International Institute for
Applied Systems Analysis
Schlossplatz 1
A-2361 Laxenburg, Austria

Tel: +43 2236 807 342
Fax: +43 2236 71313
E-mail: publications@iiasa.ac.at
Web: www.iiasa.ac.at

Interim Report **IR-02-016**

A Dynamical Model of Optimal Allocation of Resources to R&D

Sergei Aseev (aseev@iiasa.ac.at)
Gernot Hutschenreiter (Gernot.Hutschenreiter@wifo.ac.at)
Arkadii Kryazhimskii (kryazhim@aha.ru)

Approved by

Arne Jernelöv (jernelov@iiasa.ac.at)
Acting Director, IIASA

March 2002

Abstract

We provide steps towards a welfare analysis of a two-country endogenous growth model where a relatively small follower absorbs part of the knowledge generated in the leading country. To solve a suitably defined dynamic optimisation problem an appropriate version of the Pontryagin maximum principle is developed. The properties of optimal controls and the corresponding optimal trajectories are charactarised by the qualitative analysis of the solutions of the Hamiltonian system arising through the implementation of the Pontryagin maximum principle. We find that for a quite small follower, optimisation produces the same asymptotic rate of innovation as the market. However, relative knowledge stocks and levels of productivity differ, in general. Thus, policy intervention has no effect on growth rates but may affect these relative levels. The results are different for not so small follower economies. The present paper provides the rigorous justification for the results presented in Aseev, Hutschenreiter and Kryazhimskii, 2002.

Key words: Endogenous Growth, R&D Spillovers, Absorptive Capacities, Optimal Control, Infinite Horizon, the Pontryagin Maximum Principle

JEL Classification: C61, O30, O40

Mathematics Subject Classification (2000): 37N40, 49KXX, 91B62

Contents

A Dynamical Model of Optimal Allocation of Resources to R&D

Sergei Aseev (aseev@iiasa.ac.at)
Gernot Hutschenreiter (Gernot.Hutschenreiter@wifo.ac.at)
Arkadii Kryazhimskii (kryazhim@aha.ru)

1 Introduction

An endogenous growth model linking a smaller follower country to a larger autarkic leader through "absorptive capacities" enabling it to tap into the knowledge generated in the leading country was introduced by Hutschenreiter, Kaniovski and Kryazhimskii, 1995. We will refer to this model as the "leader-follower" model. It is built along the lines of the basic endogenous growth model with horizontal product differentiation (Grossman and Helpman, 1991, Chapter 3), where technical progress is represented by an expanding variety of products. The leader-follower model was symmetrised to allow for knowledge flows in both directions (Borisov, Hutschenreiter and Kryazhimskii, 1999).

Based on a comprehensive analysis of the dynamic behaviour of the leader-follower model, a particular class of asymptotics was singled out. Any trajectory characterised by this asymptotics was shown to be a perfect-foresight equilibrium trajectory analogous to the one found for the basic Grossman - Helpman model. For this type of trajectory, explicit expressions in terms of model parameters for key variables such as the rate of innovation, the rate of output and productivity growth, the ratio of the stocks of knowledge of the two countries, or the amounts (shares) of labour devoted to R&D and manufacturing were given (Hutschenreiter, Kaniovski and Kryazhimskii, 1995).

The evolution of the economy represented by this model is the result of decentralised maximising behaviour of economic agents. A perfect-foresight equilibrium trajectory generated by the model can therefore be referred to as "decentralised" or "market" solution. However, a market solution is not necessarily an optimal solution. Rather, non-optimality is a common outcome in the presence of externalities of some kind. According to Grossman and Helpman, 1991, in their basic model intertemporal spillovers result in a market allocation of resources which is not Pareto-optimal since too little labour is allocated to R&D. In contrast, Benassy, 1998, finds that both underinvestment and overinvestment in R&D (in terms of the allocation of labour) are possible if returns to specialisation are seperated from the monopolistic mark up. In any case, deviations of the optimal from the market solution provide scope for welfare-enhancing policy intervention.

A welfare analysis of the leader-follower model introduced by Hutschenreiter, Kaniovski and Kryazhimskii, 1995 is missing so far. This paper provides important steps in this direction. For this purpose, we set up and analyse an optimisation problem capturing the task of intertemporal utility maximisation faced by a fictituous social planner.

*This work was partially supported by the Russian Foundation for Basic Research (projects No. 00-01-00682, No. 02-01-00334) and the Fujitsu Research Institute (contract No. 01-109).

The analysis is carried out within the framework of mathematical optimal control theory (Pontryagin, et al., 1969). An important feature of the problem under consideration is that the goal functional is defined on an infinite time interval. In problems with infinite time horizons the Pontryagin maximum principle, the key instrument in optimal control theory, is, in general, less efficient than in problems with finite time horizons. In particular, for the case of infinite time horizons the natural transversality conditions, providing as a rule essential information on the solutions, may not be valid (Halkin, 1974). Additional difficulties in our analysis arise due to non-standard logarithmic singularity in the goal functional.

In the present paper our study is based on the approximation approach to the investigation of optimal control problems with infinite time horizons developed recently by Aseev, Kryazhimskii and Tarasyev, 2001, (a)-(c). This approach provides a possibility to establish existence results for problems with infinite time horizons and to derive the appropriate versions of the Pontryagin maximum principle which contain some extra conditions on the adjoint function and the behaviour of the Hamiltonian at the infinity (in fact, this allows us, in some cases, to guarantee the normality of the problem and the validity of the additional transversality conditions at the infinity). In this paper, we apply the approximation technique to prove the existence of an optimal control and also to derive an appropriate version of the Pontryagin maximum principle. Then we qualitatively analyse the solutions of the Hamiltonian system arising through the implementation of the Pontryagin maximum principle. Namely, we find that the global optimizers are characterized by the exceptional qualitative behavior; this allows us to select the unique optimal regime in the pool of all local extremals.

The paper is organised as follows.

In section 2 we formulate the problem and discuss it in economic terms.

Section 3 is devoted to developing necessary mathematical tools for the problem under consideration. In particular we establish the existence of a solution and develop a relevant version of the Pontryagin maximum principle.

In section 4 we introduce new state and adjoint variables, reformulate the Pontryagin maximum principle in terms of these variables and construct the associated Hamiltonian system.

Starting from section 5, we restrict our analysis to the case where the follower country is quite small relative to the leader. In section 5 we classify the qualitative behaviours of the solutions of the Hamiltonian system.

In section 6 we focus on the solutions of the Hamiltonian system which exhibit an exceptional behaviour (we call them equilibrium solutions). We show that a global optimizer is described by an equilibrium solution and state the uniqueness of this solution. Based on these results, we give the final description of an optimal process and prove its uniqueness.

In section 7 we consider the family of the original problems parametrized by the initial state and describe an optimal synthesis for this family i.e., define a feedback which solves the problem with any initial state.

The final section 8 interprets main results of the mathematical analysis in economic terms.

2 Optimal control problem

In the model we analyse, an economy's labour resources can be used in two different ways, either for manufacturing intermediate goods (which enter final output) or in the production of blueprints for new intermediate goods which permanently raises productivity in final goods production. The optimisation problem faced by a fictitious social planner maximising utility allocating resources to R&D or manufacturing is the following:

$$J(n^B(t), L_n^B(t)) = \int_0^\infty e^{-\rho t}[(\frac{1}{\alpha} - 1)\ln n^B(t) + \ln(L^B - L_n^B(t))]dt \to \max\,, \tag{2.1}$$

$$\dot{n}^B(t) = \frac{L_n^B(t)}{a}(n^B(t) + \gamma n^A(t)), \tag{2.2}$$

$$\dot{n}^A(t) = \bar{g}^A n^A(t), \tag{2.3}$$

$$n^B(0) = n_0^B, \qquad n^A(0) = n_0^A, \tag{2.4}$$

$$L_n^B(t) \in [0, L^B). \tag{2.5}$$

Let us state for now that the model parameters $\rho, \alpha, L^B, a, \gamma, \bar{g}^A$ are all positive. Also note that the objective function (2.1) is the same as in the social planning problem formulated by Grossman and Helpman, 1991. Let us first comment on this objective function. Recall that production of final output is represented by a Dixit - Stiglitz - Ethier production function $Y^B(t)$ where final output is produced by a set of differentiated intermediate goods (Dixit and Stiglitz, 1977, Ethier, 1982)

$$Y^B(t) = [\int_0^{n^B(t)} x(j)^\alpha dj]^{1/\alpha},$$

where $n^B(t)$ is the number of these goods invented up to time t and $x(j)$ represents the output of intermediate good of variety j. The parameter $0 < \alpha < 1$ is related to the (constant) elasticity of substitution $\varepsilon = 1/(1-\alpha)$. In Grossman and Helpman, 1991, there is a dual interpretation of the function $Y^B(t)$ as an index of utility ("love of variety" approach) which we will not take up here. See on this issue Barro and Sala-i-Martin, 1995.

It is a well-known feature of the basic Grossman - Helpman model that in a momentary, symmetric equilibrium, all types of intermediates are produced in the same quantities. If $x^B(t)$ denotes this uniform output per brand, aggregate output of intermediates is given by $X^B(t) = n^B(t)\, x^B(t)$. Consequently, for the production function $Y^B(t)$, final output at time t is given by

$$Y^B(t) = (n^B(t))^{1/\alpha} x^B(t) = (n^B(t))^{1/\alpha - 1} X^B(t). \tag{2.6}$$

Obviously, total factor productivity (TFP) at time t is an increasing function of the country's stock of knowledge:

$$\frac{Y^B(t)}{X^B(t)} = (n^B(t))^{1/\alpha - 1}.$$

With steady growth, where the allocation of labour to manufacturing and R&D is constant, the growth rate of final output and TFP is identically $(1/\alpha - 1)\bar{g}^B(t)$, where $\bar{g}^B(t)$ denotes the steady-state rate of growth of the country's knowledge stock.

In the basic Grossman - Helpman model, each intermediate good is produced by a constant-returns-to-scale technology where one unit of labour is required to turn out one additional unit of output. Consequently, aggregate output of intermediate goods equals total labour allocated to manufacturing,

$$X^B(t) = L^B - L_n^B(t), \tag{2.7}$$

where L^B represents the economy's constant supply of homogenous labour and $L_n^B(t)$ the amount of this pool of labour allocated to R&D.

At any moment of time, the market for final goods is assumed to be in equilibrium so that consumption $C^B(t)$ equals the flow of final output

$$C^B(t) = Y^B(t).$$

In the following analysis we assume that instantaneous utility is given by

$$U(t) = \ln C^B(t). \tag{2.8}$$

Of course, one could work with a more general utility function. In fact (2.8) is a limiting case for the widely-used constant elasticity of intertemporal substitution utility function

$$U(t) = \frac{C^B(t)^{1-\theta} - 1}{1-\theta}$$

as $\theta \to 1$. For simplicity, we restrict ourselves to this limiting case.

Combining (2.6), (2.7) and (2.8) and discounting by the time preference rate ρ we obtain the expression in the integral defining the objective function.

Let us next turn to equation (2.2) in the above optimal control problem. In the spirit of Romer, 1990, we employ a production function for developing blueprints for novel intermediates where the productivity of resources devoted to R&D is enhanced by the accumulated stock of knowledge capital. A distinguishing feature of the leader-follower model is that the knowledge stock available in country B at time t is assumed to consist of the sum of the knowledge accumulated in country B which is represented by the number of differentiated inputs developed so far domestically, $n^B(t)$, and a term comprising externally produced knowledge appropriated by country B. More specifically, a fraction $0 \leq \gamma(n^B) \leq 1$ of the knowledge stock produced in country A is absorbed into the knowledge stock of country B. Function $\gamma(n^B)$ represents the absorptive capacities (see Cohen and Levinthal, 1989) of the follower (determined by its capabilities but also by barriers to international communication or the extent of redundant knowledge which will not be targeted by the follower). For simplicity, in the present optimisation problem we treat the absorptive capacities of the follower country as a parameter γ. We assume $\gamma > 0$. Parameter a reflects productivity in R&D.

Equation (2.3) tells us that the autarkic leading country's stock of knowledge grows exponentially at the steady rate of innovation $\bar{g}^A > 0$. If the leading country evolves in its steady state, we know from Grossman and Helpman, 1991 that its exponential rate of innovation is given by

$$\bar{g}^A = (1-\alpha)\frac{L^A}{a} - \alpha\rho > 0. \tag{2.9}$$

Equation (2.4) fixes initial conditions. Finally (see (2.5)), it is assumed that the follower country's R&D labour does not exhaust its total labour force and thus manufacturing activity does not vanish at any instant of time.

In this paper (starting from section 5) the analyis is restricted to the case

$$a\bar{g}^A > L^B.$$

This inequality has the straightforward interpretation that the amount of labour allocated to R&D in the leading country exceeds the total labour force in the follower country. This suggests that the follower country is quite small relative to the leader. As shown in Hutschenreiter, Kaniovski and Kryazhimskii, 1995, the opposite inequality must hold for the follower country to be able to catch up with the leader in terms of knowledge stocks in the market economy.

At present, we have some tentative results for the slightly relaxed case

$$a\bar{g}^A + \rho > L^B$$

as well as for the opposite case

$$a\bar{g}^A + \rho \le L^B.$$

Clearly, this latter case fulfills the necessary condition identified for the follower country to catch up with the leader in terms of knowledge stocks.

3 Existence of an optimal control and the Pontryagin maximum principle

To simplify the analytic expressions dealt with in the sequel, we use the following notations in sections 3-7:

$$x(t) = n^B(t),$$
$$y(t) = n^A(t),$$
$$u(t) = \frac{L^B(t)}{a},$$
$$b = \frac{L^B}{a},$$
$$\nu = \bar{g}^A,$$
$$\kappa = \frac{1}{\alpha} - 1.$$

Using these notations, we rewrite problem (2.1)-(2.5) as the following problem (P):

$$J(x(t), u(t)) = \int_0^\infty e^{-\rho t}[\kappa \ln x(t) + \ln(b - u(t))]dt \to \max, \tag{3.1}$$

$$\dot{x}(t) = u(t)(x(t) + \gamma y(t)), \tag{3.2}$$

$$\dot{y}(t) = \nu y(t), \tag{3.3}$$

$$x(0) = x_0, \quad y(0) = y_0, \tag{3.4}$$

$$u(t) \in [0, b). \tag{3.5}$$

Here b, γ, ρ, ν and κ are positive parameters, and x_0 and y_0 are positive initial values for the state variables.

Let us recall several standard definitions of optimal control theory in the context of problem (P). A *control* for system (3.2), (3.3) is identified with any measurable function $u(t) : [0, \infty) \to R^1$, which is bounded on arbitrary finite time interval $[0, T]$. We define the *trajectory* under control $u(t)$ to be the component $x(t)$ of the (unique) Caratheodory solution $(x(t), y(t))$ on $[0, \infty)$ of differential equation (3.2), (3.3) with the initial state (x_0, y_0). A *control process* for system (3.2), (3.3) is a pair $(x(t), u(t))$ where $u(t)$ is a control and $x(t)$ is the trajectory corresponding to $u(t)$. A control $u(t)$ is an *admissible* one in problem (P) if $u(t)$ satisfies (3.5) for all $t \ge 0$. A control process $(x(t), u(t))$ is an *admissible control process* in problem (P) if $u(t)$ is an admissible control.

Remark 3.1 The trajectory does not include the component $y(t)$ of the solution $(x(t), y(t))$ of the system (3.2), (3.3). This simplification of the definition (and in further notations) does not lead to any ambiguity, since $y(t) = y_0 e^{\nu t}$ does not depend on $u(t)$.

An accurate formulation of problem (P) is as follows: maximize $J(x(t), u(t))$ over the set of all admissible control processes $(x(t), u(t))$ in problem (P). An *optimal control* in problem (P) is defined to be an admissible control $u_*(t)$ such that the associated control process $(x_*(t), u_*(t))$ satisfies $J(x_*(t), u_*(t)) = J_*$ where J_* is the maximal (*optimal*) value in problem (P).

Remark 3.2 The non-closedness of the interval $[0, b)$ for admissible controls (see (3.5)) and the associated logarithmic singuliarity $\ln(b - u(t))$ in the goal functional (3.1) ($\ln(b - u(t))$ approaches $-\infty$ as $u(t)$ approaches b) prevent us from referring to standard theorems stating the existence of optimal controls (see Balder, 1983) and from using the modified Pontryagin maximum principle suggested in Aseev, Kryazhimskii and Tarasyev, 2001, (a)-(c), directly. We study problem (P) in-depth, based on the approximation methodology of Aseev, Kryazhimskii and Tarasyev, 2001, (a)-(c).

Our goal in this section is to prove the existence of an optimal control in problem (P) and to develop an appropriate version of the Pontryagin maximum principle for this problem,

For technical reasons we will start with the consideration of the following (slightly more general) optimal control problem (Q):

$$\tilde{J}(x(t), u(t)) = \int_0^\infty e^{-\rho t}[\kappa \ln x(t) + \ln(b - u(t)) + \phi(u(t), t)]dt \to \max\,, \tag{3.6}$$

$$\dot{x}(t) = u(t)(x(t) + \gamma y(t)), \tag{3.7}$$

$$\dot{y}(t) = \nu y(t), \tag{3.8}$$

$$x(0) = x_0, \quad y(0) = y_0, \tag{3.9}$$

$$u(t) \in [0, b). \tag{3.10}$$

Here function $\phi(u, t)$ is continuous on $[0, b] \times [0, \infty)$ and concave in u. We assume that the function $\phi(u, t)$ is bounded, i.e. there exists a constant $K_0 > 0$ such that $|\phi(u, t)| \leq K_0$ $\forall u \in [0, b]$, $\forall t \geq 0$. All other data of problem (Q) are assumed to be the same as in the initial problem (P). As in the initial problem (P) the set of admissible controls $u(t)$ for problem (Q) consists of all measurable functions $u(t) : [0, \infty) \to [0, b)$.

For arbitrary $T > 0$ let us consider the following auxilary optimal control problem (Q_T) on the final time interval $[0, T]$:

$$\tilde{J}_T(x(t), u(t)) = \int_0^T e^{-\rho t}[\kappa \ln x(t) + \ln(b - u(t)) + \phi(u(t), t)]dt \to \max\,, \tag{3.11}$$

$$\dot{x}(t) = u(t)(x(t) + \gamma y(t)), \tag{3.12}$$

$$\dot{y}(t) = \nu y(t), \tag{3.13}$$

$$x(0) = x_0, \quad y(0) = y_0, \tag{3.14}$$

$$u(t) \in [0, b). \tag{3.15}$$

Here all data of problem (Q_T) are the same as in problem (Q) and as usual the set of admissible controls $u(t)$ for problem (Q_T) consist of all measurable functions $u(t)$: $[0, \infty) \to [0, b)$.

Lemma 3.1 *For arbitrary $T > 0$ there exists a constant $l(T)$, $0 < l(T) < b$ such that for all $0 < l \leq l(T)$ condition (3.15) can be replaced by the condition*

$$u(t) \in [0, b - l] \tag{3.16}$$

in the formulation of the problem (Q_T) without changing the optimal value and the set of controls which can be optimal in the problem.

Proof. For arbitrary $0 < l < b$ and arbitrary admissible control $u(t) : [0, T] \to [0, b)$ define a set

$$M_l = \{t \in [0, T] : \, u(t) > b - l\}.$$

Let $x(t)$ be the trajectory of the system (3.12), (3.13) corresponding to the control $u(t)$ with initial conditions (3.14). Consider now an admissible control $u_0(t) : [0, T] \to [0, b - l]$ which is equal 0 if $t \in M_l$ and $u(t)$ if $t \notin M_l$. Let $x_0(t)$ be the trajectory of the system (3.12), (3.13) corresponding to the control $u_0(t)$ with initial conditions (3.14). Due to the definition of the control $u_0(t)$ and the boundedness of the function $\phi(u, t)$ the following inequalities hold:

$$\int_0^T e^{-\rho t}[\ln(b - u(t)) - \ln(b - u_0(t))]dt \leq (\ln l - \ln b)e^{-\rho T}\text{meas}\,(M_l), \tag{3.17}$$

$$\int_0^t e^{-\rho T}[\phi(u(t), t) - \phi(u_0(t), t)]dt \leq 2K_0\text{meas}\,(M_l). \tag{3.18}$$

Further, for arbitrary $t \in [0, T]$ we have

$$x(t) - x_0(t) = \int_0^t u(s)(x(s) + \gamma y(s))ds - \int_0^t u_0(s)(x_0(s) + \gamma y(s))ds =$$

$$\int_{M_l \cap [0,t]} u(s)(x(s) + \gamma y(s))ds + \int_{[0,t]\setminus M_l} u(s)(x(s) - x_0(s))ds \leq$$

$$\leq bK_1(T)\text{meas}\,(M_l) + b\int_0^t (x(s) - x_0(s))ds,$$

where $K_1(T)$ is a positive constant such that $x(t) + \gamma y(t) \leq K_1(T)$ for an arbitrary admissible trajectory $x(t)$ of the control system under consideration. Due to the Bellman-Gronwall inequality (Hartman, 1964) for every $t \in [0, T]$ we have

$$x(t) - x_0(t) \leq bK_1(T)\text{meas}\,(M_l)e^{bt}.$$

Hence the following inequalities take place:

$$\int_0^T e^{-\rho t}[\kappa \ln x(t) - \kappa \ln x_0(t)]dt \leq \kappa \int_0^T e^{-\rho t}\frac{x(t) - x_0(t)}{x_0}dt \leq K_2(T)\text{meas}\,(M_l), \tag{3.19}$$

where $K_2(T)$ is a positive constant which is independent from the trajectory $x(t)$.

Combining now inequalities (3.17)-(3.19) we get

$$\tilde{J}_T(x(t), u(t)) - \tilde{J}_T(x_0(t), u_0(t)) =$$

$$= \int_0^T e^{-\rho t}[\kappa \ln x(t) + \ln(b - u(t)) + \phi(u(t), t)]dt - \int_0^T e^{-\rho t}[\kappa \ln x_0(t) + \ln(b - u_0(t)) + \phi(u_0(t), t)]dt \leq$$

$$\leq ((\ln l - \ln b)e^{-\rho T} + 2K_0 + bK_2(T))\text{meas}\,(M_l).$$

Hence there exists a constant $l(T)$, $0 < l(T) < b$ such that for all $0 < l \leq l(T)$

$$\tilde{J}_T(x(t), u(t)) - \tilde{J}_T(x_0(t), u_0(t)) < 0$$

if $\text{meas}\, M_l > 0$. The Lemma is proved.

Corollary 3.1 *For arbitrary $T > 0$ there exists an optimal control $u_T(t)$ in problem (Q_T).*

Indeed due to the Lemma 3.1 the condition $u \in [0, b)$ (see (3.15)) in the statement of problem (Q_T) may be replaced by the condition $u \in [0, b-l]$ (see (3.16)) where l is a small enough positive number. Now, the existence of an optimal control $u_T(t)$ is a consequence of the Fillipov existence theorem (Cesari, 1983).

Corollary 3.2 *An optimal control process $(x_T(t), u_T(t))$ in problem (Q_T) satisfies the Pontryagin maximum principle (Pontryagin, et al., 1969), i.e. there exists an absolutely continuous vector function $\psi(t) : [0, T] \to R^2$, $\psi(t) = (\psi^1(t), \psi^2(t))$ such that the following conditions hold:*

1) The vector function $\psi(t)$ is a solution to the adjoint system

$$\dot{\psi}^1(t) = -u_T(t)\psi^1(t) - \kappa\frac{e^{-\rho t}}{x_T(t)}, \tag{3.20}$$

$$\dot{\psi}^2(t) = -\gamma u_T(t)\psi^1(t) - \nu\psi^2(t); \tag{3.21}$$

2) For almost all $t \in [0, T]$ the maximum condition takes place:

$$u_T(t)(x_T(t) + \gamma y(t))\psi^1(t) + e^{-\rho t}(\ln(b - u_T(t)) + \phi(u_T(t), t)) =$$

$$= \sup_{u \in [0,b)} [u(x_T(t) + \gamma y(t))\psi^1(t) + e^{-\rho t}(\ln(b-u) + \phi(u, t))]; \tag{3.22}$$

3) The transversality conditions hold:

$$\psi^1(T) = 0, \qquad \psi^2(T) = 0. \tag{3.23}$$

Indeed, let us take an arbitrary sequence $\{l_i\}$, $i = 1, 2, \ldots$ such that $0 < l_{i+1} < l_i < l(T)$, $\forall i = 1, 2, \ldots$ and $l_i \to +0$ as $i \to \infty$. Then for arbitrary $i = 1, 2, \ldots$ due to the Lemma 3.1 the optimal control process $(x_T(t), u_T(t))$ is an optimal one in problem $(Q_{i,T})$ which is different from (Q_T) only by the constraints on controls: the condition (3.15) is replaced by $u(t) \in [0, b - l_i]$. The last problem $(Q_{i,T})$ satisfies all the assumptions of the Pontryagin maximum principle for free end point optimal control problems on fixed time interval $[0, T]$ (Pontryagin, et al., 1969). Hence for arbitrary $i = 1, 2, \ldots$ there exists an absolutely continuous vector function $\psi_i(t) : [0, T] \to R^2$, $\psi_i(t) = (\psi_i^1(t), \psi_i^2(t))$ such that the following conditions hold:

1) The vector function $\psi_i(t)$ is a solution to the adjoint system

$$\dot{\psi}_i^1(t) = -u_T(t)\psi_i^1(t) - \kappa\frac{e^{-\rho t}}{x_T(t)}, \tag{3.24}$$

$$\dot{\psi}_i^2(t) = -\gamma u_T(t)\psi_i^1(t) - \nu\psi_i^2(t); \tag{3.25}$$

2) For almost all $t \in [0, T]$ the maximum condition takes place:

$$u_T(t)(x_T(t) + \gamma y(t))\psi_i^1(t) + e^{-\rho t}(\ln(b - u_T(t)) + \phi(u_T(t), t)) =$$

$$= \sup_{u \in [0,b-l_i)} [u(x_T(t) + \gamma y(t))\psi_i^1(t) + e^{-\rho t}(\ln(b-u) + \phi(u, t))]; \tag{3.26}$$

3) The transversality conditions hold:

$$\psi_i^1(T) = 0, \qquad \psi_i^2(T) = 0. \tag{3.27}$$

It is easy to see that the sequence $\{\psi_i(t)\}$, $i = 1, 2, \dots$ is a precompact in $C[0, T]$. Hence passing if necessary to a subsequence we can assume that there exists an absolutely continous vector function $\psi : [0, T] \to R^2$, $\psi(t) = (\psi^1(t), \psi^2(t))$ such that $\psi_i(t) \rightrightarrows \psi(t)$ as $i \to \infty$ and $\dot{\psi}_i(t) \to \dot{\psi}(t)$ weakly in $L^1[0, T]$ as $i \to \infty$. It is easy to see that due to (3.27) condition (3.23) holds. Due to the Mazur theorem (Mordukhovich, 1988) passing to a limit in (3.24), (3.25) as $i \to \infty$ we get that the function $\psi(t)$ satisfies the adjoint system (3.20), (3.21) for problem (Q_T). Finally as far as $l_i \to +0$ as $i \to \infty$, the maximum condition (3.22) follows from (3.26).

The following two results provide estimations on the "tail" of the integral goal functional in the basic problem (Q) and the approximation problems (Q_T) $(T > 0)$.

Lemma 3.2 *There exists a nonincreasing positive function $\omega_0(t)$; $\omega_0(t) \to +0$ as $t \to \infty$ such that for arbitrary admissible control process $(x(t), u(t))$ in problem (Q) and all $0 < \tau < \xi$ and the following inequalities hold:*

$$\int_\tau^\xi e^{-\rho s}[\kappa \ln x(s) + \ln(b - u(s)) + \phi(u(s), s)]ds \le \omega_0(\tau), \tag{3.28}$$

$$\int_\tau^\infty e^{-\rho s}[\kappa \ln x(s) + \ln(b - u(s)) + \phi(u(s), s)]ds \le \omega_0(\tau). \tag{3.29}$$

The proof follows directly from the boundedness of $\phi(u, t)$, the uniform exponentional boundedness of trajectories of the control system (3.7), (3.8) with initial conditions (3.9) and the logarithmic form of the corresponding term in the goal functional (3.6).

Lemma 3.3 *There exists a nondecreasing negative function $\omega_1(t)$; $\omega_1(t) \to -0$ as $t \to \infty$ such that for arbitrary $T > 0$, arbitrary optimal control process $(x_T(t), u_T(t))$ in problem (Q_T) and all $0 < \tau < \xi \le T$ the following inequality holds:*

$$\int_\tau^\xi e^{-\rho s}[\kappa \ln x_T(s) + \ln(b - u_T(s)) + \phi(u_T(s), s)]ds \ge \omega_1(\tau). \tag{3.30}$$

Proof. Due to the optimality of the process $(x_T(t), u_T(t))$ in problem (Q_T) the restriction of this process on the time interval $[\tau, T]$ is an optimal one in the corresponding optimal control problem on this time interval $[\tau, T]$. Hence we have

$$\int_\tau^T e^{-\rho s}[\kappa \ln x_T(s) + \ln(b - u_T(s)) + \phi(u_T(s), s)]ds = \int_\tau^\xi e^{-\rho s}[\kappa \ln x_T(s) + \ln(b - u_T(s)) +$$

$$+\phi(u_T(s), s)]ds + \int_\xi^T e^{-\rho s}[\kappa \ln x_T(s) + \ln(b - u_T(s)) + \phi(u_T(s), s)]ds \ge$$

$$\ge \int_\tau^T e^{-\rho s}[\kappa \ln x_T(\tau) + \ln b + \phi(0, s)]ds.$$

Hence (due to (3.28))

$$\int_\tau^\xi e^{-\rho s}[\kappa \ln x_T(s) + \ln(b - u_T(s)) + \phi(u_T(s), s)]ds \ge \int_\tau^T e^{-\rho s}[\kappa \ln x_T(t) + \ln b + \phi(0, s)]ds -$$

$$- \int_\xi^T e^{-\rho s}[\kappa \ln x_T(\tau) + \ln(b - u_T(s)) + \phi(u_T(s), s)]ds \ge \omega_1(\tau),$$

where

$$\omega_1(\tau) = -\int_\tau^\infty e^{-\rho s}[\kappa |\ln x_0| + |\ln b| + K_0]ds - \omega_0(\tau).$$

The Lemma is proved.

Theorem 3.1 *There exists an optimal control in problem (Q).*

Proof. Let $\{T_k\}$, $k = 1, 2, \ldots$ be a sequence of positive numbers such that $T_k < T_{k+1}$, $k = 1, 2, \ldots$ and $T_k \to \infty$ as $k \to \infty$. Consider a corresponding sequence of optimal control problems $\{(Q_{T_k})\}$ each of which defined on its own fixed time interval $[0, T_k]$ (see (3.11)-(3.15)). For brevity we will write below (Q_k) instead of (Q_{T_k}).

Due to Corollary 3.1 for any $k = 1, 2, \ldots$ there exists an optimal process $(x_k(t), u_k(t))$ in problem (Q_k).

Consider now the sequence of controls $\{u_k(t)\}$, $k = 1, 2, \ldots$ on the time interval $[0, T_1]$. Due to the convexity and compactness of the interval $[0, b]$ one can choose a subsequence $\{u_{1,k}(t)\}$ of $\{u_k(t)\}$ such that $u_{1,k}(t) \to u_*(t)$ weakly in $L^1[0, T_1]$ as $k \to \infty$ where $u_*(t)$ is a measurable function on the time interval $[0, T_1]$ with values lies in $[0, b]$. Note, that by the construction each control $u_{1,k}(t)$, $k = 1, 2, \ldots$ is an optimal one in a corresponding problem $(Q_{m(1,k)})$ on the time interval $[0, T_{m(1,k)}]$, $T_{m(1,k)} \geq T_1$ for some number $m(1, k) \geq 1$. Assume $x_{1,k}(t)$ is the optimal trajectory corresponding to $u_{1,k}(t)$ on the time interval $[0, T_{m(1,k)}]$, $k = 1, 2, \ldots$, and $x_*(t)$ denotes the trajectory of the system (3.7), (3.8) corresponding to control $u_*(t)$ on the time interval $[0, T_1]$.

Due to the linearity in respect to control of the system (3.12), (3.13) we have $x_{1,k}(t) \rightrightarrows x_*(t)$ on $[0, T_1]$ as $k \to \infty$. Obviously, $\dot{x}_{1,k}(t) \to \dot{x}_*(t)$ weakly in $L^1[0, T_1]$ as $k \to \infty$.

Consider now the sequence $\{u_{1,k}(t)\}$, $k = 1, 2, \ldots$ on the time interval $[0, T_2]$ for $k \geq 2$.

Analogously to the previous step there exists a subsequence $\{u_{2,k}(t)\}$, $k = 1, 2, \ldots$ of the sequence $\{u_{1,k}(t)\}$, $k = 2, 3, \ldots$ such that $\{u_{2,k}(t)\}$ converges weakly in $L^1[0, T_2]$ to a measurable function is defined on the time interval $[0, T_2]$ with values lies in $[0, b]$ and coincide with $u_*(t)$ on $[0, T_1]$. Let us denote the control constructed by this procedure on $[0, T_2]$ again by $u_*(t)$.

By the construction each control $u_{2,k}(t)$, $k = 1, 2, \ldots$ is an optimal one in a corresponding problem $(Q_{m(2,k)})$ on the time interval $[0.T_{m(2,k)}]$, $T_{m(2,k)} \geq T_2$ for some number $m(2, k) \geq 2$. Let $x_{2,k}(t)$ be the corresponding to $u_{2,k}(t)$ optimal trajectory on the time interval $[0, T_{m(2,k)}]$, $k = 1, 2, \ldots$ and let $x_*(t)$ be the trajectory of the system (3.7), (3.8) corresponding to control $u_*(t)$ on the time interval $[0, T_2]$.

Analogously to the previous step we have $x_{2,k}(t) \rightrightarrows x_*(t)$ on $[0, T_2]$ as $k \to \infty$ and $\dot{x}_{2,k}(t) \to \dot{x}_*(t)$ weakly in $L^1[0, T_2]$ as $k \to \infty$.

Repeating this procedure we construct step by step a measurable function $u_* : [0, \infty) \to [0, b]$ and the corresponding trajectory $x_*(t)$ of the system (3.7), (3.8). Simultaneously we construct a countable family of controls $\{u_{i,k}(t)\}$, $i = 1, 2, \ldots$, $k = 1, 2, \ldots$ and the corresponding family of trajectories $\{x_{i,k}(t)\}$, $i = 1, 2, \ldots$, $k = 1, 2, \ldots$. Furthermore, for all $i = 1, 2, \ldots$, $k = 1, 2, \ldots$ the control $u_{i,k}(t)$ which is defined by this procedure, is an optimal one in optimal control problem $(Q_{m(i,k)})$, $m(i, k) \geq i$ on the corresponding time interval $[0, T_{m(i,k)}]$ where $T_{m(i,k)} \geq T_i$, $i = 1, 2, \ldots$ Moreover, for all $i = 1, 2, \ldots$ we have

$$u_{i,k}(t) \to u_*(t) \qquad \text{weakly in} \quad L^1[0, T_i] \quad \text{as} \quad k \to \infty;$$

$$x_{i,k}(t) \rightrightarrows x_*(t), \qquad \text{on} \quad [0, T_i] \quad \text{as} \quad k \to \infty;$$

$$\dot{x}_{i,k}(t) \to \dot{x}_*(t) \qquad \text{weakly in} \quad L^1[0, T_i] \quad \text{as} \quad k \to \infty.$$

Let us take the diagonal sequence $\{u_{k,k}(t)\}$, $k = 1, 2, \ldots$ from the constructed family $\{u_{i,k}(t)\}$, $i = 1, 2 \ldots$, $k = 1, 2 \ldots$ and denote $\bar{u}_k(t) = u_{k,k}(t)$, $\bar{x}_k(t) = x_{k,k}(t)$, and $m(k) = m(k, k)$, $k = 1, 2, \ldots$.

Constructed by this procedure pair $(u_*(t), x_*(t))$, and sequences of controls $\{\bar{u}_k(t)\}$, $k = 1, 2, \ldots$ and corresponding trajectories $\{\bar{x}_k(t)\}$, $k = 1, 2, \ldots$ satisfy the following properties:

a) $\forall k = 1, 2, \dots$ the control $\bar{u}_k(t)$ is defined on the time interval $[0, T_{m(k)}]$, $m(k) \geq k$ and $\bar{u}_k(t)$ is an optimal control in problem $(Q_{m(k)})$.

b) $\forall T > 0$ we have

$$\bar{u}_k(t) \to u_*(t) \qquad \text{weakly in} \quad L^1[0,T] \quad \text{as} \quad k \to \infty;$$

$$\bar{x}_k(t) \rightrightarrows x_*(t) \qquad \text{on} \quad [0,T] \quad \text{as} \quad k \to \infty;$$

$$\dot{\bar{x}}_k(t) \to \dot{x}_*(t) \qquad \text{weakly in} \quad L^1[0,T] \quad \text{as} \quad k \to \infty.$$

Let us prove that the control $u_*(t)$ constructed above is an admissible one in problem (Q), i.e. that (3.5) holds for almost all $t \in [0, \infty)$.

Due to the Lemma 3.3 there exists a negative nondecreasing function $\omega_1(t)$; $\omega_1(t) \to -0$ as $t \to \infty$ such that for all $T > 0$, all $0 < \tau < T$ and the optimal pair $(\bar{x}_k(t), \bar{u}_k(t))$ in problem $(Q_{m(k)})$: $T_{m(k)} > T$ the following inequality (3.30) holds:

$$\int_\tau^T e^{-\rho s}[\kappa \ln \bar{x}_k(s) + \ln(b - \bar{u}_k(s)) + \phi(\bar{u}_k(t), t)]ds \geq \omega_1(\tau).$$

Hence, for any $c > 0$ and for arbitrary $\epsilon > 0$ due to the upper semicontinuity of the integral functional

$$\tilde{J}_{\tau,T}(x(t), u(t)) = \int_\tau^T e^{-\rho s}[\kappa \ln x(s) + \ln(b - u(s) + c) + \phi(u(s), s)]ds$$

(see Theorem 10.8.ii in Cesari, 1983), there exists $N > 0$ such that $\forall k \geq N$ the following inequality takes place:

$$\omega_1(\tau) \leq \int_\tau^T e^{-\rho s}[\kappa \ln \bar{x}_k(s) + \ln(b - \bar{u}_k(s) + c)) + \phi(\bar{u}_k(s), s)]ds \leq$$

$$\leq \int_\tau^T e^{-\rho s}[\kappa \ln x_*(s) + \ln(b - u_*(s) + c) + \phi(u_*(s), s)]ds + \epsilon.$$

Whence, for all $T > 0$ and all $0 < \tau < T$ we get

$$\int_\tau^T e^{-\rho s}[\kappa \ln x_*(s) + \ln(b - u_*(s) + c) + \phi(u_*(s), s)]ds \geq \omega_1(\tau) \tag{3.31}$$

and, as far as function $\phi(u, t)$ is bounded and the set of all trajectories of the system (3.7), (3.8) is uniformly bounded on $[0, T]$, there exists a constant $K_3(T) > 0$ such that

$$\int_0^T e^{-\rho s} \ln(b - u_*(s) + c)ds \geq \omega_1(0) - K_3(T).$$

It follows immidiately from the last inequality and arbitrariness of $c > 0$ that for almost all $t \in [0, T]$ the constructed control $u_*(t)$ satisfies to inequality $u_*(t) < b$. Hence, $u_*(t) < b$ almost everywhere on $[0, \infty)$.

Let us prove now that the constructed admissible control $u_*(t)$ is an optimal one in problem (Q).

First of all note that for arbitrary $\tau > 0$ the following inequality

$$\int_\tau^\infty e^{-\rho s}[\kappa \ln x_*(s) + \ln(b - u_*(s)) + \phi(u_*(s), s)]ds \geq \omega_1(\tau)$$

follows from the validity of (3.31) for arbitrary $c > 0$. This inequality and Lemma 3.2 (see (3.29)) imply the existence of a positive nonincreasing function $\omega(t)$; $\omega(t) \geq \omega_0(t)$ $\forall t \geq 0$ and $\omega(t) \to +0$ as $t \to \infty$ such that

$$|\int_{\tau}^{\infty} e^{-\rho s}[\kappa \ln x_*(s) + \ln(b - u_*(s)) + \phi(u_*(s), s)]ds| \leq \omega(\tau) \qquad \forall \tau > 0. \tag{3.32}$$

Note, that due to Lemmas 3.2 and 3.3 without loss of generality it is possible to assume also that for arbitrary $T > 0$, all $0 < \tau < T$ and all $k : T_k \geq T$ the analogous inequality holds for all pairs $(\bar{u}_k(t), \bar{x}_k(t))$, which are optimal in corresponding problems $(Q_{m(k)})$:

$$|\int_{\tau}^{T} e^{-\rho s}[\kappa \ln \bar{x}_k(s) + \ln(b - \bar{u}_k(s)) + \phi(\bar{u}_k(s), s)]ds| \leq \omega(\tau).$$

Assume that $u_*(t)$ is not optimal in problem (Q). Then there exist $\epsilon > 0$ and an admissible pair $(\tilde{u}(t), \tilde{x}(t))$ such that

$$\tilde{J}(x_*(t), u_*(t)) < \tilde{J}(\tilde{x}(t), \tilde{u}(t)) - \epsilon. \tag{3.33}$$

Further, due to the the properties of the function $\omega(t)$ there exists k_1 such that $\forall T \geq T_{k_1}$ we have

$$\omega(T) < \frac{\epsilon}{4}. \tag{3.34}$$

Consider now the above constructed sequences $\{\bar{u}_k(t)\}$, $\{\bar{x}_k(t)\}$ on the time interval $[0, T_{m(k_1)}]$ for $k \geq k_1$.

On this time interval $[0, T_{m(k_1)}]$ we have

$$\bar{u}_k(t) \to u_*(t) \qquad \text{weakly in} \quad L^1[0, T_{m(k_1)}] \quad \text{as} \quad k \to \infty;$$

$$\bar{x}_k(t) \rightrightarrows x_*(t) \qquad \text{on} \quad [0, T_{m(k_1)}] \quad \text{as} \quad k \to \infty;$$

$$\dot{\bar{x}}_k(t) \to \dot{x}_*(t) \qquad \text{weakly in} \quad L^1[0, T_{m(k_1)}] \quad as \quad k \to \infty.$$

Further, due to the upper semicontinuity of the functional $\tilde{J}_{T_{m(k_1)}}(x(t), u(t))$ (see Theorem 10.8.ii in Cesari, 1983) there exists $k_2 \geq k_1$ such that $\forall k \geq k_2$ the following inequality holds:

$$\tilde{J}_{T_{m(k_1)}}(\bar{x}_k(t), \bar{u}_k(t)) \leq \tilde{J}_{T_{m(k_1)}}(x_*(t), u_*(t)) + \frac{\epsilon}{4} \tag{3.35}$$

Consider now the admissible pair $(\bar{u}_{k_2}(t), \bar{x}_{k_2}(t))$ on the corresponding time interval $[0, T_{m(k_2)}]$. By the construction $\bar{u}_{k_2}(t)$ is an optimal control in optimal control problem $(Q_{m(k_2)})$ on the time interval $[0, T_{m(k_2)}]$. Hence, due to (3.34) and inequality (3.29) we have

$$\tilde{J}_{T_{m(k_2)}}(\bar{x}_{k_2}(t), \bar{u}_{k_2}(t)) \geq \int_0^{T_{m(k_2)}} e^{-\rho t}[\kappa \ln \tilde{x}(t) + \ln(b - \tilde{u}(t)) + \phi(\tilde{u}(t), t)]dt \geq$$

$$\geq \int_0^{\infty} e^{-\rho t}[\kappa \ln \tilde{x}(t) + \ln(b - \tilde{u}(t)) + \phi(\tilde{u}(t), t)]dt - \frac{1}{4}\epsilon = J(\tilde{x}(t), \tilde{u}(t)) - \frac{1}{4}\epsilon.$$

Whence due to (3.34), inequality (3.28) and (3.35) we get

$$J(\tilde{x}(t), \tilde{u}(t)) \leq \tilde{J}_{T_{m(k_2)}}(\bar{x}_{k_2}(t), \bar{u}_{k_2}(t)) + \frac{1}{4}\epsilon = \int_0^{T_{m(k_1)}} e^{-\rho t}[\kappa \ln \bar{x}_{k_2}(t) + \ln(b - \bar{u}_{k_2}(t)) +$$

$$+\phi(\bar{u}_{k_2}(t), t)]dt + \int_{T_{m(k_1)}}^{T_{m(k_2)}} e^{-\rho t}[\kappa \ln \bar{x}_{k_2}(t) + \ln(b - \bar{u}_{k_2}(t)) + \phi(\bar{u}_{k_2}(t), t)]dt + \frac{1}{4}\epsilon \leq$$

$$\leq \tilde{J}_{T_{m(k_1)}}(x_*(t), u_*(t)) + \frac{3}{4}\epsilon \leq J(x_*(t), u_*(t)) + \epsilon,$$

which contradicts (3.33). Hence $u_*(t)$ is an optimal control in (Q). The Theorem is proved.

Corollary 3.3 *There exists an optimal control $u_*(t)$ in problem (P).*

Indeed, problem (P) is a particular case of (Q) with $\phi(u,t) \equiv 0$.

Now we concretize the function $\phi(u,t)$ in auxiliary problems (Q_k), $k = 1, 2, \dots$ used in the proof of the Theorem 3.1 by such a way that the corresponding sequence $\{u_k(t)\}$, $k = 1, 2, \dots$ of their optimal controls will provide an appropriate (strong in $L_2[0,T]$, $\forall T > 0$) approximation of the given optimal control $u_*(t)$ of problem (P). We need such a strong approximation to derive the desirable necessary optimality conditions for problem (P). As above in the realization of the approximation approach we follow closely the constructions developed in Aseev, Krayazhimskii and Tarasyev, 2001, (a)-(c).

Assume $u_*(t)$ is an optimal control in initial problem (P) and $x_*(t)$ is the corresponding optimal trajectory. Referring to the proof of Theorem 3.1 (stated for problem (Q) more general than problem (P)) we assume that the pair $(x_*(t),\, u_*(t))$ and a positive non-increasing function $\omega(t)$; $\omega(t) \to 0$ as $t \to \infty$ satisfy the property (3.32) with $\phi(u,t) \equiv 0$.

For $k = 1, 2, \dots$ we fix a continuously differentiable function $v_k : [0,\infty) \to R^1$ such that

$$\sup_{t\in[0,\infty)} |v_k(t)| \le b + 1, \tag{3.36}$$

$$\int_0^\infty e^{-\rho t}(v_k(t) - u_*(t))^2 dt \le \frac{1}{k}, \tag{3.37}$$

$$\sup_{t\in[0,\infty)} |\dot{v}_k(t)| \le \sigma_k < \infty. \tag{3.38}$$

It is easy to see that such sequence $\{v_k(t)\}$, $k = 1, 2, \dots$ of continuously differentiable functions $v_k(t)$ exists. Without loss of generality we can assume that $\sigma_k \to \infty$ as $k \to \infty$.

Let us take now a sequence of positive numbers $\{T_k\}$, $k = 1, 2, \dots$ such that $T_k < T_{k+1}$ $\forall k$, $T_k \to \infty$ as $k \to \infty$, and for arbitrary admissible trajectory $x(t)$ of the control system (3.2), (3.3) and $\forall k = 1, 2, \dots$ we have

$$\int_{T_k}^\infty e^{-\rho t}(\kappa \ln x(t) + \ln b) dt \le \frac{1}{k(1+\sigma_k)}, \tag{3.39}$$

$$\omega(T_k) \le \frac{1}{k(1+\sigma_k)}. \tag{3.40}$$

Consider now the sequence of the following auxiliary optimal control problems (P_k), $k = 1, 2, \dots$ each of which is defined on its own time interval $[0, T_k]$:

$$J_k(x(t), u(t)) = \int_0^{T_k} e^{-\rho t}[\kappa \ln x(t) + \ln(b - u(t)) - \frac{(u - v_k(t))^2}{1+\sigma_k}]dt \to \max,$$

$$\dot{x}(t) = u(t)(x(t) + \gamma y(t)),$$

$$\dot{y}(t) = \nu y(t),$$

$$x(0) = x_0, \quad y(0) = y_0,$$

$$u(t) \in [0, b).$$

Here all data of problem (P_k) are the same as in initial problem (P). As usual we are searching for a minimizer of problem (P_k) in a class of all measurable bounded functions $u : [0, T_k] \to [0, b)$.

Note that for each $k = 1, 2, \dots$ problem (P_k) is a particular case of problem (Q_k) with $\phi(u,t) = -(u - v_k(t))^2/(1 + \sigma_k)$ and, hence, due to Corollary 3.1 for every $k = 1, 2, \dots$ there exists an optimal control $u_k(t)$ in problem (P_k).

In what follows we shall assume that for any $k = 1, 2, \ldots$ the control $u_k(t)$ is continued by 0 on the whole time interval $[0, \infty)$. Denote by $x_k(t)$ the trajectory corresponding to $u_k(t)$ on $[0, \infty)$.

Lemma 3.4 *$\forall T > 0$ we have*

$$u_k(t) \to u_*(t) \qquad in \quad L^2[0,T] \quad as \quad k \to \infty.$$

Proof. Let $T > 0$ and let us take a number k_1 such that $T_{k_1} \geq T$. Obviously, for any $k = 1, 2, \ldots$ we have

$$J_k(x_k(t), u_k(t)) = \int_0^{T_k} e^{-\rho t}[\kappa \ln x_k(t) + \ln(b - u_k(t)) - \frac{(u_k(t) - v_k(t))^2}{1 + \sigma_k}]dt \leq$$

$$\leq \int_0^{T_k} e^{-\rho t}[\kappa \ln x_k(t) + \ln(b - u_k(t))]dt - \frac{e^{-\rho T}}{1 + \sigma_k} \int_0^T (u_k(t) - v_k(t))^2 dt.$$

Hence, due to the optimality of $u_k(t)$ in problem (P_k), $k \geq k_1$, optimality of $u_*(t)$ in problem (P), and due to (3.37), (3.39), (3.40) and (3.32) we get

$$\frac{e^{-\rho T}}{1 + \sigma_k} \int_0^T (u_k(t) - v_k(t))^2 dt \leq \int_0^{T_k} e^{-\rho t}[\kappa \ln x_k(t) + \ln(b - u_k(t))]dt - J_k(x_*(t), u_*(t)) \leq$$

$$\leq \int_0^{\infty} e^{-\rho t}[\kappa \ln x_k(t) + \ln(b - u_k(t))]dt - \int_{T_k}^{\infty} e^{-\rho t}[\kappa \ln x_k(t) + \ln b]dt - J_k(x_*(t), u_*(t)) \leq$$

$$\leq J(x_k(t), u_k(t)) - J(x_*(t), u_*(t)) + \omega(T_k) + \frac{1}{k(1 + \sigma_k)} +$$

$$+ \int_0^{\infty} \frac{e^{-\rho t}}{1 + \sigma_k}(v_k(t) - u_*(t))^2 dt \leq \frac{3}{k(1 + \sigma_k)}.$$

Hence we get

$$\int_0^T (u_k(t) - v_k(t))^2 dt \leq \frac{3e^{\rho T}}{k}.$$

Hence

$$(\int_0^T (u_k(t) - u_*(t))^2 dt)^{\frac{1}{2}} \leq (\int_0^T (u_*(t) - v_k(t))^2 dt)^{\frac{1}{2}} +$$

$$+(\int_0^T (u_k(t) - v_k(t))^2 dt)^{\frac{1}{2}} \leq \sqrt{\frac{e^{\rho T}}{k}} + \sqrt{\frac{3e^{\rho T}}{k}} = (\sqrt{3} + 1)\sqrt{\frac{e^{\rho T}}{k}}.$$

Hence $\forall \epsilon > 0$ $\exists k_2 \geq k_1$ such that $\forall k \geq k_2$ the following condition holds:

$$\|u_k(t) - u_*(t)\|_{L^2[0,T]} dt < \epsilon.$$

Hence the assertion of the Lemma holds. The Lemma is proved.

Remark 3.3 *It follows immediately from the assertion of the Lemma* 3.4 *that without loss of generality we can assume that for arbitrary $T > 0$ we have*

$$u_k(t) \to u_*(t) \qquad in \quad L^2[0,T] \quad as \quad k \to \infty;$$

$$x_k(t) \rightrightarrows x_*(t) \qquad on \quad [0,T] \quad as \quad k \to \infty;$$

$$\dot{x}_k(t) \to \dot{x}_*(t) \qquad in \quad L^2[0,T] \quad as \quad k \to \infty.$$

Passing if necessary to subsequence we can assume also that

$$u_k(t) \to u_*(t) \qquad for\ almost\ all \quad t \in [0,T] \quad as \quad k \to \infty.$$

Now we develop an appropriate version of the Pontryagin maximum principle for problem (P) using the limit procedure in the relations of the Pontryagin maximum principle for problem (P_k) as $k \to \infty$.

Note, that due to the logarithmic singularity in the goal functional (3.1) problem (P) does not satisfy to the assumptions of the maximum principle developed in Aseev, Kryazhimskii and Tarasyev, 2001, (a)-(c). First let us introduce some standard notations.

Let

$$\mathcal{H}(x, y, t, u, \psi^1, \psi^2) = u(x + \gamma y)\psi^1 + \nu y \psi^2 + e^{-\rho t}(\kappa \ln x + \ln(b - u))$$

and

$$H(x, y, t, \psi^1, \psi^2) = \sup_{u \in [0,b)} \mathcal{H}(x, y, t, u, \psi)$$

denote the Hamilton–Pontryagin function and the Hamiltonian (maximum function) respectively for problem (P) with the Lagrange multiplier ψ^0 corresponding to the maximized functional $J(x(t), u(t))$ equal 1.

Theorem 3.2 *Let $u_*(t)$ be an optimal control in problem (P) and $x_*(t)$ be the corresponding optimal trajectory. Then there exists an absolutely continuous vector function $\psi(t) : [0, \infty) \to R^2$, $\psi(t) = (\psi^1(t), \psi^2(t))$ such that the following conditions hold:*

1) The vector function $\psi(t)$ is a solution to the adjoint system

$$\dot{\psi}^1(t) = -u_*(t)\psi^1(t) - \kappa \frac{e^{-\rho t}}{x_*(t)}, \tag{3.41}$$

$$\dot{\psi}^2(t) = -\gamma u_*(t)\psi^1(t) - \nu \psi^2(t); \tag{3.42}$$

2) For almost all $t \in [0, \infty)$ the maximum condition takes place:

$$\mathcal{H}(x_*(t), y_*(t), t, u_*(t), \psi^1(t), \psi^2(t)) = H(x_*(t), y_*(t), t, \psi^1(t), \psi^2(t)); \tag{3.43}$$

3) The condition of the asymptotic stationarity of the Hamiltonian is valid:

$$\lim_{t \to \infty} H(x_*(t), y_*(t), t, \psi^1(t), \psi^2(t)) = 0; \tag{3.44}$$

4) The vector function $\psi(t)$ is strictly positive, i.e.

$$\psi^1(t) > 0, \quad \psi^2(t) > 0 \qquad \forall t \geq 0. \tag{3.45}$$

Remark 3.4 *Note, that Theorem* 3.2 *formulated above is identical to the version of the Pontryagin maximum principle for a class of problems with infinite time horizons developed by Aseev, Kryazhimskii and Tarasyev, 2001, (a)-(c). Theorem* 3.2 *is a version of the Pontryagin maximum principle in the so-called normal form. It asserts that the Lagrange multiplier ψ^0 corresponding to the maximizing functional* (3.1) *is strictly positive and hence may be taken to equal* 1*. Further, this result incorporates additional conditions* (3.44) *and* (3.45)*, where the stationarity condition* (3.44) *(which was introduced by Michel, 1982) is analogous to the transversality condition with respect to time in the formulation of the Pontryagin maximum principle for a free time finite horizon optimal control problem (Pontryagin et.al., 1969). Condition* (3.45) *is not standard for the Pontryagin maximum principle; it arises in problems of optimal growth and plays a serious role in our analysis.*

Proof. Let us consider the sequence of auxiliary problems (P_k), $k = 1, 2, \dots$ constructed above. Let $u_k(t)$ be an optimal control in problem (P_k) and let $x_k(t)$ be the corresponding optimal trajectory, $k = 1, 2, \dots$. As shown above (see Remark 3.3) we can assume that $\forall T > 0$

$$u_k(t) \to u_*(t) \quad \text{in} \quad L^2[0, T] \quad \text{as} \quad k \to \infty;$$

$$x_k(t) \rightrightarrows x_*(t) \quad \text{on} \quad [0, T] \quad \text{as} \quad k \to \infty;$$

$$\dot{x}_k(t) \to \dot{x}_*(t) \quad \text{in} \quad L^2[0, T] \quad \text{as} \quad k \to \infty$$

and

$$u_k(t) \to u_*(t) \quad \text{for almost all} \quad t \in [0, T].$$

Due to the Pontryagin maximum principle (Pontryagin et. al., 1969) and Lemma 3.1 (se Corollary 3.1 for problem (P_k), $k = 1, 2, \dots$ there exists an absolutely continuous vector function $\psi_k(t) : [0, T_k] \to R^2$, $\psi_k(t) = (\psi_k^1(t), \psi_k^2(t))$ such that for almost all $t \in [0, T_k]$ the following conditions hold:

$$\dot{\psi}_k^1(t) = -u_k(t)\psi_k^1(t) - \kappa \frac{e^{-\rho t}}{x_k(t)}, \tag{3.46}$$

$$\dot{\psi}_k^2(t) = -\gamma u_k(t)\psi_k^1(t) - \nu \psi_k^2(t), \tag{3.47}$$

$$\mathcal{H}_k(x_k(t), y_k(t), t, u_k(t), \psi_k^1(t), \psi_k^2(t)) = H_k(x_k(t), t, \psi_k^1(t), \psi_k^2(t)) \tag{3.48}$$

and

$$\psi_k^1(T_k) = 0, \qquad \psi_k^2(T_k) = 0. \tag{3.49}$$

Here

$$\mathcal{H}_k(x, y, t, u, \psi^1, \psi^2) = u(x + \gamma y)\psi^1 + \nu y \psi^2 + e^{-\rho t}[\kappa \ln x + \ln(b - u) - \frac{(u - v_k(t))^2}{1 + \sigma_k}]$$

and

$$H_k(x, y, t, \psi^1, \psi^2) = \sup_{u \in [0, b)} \mathcal{H}_k(x, t, u, \psi^1, \psi^2)$$

are the Hamilton–Pontryagin function and the Hamiltonian (maximum function) for problem (P_k), $k = 1, 2, \dots$ in a normal form.[1]

Note that due to relations (3.46), (3.47) and (3.48) of the Pontryagin maximum principle for problem (P_k) the following condition holds for $k = 1, 2, \dots$:

$$\frac{dH_k(x_k(t), y_k(t), t, \psi_k^1(t), \psi_k^2(t))}{dt} \overset{a.e.}{=} \frac{\partial \mathcal{H}_k}{\partial t}(x_k(t), y_k(t), t, u_k(t), \psi_k^1(t), \psi_k^2(t)). \tag{3.50}$$

Further, due to (3.46), (3.47) and (3.49) we have immediately that $\psi_k^i(t) > 0$ $\forall t \in [0, T_k)$, $i = 1, 2$.

Now we show that the sequences $\{|\psi_k^i(0)|\}$, $k = 1, 2, \dots$, $i = 1, 2$ are bounded. For this purpose let us integrate the equality (3.50) on the time interval $[0, T_k]$, $k = 1, 2, \dots$.

Using (3.50) we get

$$H(x_0, 0, \psi_k^1(0), \psi_k^2(0)) = e^{-\rho T_k}[\kappa \ln x_k(T_k) + \sup_{u \in [0, b)} (\ln(b - u) - \frac{(u - v_k(T_k))^2}{1 + \sigma_k})] +$$

[1] Problem (P_k) is a free right end point optimal control problem on the fixed time interval $[0, T_k]$, $k = 1, 2, \dots$. Hence the multiplier ψ^0 can be taken to equal 1.

$$+\rho\int_0^{T_k} e^{-\rho t}[\kappa\ln x_k(t) - \frac{(u_k(t)-v_k(t))^2}{1+\sigma_k}]dt - 2\int_0^{T_k} e^{-\rho t}\frac{(u_k(t)-v_k(t))\dot{v}_k(t)}{1+\sigma_k}dt.$$

It is not difficult to see that due to the conditions (3.36)-(3.38), there exists a constant $K_4 > 0$ such that for all $k = 1, 2, \dots$ we have

$$H_k(x_0, y_0, 0, \psi_k^1(0), \psi_k^2(0)) \le K_4.$$

Hence

$$\sup_{u\in[0,b)}[u(x_0+\gamma y_0)\psi_k^1(0) + \ln(b-u) - \frac{(u-v_k(0))^2}{1+\sigma_k}] + \nu y_0\psi_k^2(0) + e^{-\rho t}\kappa\ln x_0 + \le K_4.$$

From the last inequality we derive

$$\frac{b}{2}(x_0+\gamma y_0)\psi_k^1(0) + \nu y_0\psi_k^2(0) \le K_4 - \kappa\ln x_0 - \ln\frac{b}{2} + \frac{1}{1+\sigma_k}(\frac{b}{2} - v_k(0))^2.$$

As far as $x_0 > 0$, $y_0 > 0$, $b > 0$ and $\nu > 0$ we get the boundedness of the sequences $\{|\psi_k^i(0)|\}$, $k = 1, 2, \dots$, $i = 1, 2$.

Now consider consequently time intervals $[0, T_i]$, $i = 1, 2, \dots$ and sequences $\{u_k(t)\}$, $\{x_k(t)\}$ and $\{\psi_k(t)\}$ on $[0, T_i]$ as $k \to \infty$.

Due to the Bellman–Gronwall inequality (Hartman, 1964), boundedness of the sequence $\{|\psi_k(0)|\}$, $k = 1, 2, \dots$, $i = 1, 2$ and (3.46), (3.47) we may assume that there exists an absolutely continuous vector function $\psi : [0, T_i] \to R^2$, $\psi(t) = (\psi^1(t), \psi^2(t))$ such that

$$\psi_k(t) \rightrightarrows \psi(t) \qquad \text{on} \quad [0, T_i] \quad \text{as} \quad k \to \infty,$$

and

$$\dot{\psi}_k(t) \to \dot{\psi}(t) \qquad \text{weakly in} \quad L^1[0, T_i] \quad \text{as} \quad k \to \infty.$$

Considering the sequence of increasing time intervals $[0, T_i]$ as $i \to \infty$, and passing to a subsequence of $\{\psi_k(t)\}$, $k = 1, 2, \dots$ on each of these time intervals, and taking then a diagonal subsequence we can suppose that there exists an absolutely continuous vector function $\psi : [0, \infty) \to R^n$, such that $\forall T > 0$ we have

$$\psi_k(t) \rightrightarrows \psi(t) \qquad \text{on} \quad [0, T] \quad \text{as} \quad k \to \infty,$$

and

$$\dot{\psi}_k(t) \to \dot{\psi}(t) \qquad \text{weakly in} \quad L^1[0, T] \quad \text{as} \quad k \to \infty.$$

Due to the uniform convergence of the sequence $\{x_k(t)\}$ to $x_*(t)$ as $k \to \infty$ and convergence of $u_k(t)$ to $u_*(t)$ in $L^2[0, T]$ as $k \to \infty$, passing to a limit in (3.46) for almost all $t \in [0, T]$ as $k \to \infty$ we get that due to the Mazur theorem (Mordukhovich, 1988) the absolutely continuous function $\psi(t)$ is a solution to the adjoint system (3.41), (3.42) on time interval $[0, T]$.

Hence the conditions (3.41), (3.42) are proved.

Due to the positiveness of the functions $\psi_k^i(t)$, $k = 1, 2 \dots$, $i = 1, 2$ we have $\psi^i(t) \ge 0$, $i = 1, 2$ $\forall t > 0$. Further, adjoint sysytem (3.41, (3.42) and condition $\psi(t) \ge 0$ implies $\psi^i(t) > 0$ $\forall t \ge 0$, $i = 1, 2$, i.e. condition (3.45) is proved.

Passing to the limit in (3.48) as $k \to \infty$ we get the maximum condition (3.43).

Let us prove now the asymptotic stationarity condition (3.44). To this end let us take an arbitrary $t > 0$ and integrate the equality (3.50) on the time interval $[t, T_k]$ for large numbers k such that $T_k > t$. Due to the equality (3.49) we get

$$H_k(x_k(t), t, \psi_k^1(t), \psi_k^2) = e^{-\rho T_k}[\kappa\ln x_k(T_k) + \max{}_{u\in[0,b)}(\ln(b-u) - \frac{(u-v_k(T_k))^2}{1+\sigma_k})] -$$

$$-\rho \int_t^{T_k} e^{-\rho s}[\kappa \ln x_k(s) + \ln(b - u_k(s)) - \frac{(u_k(s) - v_k(s))^2}{1+\sigma_k}]ds +$$

$$+2 \int_t^{T_k} e^{-\rho s} \frac{(u_k(s) - v_k(s))\dot{v}_k(s)}{1+\sigma_k} ds. \tag{3.51}$$

Further, passing to the limit in the equality (3.51) as $k \to \infty$ we have

$$H(x_*(t), t, \psi^1(t), \psi^2(t)) = \rho \int_t^{\infty} e^{-\rho s}[\kappa \ln x_*(s) + \ln(b - u_*(s))]ds. \tag{3.52}$$

Finally, passing to the limit in the last equality (3.52), as $t \to \infty$ we get the condition (3.44).

The Theorem is proved.

Corollary 3.4 *The following transversality conditions hold:*

$$\lim_{t\to\infty} \psi^1(t)x_*(t) = 0, \qquad \lim_{t\to\infty} \psi^2(t)y(t) = 0; \tag{3.53}$$

moreover, $\forall t \geq 0$ *the following inequality takes place:*

$$e^{\rho t}\psi^1(t)x_*(t) \leq \frac{\kappa}{\rho}. \tag{3.54}$$

Indeed, it is easy to see that the transversality conditions (3.53) are a direct consequence of the asymptotic stationarity condition (3.44), condition (3.45) and positiveness of the parameters b, γ, ν, and the trajectories $x_*(t)$ and $y(t)$.

Let us prove the validity of inequality (3.54). Differentiating the product $\psi^1(t)x_*(t)$ for almost all t we have

$$\frac{d}{dt}\psi^1(t)x_*(t) = \dot{\psi}^1(t)x_*(t) + \psi^1(t)\dot{x}_*(t) = -u_*(t)\psi^1(t)x_*(t) - \kappa e^{-\rho t} +$$

$$+u_*(t)(x_*(t) + \nu y(t))\psi^1(t) \geq -\kappa e^{-\rho t}.$$

Whence, integrating the last inequality on arbitrary time interval $[t, T]$ we get

$$\psi^1(T)x_*(T) - \psi^1(t)x_*(t) \geq -\kappa \int_t^T e^{-\rho s} ds = \frac{\kappa}{\rho}(e^{-\rho T} - e^{-\rho t}).$$

Hence, for all $0 \leq t < T$ we have

$$\psi^1(t)x_*(t) \leq \psi^1(T)x_*(T) + \frac{\kappa}{\rho}(e^{-\rho t} - e^{-\rho T}).$$

Passing to a limit in a last inequality as $T \to \infty$ for arbitrary fixed $t \geq 0$ due to the first of transversality conditions (3.53) we get the inequality (3.54).

4 Problem reformulation and construction of the associated Hamiltonian system

Now we simplify the problem formulation by reducing the dimension of the state variable for 1. Set

$$z(t) = \frac{x(t)}{y(t)}, \qquad z_0 = \frac{x_0}{y_0}.$$

If $(x(t), u(t))$ is a control process, then (see (3.2) (3.3))

$$\dot{z}(t) \stackrel{a.e.}{=} \frac{\dot{x}(t)}{y(t)} - \frac{x(t)}{y^2(t)}\dot{y}(t) \stackrel{a.e.}{=} \frac{u(t)(x(t)+\gamma y(t))}{y(t)} - \frac{x(t)}{y^2(t)}\nu y(t) =$$

$$= u(t)(z(t)+\gamma) - \nu z(t)$$

and (see (3.1))

$$J(x(t), u(t)) = \int_0^\infty e^{-\rho t}[\kappa \ln(z(t)y(t)) + \ln(b-u(t))]dt =$$

$$= \int_0^\infty e^{-\rho t}[\kappa \ln z(t) + \kappa \ln y(t) + \ln(b-u(t))]dt =$$

$$= \int_0^\infty e^{-\rho t}[\kappa \ln z(t) + \kappa \ln\left(y_0 e^{\nu t}\right) + \ln(b-u(t))]dt =$$

$$= \int_0^\infty e^{-\rho t}[\kappa \ln z(t) + \ln(b-u(t))]dt + K_5$$

where

$$K_5 = \int_0^\infty e^{-\rho t}[\kappa \ln y_0 + \kappa \nu t]dt.$$

This observation leads to the next equivalency statement.

Lemma 4.1 *Problem* (P) *is equivalent to the optimal control problem* ($\tilde{\mathrm{P}}$)

$$J(z(t), u(t)) = \int_0^\infty e^{-\rho t}[\kappa \ln z(t) + \ln(b-u(t))]dt \to \max,$$

$$\dot{z}(t) = u(t)(z(t)+\gamma) - \nu z(t), \tag{4.1}$$

$$z(0) = z_0, \tag{4.2}$$

$$u(t) \in [0, b)$$

in the following sense:

1) $u_*(t)$ is an optimal control in problem (P) *if and only if it is an optimal control in problem* ($\tilde{\mathrm{P}}$)*,*

2) the optimal values J_* and J_{**} in problems (P) *and* ($\tilde{\mathrm{P}}$) *are related to each other through $J_* = J_{**} + K_5$.*

Problem ($\tilde{\mathrm{P}}$) introduced in Lemma 4.1 is understood similarly to problem (P). Namely, the *trajectory* of system (4.1) under a control $u(t)$ with the initial state z_0 is the unique Caratheodory solution $z(t)$ on $[0, \infty)$ of equation (4.1) with the initial condition (4.2); a *control process* for system (4.1) is a pair $(z(t), u(t))$ where $u(t)$ is a control and $z(t)$ is the trajectory under $u(t)$.

Due to Lemma 4.1 and Theorem 3.1 (see Corollary 3.3) there exists an optimal control $u_*(t)$ in problem ($\tilde{P}$).

Below we will use the reduced problem ($\tilde{\mathrm{P}}$) for the qualitative analysis of the optimal processes.

Let us introduce now a new adjoint variable $p(t) = e^{\rho t}\psi^1(t)y(t)$, where $\psi^1(t)$ is the adjoint variable corresponding to the optimal trajectory $x_*(t)$ in problem (P) via the Pontryagin maximum principle (Theorem 3.2), and let $z_*(t) = \frac{x_*(t)}{y(t)}$.

Then the function $p(t)$ is strictly positive and due to the condition (3.41) for almost all $t \geq 0$ we have

$$\dot{p}(t) = \rho e^{\rho t}\psi^1(t)y(t) + e^{\rho t}(-u_*(t)\psi^1(t)y(t) - \kappa e^{-\rho t}\frac{y(t)}{x_*(t)} + \nu \psi^1(t)y(t)) =$$

$$= -(u_*(t) - \nu - \rho)p(t) - \frac{\kappa}{z_*(t)}.$$

Further, the maximum condition (3.43) can be rewritten in terms of these new variables $z_*(t)$ and $p(t)$ as follows:

$$u_*(t)p(t)(z_*(t)+\gamma) + \ln(b - u_*(t)) \overset{a.e.}{=} \sup_{u\in[0,b)} (up(t)(z_*(t)+\gamma) + \ln(b-u)).$$

Finally, the condition (3.54) can be rewritten as

$$p(t)z_*(t) \le \frac{\kappa}{\rho} \qquad \forall t \ge 0.$$

Thus, due to Lemma 4.1 we get the following version of the Pontryagin maximum principle for problem $(\tilde{P})$ in terms of variables $z_*(t)$, $p(t)$ as a consequence of Theorem 3.2.

Theorem 4.1 *Let $u_*(t)$ be an optimal control in problem $(\tilde{P})$ and $z_*(t)$ be the corresponding optimal trajectory. Then there exists an absolutely continuous strictly positive function $p(t)$ defined on $[0,\infty)$ such that the following conditions hold:*

1) The function $p(t)$ is a solution to the adjoint system

$$\dot{p}(t) = -(u_*(t) - \nu - \rho)p(t) - \frac{\kappa}{z_*(t)}; \tag{4.3}$$

2) For almost all $t \in [0,\infty)$ the maximum condition takes place:

$$u_*(t)p(t)(z_*(t)+\gamma) + \ln(b - u_*(t)) = \sup_{u\in[0,b)} (up(t)(z_*(t)+\gamma) + \ln(b-u)); \tag{4.4}$$

3) The boundedness condition is valid:

$$p(t)z_*(t) \le \frac{\kappa}{\rho} \qquad \forall t \ge 0. \tag{4.5}$$

Let us construct now the Hamiltonian system associated with the optimal control problem $(\tilde{P})$ via implementation of the Pontryagin maximum principle (Theorem 4.1).

For this we introduce the function $g(z) : (0,\infty) \to (0,\infty)$,

$$g(z) = \frac{1}{b(z+\gamma)} \tag{4.6}$$

and the sets

$$G_1 = \{(z,p) \in R^2 : z > 0, \quad p \ge g(z)\}, \tag{4.7}$$

$$G_2 = \{(z,p) \in R^2 : z > 0, \quad 0 \le p < g(z)\}. \tag{4.8}$$

Obviously, $G_1 \cup G_2 = G$ where

$$G = (0,\infty) \times [0,\infty). \tag{4.9}$$

Define functions $r(z,p) : G \to R^1$ and $s(z,p) : G \to R^1$,

$$r(z,p) = \begin{cases} (b-\nu)z + b\gamma - \dfrac{1}{p}, & \text{if} \quad (z,p) \in G_1, \\ -\nu z, & \text{if} \quad (z,p) \in G_2, \end{cases} \tag{4.10}$$

$$s(z,p) = \begin{cases} -(b-\nu-\rho)p - \dfrac{\gamma\kappa + (\kappa-1)z}{(z+\gamma)z}, & \text{if} \quad (z,p) \in G_1, \\ (\nu+\rho)p - \dfrac{\kappa}{z}, & \text{if} \quad (z,p) \in G_2. \end{cases} \tag{4.11}$$

Remark 4.1 One can easily check that both functions $r(z,p)$ and $s(z,p)$ are continuous.

Lemma 4.2 *Let $(z_*(t), u_*(t))$ be an optimal control process in problem $(\tilde{P})$. Then*

1) there exists a strictly positive function $p(t)$ defined on $[0,\infty)$ such that $(z_*(t), p(t))$ solves the equation

$$\dot{z}(t) = r(z(t), p(t)), \tag{4.12}$$

$$\dot{p}(t) = s(z(t), p(t)), \tag{4.13}$$

(in G) on $[0,\infty)$,

2) for almost all $t \geq 0$

$$u_*(t) = \begin{cases} b - \dfrac{1}{p(t)(z_*(t)+\gamma)}, & \text{if} \quad (z_*(t), p(t)) \in G_1, \\ 0, & \text{if} \quad (z_*(t), p(t)) \in G_2; \end{cases} \tag{4.14}$$

3) $\forall t \geq 0$

$$p(t) z_*(t) \leq \frac{\kappa}{\rho}. \tag{4.15}$$

Remark 4.2 Equation (4.12), (4.13) represents the stationary Hamiltonian system for problem $(\tilde{P})$. In our further analysis we will for convenience split (4.12), (4.13) in two parts:

$$\dot{z}(t) = (b-\nu)z(t) + b\gamma - \frac{1}{p(t)}, \tag{4.16}$$

$$\dot{p}(t) = -(b-\nu-\rho)p(t) - \frac{\gamma\kappa + (\kappa-1)z(t)}{(z(t)+\gamma)z(t)}, \tag{4.17}$$

$$((z(t), p(t)) \in G_1),$$

and

$$\dot{z}(t) = -\nu z(t), \tag{4.18}$$

$$\dot{p}(t) = (\nu+\rho)p(t) - \frac{\kappa}{z(t)}, \tag{4.19}$$

$$((z(t), p(t)) \in G_2).$$

We will call equation (4.16), (4.17) *nondegenerate* and equation (4.18), (4.19) *degenerate*.

Proof of Lemma 4.2. Let $(z_*(t), u_*(t))$ be an optimal control process in problem $(\tilde{P})$. Then by Theorem 4.1 there exists a strictly positive solution $p(t)$ of the adjoint equation (4.3) which satisfies the maximum condition (4.4) and the boundedness condition (4.5).

Consider the maximum condition (4.4). For a given $t \geq 0$, the differentiation of the function

$$h(u) = up(t)(z_*(t)+\gamma) + \ln(b-u)$$

at $u \in [0,b)$ yields

$$\frac{\partial}{\partial u} h(u) = p(t)(z_*(t)+\gamma) - \frac{1}{b-u}.$$

We see that the function $\frac{\partial}{\partial u}h(u)$ decreases and tends to $-\infty$ as $u \to b-0$. Hence, $h(u)$ reaches its maximum at $u \in [0,b)$ such that $\frac{\partial}{\partial u}h(u) = 0$ if $\frac{\partial}{\partial u}h(0) \geq 0$, and at 0 otherwise. Therefore, (4.4) is specified as

$$u_*(t) = \begin{cases} b - \dfrac{1}{p(t)(z_*(t)+\gamma)}, & \text{if} \quad \dfrac{1}{p(t)(z_*(t)+\gamma)} \leq b, \\ 0, & \text{if} \quad \dfrac{1}{p(t)(z_*(t)+\gamma)} > b \end{cases} \tag{4.20}$$

for almost all $t \geq 0$. Substituting (4.20) into the system equation (4.1) and the adjoint equation (4.3), we get

$$\dot{z}_*(t) = \left[b - \frac{1}{p(t)(z_*(t)+\gamma)}\right](z_*(t)+\gamma) - \nu z_*(t) =$$

$$= (b-\nu)z_*(t) + b\gamma - \frac{1}{p(t)},$$

$$\dot{p}(t) = -\left[b - \frac{1}{p(t)(z_*(t)+\gamma)} - \nu - \rho\right]p(t) - \frac{\kappa}{z_*(t)} =$$

$$= -(b-\nu-\rho)p(t) + \frac{1}{z_*(t)+\gamma} - \frac{\kappa}{z_*(t)} =$$

$$= -(b-\nu-\rho)p(t) - \frac{\gamma\kappa + (\kappa-1)z_*(t)}{(z_*(t)+\gamma)z_*(t)}$$

for almost all $t \geq 0$ such that

$$\frac{1}{p(t)(z_*(t)+\gamma)} \leq b,$$

and

$$\dot{z}_*(t) = -\nu z_*(t),$$

$$\dot{p}(t) = (\nu+\rho)p(t) - \frac{\kappa}{z_*(t)}$$

for almost all $t \geq 0$ such that

$$\frac{1}{p(t)(z_*(t)+\gamma)} > b.$$

The Lemma is proved.

5 Qualitative analysis of the Hamiltonian system

In what follows we assume that[2]:

$$\nu > b.$$

The vector field of the Hamiltonian system (4.12), (4.13) in G (see (4.9)) is the union of the vector fields of the nondegenerate equation (4.16), (4.17) in G_1 (see (4.7) and the degenerate equation (4.18), (4.19) in G_2 (see (4.8).

Let us study the structure of the vector field of the nondegenerate equation (4.16), (4.17) in G_1.

Define $h_1(z) : (0, b\gamma/(\nu-b)) \to (0,\infty)$ and $h_2(z) : (0,\infty) \to (0,\infty)$,

$$h_1(z) = \frac{1}{b\gamma - (\nu-b)z}, \tag{5.1}$$

[2]An economic interpretations of this assumption (presented in Introduction as $a\bar{g}^A > L^B$) is that the follower country B is much smaller than the leader country A; see Introducton for details.

$$h_2(z) = \frac{\gamma\kappa + (\kappa - 1)z}{(\nu + \rho - b)(z + \gamma)z}. \tag{5.2}$$

Note that $h_1(z)$ is strictly increasing on $(0, b\gamma/(\nu - b))$,

$$h_1(z) \to \infty \quad \text{as} \quad z \to \frac{b\gamma}{\nu - b} - 0, \tag{5.3}$$

$$h_1(z) > g(z) \quad (z \in (0, b\gamma/(\nu - b))) \tag{5.4}$$

(see (4.6)).

Consider function $h_2(z)$. Obviously,

$$h_2(z) \to \infty \quad \text{as} \quad z \to +0. \tag{5.5}$$

Consider derivative $\frac{d}{dz}h_2(z)$. We have (see (5.2))

$$\frac{d}{dz}h_2(z) = \frac{-(\kappa - 1)z^2 - 2\kappa\gamma z - \kappa\gamma^2}{(\nu + \rho - b)(z + \gamma)^2 z^2}$$

Obviously $\frac{d}{dz}h_2(z) < 0$ for all $z > 0$, if $\kappa \geq 1$. Hence, in this case $h_2(z)$ is positive and it strictly decreases on $(0, \infty)$.

Consider the case $0 < \kappa < 1$. As far as in this case $\kappa < (\nu + \rho)/b$ there exists a unique point $z^9 > 0$,

$$z^0 = \frac{b\gamma\kappa}{\nu - b\kappa + \rho} \tag{5.6}$$

which is a solution of the equation $h_2(z) = g(z)$, i.e. $h_2(z^0) = g(z^0)$. Let us show that function $h_2(z)$ is strictly decreasing on $(0, z^0]$ in this case.

As far as $\frac{d}{dz}h_2(z) < 0$ for all small z and $\frac{d}{dz}h_2(z)$ is continuous on $(0, \infty)$ it can change its sign only at the positive root of the equation

$$(\kappa - 1)z^2 + 2\kappa\gamma z + \kappa\gamma^2 = 0.$$

In the case $\kappa < 1$ the last equation has the unique positive root

$$\tilde{z} = \frac{\gamma\kappa}{1 - \kappa} + \gamma\frac{\sqrt{\kappa}}{1 - \kappa}.$$

Let us show that $z^0 < \tilde{z}$. We have (recall that $b < \nu$ and $\kappa < 1$)

$$z^0 - \tilde{z} = \frac{b\gamma\kappa}{\nu - b\kappa + \rho} - \frac{\gamma\kappa}{1 - \kappa} - \gamma\frac{\sqrt{\kappa}}{1 - \kappa} <$$

$$< \frac{b\gamma\kappa}{b - b\kappa} - \frac{\gamma\kappa}{1 - \kappa} - \gamma\frac{\sqrt{\kappa}}{1 - \kappa} = -\gamma\frac{\sqrt{\kappa}}{1 - \kappa} < 0.$$

Hence, for $0 < \kappa < 1$ we have $\frac{d}{dz}h_2(z) < 0$ for all $0 < z \leq z^0$. Thus the function $h_2(z)$ strictly decrease on $(0, z^0]$, if $0 < \kappa < 1$.

The right hand side of equation (4.16) (for $z(t)$) is zero on the curve

$$V_z^0 = \{(z, p) \in G_1 : p = h_1(z)\}, \tag{5.7}$$

positive in the domain

$$V_z^+ = \{(z, p) \in G_1 : p > h_1(z)\}$$

and negative in the domain

$$V_z^- = \{(z,p) \in G_1 : p < h_1(z)\} \cup \left\{(z,p) \in G_1 : z \geq \frac{b\gamma}{\nu - b}\right\}.$$

The right hand side of equation (4.17) (for $p(t)$) is zero on the curve

$$V_p^0 = \{(z,p) \in G_1 : p = h_2(z)\}, \tag{5.8}$$

positive in the domain

$$V_p^+ = \{(z,p) \in G_1 : p > h_2(z)\}$$

and negative in the domain

$$V_p^- = \{(z,p) \in G_1 : p < h_2(z)\}.$$

Thus, the vector field of the nondegenerate equation (4.16), (4.17) is

(i) positive in both coordinates in the domain

$$V^{++} = V_z^+ \times V_p^+ = \{(z,p) \in G_1 : p > h_1(z),\ p > h_2(z)\}, \tag{5.9}$$

(ii) negative in both coordinates in the domain

$$V^{--} = V_z^- \times V_p^- = \{(z,p) \in G_1 : p < h_1(z),\ p < h_2(z)\}, \tag{5.10}$$

(iii) positive in the z coordinate and negative in the p coordinate in the domain

$$V^{+-} = V_z^+ \times V_p^- = \{(z,p) \in G_1 : p > h_1(z),\ p < h_2(z)\}, \tag{5.11}$$

(iv) negative in the z coordinate and positive in the p coordinate in the domain

$$V^{-+} = V_z^- \times V_p^+ = \{(z,p) \in G_1 : p < h_1(z),\ p > h_2(z)\}. \tag{5.12}$$

The rest points of equation (4.16), (4.17) in G_1 are the solutions of the next system of algebraic equations

$$p = h_1(z), \quad p = h_2(z). \tag{5.13}$$

Relations (5.3), (5.4), (5.5) imply that (5.13) has a solution in G_1. The fact that $h_1(z)$ is strictly increasing and $h_2(z)$ strictly decreasing (at least in the domain where $h_2(z) > g(z)$) impliy that the solution (z^*, p^*) of (5.13) in G_1 is unique. Using the definitions of $h_1(z)$ and $h_2(z)$ (see (5.1) and (5.2)), we find z^* through the next series of equivalent transformations:

$$\frac{\gamma\kappa + (\kappa - 1)z^*}{(\nu + \rho - b)(z^* + \gamma)z^*} = \frac{1}{b\gamma - (\nu - b)z^*},$$

$$(\gamma\kappa + (\kappa - 1)z^*)(b\gamma - (\nu - b)z^*) = (\nu + \rho - b)(z^* + \gamma)z^*,$$

$$\gamma^2 b\kappa - \gamma\kappa(\nu - b)z^* + b\gamma(\kappa - 1)z^* = (\nu + \rho - b + \nu\kappa - b\kappa - \nu + b)z^{*2} + (\nu + \rho - b)\gamma z^*,$$

$$(\kappa(\nu - b) + \rho)z^{*2} + (\kappa\nu - 2\kappa b + \nu + \rho)\gamma z^* - \gamma^2 b\kappa = 0,$$

and finally

$$z^* \in \{z_1^*, z_2^*\}$$

where

$$z_1^* = \frac{\gamma(2b\kappa - \kappa\nu - \nu - \rho) + \gamma[(2b\kappa - \kappa\nu - \nu - \rho)^2 + 4b\kappa(\kappa(\nu - b) + \rho)]^{1/2}}{2(\kappa(\nu - b) + \rho)},$$

$$z_2^* = \frac{\gamma(2b\kappa - \kappa\nu - \nu - \rho) - \gamma[(2b\kappa - \kappa\nu - \nu - \rho)^2 + 4b\kappa(\kappa(\nu - b) + \rho)]^{1/2}}{2(\kappa(\nu - b) + \rho)}.$$

We have $z^* = z_1^*$, for $z_2^* < 0$. Employing the first equation in (5.13), we provide the final formulas for the unique rest point of the nondegenerate equation (4.16), (4.17) in domain G_1:

$$z^* = \frac{\gamma(2b\kappa - \kappa\nu - \nu - \rho) + \gamma[(2b\kappa - \kappa\nu - \nu - \rho)^2 + 4b\kappa(\kappa(\nu - b) + \rho)]^{1/2}}{2(\kappa(\nu - b) + \rho)} =$$

$$= \gamma\frac{2b\kappa - (\kappa+1)\nu - \rho + [((\kappa+1)\nu + \rho)^2 - 4b\kappa^2\nu]^{1/2}}{2(\kappa(\nu - b) + \rho)}, \tag{5.14}$$

$$p^* = h_1(z^*) = \frac{1}{b\gamma - (\nu - b)z^*}. \tag{5.15}$$

Note that due to (5.4) we have

$$p^* = h_1(z^*) > g(z^*), \tag{5.16}$$

i.e., (z^*, p^*) lies in the interior of G_1 (see (4.7)).

Now let us describe the structure of the vector field of the degenerate system (4.18), (4.19) in G_2

Define $h(z) : (0, \infty) \to (0, \infty)$,

$$h(z) = \frac{\kappa}{(\nu + \rho)z}. \tag{5.17}$$

The vector field of the degenerate equation (4.18), (4.19) is

(i) negative in the z coordinate and zero in the p coordinate in the domain

$$W_p^0 = \{(z,p) \in G_2 : p = h(z)\}, \tag{5.18}$$

(ii) negative in both coordinates in the domain

$$W^{--} = \{(z,p) \in G_2 : p < h(z)\}, \tag{5.19}$$

(iii) negative in the z coordinate and positive in the p coordinate in the domain

$$W^{-+} = \{(z,p) \in G_2 : p > h(z)\}. \tag{5.20}$$

Let us analyze how the vector fields of the nondegenerate equation (in G_1) and degenerate equation (in G_2) are pasted together. Note that G_1 and G_2 are separated by the curve

$$G^0 = \{(z,p) \in R^2 : z > 0,\ p = g(z)\}$$

(see (4.7) and (4.8)). Inequality (5.4) shows that curve V_z^0 (5.7) does not intersect G^0.

Note that, two cases are possible depending on the values of the parameters b, κ, ν and ρ:

$$a)\ \kappa < \frac{\nu + \rho}{b}$$

or

$$b)\ \kappa \geq \frac{\nu + \rho}{b}.$$

Consider case (a). In this case curve V_p^0 intersects G^0 at point $(z^0, g(z^0))$ (see (5.6)), it goes down and lies above G^0 (on the (z,p) plane) in the strip $\{(z,p) : 0 < z < z^0,\ p \geq 0\}$; more accurately,

$$h_2(z) > g(z)\ \ (z < z^0), \quad h_2(z^0) = g(z^0), \quad h_2(z) < g(z)\ \ (z > z^0). \tag{5.21}$$

Indeed, using (5.2) and (4.6), we get the next sequence of equivalent transformations:

$$h_2(z) > g(z),$$

$$\frac{\gamma\kappa + (\kappa - 1)z}{(\nu - b + \rho)(z + \gamma)z} > \frac{1}{b(z+\gamma)},$$

$$\frac{\gamma\kappa + (\kappa - 1)z}{(\nu - b + \rho)z} > \frac{1}{b},$$

$$b\gamma\kappa + b(\kappa - 1)z > (\kappa - b + \rho)z,$$

$$z^0 > z.$$

Note that (5.16) implies $p^* = h_2(z^*) > g(z^*)$; consequently by (5.21)

$$z^0 > z^*. \tag{5.22}$$

Formulas (4.6) and (5.17) show that in this case curve W_p^0 (5.18) also intersects G^0 at point $(z^0, g(z^0))$ and lies below it in the strip $\{(z,p) : z > z^0,\ p \geq 0\}$; more accurately,

$$h(z) > g(z) \quad (z < z^0), \quad h(z^0) = g(z^0), \quad h(z) < g(z) \quad (z > z^0). \tag{5.23}$$

Indeed,

$$h(z) < g(z)$$

is equivalently transformed as follows:

$$\frac{\kappa}{z(\nu + \rho)} < \frac{1}{b(z+\gamma)},$$

$$b\kappa(z + \gamma) < z(\nu + \rho),$$

$$b\gamma\kappa < (\nu - b\kappa + \rho),$$

$$z^0 < z.$$

Therefore,

$$\inf\{z : (z,p) \in W^{-+}\} = z^0. \tag{5.24}$$

Relations (5.23) show that in case (a) the vector field of the entire Hamiltonian system (4.12), (4.13) (in G (4.9)) changes the sign in the p coordinate on the (continuous) curve

$$L_p^0 = \{(z,p) : p = h_2(p),\ 0 < z \leq z^0\} \cup \{(z,p) : p = h(p),\ z > z^0\}. \tag{5.25}$$

Consider now the case (b), $\kappa \geq (\nu + \rho)/b$. Due to (4.6), (5.2) and (5.17) for all $z > 0$ we have

$$h_2(z) > g(z), \qquad h(z) > g(z).$$

Thus, in this case both curves V_p^0 and W_p^0 lie above the curve G^0, the set G_2 coincides with W^{--} and $W^{-+} = \emptyset$. Hence, in this case the vector field of the entire Hamiltonian system (4.12), (4.13) (in G (4.9)) changes its sign in the p coordinate at the (continuous) curve V_p^0 (5.8).

We end up with the next description of the vector field of (4.12), (4.13).

Lemma 5.1 *The vector field of the Hamiltonian system* (4.12), (4.13) *(in G) is*

(i) positive in both coordinates in domain V^{++} (see (5.9)*),*

(ii) negative in both coordinates in domain $V^{--}\cup W^{--}$ (see (5.10) *and* (5.19)*),*

(iii) positive in the z coordinate and negative in the p coordinate in domain V^{+-} (see (5.11)*),*

(iv) negative in the z coordinate and positive in the p coordinate in domain $V^{-+}\cup W^{-+}$ (see (5.12) *and* (5.20)*), if $\kappa<\frac{\nu+\rho}{b}$, or in V^{-+}, if $\kappa\geq\frac{\nu+\rho}{b}$ (in this case $W^{-+}=\emptyset$).*

(v) zero in the z coordinate on curve V_z^0 (see (5.7)*), and*

(vi) zero in the p coordinate on curve L_p^0 (see (5.25)*) if $\kappa<\frac{\nu+\rho}{b}$. In this case curve V_p^0 intersects G_0 at point $(z^0,g(z^0))$.*

(vii) zero in the p coordinate on curve V_p^0 (see (5.8) *if $\kappa\geq\frac{\nu+\rho}{b}$. In this case curve V_p^0 lies strictly above G_0 and $W^{-+}=\emptyset$.*

The rest point (z^*,p^*) of (4.12), (4.13) *in G is unique; it is defined by* (5.14), (5.15) *and lies in the interior of G_1.*

The vector field of system (4.12), (4.13) in case (a) is shown in Fig. 1.

The vector field of system (4.12), (4.13) in case (b) is similar. In this case curve V_p^0 lies strictly above G_0 and $W^{-+}=\emptyset$.

Lemma 5.1 allows us to give a full classification of the qualitative behaviors of the solutions of the Hamiltonian system (4.12), (4.13) in G (see also Fig. 1).

In what follows, cl$E$ denotes the closure of a set $E\subset R^2$.

Lemma 5.2 *Let $(z(t),p(t))$ be a solution of* (4.12), (4.13) *in G, which is nonextendable to the right, Δ be the interval of its definition, $t_*\in\Delta$, and $(z(t_*),p(t_*))\neq(z^*,p^*)$.*

The following statements are true.

1. If $(z(t_*),p(t_*))\in\mathrm{cl}(V^{--}\cup W^{--})$, then Δ is bounded, $(z(t),p(t))\in V^{--}\cup W^{--}$ for all $t\in\Delta\cap(t_*,\infty)$, and $p(\vartheta)=0$ where $\vartheta=\sup\Delta$.

2. If $(z(t_*),p(t_*))\in\mathrm{cl}V^{++}$, then Δ is unbounded, $(z(t),p(t))\in V^{++}$ for all $t\in(t_*,\infty)$ and

$$\lim_{t\to\infty}z(t)=\frac{b\gamma}{\nu-b},\tag{5.26}$$

$$\lim_{t\to\infty}p(t)=\infty.\tag{5.27}$$

3. If $(z(t_*),p(t_*))\in V^{+-}$, then one of the next cases (i), (ii), (iii) takes place:

(i) Δ is bounded and $(z(t),p(t))\in\mathrm{cl}(V^{--}\cup W^{--})$ for all $t\in\Delta\cap[t^*,\infty)$ with some $t^*>t_*$;

(ii) Δ is unbounded, $(z(t),p(t))\in\mathrm{cl}V^{++}$ for all $t\geq t^*$ with some $t^*>t_*$ and relations (5.26) *and* (5.27) *hold;*

(iii) Δ is unbounded, $(z(t),p(t))\in\mathrm{cl}V^{+-}$ for all $t\geq t_*$ and

$$\lim_{t\to\infty}z(t)=z^*,\tag{5.28}$$

$$\lim_{t\to\infty}p(t)=p^*.\tag{5.29}$$

4. If $(z(t_*),p(t_*))\in V^{-+}\cup W^{-+}$, then one of the next cases (i), (ii), (iii) takes place:

(i) Δ is bounded and $(z(t),p(t))\in\mathrm{cl}(V^{--}\cup W^{--})$ for all $t\in\Delta\cap(t^*,\infty)$ with some $t^*>t_*$;

(ii) Δ is unbounded, $(z(t),p(t))\in\mathrm{cl}V^{++}$ for all $t\in[t_*,\infty)$ with some $t^*\in\Delta\cup[t_*,\infty)$ and relations (5.26) *and* (5.27) *hold;*

(iii) Δ is unbounded, $(z(t),p(t))\in\mathrm{cl}V^{-+}$ for all $t\in[t_*,\infty)$ and relations (5.28) *and* (5.29) *hold.*

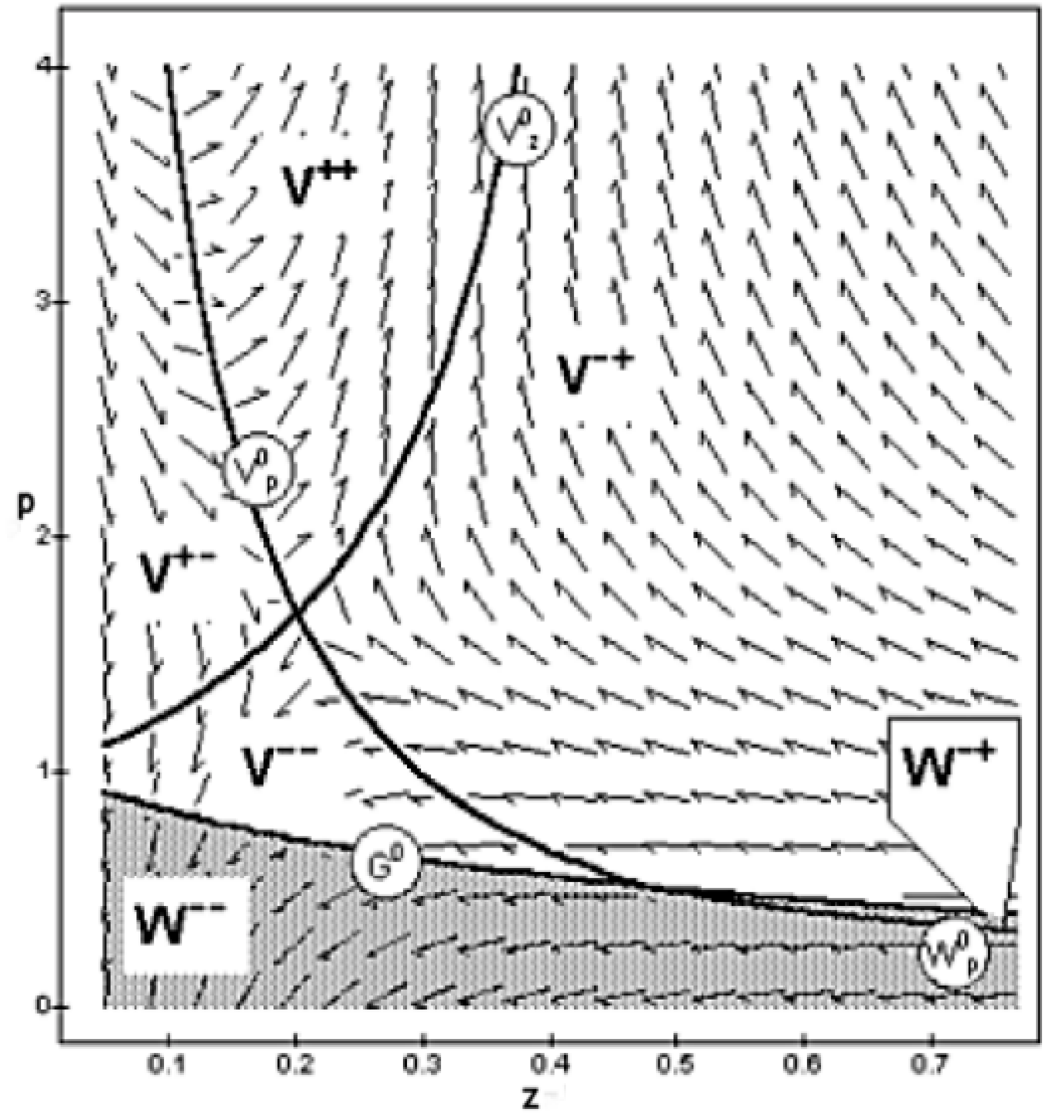

Figure 1: The vector field of the Hamiltonian system (4.12), (4.13) for $\nu = 4$, $b = 2$, $\kappa = 1$, $\rho = 0.1$, $\gamma = 0.5$ (a Maple simulation).

Proof. 1. Let $(z(t_*), p(t_*)) \in \mathrm{cl}(V^{--} \cup W^{--})$. The fact that the vector field of (4.12), (4.13) is negative in both coordinates in $V^{--} \cup W^{--}$ (Lemma 5.1) and the locations of V^{--} and W^{--} in G (see (5.10) and (5.19)) imply that the set $\mathrm{cl}(V^{--} \cup W^{--}) \cap \{(z,p) \in G : p \leq p(t_*)\}$ is invariant for (4.12), (4.13); moreover, $(z(t), p(t)) \in V^{--} \cup W^{--}$ for all $t \in \Delta \cap (t_*, \infty)$ and there are $\delta > 0$ and $t^* \in \Delta \cap [t^*, \infty)$ such that $\dot{p}(t) \leq -\delta$ for all $t \geq \Delta \cap [t^*, \infty)$. Hence, $p(\vartheta) = 0$ for some finite ϑ, i.e., $(z(t), p(t))$ is nonextendable to the right in G and $\vartheta = \sup \Delta$.

2. Let $(z(t_*), p(t_*)) \in \mathrm{cl}V^{++}$. The fact that the vector field of (4.12), (4.13) is positive in both coordinates in V^{++} (Lemma 5.1) and the location of V^{++} in G (see (5.9) imply that the set $\mathrm{cl}V^{++} \cap \{(z,p) \in G : p \geq p(t_*)\}$ is invariant for (4.12), (4.13); moreover, $(z(t), p(t)) \in V^{++}$ for all $t \in \Delta \cap (t_*, \infty)$ and there are a $\delta > 0$ and a $t^* \in \Delta \cap [t^*, \infty)$ such that $\dot{p}(t) \geq \delta$ and $\dot{z}(t) > 0$ for all $t \geq \Delta \cap [t^*, \infty)$. Therefore, Δ is unbounded and (5.27) holds. Now (5.3) and the fact that $\dot{z}(t) > 0$ for all $t \geq \Delta \cap [t^*, \infty)$ imply (5.26).

3. Let $(z(t_*), p(t_*)) \in V^{+-}$. Due to the definitions of V^{+-}, V_z^0 and V_p^0 (see (5.11), (5.7 and (5.8), three cases are admissible: $(z(t^*), p(t^*)) \in V_z^0$ for some $t^* \geq t_*$ (case 1), $(z(t^*), p(t^*)) \in V_p^0$ for some $t^* \geq t_*$ (case 2), and $(z(t^*), p(t^*)) \in V^{+-}$ for all $t \in \Delta$ (case 3). Note that assumption $(z(t_*), p(t_*)) \neq (z^*, p^*)$ implies that $(z(t), p(t)) \neq (z^*, p^*)$ for all $t \in \Delta$ (we refer to the theorem of the uniqueness of the solution of a Cauchy problem for a differential equation with a Lipschitz right hand side). Therefore, in case 1 we have the situation described in statement 1 (with t_* replaced by t^*); hence, (i) holds. Smilarly, in case 2 (ii) holds due to statement 2. Let case 3 take place. If $\vartheta = \sup \Delta < \infty$, then $(z(\vartheta), p(\vartheta))$ belongs to the interior of G; hence, $(z(t), p(t))$ is extendable to the right, which contradicts the assumption that $(z(t), p(t))$ is nonextendable to the right. Therefore, Δ is unbounded. Functions $z(t)$ and $p(t)$ are increasing and bounded; consequently,

$$z(t) \to z_1 \quad \text{as} \quad t \to \infty, \tag{5.30}$$

$$z(t) \leq z_1 \quad (t \in \Delta), \tag{5.31}$$

$$p(t) \to p_1 \quad \text{as} \quad t \to \infty,$$

$$p(t) \leq p_1 \quad (t \in \Delta).$$

Suppose $(z_1, p_1) \neq (z^*, p^*)$. Then one of the right hand sides $r(z_1, p_1)$, $s(z_1, p_1)$ of the Hamiltonian system (4.12), (4.13) is positive at point (z_1, p_1). Let, for example $r(z_1, p_1) > \delta > 0$. By (5.30) $\dot{z}(t) = r(z(t), p(t)) > \delta/2$ for all sufficiently large t. Then, referring to (5.30) again, we find that $z(t) > z_1$ for all sufficiently large t, which contradicts (5.31). Similarly, we arrive at a contradiction if we assume $s(z_1, p_1) > \delta > 0$. Thus, $(z_1, p_1) = (z^*, p^*)$. and we get (5.28) and (5.29). Statement 3 is proved.

4. A justification of statement 4 is similar to that of statement 3.

The Lemma is proved.

6 Optimal control process

In this section we give an entire description of a solution of problem $(\tilde{P})$ and state its uniqueness.

The core of the analysis is Lemma 6.2 which selects solutions of the Hamiltonian system (4.12), (4.13) (we call them equilibrium solutions) whose qualitative behavior agrees with the Pontryagin maximum principle (Theorem 4.1) and also acts as a necessary condition for the global optimality in problem $(\tilde{P})$.

We call a solution $(z(t), p(t))$ (in G) of the Hamiltonian system (4.12) (4.13) an *equilibrium solution* if it is defined on $[0, \infty)$ and converges to the rest point (z^*, p^*), i.e. satisfies

(5.28) and (5.29). Let us formulate additional properties of equilibrium solutions based on Lemma 5.2.

Lemma 6.1 *Let* $(z(t), p(t))$ *be an equilibrium solution of the Hamiltonian system* (4.12) (4.13). *Then*

(i) $z(0) = z^*$ *implies that* $(z(t), p(t)) = (z^*, p^*)$ *for all* $t \geq 0$,

(ii) $z(0) < z^*$ *implies that* $(z(t), p(t)) \in V^{+-}$ *for all* $t \geq 0$,

(iii) $z(0) > z^*$ *implies that* $(z(t), p(t)) \in V^{-+} \cup W^{-+}$ *for all* $t \geq 0$.

Proof. Prove (i). Let $z(0) = z^*$. If $p(0) < p^*$, then $(z(0), p(0)) \in \mathrm{cl}V^{--} \cup \mathrm{cl}W^{--}$ (see (5.10)). Hence, by Lemma 5.2, (statement 1) the interval of definition of $(z(t), p(t))$ is bounded, which is not the case. If $p(0) > p^*$, then $(z(0), p(0)) \in V^{++}$ (see (5.9)). Hence, by Lemma 5.2 (statement 2) (5.26), (5.27), hold, which contradicts (5.28), (5.29). Thus, $p(0) = p^*$. Due to the uniqueness of the solution of a Cauchy problem for system (4.12), (4.13) we have $(z(t), p(t)) = (z^*, p^*)$ for all $t \geq 0$.

Prove (ii). Let $z(0) < z^*$. Then

$$(z(0), p(0)) \notin \mathrm{cl}V^{-+} \cup \mathrm{cl}W^{-+}. \tag{6.1}$$

Indeed, the definition of V^{-+} (5.12), the facts that $h_1(t)$ is strictly increasing and $h_2(t)$ strictly decreasing (at least in the domain where $h_2(z) > g(z)$; see Section 5) and equality $p^* = h_1(z^*) = h_2(z^*)$ imply that $z \geq z^*$ for every $(z, p) \in \mathrm{cl}V^{-+}$. Furthermore, by (5.24) and (5.22) $z(0) \notin W^{-+}$ In view of (6.1) three cases are admissible: $(z(0), p(0)) \in \mathrm{cl}V^{--} \cup \mathrm{cl}W^{--}$ (case 1), $(z(0), p(0)) \in \mathrm{cl}V^{++}$ (case 2), and $(z(0), p(0)) \in V^{+-}$. In case 1 by Lemma 5.2 (statement 1) the interval of definition of $(z(t), p(t))$ is bounded, which is a contradiction. In case 2 by Lemma 5.2 (statement 2) we have (5.26), (5.27), which contradicts (5.28), (5.29). Therefore, case 3 takes place. For this case statement 3 of Lemma 5.2 holds. Situations (i) and (ii) of this statement do not take place (see the above argument). Therefore, we have situation (iii) of this statement, which proves (ii) in the present Lemma.

Statement (iii) is proved similarly.

The Lemma is proved.

Lemma 6.2 *Let* $(z_*(t), u_*(t))$ *be an optimal control process in problem* $(\tilde{P})$. *Then*

(i) there exists a strictly positive function $p(t)$ *such that* $(z(t), p(t))$ *is an equilibrium solution of the Hamiltonian system* (4.12), (4.13);

(ii) for almost all $t \geq 0$ (4.14) *holds,*

(iii) for all $t \geq 0$

$$p(t) z_*(t) \leq \frac{\kappa}{\rho}.$$

Proof. By Lemma 4.2 there exists a strictly positive function $p(t)$ defined on $[0, \infty)$ such that $(z(t), p(t))$ solves (4.12) (4.13) (in G) on $[0, \infty)$, for almost all $t \geq 0$ (4.14) holds and $p(t) z_*(t) \leq \kappa/\rho$ for all $t \geq 0$. These facts prove statements (i) and (iii). In order to prove (ii) it remains to show that $z(t)$, $p(t)$ is an equilibrium solution of the Hamiltonian system (4.12), (4.13).

According to Lemma 5.2 three cases are admissible:

Case 1: the interval of definition of $(z(t), p(t))$ is bounded (Lemma 5.2, statement 1, statement 3, (i), and statement 4, (i)).

Case 2: relations (5.26), (5.27) hold (Lemma 5.2, statement 2, statement 3, (ii), and statement 4, (ii)).

Case 3: relations (5.28), (5.29) hold, i.e., $(z(t),p(t))$ is equilibrium (Lemma 5.2, statement 3, (iii), and statement 4, (iii)).

Case 1 is not possible since $(z(t),p(t))$ is defined on $[0,\infty)$.

Statement (iii) shows that case 2 is not possible either. By excluding cases 1 and 2 we state that case 3 takes place.

The Lemma is proved.

Lemma 6.3 *For every $z_0 > 0$ there exists a unique equilibrium solution $(z_0(t),p(t))$ of the Hamiltonian system* (4.12), (4.13), *which satisfies $z_0(0) = z_0$.*

Proof. Suppose $z_0 = z^*$. By Lemma 6.1 for any equilibrium solution $(z(t),p(t))$ of (4.12), (4.13) such that $z(0) = z_0 = z^*$ we have $(z(t),p(t)) = (z^*,p^*)$ $(t \geq 0)$, which completes the proof.

Let $z_0 < z^*$. The existence of a desired equilibrium solution follows from the existence of an optimal control process. Indeed, by Theorem 3.1 (see Corollary 3.3) there exists an optimal control process $(z_*(t),u_*(t))$ in problem $(\tilde{P})$ with initial condition $z_*(0) = z_0$.

By Lemma 6.2, (i), there exists a function $p(t)$ such that $(z_*(t),p(t))$ is an equilibrium solution of the Hamiltonian system (4.12) (4.13). (Note that the existence of a desired equilibrium solution can also be proved explicitly).

Let us state the uniqueness of the considered equilibrium solution. Suppose there are two different equilibrium solutions of (4.12), (4.13), $(z_1(t),p_1(t))$ and $(z_2(t),p_2(t))$, such that $z_1(0) = z_2(0) = z_0$. Then

$$\lim_{t\to\infty} z_i(t) = z^*, \quad \lim_{t\to\infty} p_i(t) = p^*, \tag{6.2}$$

$i = 1,2$, and $p_2(0) \neq p_1(0)$ (otherwise $(z_2(t),p_2(t))$ and $(z_1(t),p_1(t))$ coincide due to the uniqueness of the solution of a Cauchy problem for equation (4.12), (4.13)). Denote $p_i^0 = p_i(0)$, $i = 1,2$. Without loss of generality assume

$$p_2^0 > p_1^0. \tag{6.3}$$

By Lemma 6.1

$$(z_i(t),p_i(t)) \in V^{+-} \quad (t \geq 0),$$

$i = 1,2$. Hence, $\dot{z}_i(t) > 0$ $(t > 0)$, $i = 1,2$. Define $\bar{p}_i : [z_0,z^*) \to [p_i(0),\infty)$ by $\bar{p}_i(\zeta) = p_i(z_i^{-1}(\zeta))$ Due to (6.2) $\lim_{z\to z_*} \bar{p}_i(z) = p^*$, $i = 1,2$, in particular,

$$\lim_{z\to z_*} (\bar{p}_2(z) - \bar{p}_1(z)) = 0. \tag{6.4}$$

We have

$$\frac{d}{dz}\bar{p}_i(z) = f(z,\bar{p}_i(z)), \quad (z \in [z_0,z^*)), \quad p(z_0) = p_i(0), \tag{6.5}$$

$i = 1,2$, where

$$f(z,p) = \frac{s(z,p)}{r(z,p)} \tag{6.6}$$

(recall that $r(z,p)$ and $s(z,p)$ determine the right hand side of the Hamiltonian system (4.12), (4.13)). Due to (6.3)

$$\bar{p}_2(z) > \bar{p}_1(z) \quad (z \in [z_0,z^*)). \tag{6.7}$$

For $(z,p) \in V^{+-} \subset G_1$ (see (4.7), (4.10), (4.11), (5.4)), (5.11)) we have

$$r(z,p) = (b-\nu)z + b\gamma - \frac{1}{p} > 0,$$

$$s(z,p) = (\nu - b + \rho)p - \frac{\gamma\kappa + (\kappa-1)z}{(z+\gamma)z} < 0;$$

hence,

$$\begin{aligned}\frac{\partial f(z,p)}{\partial p} &= \left(\frac{\partial s(z,p)}{\partial p} r(z,p) - \frac{\partial r(z,p)}{\partial p} s(z,p)\right)\frac{1}{r^2(z,p)} \\ &= \left((\nu - b + \rho) r(z,p) - \frac{1}{p^2} s(z,p)\right)\frac{1}{r^2(z,p)} > 0.\end{aligned}$$

Then, in view of (6.7) and (6.5),

$$\frac{d}{dz}\bar{p}_2(z) - \frac{d}{dz}\bar{p}_1(z) \geq 0 \quad (z \in [z_0, z^*)).$$

Hence (see (6.5) again),

$$\bar{p}_2(z) - \bar{p}_1(z) \geq p_2^0 - p_1^0 \quad (z \in [z_0, z^*)),$$

which contradicts (6.4). The contradiction completes the proof for $z_0 < z^*$.

The case $z_0 > z^*$ is treated similarly.

The Lemma is proved.

Given a $z_0 > 0$, the equilibrium solution $(z_0(t), p(t))$ of the Hamiltonian system (4.12), (4.13), which satisfies $z_0(0) = z_0$ (and whose uniqueness has been stated in Lemma 6.3) will further be said to be *determined by* z_0.

Lemmas 6.2 and 6.3 yield the next characterization of a solution of problem $(\tilde{P})$.

Theorem 6.1 *Let $(z_0(t), p(t))$ be the equilibrium solution of the Hamiltonian system* (4.12), (4.13) *which is determined by z_0. A control process $(z_*(t), u_*(t))$ is optimal in problem $(\tilde{P})$ if and only if $z_*(t) \equiv z_0(t)$ and* (4.14) *holds for almost all $t \geq 0$.*

Proof. Necessity. Let a control process $(z_*(t), u_*(t))$ be optimal in problem $(\tilde{P})$. By Lemmas 6.2 and 6.3 $z_*(t) \equiv z_0(t)$ and (4.14) holds for almost all $t \geq 0$.

Sufficiency. Consider a control process $(z_0(t), u_0(t))$ $t \geq 0$. Suppose $(z_0(t), u_0(t))$ is not optimal in problem $(\tilde{P})$. By Theorem 3.1 there exists an optimal control process $(z_*(t), u_*(t))$. By Lemmas 6.2 and 6.3 $z_*(t) = z_0(t)$ and (4.14), where $u_*(t)$ is replaced by $u_0(t)$, holds for almost all $t \geq 0$. Hence, $z_0(t) \equiv z_*(t)$ and $u_0(t) = u_*(t)$ for almost all $t \geq 0$. Therefore, $(z_0(t), u_0(t))$ is optimal, which contradicts the assumption. The contradiction completes the proof.

Theorem 6.1 and Lemma 6.3 imply the next uniqueness result.

Corollary 6.1 *The optimal control process in problem $(\tilde{P})$ is unique in the following sense: if $(z_1(t), u_1(t))$ and $(z_2(t), u_2(t))$ are optimal control processes in problem $(\tilde{P})$, then $z_1(t) = z_2(t)$ for all $t \geq 0$ and $u_1(t) = u_2(t)$ for almost all $t \geq 0$.*

Theorem 6.1 provides the next algorithm to finding the solution $(z_*(t), u_*(t))$ of problem $(\tilde{P})$:

1. Find the equilibrium solution $(z_0(t), p(t))$ of the Hamiltonian system (4.12), (4.13) which is determined by z_0.
2. Set $z_*(t) \equiv z_0(t)$ and define $u_*(t)$ by (4.14) $(t \geq 0)$.

7 Optimal synthesis

In this section we consider the family of problems $(\tilde{P})$ parametrized by the initial state $z_0 > 0$ and describe an optimal synthesis for this family (see Pontryagin, et. al., 1969, p. 51), i.e., define a feedback which solves problem $(\tilde{P})$ with arbitrary z_0.

In this paper we define a *feedback* to be an arbitrary continuous function $U(z)$: $(0, \infty) \to [0, b)$ such that for every $z_0 > 0$ the equation

$$\dot{z}(t) = U(z(t))(z(t) + \gamma) - \nu z(t) \tag{7.1}$$

has the unique solution $z(t)$ defined on $[0, \infty)$ and satisfying $z(0) = z_0$; we call $z(t)$ the *motion* (of system (4.1)) under feedback $U(z)$ with the initial state z_0.

Remark 7.1 Equation (7.1) represents the original control system (4.1) with control values $u(t)$ formed on the basis of current states $z(t)$ via feedback $U(z)$: $u(t) = U(z(t))$ $(t \geq 0)$. According to a terminology oftenly used in control theory (7.1) is the control system (4.1) *closed* with feedback $U(z)$.

Given a feedback $U(z)$ and a $z_0 > 0$, we define the *control process* under $U(z)$ with the initial state z_0 to be the pair $(z(t), u(t))$ where $z(t)$ is the motion under feedback $U(z)$ with the initial state z_0 and $u(t) = U(z(t))$; obviously, $(z(t), u(t))$ is a control process for system (4.1) (see the definition of a control process in section 3). We call a feedback $U_*(z)$ an *optimal synthesis* if for every $z_0 > 0$ the control process under $U_*(z)$ with the initial state z_0 is an optimal control process in problem $(\tilde{P})$.

In the construction of an optimal feedback, our main instrument will be one-dimensional representations of the equilibrium solutions of the Hamiltonian system (4.12), (4.13). These are functions $\bar{p}(z)$ solving the one-dimensional equation

$$\frac{d}{dz}\bar{p}(z) = f(z, \bar{p}(z)) \tag{7.2}$$

which is derived from (4.12), (4.13) by deviding its second component by the first one. Thus, in (7.2) and in what follows $f(z, p)$ is defined by (6.6). Note that the domain of definition of $f(z, p)$ is $\mathrm{dom} f = G \setminus V_z^0$ (see (4.9) and (5.7)); therefore, solutions of (7.2) are understood as those in $\mathrm{dom} f = G \setminus V_z^0$ (i.e., by definition every solution $\bar{p}(z)$ of (7.2) satisfies $(z, \bar{p}(z)) \in G \setminus V_z^0$ for any z from the domain of its definition). A positive solution $\bar{p}(z)$ of (7.2) (in $G \setminus V_z^0$) will be called

(i) a *left equilibrium solution* if $\bar{p}(z)$ is defined on $(0, z^*)$ and

$$\lim_{z \to z^*} \bar{p}(z) = p^*, \tag{7.3}$$

(ii) a *right equilibrium solution* if $\bar{p}(z)$ is defined on (z^*, ∞) and (7.3) holds.

Lemma 7.1 *1. Let $\bar{p}(z)$ be a left equilibrium solution of* (7.2). *Then*

$$(z, \bar{p}(z)) \in V^{+-} \quad (z \in (0, z^*)).$$

2. Let $\bar{p}(z)$ be a right equilibrium solution of (7.2). *Then*

$$(z, \bar{p}(z)) \in V^{-+} \cup W^{-+} \quad (z \in (z^*, \infty)).$$

Proof. We will prove statement 1 only (the proof of statement 2 is similar). Suppose statement 1 is not true, i.e., $(z_0, \bar{p}(z_0)) \notin V^{+-}$ for some $z_0 \in (0, z^*)$. Let $(z(t), p(t))$ be the nonextendable solution of the Hamiltonian system (4.12), (4.13) (in G), which satisfies $(z(0), p(0)) = (z_0, \bar{p}(z_0))$. The definition of V^{-+} (5.12) and relations (5.22), (5.24) show that $z_0 < z^*$ yields $(z(0), p(0)) \notin V^{-+} \cup W^{-+}$. Therefore,

$$(z(0), p(0)) = (z_0, \bar{p}(z_0)) \in \mathrm{cl}(V^{--} \cup W^{--}) \cup \mathrm{cl}V^{++}.$$

Note that $(z_0, \bar{p}(z_0))$ lies in $\mathrm{dom}f = G \setminus V_z^0$ of $f(z, p)$ (see (6.6)), i.e., $(z_0, \bar{p}(z_0)) \notin V_z^0$. We consider separetely the cases

$$(z(0), p(0)) = (z_0, \bar{p}(z_0)) \in \mathrm{cl}(V^{--} \cup W^{--}) \tag{7.4}$$

and

$$(z(0), p(0)) = (z_0, \bar{p}(z_0)) \in \mathrm{cl}V^{++}. \tag{7.5}$$

Let (7.4) hold. Then by Lemma 5.2

$$(z(t), p(t)) \in V^{--} \cup W^{--} \quad (t \in \Delta \cap [0, \infty)) \tag{7.6}$$

where Δ is the domain of definition of $(z(t), p(t))$; moreover, Δ is bounded and

$$p(\vartheta) = 0 \tag{7.7}$$

where $\vartheta = \sup \Delta$. By (7.6) $(z(t), p(t))$ lies in $\mathrm{dom}f = G \setminus V_z^0$ for all $t \in \Delta$ and $z(t)$ is strictly decreasing. Hence, $\hat{p}(z(\vartheta))$ defined by $\hat{p}(\zeta) = p(z^{-1}(\zeta))$ solves equation (7.2). Since $\hat{p}(z_0) = p(0) = \bar{p}(z_0)$, and due to the uniqueness of the solution of a Cauchy problem for equation (7.2), we get $\bar{p}(z) = \hat{p}(z)$ for all $z \in (z(\vartheta), z_0)$. In particular,

$$\bar{p}(z(\vartheta)) = \hat{p}(z(\vartheta)) = p(z^{-1}(z(\vartheta)) = p(\vartheta) = 0$$

(see (7.7)). By the definition of $f(z, p)$ (see (6.6), (4.10), (4.11)) we have $f(z(\vartheta), 0) > 0$. Hence, solution $\bar{p}(z)$ of (7.2) is nonextendable to the left of $z(\vartheta) > 0$ in G (see (4.9)), which contradicts the assumption that $\bar{p}(z)$ is defined on $(0, z^*)$. Thus, (7.4) is untrue.

Suppose (7.5) holds. Then by Lemma 5.2

$$(z(t), p(t)) \in V^{++} \quad (t \geq 0) \tag{7.8}$$

and relations (5.26), (5.27) hold. By (7.8) $(z(t), p(t))$ lies in $\mathrm{dom}f = G \setminus V_z^0$ for all $t \geq 0$ and $z(t)$ is strictly increasing. Hence, $\hat{p}(z(\vartheta))$ defined by $\hat{p}(\zeta) = p(z^{-1}(\zeta))$ solves equation (7.2). Since $\hat{p}(z_0) = p(0) = \bar{p}(z_0)$, and due to the uniqueness of the solution of a Cauchy problem for equation (7.2), we get $\bar{p}(z) = \hat{p}(z)$ for all $z \in (z_0, z^*)$. Then by (5.26) and (5.27) $\bar{p}(z) = \hat{p}(z) \to \infty$ as $z \to z^*$, which in not possible, for the left equilibrium solution $\bar{p}(z)$ of (7.2) satisfies (7.3). The contradiction eliminates case (7.5) and completes the proof.

We will use Lemma 7.1 for proving the uniqueness part of the next existence and uniqueness theorem.

Theorem 7.1 *There exist a unique left equilibrium solution of* (7.2) *and a unique right equilibrium solution of* (7.2).

Proof. We will prove the existence and uniqueness of the left equilibrium solution of (7.2) (the existence and uniqueness of the right equilibrium solution is stated similarly).

Take a $z_0 \in (0, z^*)$. By Lemma 6.3 there exists an equilibrium solution $(z(t), p(t))$ of the Hamiltonian system (4.12), (4.13), which satisfies $z(0) = z_0$. By Lemma 6.1, (ii),

$$(z(t), p(t)) \in V^{+-} \quad (t \geq 0). \tag{7.9}$$

Hence, $\dot{z}(t) = r(z(t), p(t)) > 0$ $(t > 0)$ and $\bar{p} : [z_0, z^*) \mapsto [p(0), \infty)$ defined by $\bar{p}(z) = p(z^{-1}(z))$ solves (7.2). By definition the equilibrium solution $(z(t), p(t))$ satisfies (5.28), (5.29), which implies (7.3). By (7.9) $(z, \bar{p}(z)) \in V^{+-}$ for all $z \in [z_0, z^*)$. Now consider a solution $\hat{p}(z)$ of (7.2) in V^{+-}, which is nonextendable to the left and satisfies $\hat{p}(z) = \bar{p}(z)$ for all $z \in [z_0, z^*)$. Let us fix the fact that

$$(z, \hat{p}(z)) \in V^{+-} \quad (z \in [z^0, z^*)). \tag{7.10}$$

In order to state that $\hat{p}(z)$ is a left equilibrium solution of (7.2), it is sufficient to show that its domain of definition is $(0, z^*)$. Suppose the domain of definition of $\hat{p}(z)$ is (ζ, z^*) where $\zeta > 0$. For all $z \in (\zeta, z^*)$, we have $(z, \hat{p}(z)) \in V^{+-}$ and hence, $f(z, \hat{p}(z)) > 0$. Therefore, $\hat{p}(z)$ is decreasing and there is the limit

$$\pi = \lim_{z \to \zeta} \hat{p}(z)$$

satisfying

$$\pi > p^*. \tag{7.11}$$

By the definition of V^{+-} (see (5.11)) the set $\{(z, p) \in V^{+-} : z \geq \zeta\}$ is bounded. Consequently, π is finite and (ζ, π) lies on the boundary of V^{+-}. Two cases are admissible:

$$(\zeta, \pi) \in V_z^0 \tag{7.12}$$

(see (5.7)) and

$$(\zeta, \pi) \in V_p^0 \tag{7.13}$$

(see (5.8). If (7.12) holds, then $\pi = h_1(\zeta) < h_1(z^*) = p^*$ (recall that $\zeta < z^*$ and $h_1(z)$ is strictly increasing); we get a contradiction with (7.11). Thus, (7.12) is not possible. Suppose (7.13) holds. Then $\pi = h_2(\zeta)$ and $s(\zeta, \pi) = 0$; the latter implies $f(\zeta, \pi) = 0$ (see (6.6)). Take an $\epsilon > 0$. There is a $\delta > 0$ such that

$$\left| \frac{d}{dz} \bar{p}(z) \right| < \epsilon \quad (z \in (\zeta, \zeta + \delta]).$$

Let

$$2\epsilon < K_5 = \inf_{z \in [\zeta, z^*]} |h_2'(z)|$$

(recall that $h_2(z)$ is strictly decreasing) and

$$\zeta_1 \in (\zeta, \zeta + \epsilon\delta/2]$$

satisfy $\zeta_1 < z^* - \delta$ (with no loss of generality we assume that δ is small enough, for example, $\delta < (z^* - \zeta)/2$) and

$$|\hat{p}(\zeta_1) - \pi| < \epsilon\delta/2.$$

Then using the fact that $h_2(z)$ is decreasing, we get

$$\begin{aligned} \hat{p}(\zeta_1 + \delta/2) &> \hat{p}(\zeta_1) - \epsilon\delta/2 > \pi - \epsilon\delta \\ &= h_2(\zeta) - \epsilon\delta > h_2(\zeta) - \kappa\delta/2 > h_2(\zeta + \delta/2) \\ &> h_2(\zeta_1 + \delta/2). \end{aligned}$$

Hence, $(\zeta_1 + \delta/2, \hat{p}(\zeta_1 + \delta/2)) \notin V^{+-}$ (see (5.11)), which contradicts (7.10). Thus, (7.13) is not possible. We have proved that $\hat{p}(z)$ is defined on $(0, z^*)$. Consequently, $\hat{p}(z)$ is a left equilibrium solution of (7.2).

It remains to prove that the left equilibrium solution of (7.2) is unique. Suppose there are two left equilibrium solutions of (7.2), $\hat{p}_1(z)$ and $\hat{p}_2(z)$. So,

$$\hat{p}_1(z_0) \neq \hat{p}_2(z_0) \tag{7.14}$$

for some $z^0 \in (0, z^*)$. By Lemma 7.1

$$(z, \hat{p}_i(z)) \in V^{+-} \quad (z \in (0, z^*)),$$

$i = 1, 2$. Let $(z_i(t), p_i(t))$ be the nonextendable solution of the Hamiltonian system (4.12), (4.13) (in G), which satisfies

$$(z_i(0), p_i(0)) = (z_0, \hat{p}_i(z_0)), \tag{7.15}$$

$i = 1, 2$. Take an $i \in \{1, 2\}$. Point $(z_i(0), p_i(0)) \in V^{-+}$ lies in dom$f = G \setminus V_z^0$; therefore, $(z_i(t), p_i(t)) \in G \setminus V_z^0$ for all t from a right neighborhood of 0. Let ϑ_i be the supremum of all $\tau \geq 0$ such that $(z_i(t), p_i(t)) \in$ dom$f$ for every $t \in [0, \tau]$, $i = 1, 2$. Then necessarily

$$\dot{z}_i(t) = r(z_i(t), p_i(t)) > 0 \quad (t \in [0, \vartheta_i)); \tag{7.16}$$

hence, setting

$$\xi_i = \lim_{t \to \vartheta_i} z_i(t),$$

we find that $\bar{p}_i(z) : [z^0, \xi_i) \mapsto [0, \infty)$ defined by

$$\bar{p}_i(\zeta) = p_i(z_i^{-1}(\zeta)) \tag{7.17}$$

solves (7.2) Consequently, $\bar{p}_i(z) = \hat{p}_i(z)$ for all $z \in [z^0, \min\{z^*, \xi_i\})$.

Suppose $\xi_i < z^*$. By the definition of ϑ_i

$$\lim_{\zeta \to \xi_i} (\zeta, \bar{p}_i(z_i(\zeta)) = \lim_{\zeta \to \xi_i} (\zeta, \hat{p}_i(z_i(\zeta)) = (\xi_i, \hat{p}_i(\xi_i)) \notin \text{dom} f,$$

which is not possible, for $(z, \hat{p}_i(z)) \in$ dom$f$ for all $z \in (0, z^*)$. Thus,

$$\xi_i \geq z^*. \tag{7.18}$$

As soon as $\hat{p}_i(z)$ is a left equilibrium solution of (7.2), we have

$$\hat{p}_i(\zeta) = \bar{p}_i(\zeta) \to p^* \quad \text{as} \quad \zeta \to z^*. \tag{7.19}$$

Suppose inequality (7.18) is strict, i.e., $\xi_i > z^*$. Then $\bar{p}_i(z^*) = p_*$ and $(z_i(\tau_i), p_i(\tau_i)) = (z^*, p^*)$ where $\tau_i = z_i^{-1}(z^*)$; consequently, $(z_i(t), p_i(t))$ is the stationary solution of the Hamiltonian system (4.12), (4.13), i.e., $(z_i(t), p_i(t)) = (z^*, p^*)$ for all t from its domain of definition, which contradicts (7.15) (recall that $z^0 < z^*$. Thus, (7.18) is in fact the equality, $\xi_i = z^*$. Then referring to (7.19), (7.16), (7.17), we find that

$$\lim_{t \to \vartheta_i} p_i(t) = \lim_{z \to z^*} \bar{p}_i(\zeta) = p^* \tag{7.20}$$

Recall that $(z_i(t), p_i(t))$ is not the stationary solution of the Hamiltonian system (4.12), (4.13), i.e., $(z_i(t), p_i(t)) \neq (z^*, p^*)$ for all t. Then by Lemma 5.2 (7.20) yields $\vartheta_i = \infty$. Therefore, $(z_i(t), p_i(t))$ is an equilibrium solution of the Hamiltonian system (4.12), (4.13)

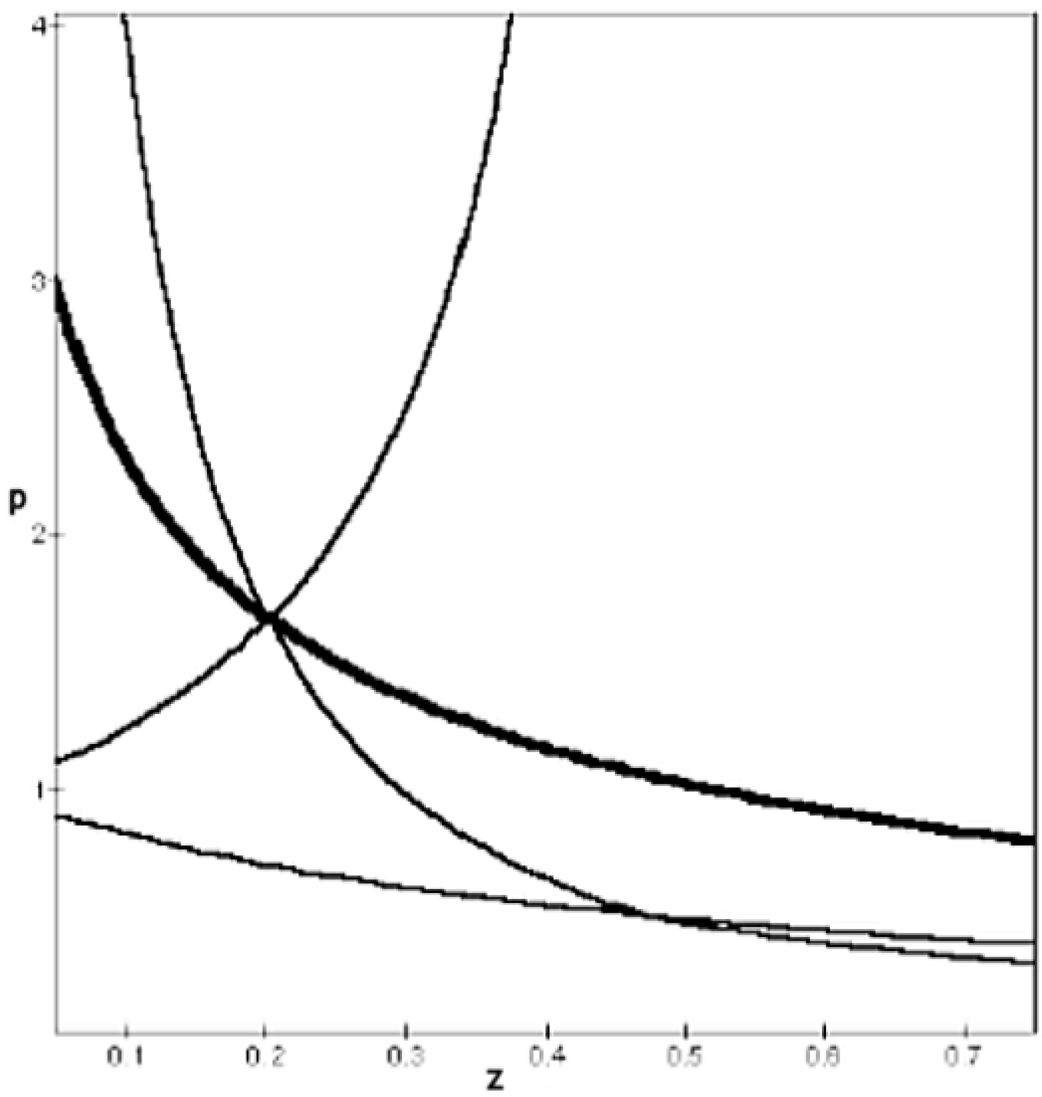

Figure 2: The left and right equilibrium solutions of (7.2) for $\nu = 4$, $b = 2$, $\kappa = 1$, $\rho = 0.1$, $\gamma = 0.5$ (a Maple simulation).

which satisfies $z_i(0) = z^0$ (see (7.15)). This holds for $i = 1, 2$. Hence, by the uniqueness Lemma 6.3 $(z_1(t), p_1(t)) = (z_2(t), p_2(t))$. However, (7.14) and (7.15) show that $p_1(0) \neq p_2(0)$. The contradiction completes the proof of the Theorem.

In what follows we denote the unique left equilibrium solution of (7.2) by $\bar{p}_-(z)$ and the unique left equilibrium solution of (7.2) by $\bar{p}_+(z)$.

In Fig. 2 the left and right equilibrium solutions of equation (7.2) are shown.

Now we are ready to construct a desired optimal synthesis $U_*(z)$. The idea is the following. In the expression (4.14) for an optimal control $u(t)$ we replace $z(t)$ by a free z and replace $p(t)$ by $\bar{p}_-(z)$ if $z < z^*$, by $\bar{p}_-(z)$ if $z > z^*$, and by p^* if $z = z^*$. Thus, we define $U_*(z) : (0, \infty) \to [0, b)$ by

$$U_*(z) = \begin{cases} b - \dfrac{1}{p^*(z^* + \gamma)}, & \text{if} \quad z = z^*, \\ b - \dfrac{1}{\bar{p}_-(z)(z + \gamma)}, & \text{if} \quad z \in (0, z^*), \\ b - \dfrac{1}{\bar{p}_+(z)(z + \gamma)}, & \text{if} \quad z \in (0, z^*), \ (z, \bar{p}_+(z)) \in G_1, \\ 0, & \text{if} \quad z \in (0, z^*), \ (z, \bar{p}_+(z)) \in G_2; \end{cases} \tag{7.21}$$

note that by Lemma 7.1 we have $(z, \bar{p}_-(z)) \in V^{+-}$ for $z \in (0, z^*)$; as soon as $V^{+-} \subset G_1$ (see (5.11), (4.7), (5.4)), $U_*(z)$ is given by the single formula for $z \in (0, z^*)$.

Lemma 7.2 *Function $U_*(z)$ (7.21) is a feedback.*

Proof. Obviously, $U_*(z)$ is continuous at every $z \neq z^*$. The fact that $\bar{p}_-(z)$ is the left equilibrium solution and $\bar{p}_+(z)$ is the right equilibrium solution of (7.2) implies that $U_*(z)$ is continuous at z^* as well. Moreover, the right hand side of equation (7.1) (for the "closed" system) is, obviously, Lipschitz on every bounded interval in $(0, \infty)$ which does not intersect a neighborhood of z^*. If it is also Lipschitz in a neighborhood of z^*, then for every $z_0 > 0$, equation (7.1) has the unique solution $z(t)$ defined on $[0, \infty)$ and satisfying $z(0) = z_0$, which proves that $U_*(z)$ is a feedback. Now we will state that the right hand side of (7.1) is Lipschitz in a neighborhood of z^*. It is sufficient to show that $U_*(z)$ is Lipschitz in a neighborhood of z^*; this is so if, in turn, $\bar{p}_-(z)$ and $\bar{p}_+(z)$ are Lipschitz in a neighborhood of z^* (see formula (7.21)). To prove the Lipschitz character of $\bar{p}_-(z)$ and $\bar{p}_+(z)$ in a neighborhood of z^* it is enough to verify that

$$\limsup_{z \to z^*} \frac{d}{dz} \bar{p}_-(z) < \infty, \tag{7.22}$$

$$\liminf_{z \to z^*} \frac{d}{dz} \bar{p}_-(z) > -\infty, \tag{7.23}$$

$$\limsup_{z \to z^*} \frac{d}{dz} \bar{p}_+(z) < \infty, \tag{7.24}$$

$$\liminf_{z \to z^*} \frac{d}{dz} \bar{p}_+(z) > -\infty. \tag{7.25}$$

By Lemma 7.1

$$\bar{p}_-(z) \in V^{+-} \quad (z \in (0, z^*)), \tag{7.26}$$

$$\bar{p}_+(z) \in V^{-+} \cup W^{-+} \quad (z \in (z^*, \infty)).$$

Hence, $d\bar{p}_-(z)/dz < 0$ $(z \in (0, z^*)$ and $d\bar{p}_+(z)/dz < 0$ $(z \in (z^*, \infty)$. Thus, (7.22) and (7.24) hold.

Let us show (7.23). Take $z \in (0, z^*)$. We have $(z, \bar{p}_-(z)) \in G_1$ and (see (7.2), (6.6), (4.10), (4.11))

$$\frac{d}{dz}\bar{p}_-(z) = f(z, \bar{p}_-(z)) = \frac{s(z, \bar{p}_-(z))}{r(z, \bar{p}_-(z))},$$

$$s(z, \bar{p}_-(z)) < 0, \quad r(z, \bar{p}_-(z)) > 0.$$

By (7.26)

$$\bar{p}_-(z) > \lim_{z \to z^*} \bar{p}_-(z) = p^*.$$

Then

$$0 > s(z, \bar{p}_-(z)) = (\nu - b + \rho)\bar{p}_-(z) - \frac{\gamma\kappa + (\kappa - 1)z}{(z+\gamma)z} > (\nu - b + \rho)p^* - \frac{\gamma\kappa + (\kappa - 1)z}{(z+\gamma)z},$$

$$r(z, \bar{p}_-(z)) = (b - \nu)z + b\gamma - \frac{1}{\bar{p}_-(z)} > (b - \nu)z + b\gamma - \frac{1}{p^*} > 0.$$

Using the L'Hopital theorem, we get

$$\liminf_{z \to z^*} \frac{d}{dz}\bar{p}_-(z) \geq \lim_{z \to z^*} \frac{(\nu - b + \rho)p^* - \frac{\gamma\kappa + (\kappa-1)z}{(z+\gamma)z}}{(b - \nu)z + b\gamma - \frac{1}{p^*}} =$$

$$= \frac{\kappa - 1}{(z^* + \gamma)^2 z^*(b - \nu)}.$$

Inequality (7.23) is proved.

Let us show (7.25). Take a $z > z^*$ sufficiently close to z^*. Since (z^*, p^*) lies in the interior of G_1, we have $(z, \bar{p}(z)_+) \in G_1$.

$$\frac{d}{dz}\bar{p}_+(z) = f(z, \bar{p}_-(z)) = \frac{s(z, \bar{p}_+(z))}{r(z, \bar{p}_+(z))},$$

$$s(z, \bar{p}_+(z)) > 0, \quad r(z, \bar{p}_-(z)) < 0.$$

By (7.26)

$$\bar{p}_+(z) < \lim_{z \to z^*} \bar{p}_+(z) = p^*.$$

Then

$$0 > s(z, \bar{p}_+(z)) = (\nu - b + \rho)\bar{p}_+(z) - \frac{\gamma\kappa + (\kappa = 1)z}{(z+\gamma)z} > (\nu - b + \rho)p^* - \frac{\gamma\kappa + (\kappa - 1)z}{(z+\gamma)z},$$

$$r(z, \bar{p}_+(z)) = (b - \nu)z + b\gamma - \frac{1}{\bar{p}_+(z)} < (b - \nu)z + b\gamma - \frac{1}{p^*} < 0.$$

Using the L'Hopital theorem, we get

$$\liminf_{z \to z^*} \frac{d}{dz}\bar{p}_+(z) \geq \lim_{z \to z^*} \frac{(\nu - b + \rho)p^* - \frac{\gamma\kappa + (\kappa-1)z}{(z+\gamma)z}}{(b - a)z + b\gamma - \frac{1}{p^*}} = \frac{\kappa - 1}{(z^* + \gamma)^2 z^*(b - \nu)}$$

which proves (7.25).

The Theorem is proved.

The next Theorem presents our final result.

Theorem 7.2 *Feedback* $U_*(z)$ (7.21) *is an optimal synthesis.*

Proof. Take a $z_0 > 0$. We must show that the control process under feedback $U_*(z)$ with the initial state z_0 is an optimal control process in problem $(\tilde{P})$.

Consider the equilibrium solution $((z(t), p(t))$ of the Hamiltonian system (4.12), (4.13) which satisfies $z(0) = z_0$. By Lemma 6.3 this solution is unique and by Theorem 6.1 the pair $z(t), u(t))$ where $u(t)$ is given by (4.14) is an optimal control process in problem $(\tilde{P})$. Therefore, in order to complete the proof of the Theorem, it is sufficient to state that $z(t), u(t))$ is the control process under feedback $U_*(z)$ with the initial state z_0.

Suppose $z_0 = z^*$. Then by Lemma 5.2 $(z(t), p(t)) = (z^*, p^*)$ $(t \geq 0)$ and by (4.14) and (7.21) $u(t) = U_*(z^*)$ $(t \geq 0)$. For $z(t) = z^*$ the right hand side of equation (7.1) for the "closed" system is zero; Thus, $z(t), u(t))$ is the control process under feedback $U_*(z)$ with the initial state $z_0 = z^*$.

Consider the case $z_0 < z^*$ (the case $z_0 > z^*$ is treated similarly). By Lemma 6.1 we have

$$(z(t), p(t)) \in V^{+-} \quad (t \geq 0).$$

Consequently, $(z(t), p(t)) \in \mathrm{dom} f = G \setminus V_z^0$ for all $t \geq 0$ and $z(t)$ is strictly increasing. Since $(z(t), p(t))$ is an equilibrium solution of (4.12), (4.13), the limit relations (5.28), (5.29) hold. Hence, the function $\bar{p} : \zeta \mapsto \bar{p}(\zeta) = p(z^{-1}(\zeta))$ is defined on $[z_0, z^*)$ and solves equation (7.2) on this interval. Due to (5.28), (5.29) $\bar{p}(z)$ satisfies the limit relation (7.3). Therefore, $\bar{p}(z)$ is the restriction to $[z^0, z^*)$ of the (unique) left equilibrium solution $\bar{p}_-(z)$ of (7.2), and we have

$$p(t) = \bar{p}(z(t)) = \bar{p}_-(z(t)) \quad (t \geq 0). \tag{7.27}$$

By Theorem 6.1 $(z(t), u(t))$ where $u(t)$ is defined by (4.14) is an optimal control process in problem $(\tilde{P})$.

Now we replace $p(t)$ in (4.14) by $\bar{p}(z(t))$ (see (7.27)). Comparing with (7.21), we find that $u(t) = U_*(z(t))$ $(t \geq 0)$. Then

$$\dot{z}(t) = r(z(t), p(t)) = r(z(t), \bar{p}(z(t))) = U(z(t))(z(t) + \gamma) - \nu z(t)$$

$(t \geq 0)$, i.e., $z(t)$ solves equation (7.1) for the "closed" system on $[0, \infty)$. Hence, $(z(t), u(t))$ is the control process under feedback $U_*(z)$ with the initial state z_0. The Theorem is proved.

In Fig. 3 the shape of the optimal synthesis $U_*(z)$ is illustrated.

Theorem 7.2 provides the next algorithm for the construction of solutions in the family of problems $(\tilde{P})$ parametrized by the initial state:

1. Find the left equilibrium solution $\bar{p}_-(z)$ and the right equilibrium solution $\bar{p}_+(z)$ of equation (7.2).

2. Given a $z_0 > 0$, find the optimal control process $(z(t), u(t))$ in problem $(\tilde{P})$ as the control process under feedback $U_*(z)$ (7.21) with the initial state z_0.

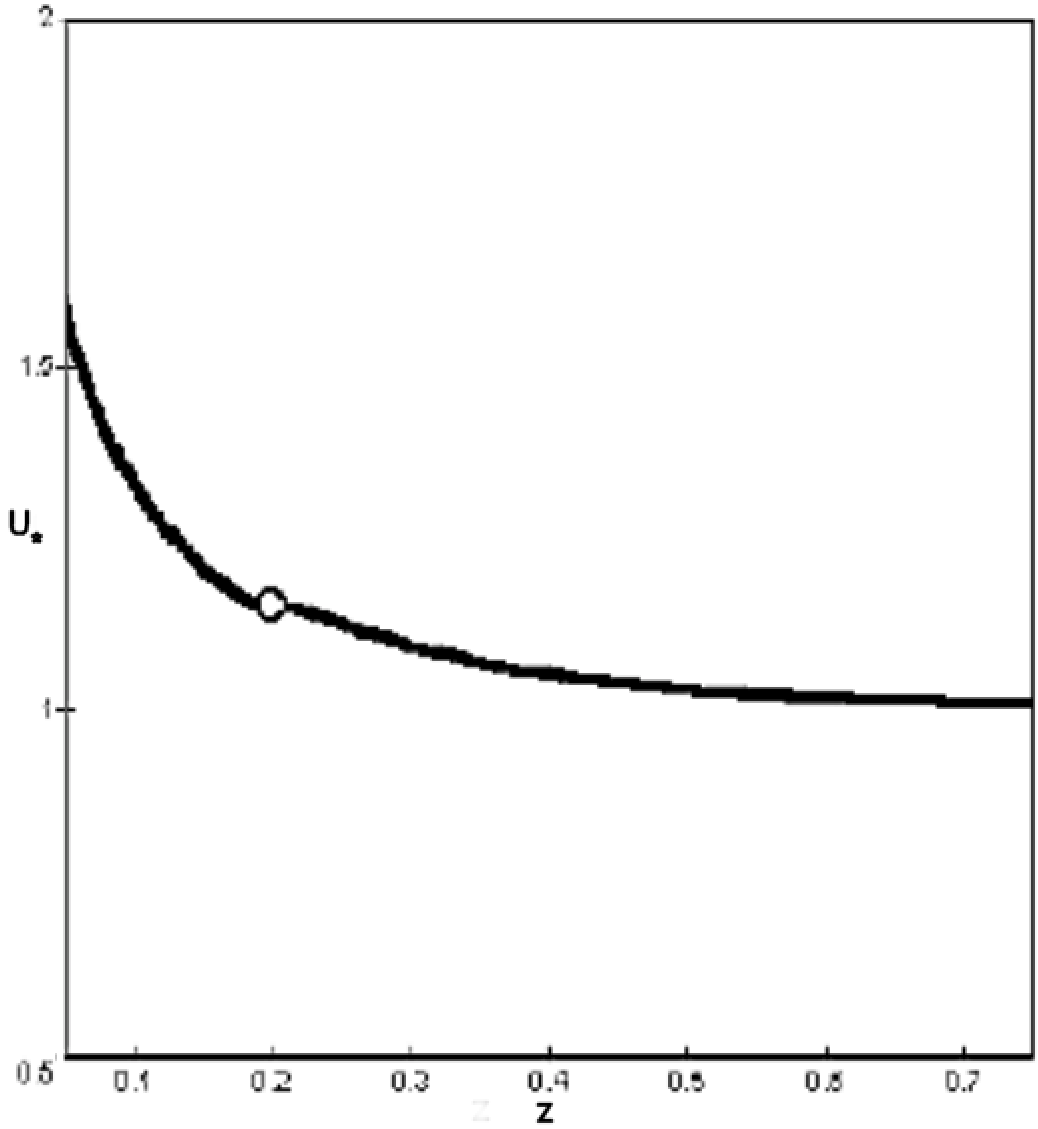

Figure 3: The optimal synthesis for $\nu = 4$, $b = 2$, $\kappa = 1$ $\rho = 0.1$, $\gamma = 0.5$ (a Maple simulation).

8 Interpretation of results

A major result of the optimisation performed in this paper is that, under the present assumptions, there is an unique constant optimal asymptotic ratio of the stocks of knowledge of the two countries. This result implies that the knowledge stocks of the two countries grow at an identical exponential rate as time goes to infinity. Since the rate of innovation of the leading country is a parameter in our maximisation problem, the asymptotic rate of innovation of the follower country equals that of the leading country.

This result was also found to hold asymptotically for a perfect-foresight equilibrium trajectory in the leader-follower model (Hutschenreiter, Kaniovski and Kryazhimskii, 1995). Thus we conclude that in terms of the asymptotic rate of innovation, social planning produces the same result as the market economy. This unsettles a basic tenet which is often but - as, e.g., pointed out by Aghion and Howitt, 1998 - nevertheless wrongly ascribed to endogenous growth theory.

However, it may be possible that the long-run ratio of stocks of knowledge associated with the market outcome differs from that in the optimal solution. Indeed, according to Hutschenreiter, Kaniovski and Kryazhimskii, 1995, in the market solution the asymptotic stock of knowledge is a simple function of the absorptive capacities of the follower and relative country size. Specifically, a perfect-foresight equilibrium trajectory was shown to be characterised by $n^B(t)$ growing to infinity, and the asymptotic ratio of knowledge stocks of the two countries, $r(t) = n^B(t)/n^A(t)$, approaching a positive constant

$$r_\infty = \frac{\gamma}{L^A/L^B - 1}, \tag{8.1}$$

where $\lim_{t\to\infty} r(t) = r_\infty$. Thus, e.g., in the case where country A is twice the size of country B, the relative knowledge stock of the latter simply equals its absorptive capacity. In contrast, solving the above optimisation problem we arrive at a much more complex expression for the ratio of knowledge stocks. Thus, in general (except for particular parameter constellations), the asymptotic relative knowledge stocks derived from the two models are not the same. This implies that the long-term relative levels of productivity or output per capita differ in the two solutions.

Thus we arrive at the "Solowian" conclusion that policy intervention does not affect the long-run growth rate but is likely to affect relative levels of productivity, output per capita etc.

We can carry this reasoning one step forward. If, in the two models under consideration, the rate of innovation is the same (as we have established) and the ratio of the stocks of knowledge differs (as is the case in general), then the amount of labour asymptotically allocated to R&D also differs from the one accomplished by the market mechanism.

To illustrate this, let us divide both sides of equation (2.2) by $n^B(t)$. Using notation $\bar{g}^B(t) = \dot{n}^B(t)/n^B(t)$ we get

$$\bar{g}^B(t) = \frac{L_n^B(t)}{a}\left(1 + \frac{\gamma}{r(t)}\right).$$

Resolving this equation for $L_n^B(t)$ yields

$$L_n^B(t) = \frac{a\bar{g}^B(t)}{1 + \frac{\gamma}{r(t)}}. \tag{8.2}$$

We have shown that $\bar{g}^A$ is the asymtotic rate of innovation of the leading and the follower country in both the market and the optimal solution. Fixing the rate of innovation at this value and passing to the limit (8.2) becomes

$$L^B_{n\infty} = \frac{a\bar{g}^A}{1 + \frac{\gamma}{r_\infty}}, \tag{8.3}$$

where $L^B_{n\infty} = \lim_{t\to\infty} L^B_n(t)$ and $r_\infty = \lim_{t\to\infty} r(t)$ (either for the market system or the social planning model).

In the case that, for a given rate of innovation, the asymptotic ratio of knowledge stocks is small, i.e. the leading country approaches a relatively large stock of knowledge, the follower will devote only little resources to its own R&D activities. The reason for this is that the productivity of the follower country's researchers is strongly boosted by knowledge absorbed by the leader so that it has to spend relatively little on R&D in order to reach the leader's rate of innovation. On the other hand, if the knowledge stock the follower country achieves in the long run gets large relative to that of country A, its R&D labour input approaches $a\bar{g}^A = \bar{L}^A_n$, i.e. the steady-state R&D labour input of the leading country from below.

Since in the market outcome the asymptotic ratio of knowledge stocks is given by (8.1), and taking into account equation (8.3), the amount of labour allocated to R&D approaches

$$L^B_{n\infty} = \frac{L^B}{L^A} a\bar{g}^A. \tag{8.4}$$

It follows that the shares of R&D employment in the total labour force are the same in both countries in the long run, namely $a\bar{g}^A/L^A$.

To summarise, in the market solution of the leader-follower model, the long-run values of the pair $r(t)$, $L^B_n(t)$ consistent with the asymptotic rate of innovation $\bar{g}^A$ is given by equations (8.1) and (8.4).

If the welfare analysis shows that the market allocates too little (too much) labour to R&D then it follows from relation (8.3) that the optimal ratio of knowledge stocks will be higher (lower) than the respective market outcome.

Now our task will be to compare r_∞ and z^*. Here we call r_∞ the *market limit* and z^* the *optimal limit*.

According to Hutschenreiter, Kaniovskii and Kryazhimskii, 1995 (see (8.1)),

$$r_\infty = \gamma \frac{1}{L^A/L^B - 1} = \gamma \frac{L^B/a}{L^A/a - L^B/a}. \tag{8.5}$$

In terms of the notations used in the present paper ($\nu = \bar{g}^A$), condition (2.9) can be rewritten as

$$\nu = (1-\alpha)\frac{L^A}{a} - \alpha\rho.$$

Further, as far as $\kappa = 1/\alpha - 1$ we have

$$\frac{L^A}{a} = \frac{(\kappa+1)\nu + \rho}{\kappa}$$

and due to (8.5), taking into account that $b = L^B/a$, we get

$$r_\infty = \gamma \frac{b\kappa}{(\kappa+1)\nu + \rho - b\kappa}.$$

Let us introduce the ratio

$$\sigma = \frac{z^*}{r_\infty}.$$

We will show that

$$\sigma > 1 \tag{8.6}$$

which is equivalent to the fact that the optimal limit z^* is greater than the market limit r_∞.

Using the expressions for z^* (see (5.14)) and r_∞ (see (8.5)), we find that

$$\sigma = \left(\frac{2b\kappa - (\kappa+1)\nu - \rho + [((\kappa+1)\nu+\rho)^2 - 4b\kappa\nu]^{1/2}}{2b\kappa} \right) \times$$

$$\times \left(\frac{(\kappa+1)\nu + \rho - b\kappa}{\kappa(\nu - b) + \rho} \right). \tag{8.7}$$

Denote $\eta = (\kappa+1)\nu + \rho$ and $\mu = b\kappa/\eta^2$ (note that $0 < b < \nu$ implies $0 < \mu < \nu\kappa/\eta^2$). Then condition (8.6) can be rewritten in terms of η, μ as (see (8.7))

$$\frac{(2\mu\eta^2 - \eta + \eta(1-4\mu\nu)^{\frac{1}{2}})}{2\mu\eta^2} \times \frac{(\eta - \mu\eta^2)}{(\eta - \mu\eta^2 - \nu)} > 1$$

or

$$\frac{(2\mu\eta - 1 + (1-4\mu\nu)^{\frac{1}{2}})}{2\mu} \times \frac{(1 - \mu\eta)}{(\eta - \mu\eta^2 - \nu)} > 1.$$

As far as $1 - \mu\eta > 0$ and $\eta - \mu\eta^2 - \nu > 0$ the last inequality is equivalent to

$$(1-4\mu\nu)^{\frac{1}{2}} > 1 - \frac{2\mu\nu}{1-\mu\eta}.$$

Squaring it we get

$$1 - 4\mu\nu > 1 - \frac{4\mu\nu}{1-\mu\eta} + \frac{4\mu^2\nu^2}{(1-\mu\eta)^2}$$

or

$$-4\mu\nu + 8\mu^2\eta\nu - 4\mu^3\eta^2\nu > -4\mu\nu + 4\mu^2\eta\nu + 4\mu^2\nu^2$$

and it is equivalent to

$$\eta - \mu\eta^2 > \nu.$$

The last inequality holds because $\mu < \nu\kappa/\eta^2$ and $\eta - \nu\kappa = \nu + \rho > \nu$. Hence condition (8.6) is proved.

References

1. Aghion, Ph., and Howitt, P., 1998, Endogenous Growth Theory, M.I.T. Press, Cambridge.

2. Aseev, S.M., Hutschenreiter, G., Kryazhimskii, A.V., 2002, Optimal Investment in R&D with International Knowledge Spillovers, WIFO Working Paper 175, Austrian Institute of Economic Research, Vienna, Austria.

3. Aseev, S.M., Kryazhimskii, A.V., and Tarasyev, A.M., 2001,

   (a) First Order Necessary Optimality Conditions for a Class of Infinite-Horizon Optimal Control Problems, International Institute for Applied Systems Analysis, Interim Report, IR-01-007, Laxenburg, Austria.

(b) The Pontryagin Maximum Principle and Transversality Conditions for an Optimal Control Problem with infinite Time Interval, Proceedings of the Steklov Institute of Mathematics, Vol. 233, 64-80.

(c) The Maximum Principle and Transversality Conditions for a Class of Optimal Economic Growth Problems, Preprints of the 5th IFAC Symposium "Nonlinear Control Systems", St.-Petersburg, Russia, July 4-6, 2001, 64-68.

4. Balder, E.J., 1983, An Existence Result for Optimal Economic Growth Problems, J. of Math. Analysis and Applications, Vol. 95, 195–213.
5. Barro, R., and Sala-i-Martin, X., 1995, Economic Growth, McGraw Hill, New York.
6. Benassy, J.-P., 1998, Is there Always too Little Research in Endogenous Growth with Expanding Product Variety?, European Economic Review, Vol. 42, 61-69.
7. Borisov, V.F., Hutschenreiter, G., and Kryazhimskii, A.V., 1999, Asymptotic Growth Rates in Knowledge-Exchanging Economies, Annals of Operations Research, Vol. 89, 61-73.
8. Cesari, L., 1983, Optimization - Theory and Applications: Problems with Ordinary Differential Equations, Springer, Berlin etc.
9. Cohen, W.M., and Levinthal, D.A., 1989, Innovation and Learning: The Two Faces of R&D, Economic Journal, Vol. 99 (September), 569-596.
10. Dixit, A., and Stiglitz, J.E., 1977, Monopolistic Competition and Optimum Product Diversity, American Economic Review, Vol. 67 (3), 297-308.
11. Ethier, W., 1982, National and International Returns to Scale in the Modern Theory of International Trade, American Economic Review, Vol. 72 (3), 389-405.
12. Grossman, G.M., and Helpman, E., 1991, Innovation and Growth in the Global Economy, M.I.T. Press, Cambridge.
13. Halkin, H., 1974, Necessary Conditions for Optimal Control Problems with Infinite Horizons, Econometrica, Vol. 42, No. 2, 267-272.
14. Hartman, Ph., 1964, Ordinary Differential Equations, New York etc., J.Wiley & Sons.
15. Hutschenreiter, G., Kaniovski, Yu.M., and Kryazhimskii, A.V., 1995, Endogenous Growth, Absorptive Capacities, and International R&D Spillovers, International Institute for Applied Systems Analysis, Working Paper, WP-95-92, Laxenburg, Austria.
16. Michel, P., 1982, On the Transversality Condition in Infinite Horizon Optimal Problems, Econometrica, Vol. 50, No. 4, 975-985.
17. Mordukhovich, B.S., 1988, Methods of Approximations in Problems of Optimization and Control, Moscow, Nauka.
18. Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V., Mishchenko, E.F., 1969, The Mathematical Theory of Optimal Processes, Moscow, Nauka.
19. Romer, P.M., 1990, Endogenous Technological Change, Journal of Political Economy, Vol. 98 (5), 71-102.