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# First Order Necessary Optimality Conditions for a Class of Infinite Horizon Optimal Control Problems

*Sergei Aseev (aseev@iiasa.ac.at)*
*Arkadii Kryazhimskii (kryazhim@aha.ru)*
*Alexander Tarasyev (tam@imm.uran.ru)*

**Approved by**

# Abstract

In this paper we investigate a class of nonlinear infinite horizon optimal control problems arising in mathematical economics in consideration of economic growth problems and problems of innovations dynamics. First order necessary optimality conditions in a form of the Pontryagin maximum principle are developed together with some extra conditions on the adjoint function and the behaviour of the Hamiltonian at the infinity. These conditions allow us to guarantee in some cases the validity of the standard transversality conditions at the infinity.

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*Arkadii Kryazhimskii (kryazhim@aha.ru)*
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## 1. Statement of the problem

Consider the following optimal control problem $(P)$:

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x) u^i, \qquad u \in U; \tag{1}$$

$$x(0) = x_0; \tag{2}$$

$$J(x, u) = \int_0^\infty e^{-\rho t} \left( \sum_{i=1}^{n} \gamma_i \ln x^i + g(u) \right) dt \to \max, \tag{3}$$

Here $x = (x^1, \ldots, x^n) \in R^n$; $u = (u^1, \ldots, u^m) \in R^m$; $U$ is a convex compact subset of $R^m$; $f_i(x)$, $i = 0, \ldots, m$ are continuously differentiable vector functions; $x_0$ is a fixed initial point with all strictly positive coordinates $x_0^i > 0$, $i = 1, 2, \ldots, n$; $\rho > 0$; $\gamma_i > 0$, $i = 1, \ldots, n$; $g$ is a concave continuous function on $U$. We search for a minimizer of the problem $(P)$ in a class of all measurable vector functions $u : [0.\infty) \to R^m$ which are bounded on each finite time interval $[0, T]$, $\forall T > 0$.

Optimal control problem $(P)$ naturally arises in mathematical economics in consideration of economic growth problems and problems of innovations dynamics [1], [14], [20], [21], [22]. In the present paper we shall not touch upon the economic motivations for consideration of the problem $(P)$. Our main goal here consists in developing of the mathematical tools for investigation of this problem. Namely, in this paper we are concerned mostly in development of the first order necessary optimality conditions for the problem $(P)$.

Note, that the main distinction of the problem $(P)$ from the classical optimal control problem [19] consists in infinity of the time interval on which we consider the behavior of the control system. The important features of this problem incorporate a special type of the integral functional which contains a discounting multiplier $e^{-\rho t}$ and a logariphmic function of the state vector coordinates. Another important feature of the problem (P) consists in the absence of any a priory assumptions concerning the behavior of an optimal trajectory at the infinity.

For the first time, the necessary optimality condition for problems with infinite horizon in a form of the Pontryagin maximum principle were obtained in [19] under additional assumption on the behavior of the optimal trajectory $x_*$ of the form $\lim_{t \to \infty} x_*(t) = x_1$, where $x_1$ is a given point of the state space $R^n$. It has been shown in [19] that under

this additional assumption a minor modification of the standard proof of the Pontryagin maximum principle [19] provides its validity for the problems with infinite horizon. We should note that the reasonings given in [19] are applicable also in the case of the free right end point infinite horizon problems (in particular in the case of the problem $(P)$). But in this case these reasonings provide an incomplete version of the maximum principle without transversality conditions at the infinity.

We remind that in the case of the free right end point optimal control problem on a finite time interval $[0, T]$ the transversality conditions at the right end point have a form

$$\psi^0 = 1, \qquad \psi(T) = 0,$$

where $\psi$ is a solution of the adjoint system from the relations of the Pontryagin maximum principle and $\psi^0$ is a Lagrange multiplier which corresponds to the maximized functional[1] $J$. Due to this circumstance it was natural to expect that in the case of infinite horizon problems the transversality conditions at the infinity should have an analogous form

$$\psi^0 = 1, \qquad \lim_{t \to \infty} \psi(t) = 0. \tag{4}$$

However, as it was first noted in [15] in a general case of infinite horizon optimal control problems "natural" transversality conditions (4) are failed. See [15] for examples of such kind of pathology. Note that the transversality conditions at the infinity plays an important role in the studies of the infinite horizon optimal control problems via the Pontryagin maximum principle. The relations of the maximum principle are incomplete without these conditions and they select in this case too wide set of admissible controls which are suspectable for optimality.

In this paper under some additional assumptions we obtain a new version of the Pontryagin maximum principle for the problem $(P)$, which contains an additional information concerning the adjoint function $\psi$ and the behavior of the Hamiltonian at the infinity. In some cases this additional information allows us to guarantee the validity of the "natural" transversality conditions (4). We should note that earlier in [9] the maximum principle was also obtained together with additional transversality conditions in the case when the control system (1) is linear and some extra assumptions on the discount parameter $\rho$ and other data of the problem are valid.

The main method which we use in the present paper for the investigation of the problem $(P)$ is the method of approximations. We approximate the initial infinite horizon problem (P) by a sequence of classical optimal control problems, each of which is considered on its own fixed finite time interval. This method allows us to obtain the necessary optimality conditions for the problem $(P)$ using the standard limit procedure in the relations of the Pontryagin maximum principle for the approximating problems. Earlier such approximations approach for the derivation of the necessary optimality conditions for the different nonclassical optimal control problems (problems with state constraints, optimal control problems for differential inclusions, nonsmooth optimal control problems) was used in [3], [4], [5], [6], [7], [18]. The review of the approximations methods of this type is given in [8]. Here we note only that using this approach we are not doing any variational analysis of the approximating problems and the necessary optimality conditions for the initial problem (P) are obtained here as a direct consequence of the classical Pontryagin maximum principle [19].

In what follows, we assume that the inequalities for the vectors (matrixes) are understood as carried out for all their coordinates (components).

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[1] In the present paper we assume that optimal control problems are the maximization ones. In the case of the problems of minimization the adjoint variable $\psi^0$ will have an opposite sign.

An admissible pair $u$, $x$ is assumed to be an arbitrary measurable control $u$ which is given on its own finite or infinite time interval and bounded on each finite time interval and satisfies $u(t) \in U$ for almost all $t$, and the corresponding trajectory $x$ of the system (1) satisfying to the initial condition (2). If a pair $u$, $x$ is defined on a finite time interval $[0, T]$ then we shall assume that it is continued to an admissible pair $u$, $x$ defined on the whole time interval $[0, \infty)$ by an arbitrary way.

We shall assume also that the data of the problem $(P)$ satisfy the following assumptions:

**(H1)** $f_0(x) + \sum_{i=1}^m f_i(x)u^i \geq 0 \ \forall x \geq x_0, \ \forall u \in U$;

**(H2)** $\exists C > 0$: $\langle x, f_0(x) + \sum_{i=1}^m f_i(x)u^i \rangle \leq C(1 + \|x\|^2) \ \forall x > x_0, \ \forall u \in U$.

Condition $(H2)$ is a standard boundedness condition of the different existence theorems of the optimal control theory [11], [13]. Due to this conditions and $(H1)$ all admissible trajectories of the control system (1) with initial condition (2) have positive coordinates and defined for all $t \geq 0$. Due to the assumption $(H2)$, and convexity and compactness of the set $U$ the set of all admissible trajectories is a compact set in $C[0, T] \ \forall T > 0$. Further, due to the condition $(H2)$ the integral (3) converges absolutely for any admissible pair $u$, $x$.

It is easy to see that due to the condition $(H2)$ there exists a nonnegative nonincreasing function $\omega : [0, \infty) \to R^1$ such that $\omega(t) \to 0$, as $t \to \infty$, and for any admissible pair $u$, $x$ of the system (1) with initial condition (2) and arbitrary $T > 0$ the following inequality holds:

$$\int_T^\infty e^{-\rho t} |\sum_{i=1}^n \gamma_i \ln x^i(t) + g(u(t))| dt \leq \omega(T). \tag{5}$$

## 2. Construction of approximating problems and auxiliary results

We start from the existence result for the problem $(P)$. Actually, this result is a particular case of the existence theorem 3.6 [10]. Nevertheless we include a simplified proof of this result in the papper for the illustration of our approximation approach and for completeness of the presentation.

**Theorem 1.** *There exists an optimal control $u_*$ in the problem $(P)$.*

**Proof.** Let $\{T_k\}$, $k = 1, 2, \ldots$ be an arbitrary sequence of positive numbers such that $T_k < T_{k+1} \ \forall k$ and $T_k \to \infty$, as $k \to \infty$.

For $k = 1, 2, \ldots$ let us consider now the following sequence of optimal control problems $(Q_k)$ each of which is defined on its own finite time interval $[0, T_k]$:

$$\dot{x} = f_0(x) + \sum_{i=1}^m f_i(x)u^i, \qquad u \in U; \tag{6}$$

$$x(0) = x_0; \tag{7}$$

$$\hat{J}_k(x, u) = \int_0^{T_k} e^{-\rho t} [\sum_{i=1}^n \gamma_i \ln x^i + g(u)] dt \to \max. \tag{8}$$

Here function $g$, vector functions $f_i$, $i = 1, 2, \ldots, m$, set $U$, vector $x_0$ and constants $\rho$, $\gamma_i$, $i = 1, 2, \ldots, n$ are the same as in the initial problem $(P)$. We are searching for a minimizer of the problem $(Q_k)$ in a class of all measurable bounded functions $u : [0, T_k] \to R^m$.

Due to the theorem 9.3.i [11] there exists an optimal control $u_k$ in the problem $(Q_k)$ for all $k = 1, 2, \ldots$. Denote by $x_k$ the trajectory corresponding to $u_k$, $k = 1, 2, \ldots$.

Consider now the sequence of controls $\{u_k\}$, $k = 1, 2, \ldots$ on the time interval $[0, T_1]$. Due to the convexity and compactness of the set $U$ one can choose a subsequence $\{u_{1,k}\}$ of $\{u_k\}$ such that $u_{1,k} \to u_*$ weakly in $L^1[0, T_1]$, as $k \to \infty$ where $u_*$ is an admissible control on the time interval $[0, T_1]$. Note that by the construction each control $u_{1,k}$, $k = 1, 2, \ldots$ is an optimal one in a corresponding problem $(Q_{m(1,k)})$ of the form (6)–(8) for some number $m(1, k) \geq 1$ on the time interval $[0, T_{m(1,k)}]$ where $T_{m(1,k)} \geq T_1$. Assume $x_{1,k}$ is the optimal trajectory corresponding to $u_{1,k}$ on the time interval $[0, T_{m(1,k)}]$, $k = 1, 2, \ldots$, and $x_*$ denotes the trajectory of the system (6) corresponding to control $u_*$ on the time interval $[0, T_1]$ with initial condition (7).

Due to the linearity in respect to control of the system (6) we have $x_{1,k} \rightrightarrows x_*$ on $[0, T_1]$, as $k \to \infty$. Obviously, $\dot{x}_{1,k} \to \dot{x}_*$ weakly in $L^1[0, T_1]$, as $k \to \infty$.

Consider now the sequence $\{u_{1,k}\}$, $k = 1, 2, \ldots$ on the time interval $[0, T_2]$ for $k \geq 2$.

Analogously to the previous case there exists a subsequence $\{u_{2,k}\}$ of the sequence $\{u_{1,k}\}$ such that $\{u_{2,k}\}$ converges weakly in $L^1[0, T_2]$ to an admissible control which is defined on the time interval $[0, T_2]$ and coincide with $u_*$ on $[0, T_1]$. Let us denote the control constructed by this procedure on $[0, T_2]$ again by symbol $u_*$.

By the construction each control $u_{2,k}$, $k = 1, 2, \ldots$ is an optimal one in a corresponding problem $(Q_{m(2,k)})$ on the time interval $[0. T_{m(2,k)}]$, $T_{m(2,k)} \geq T_2$ of the type (6)–(8) for some number $m(2, k) \geq 2$. Let $x_{2,k}$ is the corresponding to $u_{2,k}$ optimal trajectory on the time interval $[0, T_{m(2,k)}]$, $k = 1, 2, \ldots$ and let $x_*$ be the trajectory of the system (6) corresponding to control $u_*$ on the time interval $[0, T_2]$ with the initial condition (7).

Analogously to the previous step we have $x_{2,k} \rightrightarrows x_*$ on $[0, T_2]$, as $k \to \infty$ and $\dot{x}_{2,k} \to \dot{x}_*$ weakly in $L^1[0, T_2]$, as $k \to \infty$.

Repeating this procedure we construct step by step an admissible control $u_*$ on the infinite time interval $[0, \infty)$ and the corresponding trajectory $x_*$. Simultaneously we construct a countable family of controls $\{u_{i,k}\}$, $i = 1, 2, \ldots$, $k = 1, 2, \ldots$ and the corresponding family of trajectories $\{x_{i,k}\}$, $i = 1, 2, \ldots$, $k = 1, 2, \ldots$. Furthermore, for all $i = 1, 2, \ldots$, $k = 1, 2, \ldots$ the control $u_{i,k}$ which is defined by this procedure, is an optimal one in an optimal control problem $(Q_{m(i,k)})$, $m(i, k) \geq i$ on the corresponding time interval $[0, T_{m(i,k)}]$ where $T_{m(i,k)} \geq T_i$, $i = 1, 2, \ldots$ Moreover, for all $i = 1, 2, \ldots$ we have

$$u_{i,k} \to u_* \qquad \text{weakly in} \quad L^1[0, T_i], \quad \text{as} \quad k \to \infty;$$

$$x_{i,k} \rightrightarrows x_*, \qquad \text{on} \quad [0, T_i], \quad \text{as} \quad k \to \infty;$$

$$\dot{x}_{i,k} \to \dot{x}_* \qquad \text{weakly in} \quad L^1[0, T_i], \quad \text{as} \quad k \to \infty.$$

Let us take the diagonal sequence $\{u_{k,k}\}$, $k = 1, 2, \ldots$ from the constructed family $\{u_{i,k}\}$, $i = 1, 2 \ldots$, $k = 1, 2 \ldots$ and denote $v_k = u_{k,k}$, $y_k = x_{k,k}$, and $m(k) = m(k, k)$, $k = 1, 2, \ldots$.

Constructed by this procedure admissible pair $u_*$, $x_*$, and sequences of controls $\{v_k\}$, $k = 1, 2, \ldots$ and corresponding trajectories $\{y_k\}$, $k = 1, 2, \ldots$ satisfy to the following properties:

a) $\forall k = 1, 2, \ldots$ the control $v_k$ is defined on the time interval $[0, T_{m(k)}]$, $m(k) \geq k$ and $v_k$ is an optimal control in the problem $(Q_{m(k)})$ of the form (6)–(8).

b) $\forall i = 1, 2, \ldots$ we have

$$v_k \to u_* \qquad \text{weakly in} \quad L^1[0, T_i], \quad \text{as} \quad k \to \infty;$$

$$y_k \rightrightarrows x_* \qquad \text{on} \quad [0, T_i], \quad \text{as} \quad k \to \infty;$$

$$\dot{y}_k \to \dot{x}_* \qquad \text{weakly in} \quad L^1[0, T_i], \quad \text{as} \quad k \to \infty.$$

Let us prove that the constructed above control $u_*$ is an optimal one in the problem $(P)$.

Assume that the control $u_*$ is not optimal in the problem $(P)$. Then there exist $\epsilon > 0$ and an admissible pair $\tilde{u}, \tilde{x}$ such that

$$J(x_*, u_*) < J(\tilde{x}, \tilde{u}) - \epsilon. \tag{9}$$

Further, due to the the properties of the function $\omega$ there exists $k_1$ such that $\forall T \geq T_{k_1}$ we have

$$\omega(T) < \frac{\epsilon}{4}. \tag{10}$$

Consider now the above constructed sequences $\{v_k\}$, $\{y_k\}$ on the time interval $[0, T_{k_1}]$ for $k \geq k_1$.

On this time interval $[0, T_{k_1}]$ we have

$$v_k \to u_* \qquad \text{weakly in} \quad L^1[0, T_{k_1}], \quad \text{as} \quad k \to \infty;$$

$$y_k \rightrightarrows x_* \qquad \text{on} \quad [0, T_{k_1}], \quad \text{as} \quad k \to \infty;$$

$$\dot{y}_k \to \dot{x}_* \qquad \text{weakly in} \quad L^1[0, T_{k_1}], \quad as \quad k \to \infty.$$

Further, due to the upper semicontinuity of the functional $\hat{J}_{k_1}$ (see theorem 10.8.ii in [11]) there exists $k_2 \geq k_1$ such that $\forall k \geq k_2$ the following inequality holds:

$$\hat{J}_{k_1}(y_k, v_k) \leq \hat{J}_{k_1}(x_*, u_*) + \frac{\epsilon}{4} \tag{11}$$

Consider now the admissible pair $v_{k_2}$, $y_{k_2}$ on the corresponding time interval $[0, T_{m(k_2)}]$. By the construction $v_{k_2}$ is an optimal control in the optimal control problem $(Q_{m(k_2)})$ on the time interval $[0, T_{m(k_2)}]$. Hence, due to (10) and inequality (5) we have

$$\hat{J}_{m(k_2)}(y_{k_2}, v_{k_2}) \geq \int_0^{T_{m(k_2)}} e^{-\rho t} [\sum_{i=1}^n \gamma_i \ln \tilde{x}^i(t) + g(\tilde{u}(t))] dt \geq$$

$$\geq \int_0^\infty e^{-\rho t} [\sum_{i=1}^n \gamma_i \ln \tilde{x}^i(t) + g(\tilde{u}(t))] dt - \frac{1}{4}\epsilon = J(\tilde{x}, \tilde{u}) - \frac{1}{4}\epsilon.$$

Whence due to (10), inequality (5) and (11) we get

$$J(\tilde{x}, \tilde{u}) \leq \hat{J}_{m(k_2)}(y_{k_2}, v_{k_2}) + \frac{1}{4}\epsilon = \int_0^{T_{m(k_1)}} e^{-\rho t} [\sum_{i=1}^n \gamma_i \ln y_{k_2}^i(t) + g(v_{k_2}(t))] dt +$$

$$+ \int_{T_{m(k_1)}}^{T_{m(k_2)}} e^{-\rho t} [\sum_{i=1}^n \gamma_i \ln y_{k_2}^i(t) + g(v_{k_2}(t))] dt + \frac{1}{4}\epsilon \leq \hat{J}_{m(k_1)}(x_*, u_*) + \frac{3}{4}\epsilon \leq J(x_*, u_*) + \epsilon,$$

that contradicts (9). Hence $u_*$ is an optimal control in $(P)$. The theorem 1 is proved.

Now we shall modify the auxiliary problems $(Q_k)$, $k = 1, 2, \ldots$ used in the proof of the theorem 1 by such a way that the corresponding sequence $\{u_k\}$, $k = 1, 2, \ldots$ of their optimal controls will provide an appropriate (strong in $L_2[0, T]$, $\forall T > 0$) approximation of the given optimal control $u_*$ of the problem $(P)$. We need such a strong approximation to derive the desirable necessary optimality conditions for the problem $(P)$.

Assume $u_*$ is an optimal control in the initial problem $(P)$ and $x_*$ is the corresponding optimal trajectory.

For $k = 1, 2, \ldots$ let us fix a continuously differentiable vector function $z_k : [0, \infty) \to R^n$ such that

$$\sup_{t \in [0,\infty)} \|z_k(t)\| \leq \max_{u \in U} \|u\| + 1, \tag{12}$$

$$\int_0^\infty e^{-\rho t} \|z_k(t) - u_*(t)\|^2 dt \leq \frac{1}{k}, \tag{13}$$

$$\sup_{t \in [0,\infty)} \|\dot{z}_k(t)\| \leq \sigma_k < \infty. \tag{14}$$

It is easy to see that such sequence $\{z_k\}$, $k = 1, 2, \ldots$ of continuously differentiable vector functions $z_k$ exists. Without loss of generality we can assume that $\sigma_k \to \infty$, as $k \to \infty$.

Let us take now a sequence of positive numbers $\{T_k\}$, $k = 1, 2, \ldots$ such that $T_k < T_{k+1}$ $\forall k$; $T_k \to \infty$, as $k \to \infty$, and $\forall k = 1, 2, \ldots$ we have

$$\omega(T_k) \leq \frac{1}{k(1 + \sigma_k)}. \tag{15}$$

Consider now the sequence of the following auxiliary optimal control problems $(P_k)$, $k = 1, 2, \ldots$ each of which is defined on its own time interval $[0, T_k]$:

$$\dot{x} = f_0(x) + \sum_{i=1}^m f_i(x) u^i, \qquad u \in U; \tag{16}$$

$$x(0) = x_0; \tag{17}$$

$$J_k(x, u) = \int_0^{T_k} e^{-\rho t} \left[ \sum_{i=1}^n \gamma_i \ln x^i + g(u) - \frac{\|u - z_k(t)\|^2}{1 + \sigma_k} \right] dt \to \max. \tag{18}$$

Here function $g$, vector functions $f_i$, $i = 1, 2, \ldots, m$, set $U$, vector $x_0$ and constants $\rho$, $\gamma_i$, $i = 1, 2, \ldots, n$ are the same as in the initial problem $(P)$. We are searching for a minimizer of the problem (16)–(18) in a class of all measurable bounded functions $u : [0, T_k] \to R^m$.

Due to the theorem 9.3.i [11] there is an optimal control $u_k$ in the problem $(P_k)$ for all $k = 1, 2, \ldots$. Denote by $x_k$ the trajectory corresponding to $u_k$, $k = 1, 2, \ldots$.

As usually in what follows we shall assume that for any $k = 1, 2, \ldots$ the pair $u_k$, $x_k$ is continued by an arbitrary way to an admissible pair $u_k$, $x_k$ on the whole time interval $[0, \infty)$.

**Lemma** $\forall T > 0$ *we have*

$$u_k \to u_* \qquad in \quad L^2[0, T], \quad as \quad k \to \infty.$$

**Proof.** Let $T > 0$ and let us take a number $k_1$ such that $T_{k_1} \geq T$. Obviously, for any $k = 1, 2, \ldots$ we have

$$J_k(x_k, u_k) = \int_0^{T_k} e^{-\rho t} \left[ \sum_{i=1}^n \gamma_i \ln x_k^i(t) + g(u_k(t)) - \frac{\|u_k(t) - z_k(t)\|^2}{1 + \sigma_k} \right] dt \leq$$

$$\leq \int_0^{T_k} e^{-\rho t} \left[ \sum_{i=1}^n \gamma_i \ln x_k^i(t) + g(u_k(t)) \right] dt - \frac{e^{-\rho T}}{1 + \sigma_k} \int_0^T \|u_k(t) - z_k(t)\|^2 dt.$$

Hence, due to the optimality of $u_k$ in the problem $(P_k)$, $k \geq k_1$, optimality of $u_*$ in the problem $(P)$, (5), (13) and (15) we get

$$\frac{e^{-\rho T}}{1 + \sigma_k} \int_0^T \|u_k(t) - z_k(t)\|^2 dt \leq \int_0^{T_k} e^{-\rho t} \left[ \sum_{i=1}^n \gamma_i \ln x_k^i(t) + g(u_k(t)) \right] dt - J_k(x_*, u_*) \leq$$

$$\leq J(x_k, u_k) - J(x_*, u_*) + 2\omega(T_{m(k)}) + \int_0^\infty \frac{e^{-\rho t}}{1 + \sigma_k} \|z_k(t) - u_*(t)\|^2 dt \leq \frac{3}{k(1 + \sigma_k)}.$$

Whence we get

$$\int_0^T \|u_k(t) - z_k(t)\|^2 dt \leq \frac{3e^{\rho T}}{k}.$$

Hence

$$(\int_0^T \|u_k(t) - u_*(t)\|^2 dt)^{\frac{1}{2}} \leq (\int_0^T \|u_*(t) - z_k(t)\|^2 dt)^{\frac{1}{2}} +$$

$$+(\int_0^T \|u_k(t) - z_k(t)\|^2 dt)^{\frac{1}{2}} \leq \sqrt{\frac{e^{\rho T}}{k}} + \sqrt{\frac{3e^{\rho T}}{k}} = (\sqrt{3} + 1)\sqrt{\frac{e^{\rho T}}{k}}.$$

Hence $\forall \epsilon > 0 \; \exists k_2 \geq k_1$ such that $\forall k \geq k_2$ the following condition holds:

$$\|u_k - u_*\|_{L^2[0,T]} dt \leq \epsilon.$$

Hence the assertion of the lemma holds. The lemma is proved.

It follows immeadiately from the assertion of the lemma that without loss of generality we can assume that for arbitrary $T > 0$ we have

$$u_k \to u_* \quad \text{in} \quad L^2[0, T], \quad \text{as} \quad k \to \infty;$$

$$x_k \rightrightarrows x_* \quad \text{on} \quad [0, T], \quad \text{as} \quad k \to \infty;$$

$$\dot{x}_k \to \dot{x}_* \quad \text{in} \quad L^2[0, T], \quad \text{as} \quad k \to \infty.$$

## 3. The main result

In this section we develop a new version of the first order necessary optimality conditions for initial problem (P) using the limit procedure in the relations of the Pontryagin maximum principle for the problem $(P_k)$, as $k \to \infty$.

First let us introduce some standard notations.

Let

$$\mathcal{H}(x, t, u, \psi) = \langle f_0(x), \psi \rangle + \sum_{i=1}^m \langle f_i(x), \psi \rangle u^i + e^{-\rho t}(\sum_{i=1}^n \gamma_i \ln x^i + g(u))$$

and

$$H(x, t, \psi) = \sup_{u \in U} \mathcal{H}(x, t, u, \psi)$$

denote the Hamilton–Pontryagin function and the Hamiltonian (maximum function) respectively for the problem $(P)$ presented in a normal forms (i.e. the Lagrange multiplier $\psi^0$ corresponding to the maximized functional $J(x, u)$ is equal 1).

In what follows we shall assume that the following conditions hold:

**(H3)** There exist vectors $a_0 \in R^n$, $a_0 > 0$ and $u_0 \in U$ such that the following inequality holds:

$$f_0(x_0) + \sum_{i=1}^m f_i(x_0)u_0^i \geq a_0. \tag{19}$$

**(H4)** Along any admissible pair $u$, $x$ of the system (1) with initial condition (2) we have

$$\frac{\partial f_0(x(t))}{\partial x} + \sum_{i=1}^m \frac{\partial f_i(x(t))}{\partial x} u^i(t) \geq 0 \tag{20}$$

for almost all $t \geq 0$.

Our main result is the following.

**Theorem 2 (maximum principle).** *Assume that conditions* $(H1)$–$(H4)$ *are fulfilled, and* $u_*$ *is an optimal control in the problem* $(P)$ *and* $x_*$ *is the corresponding to* $u_*$ *optimal trajectory. Then there exists an absolutely continuous vector function* $\psi : [0, \infty) \to R^n$ *such that the following conditions hold:*

*1) The function* $\psi$ *is a solution to the adjoint system*[2]

$$\dot\psi \overset{a.e}{=} -[\frac{\partial f_0(x_*(t))}{\partial x} + \sum_{i=1}^{m} \frac{\partial f_i(x_*(t))}{\partial x} u_*^i(t)]^* \psi - e^{-\rho t}(\frac{\gamma}{x_*(t)}); \tag{21}$$

*2) For almost all* $t \in [0, \infty)$ *the maximum condition takes place:*

$$\mathcal{H}(x_*(t), t, u_*(t), \psi(t)) = H(x_*(t), t, \psi(t)); \tag{22}$$

*3) The condition of the asymptotic stationarity of the Hamiltonian is valid:*

$$\lim_{t \to \infty} H(x_*(t), t, \psi(t)) = 0; \tag{23}$$

*4) The vector function* $\psi$ *is nonnegative, i.e.*

$$\psi(t) \geq 0 \qquad \forall t \geq 0. \tag{24}$$

**Remark 1.** Note, that the formulated above theorem is a variant of the Pontryagin maximum principle in a normal form. It asserts that a Lagrange multiplier $\psi^0$ corresponding to the maximizing functional is strictly positive and hence may be taken equal 1. Further, this result incorporates some additional conditions (23) and (24), where the stationarity condition (23) is analogous to the transversality condition with respect to time in the formulation of the Pontryagin maximum principle for a free time finite horizon optimal control problem (see [19]).

**Proof.** Let us consider the sequence of auxiliary problems $(P_k)$, $k = 1, 2, \ldots$ constructed above in section 2. Let $u_k$ be an optimal control in the problem $(P_k)$ and let $x_k$ be the corresponding optimal trajectory, $k = 1, 2, \ldots$. As it was shown in section 2 for $i = 1, 2, \ldots$ we have

$$u_k \to u_* \quad \text{in} \quad L^2[0, T_i], \quad \text{as} \quad k \to \infty;$$

$$x_k \rightrightarrows x_* \quad \text{on} \quad [0, T_i], \quad \text{as} \quad k \to \infty;$$

$$\dot x_k \to \dot x_* \quad \text{in} \quad L^2[0, T_i], \quad \text{as} \quad k \to \infty.$$

Due to the Pontryagin maximum principle [19] for the problem $(P_k)$, $k = 1, 2, \ldots$ there exists an absolutely continuous function $\psi_k : [0, T_k] \to R^n$ such that the following conditions hold:

$$\psi_k(t) \overset{a.e.}{=} -[\frac{\partial f_0(x_k(t))}{\partial x} + \sum_{i=1}^{m} \frac{\partial f_i(x_k(t))}{\partial x} u_k^i(t)]^* \psi_k(t) - e^{-\rho t}(\frac{\gamma}{x_k(t)}); \tag{25}$$

$$\mathcal{H}_k(x_k(t), t, u_k(t), \psi(t)) \overset{a.e.}{=} H_k(x_k(t), t, \psi_k(t)); \tag{26}$$

$$\psi_k(T_k) = 0. \tag{27}$$

---

[2]Here and in what follows a symbol $(\frac{\gamma}{x})$ denote the vector $(\frac{\gamma}{x}) = (\frac{\gamma_1}{x^1}, \frac{\gamma_2}{x^2}, \ldots, \frac{\gamma_n}{x^n})$.

Here

$$\mathcal{H}_k(x,t,u,\psi) = \langle f_0(x),\psi\rangle + \sum_{i=1}^{m}\langle f_i(x),\psi\rangle u^i + e^{-\rho t}(\sum_{i=1}^{n}\gamma_i \ln x^i + g(u) - \frac{\|u - z_k(t)\|^2}{1+\sigma_k})$$

and

$$H_k(x,t,\psi) = \sup_{u \in U}\mathcal{H}_k(x,t,u,\psi)$$

are the Hamilton–Pontryagin function and the Hamiltonian (maximum function) for the problem $(P_k)$, $k = 1,2,\ldots$ in a normal form.[3]

We should note that due to relations (25), (26) of the Pontryagin maximum principle for the problem $(P_k)$ the following condition holds for $k = 1,2,\ldots$:

$$\frac{dH_k(x_k(t),t,\psi_k(t))}{dt} \overset{a.e.}{=} \frac{\partial \mathcal{H}_k}{\partial t}(x_k(t),t,u_k(t),\psi_k(t)). \tag{28}$$

Further, due to (25), (26) and (27) we have $\psi_k(t) > 0$ $\forall t \in [0,T_k]$.

Indeed, due to (25) and (27) we have the inequality $\psi_k(t) > 0$ for all $t$ from some small enough left neighborhood of the point $T_k$. Let us show now that

$$\psi_k(t) > 0, \qquad \forall t \in [0,T_k]. \tag{29}$$

Let us assume that there exists $t_* \in [0,T_*)$ such that at least one coordinate of the vector $\psi_k(t_*)$ is equal 0. Let $t_*$ be a maximal such point and let $i_*$ be a coordinate such that $\psi_k^{i_*}(t_*) = 0$. Then

$$\psi_k(t) > 0 \qquad \forall t \in (t_*,T_k) \tag{30}$$

and

$$\psi_k^{i_*}(t) = -\int_{t_*}^{t}\langle\frac{\partial f_0(x_k(s))}{\partial x}e_{i_*},\psi_k(s)\rangle ds-$$

$$-\int_{t_*}^{t}\sum_{i=1}^{n}\langle\frac{\partial f_i(x_k(s))}{\partial x}e_{i_*},\psi_k(s)\rangle ds - \int_{t_*}^{t}e^{-\rho s}\langle(\frac{\gamma}{x_k(s)}),e_{i_*}\rangle ds,$$

where $e_{i_*}$ is a vector with unite coordinate $i_*$ and vanishing all other coordinates. Now this equality and (20) imply inequality $\psi_k^{i_*}(t) \leq 0$ $\forall t \in (t_*,T_k)$ which contradicts to (30).

So, the condition (29) is proved.

Now we show that the sequence $\{\|\psi_k(0)\|\}$, $k = 1,2,\ldots$ is bounded. For this purpose let us integrate the equality (28) on the time interval $[0,T_k]$, $k = 1,2,\ldots$.

Using (28) we get

$$H(x_0,0,\psi_k(0)) = e^{-\rho T_k}[\sum_{i=1}^{n}\gamma_i \ln x_k^i(T_k) + \max_{u \in U}(g(u) - \frac{\|u - z_k(T_k)\|^2}{1+\sigma_k})]+$$

$$+\rho\int_{0}^{T_k}e^{-\rho t}[\sum_{i=1}^{n}\gamma_i \ln x_k^i(t) - \frac{\|u_k(t) - z_k(t)\|^2}{1+\sigma_k}]dt - 2\int_{0}^{T_k}e^{-\rho t}\frac{\langle u_k(t) - z_k(t),\dot{z}_k(t)\rangle}{1+\sigma_k}dt.$$

It is not difficult to see that due to the conditions (12)-(14), boundedness of the set $U$ and condition $(H_2)$ there exists a constant $M > 0$ such that for all $k = 1,2,\ldots$ we have

$$H_k(x_0,0,\psi_k(0)) \leq M.$$

---

[3] The problem $(P_k)$ is a free right end point optimal control problem on the fixed time interval $[0,T_k]$, $k = 1,2,\ldots$. Hence the multiplier $\psi^0$ can be taken equal 1.

From this inequality using (19) we derive

$$\langle a_0, \psi_k(0)\rangle \leq M + \sum_{i=1}^{n} \gamma_i \ln x_0^i + \max{}_{u \in U} g(u).$$

Now the boundedness of the sequence $\{\|\psi_k(0)\|\}$, $k = 1, 2, \ldots$ follows directly from the last inequality, strict positiveness of the vectors $a_0$, $\psi_k(0)$, $k = 1, 2, \ldots$ and boundedness of the set $U$.

Now consider consequently time intervals $[0, T_i]$, $i = 1, 2, \ldots$ and sequences $\{u_k\}$, $\{x_k\}$ and $\{\psi_k\}$ on $[0, T_i]$, as $k \to \infty$.

Due to the Bellman–Gronwall inequality [16], boundedness of the sequence $\{\|\psi_k(0)\|\}$, $k = 1, 2, \ldots$ and (25) we may assume that there exists an absolutely continuous vector function $\psi : [0, T_i] \to R^n$ such that

$$\psi_k \rightrightarrows \psi \qquad \text{on} \quad [0, T_i], \quad \text{as} \quad k \to \infty,$$

and

$$\dot{\psi}_k \to \dot{\psi} \qquad \text{weakly in} \quad L^1[0, T_i], \quad \text{as} \quad k \to \infty.$$

Considering the sequence of increasing time intervals $[0, T_i]$, as $i \to \infty$, and passing to a subsequence of $\{\psi_k\}$, $k = 1, 2, \ldots$ on each of these time intervals, and taking then a diagonal subsequence we can suppose that there exists an absolutely continuous vector function $\psi : [0, \infty) \to R^n$, such that $\forall T > 0$ we have

$$\psi_k \rightrightarrows \psi \qquad \text{on} \quad [0, T], \quad \text{as} \quad k \to \infty,$$

and

$$\dot{\psi}_k \to \dot{\psi} \qquad \text{weakly in} \quad L^1[0, T], \quad \text{as} \quad k \to \infty.$$

Due to the uniform convergence of the sequence $x_k$ to $x_*$, as $k \to \infty$ and convergence of $u_k$ to $u_*$ in $L^2[0, T]$, as $k \to \infty$, passing to a limit in (25) for almost all $t \in [0, T]$, as $k \to \infty$ we get that due to the Mazur theorem [18] the absolutely continuous function $\psi$ is a solution to the adjoint system (21) on time interval $[0, T]$.

Hence the condition (21) is proved.

Due to the positiveness of the functions $\psi_k$, $k = 1, 2 \ldots$ we have $\psi(t) \geq 0 \ \forall t > 0$, i.e. the condition (24) is proved.

Passing to the limit in (26), as $k \to \infty$ we get the maximum condition (22).

Let us prove now the asymptotic stationarity condition (23). To this end let us take an arbitrary $t > 0$ and integrate the equality (28) on the time interval $[t, T_k]$ for large numbers $k$ such that $T_k > t$. Due to the equality (27) we get

$$H_k(x_k(t), t, \psi_k(t)) = e^{-\rho T_k}[\sum_{i=1}^{n} \gamma_i \ln x_k^i(T_k) + \max{}_{u \in U}(g(u) - \frac{\|u - z_k(T_k)\|^2}{1 + \sigma_k})] -$$

$$-\rho \int_t^{T_k} e^{-\rho s}[\sum_{i=1}^{n} \gamma_i \ln x_k^i(s) + g(u_k(s)) - \frac{\|u_k(s) - z_k(s)\|^2}{1 + \sigma_k}]ds +$$

$$+2 \int_t^{T_k} e^{-\rho s} \frac{\langle u_k(s) - z_k(s), \dot{z}_k(s)\rangle}{1 + \sigma_k} ds. \tag{31}$$

Further, passing to the limit in the equality (31), as $k \to \infty$ we have

$$H(x_*(t), t, \psi(t)) = \rho \int_t^{\infty} e^{-\rho s}[\sum_{i=1}^{n} \gamma_i \ln x_*^i(s) + g(u_*(s))]ds. \tag{32}$$

Finally, passing to the limit in the last equality (32), as $t \to \infty$ we get the condition (23). The theorem 2 is proved.

**Remark 2.** It is easy to see that condition (23) immediately implies the following equality:

$$\lim_{t \to \infty} \left( \langle f_0(x_*(t)), \psi(t) \rangle + \sum_{i=1}^{m} \langle f_i(x_*(t)) u_*^i(t), \psi(t) \rangle \right) = 0.$$

**Remark 3.** In the case $n = 1$ the theorem 2 is valid without the assumption (20). Indeed, condition (20) was used in the proof of the theorem 2 only for proving the positiveness of the vector functions $\psi_k$, $k = 1, 2, \ldots$. In the case $n = 1$ the positiveness of the functions $\psi_k$, $k = 1, 2, \ldots$ is an immediate consequence of (25) and (27).

**Corollary 1** *Assume that assumptions $(H1)$–$(H4)$ are fulfilled and an admissible pair $u_*$, $x_*$ satisfy to the conditions $(21)$–$(24)$ of the maximum principle (theorem 2). Moreover, assume that there exists a vector $a_1 \in R^n$, $a_1 > 0$ such that the following inequality takes place:*

$$f_0(x(t)) + \sum_{i=1}^{m} f_i(x(t)) u^i(t) \ge a_1 \tag{33}$$

*along the pair $u_*$, $x_*$. Then the transversality condition at the infinity (4) holds.*

**Proof.** Indeed due to the condition (23) (see remark 2 above) and (33) we have

$$\lim_{t \to \infty} \langle a_1, \psi(t) \rangle \le \lim_{t \to \infty} \left[ \langle f_0(x(t)), \psi(t) \rangle + \sum_{i=1}^{m} \langle f_i(x(t)), \psi(t) \rangle u^i(t) \right] = 0.$$

From these relations due to (24) we have

$$\lim_{t \to \infty} \psi(t) = 0.$$

The corollary is proved.

**Corollary 2** *Assume that assumptions $(H1)$–$(H4)$ are fulfilled and an admissible pair $u_*$, $x_*$ satisfy to the conditions $(21)$–$(24)$ of the maximum principle (theorem 2). Moreover, assume that there exists $n \times n$ matrix $A > 0$ such that the following relation holds:*

$$\frac{\partial f_0(x_*(t))}{\partial x} + \sum_{i=1}^{m} \frac{\partial f_i(x_*(t))}{\partial x} u_*^i(t) \overset{a.e.}{\ge} A \tag{34}$$

*along the pair $u_*$, $x_*$. Then the strengthened transversality condition holds:*

$$\lim_{t \to \infty} \langle x_*(t), \psi(t) \rangle = 0. \tag{35}$$

It is easy to see that due to the positiveness of the vector $x_0$ and $(H_3)$ the relation (35) imply (4).

**Proof.** Indeed, due to the conditions of the maximum principle (theorem 2 ) and (34) we have

$$\frac{d}{dt} \langle x_*(t), \psi(t) \rangle \overset{a.e.}{=} < f_0(x_*(t)), \psi(t) > + \sum_{i=1}^{m} \langle f_i(x_*(t)) u_*^i(t), \psi(t) \rangle -$$

$$-\langle x_*(t), [\frac{\partial f_0(x_*(t))}{\partial x}]^* \psi(t)\rangle - \langle x_*(t), \sum_{i=1}^{n}[\frac{\partial}{\partial x}f_i(x_*(t))]^* u_*^i(t)\psi(t)\rangle - \langle x_*(t), e^{-\rho t}(\frac{\gamma}{x_*(t)})\rangle \overset{a.e}{\leq}$$

$$\overset{a.e}{\leq} -\langle Ax_*(t), \psi(t)\rangle + H(x_*(t), t, \psi(t)) - e^{-\rho t}[\sum_{i=1}^{n}\gamma_i \ln x_*^i(t) + g(u_*(t))] - e^{-\rho t}\sum_{i=1}^{n}\gamma_i.$$

Hence there exist constants $\mu > 0$ such that

$$\frac{d}{dt}\langle x_*(t), \psi(t)\rangle \leq -\mu\langle x_*(t), \psi(t)\rangle + \alpha(t),$$

where $\alpha(t) = H(x_*(t), t, \psi(t)) - e^{-\rho t}[\sum_{i=1}^{n}\gamma_i \ln x_0^i + \min_{u\in U}g(u)] \to 0$, as $t \to \infty$. From the last inequality we have

$$0 \leq \langle x_*(t), \psi(t)\rangle \leq e^{-\mu t}\langle x_0, \psi(0)\rangle + e^{-\mu t}\int_0^t e^{\mu s}\alpha(s)ds. \qquad (36)$$

Further, due to the relation $\frac{d}{dt}H(x_*(t), t, \psi(t)) \overset{a.e.}{=} \frac{\partial}{\partial t}\mathcal{H}(x_*(t), t, u_*(t), \psi(t))$ we have

$$\dot{\alpha}(t) \overset{a.e.}{=} -\rho e^{-\rho t}[\sum_{i=1}^{n}\gamma_i \ln x_*^i(t) + g(u_*(t)] + \rho e^{-\rho t}[\sum_{i=1}^{n}\gamma_i \ln x_0^i + \min_{u\in U}g(u)] \leq 0.$$

Whence, integrating by parts we get

$$\int_0^t e^{\mu s}\alpha(s)ds = \frac{1}{\mu}[e^{\mu t}\alpha(t) - \alpha(0)] + \frac{1}{\mu}\int_0^t e^{\mu s}\dot{\alpha}(s)ds \leq$$

$$\leq \frac{1}{\mu}[e^{\mu t}\alpha(t) - \alpha(0)].$$

Substituting the last estimation in (36) we get

$$0 \leq \langle x_*(t), \psi(t)\rangle \leq e^{-\mu t}\langle x_0, \psi(0)\rangle + e^{-\mu t}[\frac{1}{\mu}[\alpha(t) - \alpha(0)].$$

Hence $\langle x_*(t), \psi(t)\rangle \to 0$, as $t \to \infty$. The corollary is proved.

**Corollary 3** *Let assumptions of the theorem 2 are valid. Then the following equality holds:*

$$J(x_*, u_*) = \frac{1}{\rho}[\langle f_0(x_0), \psi(0)\rangle + \sum_{i=1}^{n}\gamma_i \ln x_0^i + \max_{u\in U}\{\sum_{i=1}^{m}\langle f_i(x_0), \psi(0)\rangle u^i + g(u)\}]. \quad (37)$$

**Proof.** Indeed the conditions of the Pontrygin maximum principle (21), (22) for the problem $(P)$ imply the validity of the equality

$$\frac{d}{dt}H(x_*(t), t, \psi(t)) \overset{a.e}{=} \frac{\partial\mathcal{H}}{\partial t}(x_*(t), t, u_*(t), \psi(t)) = -\rho e^{-\rho t}(\sum_{i=1}^{n}\gamma_i x_*^i(t) + g(u_*(t))).$$

Integrating this equality on the time interval $[0, \infty)$ due to (23) we get

$$H(x_0, 0, \psi(0)) = \rho\int_0^{\infty}e^{-\rho t}(\sum_{i=1}^{n}\gamma_i x_*^i(t) + g(u_*(t)))dt = \rho J(x_*, u_*).$$

So, the equality (37) is valid and the corollary is proved.

**Remark 4.** The equality (37) is related with the Hamilton-Jacobi equation for the problem $(P)$. Indeed, assume that assumptions of the theorem 2 are fulfilled and $w(t_0, x_0)$ is the value function of the following optimal control problem $(P_{(t_0,x_0)})$:

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x)u^i, \qquad u \in U;$$

$$x(t_0) = x_0;$$

$$J_{(t_0,x_0)}(x, u) = \int_{t_0}^{\infty} e^{-\rho t}\Big(\sum_{i=1}^{n} \gamma_i \ln x^i + g(u)\Big)dt \to \max.$$

Here the function $g$, vector functions $f_i$, $i = 1, 2, \ldots, m$, set $U$, constants $\rho$, $\gamma_i$, $i = 1, 2, \ldots, n$ are the same as in the initial problem $(P)$, and $t_0 \geq 0$, $x_0 > 0$ are arbitrary initial time and initial state respectively. The pair $t_0$, $x_0$ is considered in the family of problems $(P_{(t_0,x_0)})$ as a parameter. Obviously the problem $(P_{(0,x_0)})$ coinsides with the initial problem $(P)$.

Let $v(x_0) = w(0, x_0)$ be the stationary projection of the value function $w(t_0, x_0)$. One can easily prove that $w(t_0, x_0) = e^{-\rho t_0} v(x_0)$ (see [12]). Assuming that function $w(t_0, x_0)$ is continuously differentiable and substituting it into the Hamilton-Jacobi equation

$$\frac{\partial w(t_0, x_0)}{\partial t} + \max_{u \in U}\Big\{\Big\langle \frac{\partial w(t_0, x_0)}{\partial x}, f_0(x_0) + \sum_{i=1}^{m} f_i(x_0)u^i\Big\rangle + e^{-\rho t_0}\Big(\sum_{i=1}^{n} \gamma_i \ln x_0^i + g(u)\Big)\Big\} = 0$$

we obtain after contracting $e^{-\rho t_0}$ the stationary Hamilton-Jacobi equation

$$-\rho v(x_0) + \Big\langle \frac{\partial v(x_0)}{\partial x}, f_0(x_0)\Big\rangle + \sum_{i=1}^{n} \gamma_i \ln x_0^i + \max_{u \in U}\Big\{\sum_{i=1}^{m} \Big\langle \frac{\partial v(x_0)}{\partial x}, f_i(x_0)\Big\rangle u^i + g(u)\Big\} = 0.$$
(38)

Taking into account that $v(x_0) = J(x_*, u_*)$ and $\frac{\partial v(x_0)}{\partial x} = \psi(0)$ we come to the conclusion that relation (37) is a generalization of the stationary Hamilton-Jacobi equation (38).

At the end of this section we present a sufficient condition of optimality for the problem $(P)$ in a form of the Pontryagin maximum principle. Note, that results of this type for problems on the finite time interval run back to the paper [17]. In the case of the infinite horizon problems a similar result under other a priory assumptions was obtained in [2].

**Theorem 3.** *Let the assumptions (H1)–(H4) of the theorem 2 are fulfilled and a pair $u_*$, $x_*$ satisfy to the conditions (21)–(24) of the maximum principle (theorem 2) together with the adjoint function $\psi$. Assume also that there exists a matrix $A > 0$ such that the relation (34) holds along the pair $u_*$, $x_*$, and the Hamiltonian $H(x, t, \psi(t))$ is continuously differentiable and concave in $x$ for all $t \in [0, \infty)$. Then the pair $u_*$, $x_*$ is an optimal one in the problem $(P)$.*

**Proof.** Due to the definition of the Hamiltonian $H(x, t, \psi)$ for all $x$ and all $t$ we have

$$\langle f_0(x), \psi(t)\rangle + \sum_{i=1}^{m}\langle f_i(x), \psi(t)\rangle u_*^i(t) + e^{-\rho t}\Big(\sum_{i=1}^{n} \gamma_i \ln x^i + g(u_*(t))\Big) \leq H(x, t, \psi(t)).$$

Further due to the maximum condition (22) the following equality holds for almost all $t \geq 0$:

$$\langle f_0(x_*(t)), \psi(t)\rangle + \sum_{i=1}^{m}\langle f_i(x_*(t)), \psi(t)\rangle u_*^i(t) + e^{-\rho t}\Big(\sum_{i=1}^{n} \gamma_i \ln x_*(t)^i + g(u_*(t))\Big) = H(x_*(t), t, \psi(t)).$$

Hence, for almost all $t \geq 0$ we have

$$\psi(t)\frac{\partial f_0(x_*(t))}{\partial x} + \sum_{i=1}^{m} \psi(t)\frac{\partial f_i(x_*(t))}{\partial x}u_*^i(t) + e^{-\rho t}\left(\frac{\gamma}{x_*(t)}\right) = \frac{\partial H(x_*(t), t, \psi(t))}{\partial x},$$

and the adjoint equation (21) can be rewritten in this case in the following equivalent way:

$$\dot{\psi}(t) = -\frac{\partial H(x_*(t), t, \psi(t))}{\partial x}. \tag{39}$$

Let now $u$, $x$ be an arbitrary admissible pair. Then due to the concavity of the Hamiltonian $H(x, t, \psi(t))$ in $x$ we have the following inequality:

$$\left\langle \frac{\partial H(x_*(t), t, \psi(t))}{\partial x}, x_*(t) - x(t) \right\rangle \leq H(x_*(t), t, \psi(t)) - H(x(t), t, \psi(t)). \tag{40}$$

Hence, due to the conditions (39), (40) for almost all $t \geq 0$ we have

$$\langle \dot{\psi}(t), x(t) - x_*(t) \rangle \leq H(x_*(t), t, \psi(t)) - H(x(t), t, \psi(t)) \leq$$

$$\leq \langle f_0(x_*(t)), \psi(t) \rangle + \sum_{i=1}^{m}\langle f_i(x_*(t)), \psi(t) \rangle u_*^i(t) + e^{-\rho t}\left(\sum_{i=1}^{n} \gamma_i \ln x_*^i(t) + g(u_*(t))\right) -$$

$$-\langle f_0(x(t)), \psi(t) \rangle - \sum_{i=1}^{m}\langle f_i(x(t)), \psi(t) \rangle u^i(t) - e^{-\rho t}\left(\sum_{i=1}^{n} \gamma_i \ln x^i(t) + g(u(t))\right) = \langle \psi(t), \dot{x}_*(t) - \dot{x}(t) \rangle +$$

$$+ e^{-\rho t}\left(\sum_{i=1}^{n} \gamma_i \ln x_*^i(t) + g(u_*(t))\right) - e^{-\rho t}\left(\sum_{i=1}^{n} \gamma_i \ln x_*^i(t) + g(u(t))\right).$$

Hence

$$\frac{d}{dt}\langle \psi(t), x(t) - x_*(t) \rangle + e^{-\rho t}\left(\sum_{i=1}^{n} \gamma_i \ln x^i(t) + g(u(t))\right) \leq e^{-\rho t}\left(\sum_{i=1}^{n} \gamma_i \ln x_*^i(t) + g(u_*(t))\right).$$

Whence, integrating the last inequality on the arbitrary finite time interval $[0, T]$, $\forall T > 0$ we have

$$\langle \psi(T), x(T) - x_*(T) \rangle + \int_0^T e^{-\rho t}\left(\sum_{i=1}^{n} \gamma_i \ln x^i(t) + g(u(t))\right)dt \leq \int_0^T e^{-\rho t}\left(\sum_{i=1}^{n} \gamma_i \ln x_*^i(t) + g(u_*(t))\right)dt.$$

As far as $\psi(t) \geq 0$, $x(t) \geq 0$ $\forall t \geq 0$ and due to the strengthened transversality condition (35) (see corollary 2) passing to a limit in the last inequality as $T \to \infty$ we get

$$\int_0^{\infty} e^{-\rho t}\left(\sum_{i=1}^{n} \gamma_i \ln x^i(t) + g(u(t))\right)dt \leq \int_0^{\infty} e^{-\rho t}\left(\sum_{i=1}^{n} \gamma_i \ln x_*^i(t) + g(u_*(t))\right)dt.$$

Hence, the pair $u_*$, $x_*$ is an optimal one and the theorem 3 is proved.

**Remark 5.** It is easy to see that if for any admissible trajectory $x \neq x_*$ on a set of positive measure the inequality (40) holds as a strict one then the optimal trajectory $x_*$ is unique.

**Corollary 4** *Let the assumptions $(H1)$–$(H4)$ of the theorem 2 are fulfilled and there exists a matrix $A > 0$ such that the relation (34) holds for almost all $t > 0$ along any admissible pair $u_*$, $x_*$. Assume also that the Hamiltonian $H(x, t, \psi)$ is continuously differentiable and concave in $x$ for all $t \in [0, \infty)$ and all $\psi > 0$. Then the maximum principle (theorem 2) is a necessary and sufficient condition of optimality for the problem $(P)$.*

In conclusion let us give an illustrative example.

**Example.** Consider the following optimal control problem:

$$\dot{x} = x + u, \qquad u \in U = [0,1]; \tag{41}$$

$$x(0) = 1; \tag{42}$$

$$J(x,u) = \int_0^\infty e^{-\rho t} \ln x \, dt \to \max, \tag{43}$$

where $x \in R^1$, $u \in R^1$ and $\rho > 0$.

Due to the theorem 1 there exists an optimal control $u_*$ in the problem (41)–(43). Obviously, conditions $(H1)$–$(H4)$ and (34) are fulfilled in this problem and the Hamiltonian $H(x,t,\psi) = x\psi + \max_{u \in [0,1]} u\psi + e^{-\rho t} \ln x$ is continuously differentiable and concave in $x$ for all $t \geq 0$ and all $\psi \geq 0$. Hence, due to the corollary 4 the maximum principle (theorem 2) is a necessary and sufficient condition of optimality for the problem (41)–(43) and the strengthened transversality condition (35) is valid (see corollary 2). Note that necessary conditions of optimality obtained in [9] are not applicable to the problem (41)-(43) in the case $\rho \leq 1$.

The application of theorem 2 provides us immediately with the unique optimal control $u_*(t) \overset{a.e.}{=} 1$ for problem (41)–(43) $\forall \rho > 0$. Indeed, due to conditions (21), (24) we have $\psi(t) > 0 \ \forall t > 0$ and due to the maximum conditon (22) we have $u_*(t)\psi(t) \overset{a.e.}{=} \max_{u \in [0,1]} u\psi(t)$. Hence $u_*(t) \overset{a.e.}{=} 1$ is the unique optimal control and $x_*(t) = 2e^t - 1$, $t \geq 0$ is the unique optimal trajectory in this problem.

The adjoint system for the problem (41)–(43) is the following one:

$$\dot{\psi} = -\psi - \frac{e^{-\rho t}}{x_*(t)}.$$

Solving it we get

$$\psi(t) = e^{-t}[\psi(0) - \int_0^t \frac{e^{(1-\rho)s}}{2e^s - 1} ds].$$

Hence due to the strengthened transversality condition (35) ($\lim_{t \to \infty} x_*(t)\psi(t) = 0$) we have

$$\psi(0) = \int_0^\infty \frac{e^{(1-\rho)s}}{2e^s - 1} ds.$$

Thus, in this example there is a unique adjoint variable $\psi$ which corresponds to the optimal pair $u_*$, $x_*$ via the developed version of the Pontryagin maximum principle (theorem 2).

# References

[1] *Arrow K.*, Application of control theory to economic growth, in Mathematics of the Decision Sciences. Part 2. Providence, Rhode Island: American, 1968.

[2] *Arrow K., Kurz M.*, Public investment, the rate of return, and optimal fiscal policy. The Johns Hopkins Press. 1970.

[3] *Arutyunov A.V.*, Perturbations of extremal problems with constraints, and necessary optimality conditions, Itogi Nauki i Tekhniki. Ser. Mat. Analiz. Vol. 27. Pp. 147–235. Moscow: VINITI. 1989. English transl. in J. of Soviet Math. Vol. 54. 1991.

[4] *Arutyunov A.V., Aseev S.M.*, The maximum principle for optimal control problems with state constraints. Nondegeneracy and stability, Dokl. Ross. Akad. Nauk. Vol. 334. Pp. 134–137. 1994. English transl. in Russian Acad. Sci. Dokl. Math. Vol. 49. 1994.

[5] *Arutyunov A.V., Aseev S.M.*, Investigation of the degeneracy phenomenon of the maximum principle for optimal control problems with state constraints, SIAM J. Control and Optimization. Vol. 35. No. 3. Pp. 930–952. 1997.

[6] *Arutyunov A.V., Aseev S.M., Blagodatskikh V.I.*, Necessary conditions of the first order in the problem of optimal control of a differential inclusion with phase constraints, Mat. Sb. Vol. 184. No. 6. Pp. 3–32. 1993. English transl. in Russian Acad. Sci. Sb. Math. Vol. 79. 1994.

[7] *Aseev S.M.*, A method of smooth approximations in the theory of necessary optimality conditions for differential inclusions, Izvestiya RAN: Ser. Mat. Vol. 61. No. 2. 1997. English transl. in Izvestiya: Mathematics, Vol. 61, No. 2, Pp. 235–258.

[8] *Aseev S.M.*, Methods of regularization in nonsmooth problems of dynamic optimization, J. of Mathematical Sciences. Vol. 94. No. 3. Pp. 1366–1393. 1999.

[9] *Aubin J.P., Clarke F.H.*, Shadow prices and duality for a class of optimal control problems, SIAM J. Control and Opimization. Vol. 17. No. 5. Pp. 567–586. 1979.

[10] *Balder E.J.*, An existance result for optimal economic grows problems, J. of Math. Analysis and Applications. Vol. 95. Pp. 195–213. 1983.

[11] *Cesari L.*, Optimization – theory and applications. Problems with ordinary differential equations. Springer-Verlag. 1983.

[12] *Dolcetta I.C.*, On a discrete approximation of the Hamilton-Jacobi equation of dynamic programming, Applied Mathematics and Optimization. Vol. 10. Pp. 367-377. 1983.

[13] *Filippov A.F.*, On Some questions in the theory of optimal regulation, Vestnik MGU. Ser. Mat. Mekh. Astr. Fiz. Him. 1959. No. 2. Pp. 25–32. 1959. English transl. in J. Sov. Indust. Appl. Math. Ser. A: Control Vol. 1. 1962.

[14] *Grossman G.M., Helpman E.*, Innovation and growth in the global economy. The MIT Press, 1991.

[15] *Halkin H.*, Necessary conditions for optimal control problems with infinite horizons, Econometrica. Vol. 42. No. 2. Pp. 267–272. 1974.

[16] *Hartman Ph.*, Ordinary differential equations. J. Wiley & Sons. New York, London, Sydney. 1964.

[17] *Mangasarian O.L.*, Sufficient conditions for the optimal control of nonlinear systems, SIAM J. of Control. Vol. 4. No. 1. Pp. 139–152. 1966.

[18] *Mordukhovich B.Sh.*, Approximation methods in problems of optimization and control. Moscow: Nauka. 1988. (Russian)

[19] *Pontryagin L.S., Boltyanskii B.G., Gamkrelidze R.V., and E.F. Mischenko*, The mathmatical theory of optimal processes. New York: Interscience Publishers, John Wiley & Sons, Inc.. 1962.

[20] *Skiba A.K.* Optimal growth with convex-concave production function, Econometrica. Vol. 46. No. 3. Pp. 527-539. 1978.

[21] *Tarasyev A., Watanabe C.*, Optimal control of R&D investment in a techno-metabolic system. IIASA IR–1999–001, 1999.

[22] *Tarasyev A,. Watanabe C.*, Optimal dynamics of innovation in models of economic growth. Journal of Optimization Theory and Applications. Vol. 108, No. 1, 2001.