sum_j\int_X max_h\, max_y(py\,|\,H_j^h(y,x)\leq 0)dG(x)]-\sum_i\alpha_i\underline{u}_i\}, \tag{9.5}$$

where $L=\{\alpha\,|\,\sum_i\alpha_i\gamma_i\geq 1,\alpha_i\geq 0\}$. Based on (9.5), a straightforward extension of process (7.3) with welfare weight adjustment can be formulated:

$$\alpha(t+1)=Proj_L[\alpha(t)+\rho_t(\underline{u}-u^{o}(\alpha(t),p(t),x(t)))],$$
$$p(t+1)=Proj_P[p(t)+\rho_t z^{o}(\alpha(t),p(t),x(t))],\ t=1,2,.... \tag{9.6}$$

To evaluate $u^o(\alpha(t), p(t), x(t))$ and $z^o(\alpha(t), p(t), x(t))$ in (9.6) one has to solve, for current $\alpha(t), p(t)$ and sampled $x(t)$, consumer problems (6.2) for every i, and producer problems (3.2), (3.3) for all producers *j*. Instead, to ease calculations, one could select consumers *i* and producers *j* randomly, rather than evaluating the net demand of all of them (see discussion in Ermoliev et al. (2000)). Convergence conditions for (9.6) are similar to those of (7.3).

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