INTERNATIONAL SERIES ON APPLIED SYSTEMS ANALYSIS

MATERIAL ACCOUNTABILITY
Theory, Verification, Applications

RUDOLF AVENHAUS

International Institute for Applied Systems Analysis

MATERIAL ACCOUNTABILITY
Theory, Verification, Applications

Rudolf Avenhaus

Nuclear Research Center Karlsruhe
and University of Mannheim

This book is concerned with the concept of accountability for all goods used in an industrialized society. This includes basic materials, e.g. metals; production materials, e.g. glass and plastic; waste materials, e.g. sulphur, carbon, nuclear substances and other pollutants. The problem is to account carefully for these materials during their use to see where they are accumulated or disposed and in what quantities with a view to taking a significant step toward complete recycling.

This book develops the mathematical theory of material accountability and its verification in a unified way, gives a detailed example of its application in nuclear materials and shows how it can be applied to other situations.

Contents

1 Introduction

2 The Material Balance Concept
- 2.1 The material balance principle for one inventory period
- 2.2 Measurement errors, random losses and diversion: formulation of the test problem
- 2.3 Sequence of inventory periods
- 2.4 Material accountability as a tool for detecting the diversion of material
- 2.5 Frequency of taking inventory
- 2.6 Subdivision of material balance areas

3 Data Verification
- 3.1 Inventory verification: one class of material
- 3.2 Inventory verification: several classes of material
- 3.3 Flow measurement verification
- 3.4 Concluding remark

4 Systems Aspects
- 4.1 The control problem as conflict situation
- 4.2 Action levels
- 4.3 Parameters of the total control system
- 4.4 Material accountability verification system not using operator's data

5 International Nuclear Material Safeguards
- 5.1 Nonproliferation treaty and IAEA nuclear material safeguards system
- 5.2 Nuclear fuel cycle and nuclear material measurement

(contents continued on back flap)

2 International Series on Applied Systems Analysis

Material Accountability: Theory, Verification, and Applications

Rudolf Avenhaus
Nuclear Research Center Karlsruhe, and University of Mannheim

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Meiner lieben Frau
Ingeborg
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Foreword

This is the second volume in the International Series on Applied Systems Analysis established by the International Institute for Applied Systems Analysis (IIASA) in order to promote the development and application of systems analysis across national boundaries.

This volume by Dr. Rudolf Avenhaus has been selected for inclusion in the Series because it comprises a thorough review of the analytical tools that have been developed to deal with a problem of considerable international importance – keeping track of rare or dangerous materials. Such problems have arisen with particular urgency in the field of nuclear energy, where the materials of concern are plutonium and uranium as they flow through the nuclear fuel cycle. Indeed, much of the analytical apparatus that Dr. Avenhaus presents has been developed or refined with this application in mind. However, the value of this monograph arises from the fact that these techniques can also be applied to the monitoring of the flow of pollutants, such as sulfur dioxide; or potentially dangerous materials, such as carbon dioxide, that may affect the global climate; or rare materials, such as the precious metals in a mint. Furthermore, many centrally planned economies already employ material balance accounting as a part of their planning procedures. And as awareness of the finiteness of the earth's readily accessible resources and the need to utilize and reutilize them efficiently grows, the likelihood increases that other nations and institutions will adopt such techniques as well. The methodology described in this monograph can form the basis for such applications.

Dr. Avenhaus is concerned with two cases: one in which the issue is how well material can be accounted for, given the imprecision of measurement techniques; the other in which the issue is how well this can be done in the presence of an antagonist (the diverter of plutonium or the polluter) whose interest is to keep the monitor from knowing that some material has been removed. His analysis of these cases draws upon statistical methodology and, in the second case, the theory of games. In sum, he presents the basic tools needed to analyze a particular kind of system, one in which the flow and storage of one or more materials is of central importance. This volume is a significant contribution to the study of such systems because it brings together in a unique synthesis results that have been obtained in a wide range of fields of application, from nuclear energy to resource economics, and in many countries.

We take added pride in including this volume in the International Series because Dr. Avenhaus is an IIASA alumnus, having spent 2 years with the Institute from 1973 to 1975. Indeed, he was the first full-time scientist to begin work at the Institute after its charter was signed in October 1972. Some of the work on this volume was accomplished at IIASA, while the remainder was done at the Kernforschungszentrum, Karlsruhe, to which Dr. Avenhaus has returned.

This volume, which brings together an internationally developed array of analytical methods to treat a class of problems of international importance, is an excellent example of the kind of result that IIASA was established to produce.

ROGER LEVIEN
Director

Preface

The idea of writing on material accountability problems and methods for their solution grew out of three major factors:

- In the Nuclear Research Center at Karlsruhe, Federal Republic of Germany, a small group of scientists has been working since 1968 on the problem of international nuclear material safeguards. The basic idea of this work was to establish a safeguards system based primarily on material accountability.
- In 1970, Resources for the Future, Inc., in Washington, D.C., published a monograph dealing with environmental problems caused by modern technologies. The authors demonstrated that material accountability could serve very well as a tool for analyzing these problems, and they even proposed as a long-term goal the establishment of an international environmental control authority.
- In 1971, more than 40 nations, under the auspices of the International Atomic Energy Agency in Vienna, agreed upon a safeguards system in partial fulfillment of the Treaty on the Non-Proliferation of Nuclear Weapons. In this very special area the material accountability principle could be accepted as the primary safeguards tool because of its inherent "objectivity" and "formalization."

With the background of these three lines of development, and bearing in mind the wide range of mass balance considerations in science and technology, it was quite natural for me to try to describe the idea of material accountability and its far-reaching applications, especially for international control problems as exemplified by nuclear material safeguards, to a broad scientific, technical, managerial, and perhaps political audience. In this sense, the International Institute for Applied Systems Analysis (IIASA) seemed the ideal institution to assist in publishing the present monograph.

There is another, and equally important, reason that IIASA's participation in the publication of this work in its International Series on Applied Systems Analysis is appropriate: the problems and their solutions are presented primarily from a *methodological point of view* – the first half of the monograph deals with theory and methods, the second half with applications. Therefore, IIASA seemed to be the ideal sponsor, because its in-house research follows the same strategic lines of combining methodology and applications.

While the outline of the monograph was first conceived in Karlsruhe in 1972, the major part of the work was done in the stimulating atmosphere of IIASA during the author's stay there. The monograph was completed in Karlsruhe, where the Research Center's facilities provided excellent access to the newest literature in the field. In fact, work on the book had to be stopped somewhat artificially: a literature search service, using the key words "material accountability, material balance, mass balances" reports an increasing number of references each month (at the moment 5 to 10); this may give an indication that the subject has attracted considerable attraction and that – perhaps – time has come for a monograph. On the other hand, it was not possible to work out some new and promising ideas; for example, the mercury and cadmium balances in national economies, which would illustrate the ideas presented here extremely well, could only be mentioned (in section 7.3.3) but not fully analyzed. Nevertheless, it is to be hoped that the presentation of the basic methods, as well as of the applications, is clear enough that the analysis of further new (and always somewhat different) examples can be carried through by interested readers themselves.

When I first raised the idea of the present monograph, I was immediately and strongly supported by Carl Bennett of Battelle Northwest in Seattle, Washington, who, because of the dominant role he has played in this field, might be called the "father of safeguards systems analysis." In the same sense, Wolf Häfele, formerly director and project leader in the Nuclear Research Center in Karlsruhe and now Deputy Director of IIASA, provided intellectual and practical support throughout the work that produced this volume. Without the active role of these two scientists, this monograph would have been realized only with great difficulty, and I would like to thank them warmly for their support. In addition, thanks are extended to the many friends, colleagues, and visitors in Karlsruhe and Laxenburg, with whom the author worked on several aspects of this broad area and who read first drafts or listened to presentations and gave their comments and advice.

RUDOLF AVENHAUS

Contents

1 Introduction 1

2 The Material Balance Concept 8
2.1 The material balance principle for one inventory period 9
2.2 Measurement errors, random losses and diversion: formulation of the test problem 11
2.3 Sequence of inventory periods 17
2.4 Material accountability as a tool for detecting the diversion of material 19
2.5 Frequency of taking inventory 24
2.6 Subdivision of material balance areas 29

3 Data Verification 33
3.1 Inventory verification: one class of material 33
3.2 Inventory verification: several classes of material 44
3.2.1 Exact solution for a special case, 45
3.2.2 Approximate solution for the general case, 50
3.3 Flow measurement verification 58
3.3.1 Independent sequential verification, 59
3.3.2 Dependent sequential verification, 62
3.4 Concluding remark 64

4 Systems Aspects 66
4.1 The control problem as conflict situation 67
4.1.1 False alarms are not possible, 68
4.1.2 False alarms are possible, 70

4.2 Action levels 72
4.2.1 *One action level only*, 73
4.2.2 *Several action levels*, 73
4.3 Parameters of the total control system 74
4.4 Material accountability verification system not using operator's data 80
4.4.1 *One class of material*, 81
4.4.2 *Several classes of materials: optimization of verification effort*, 85

5 International Nuclear Material Safeguards 90
5.1 Nonproliferation treaty and IAEA nuclear material safeguards system 90
5.2 Nuclear fuel cycle and nuclear material measurement 95
5.3 Numerical example 97
5.3.1 *Basic data*, 97
5.3.2 *Material accountability procedures*, 100
5.3.3 *Verification procedures*, 104
5.3.4 *Determining the correlation between data verification and material balance establishment*, 106
5.3.5 *Overall probability of detection*, 109
5.4 Conclusion 111

6 Material Accountability in Technology and Economics 114
6.1 The chemical industry 115
6.1.1 *Distillation*, 115
6.1.2 *Isotope separation*, 119
6.2 The metal industry 123
6.2.1 *Metal balances*, 124
6.2.2 *Mint material accountability*, 125
6.3 The Materials-Process-Product Model 129
6.4 Use of material balances in socialist economies 132
6.5 Computer-aided material accountability procedures 134

7 Environmental Accountability 138
7.1 Man's impact on natural material balances 139
7.2 A global example: the carbon dioxide cycle of the earth 142
7.2.1 *The four boxes model*, 143
7.2.2 *Uncertainty considerations*, 145
7.2.3 *Numerical calculations and applications*, 149
7.3 Regional and local examples 153
7.3.1 *Regional sulfur dioxide accountability*, 154
7.3.2 *Sulfur accountability in power plants and refineries*, 155

7.3.3 Note on special metal balances, 156
7.4 Conclusion 158

8 Arms Control 161
8.1 Survey on arms control analyses 162
8.1.1 Models describing arms control as a conflict situation, 162
8.1.2 Material accountability as an arms control verification tool, 163
8.1.3 Direct inspections, 164
8.1.4 Conclusions, 164
8.2 Verification of Mutual Balanced Force Reduction in Central Europe 164
8.2.1 MBFR models, 165
8.2.2 Formulation of the verification problem, 167
8.2.3 Basic means and procedures, 167
8.2.4 Mathematical considerations, 169
8.2.5 System problems, 174

List of Symbols 177

Index 181

1 Introduction

The purpose of this monograph may be summarized very concisely: it is meant to demonstrate that "the law of conservation of mass is not an Einsteinian abstraction,* but rather applies in real life."[1] In order to illustrate this statement in a preliminary way, four problems will be formulated that may be viewed as representative of the kinds of application to be considered:

- As the amount of fossil fuels burned increases continuously, the amount of carbon dioxide released into the atmosphere also increases continuously. One knows that, because of exchange processes between atmosphere, hydrosphere, and biosphere, not all of the carbon dioxide remains in the atmosphere; however, one wants to know quantitatively how the carbon dioxide content of the atmosphere will develop. This is important because, among other consequences, a major change in the carbon dioxide content of the atmosphere might have a significant impact on the global climate (see, e.g., Matthews *et al.*[2] and Zimen and Altenhein[3]).
- The nuclear materials safeguards system of the International Atomic Energy Agency in Vienna, set up in partial fulfillment of the nuclear weapon nonproliferation treaty, is aimed at the "timely detection of the diversion of significant quantities of nuclear material from peaceful nuclear activities to the manufacture of nuclear weapons or of other nuclear explosive devices or for purposes unknown and deterrence of such diversion by the risk of early detection."[4] It has become clear that the only possibility of fulfilling this requirement in an objective and rational way is to devise and operate a system that allows complete accountability of all the nuclear material in the nuclear fuel cycle.

* Perhaps one should better say "Lavoisier's abstraction" because Lavoisier was one of the first scientists to clearly formulate the mass conservation law. On the other hand, Einstein formulated it in the most general form by taking into account the equivalence of mass and energy.

- Because of the scarcity of some rare metal resources, the industries processing these metals must seek ways and means of recycling those amounts of material that up to now have been thrown away as waste for economic reasons. Several proposals have already been made to offer better incentives for material recycling – e.g., imposition of taxes on materials unaccounted for. It is clear that for the enforcement of such taxes the establishment of a practicable material accountability system would again be a necessary prerequisite.
- The sulfur dioxide released into the atmosphere by the crude-oil-refining industries imposes a severe environmental burden, at least in the immediate vicinity of the refineries. Therefore, strict emission standards have been set up for the individual plants. Clearly, the problem of how to enforce or to verify these emission standards arises (see, e.g., Märzendorfer[7]), and, again, a careful accountability of the sulfur processed in the refineries has been proposed as a basis for the verification of compliance with the emission standards.

Common to all these problems is the consideration of materials with specific properties – dangerous, or unpleasant, or rare – that are handled in the course of man's industrial activities and whose flow must be observed carefully for several reasons. These reasons, which have already been mentioned, may equally well be formulated as questions to be answered by the monograph:

- Where does a specific material flow, and is our knowledge about the flow complete? Where do we have to improve our knowledge?
- What is the probability that a diversion of a given amount of material is detected in time?
- What can be done to ensure that some rare metals are used more efficiently in industry? Where is the major part of these metals lost?
- How can we guarantee that substances that pose a hazard to the environment are released only within the limits of some emission standards?

Special forms of material accountability have been used for as long as one may think back – where gold and silver are processed, in mints, or in alcohol processing, for example. In addition, in the physical and chemical sciences the mass conservation principle was formulated long ago; the so-called continuity equation

$$\frac{d\gamma}{dt} + \mathrm{div}\,(\gamma \cdot v) = 0$$

(where γ is the mass density, v the velocity vector, and "div" the divergence operator) represents the final mathematical formulation of this principle and has been used in innumerable cases. However, since the mid-1960s these principles have attracted the interest of a broader technical, economic, and scientific audience because of several general developments.

One development was the end of the "flat earth theory" of economics. It was realized that "free goods" like water, air, and other abundant resources could no longer be considered free and that in this connection a much more careful treatment of "residuals" and the possibility of recycling these residuals (i.e., feeding them back into the production process) was necessary. This had been the case for centuries in the rural economy, and this was seen as an example for modern industrial activities. Among the exponents of this view are the members of the Resources for the Future (RFF) research group, who have stressed the need for a "material balance approach" to economic problems in a series of papers and monographs, both theoretical and applied.[6,8,9] Their work may be briefly summarized as follows: first, a careful accountability of all materials handled in the course of man's activities must be developed on a worldwide basis because of our limited resources. [The imaginative scenario that posits a "World Environment Control Authority" (see Kneese *et al.*,[6] Chapter 1) is worthy of mention here.]

Second, strict application of the accountability principle may become a major tool for governments and administrative agencies and authorities in the development and enforcement of regulations aimed at achieving more efficient use of some rare materials or a better separation of some hazardous substances from the ecosphere. We will come back to these ideas in Chapters 6 and 7.

Another much more special and concrete development has occurred as a consequence of the worldwide growth of peaceful nuclear industries. With the spread of nuclear know-how, which was initiated with the first Geneva Conference in 1956 and accelerated by the fact that light-water reactors became economically competitive power generation plants, the danger of proliferation of nuclear weapons grew. Therefore, the United Kingdom, the Soviet Union, and the United States proposed a treaty for the nonproliferation of nuclear weapons; this treaty was signed in 1968 and put into force in 1970 after having been ratified by more than 40 nations. (By 1977, more than 100 nations had signed the treaty.) The important fact in our case is that this treaty foresees international controls on the nuclear material of the peaceful nuclear industry and that the major tool of these controls is strict accountability of the nuclear material.[5] Even more important than the new international political significance of material accountability principles is that more than 40 sovereign states agreed on an international control system. This unique feature of the control system suggests immediately that it could serve very well as a pilot example for other urgently needed international control systems for the atmosphere or the oceans – quite in the sense outlined by the scenario already mentioned. Without going into greater detail here, it should be stressed that acceptance of this international control system was possible because of its "objective" and "rational" nature; in other words, it was the material accountability principle that helped make this system operational, and not, for example, surveillance measures that are built on the subjective impressions of inspectors, which have been promoted strongly for some time by several groups. Therefore, it seems worthwhile (and it was in fact, one of the basic ideas for this monograph, which has grown out of long-term theoretical work in the nuclear material safeguards field) to inform a broader audience of the power of material accountability principles in solving difficult technical and political problems and, furthermore, to demonstrate the feasibility of such international control systems.

How can the material accountability principle actually be formulated in a simple and applicable way? A trivial formulation may be that any material that enters a well-defined "box" or "area" cannot simply disappear; this means that either it can be found within the box, or it has left it again. A more refined formulation may be that the so-called book inventory of a box at a given time t_1, which is defined as the starting physical inventory at time t_0 before t_1 plus the inputs into the box minus the outputs from the box between t_0 and t_1, must be equal to the physical inventory at time t_1, and furthermore that the difference between these two quantities must be the amount of material that has been lost, diverted, badly accounted for, or not accounted for at all.

In this connection it should be mentioned that it is important to make a clear distinction between those situations in which there exist only losses or bad accountancy of material that need be considered (as is the case in the first and third example given above) and those situations in which the possibility of purposeful losses or even of diversion of material cannot be dismissed (as is the case in the second example). It is clear that the latter situation is the more difficult one, especially if detection of diversion of material is the major objective of material accountability: as a material imbalance may be caused by one or more of the possibilities mentioned, the problem arises of how to identify its true source. In addition, one must take into account that if diversion of material is planned, then it will be planned in such a way that the chance for detection is low, which means that in this case various strategies have to be considered and evaluated.

Another important question that has been mentioned already is that of bad accountancy. This may result from incomplete knowledge or incomplete measurement of all material flows or inventories, or it may result from measurement errors that cannot be avoided even if a very refined measurement technique is used or that are made because the financial and personnel resources available for this purpose are limited – which will always be the case in practical situations. This question has not yet commanded the full attention of economists dealing with material accountability problems[9] because they are concerned primarily with the introduction of the basic ideas; it is of paramount importance, however, in the nuclear material case. In general, the role of this aspect will be larger, the "worse" the relations between material throughputs or inventories, measurement accuracies, and importance of the material are. If an industrial plant processing zinc has a throughput of several hundred tons per year, if the measurement accuracies are in the order of percent, and if taxes on wasted zinc become significant only if some tens of tons are wasted, then measurement accuracies are not the central problem. If, on the other hand, a plutonium-processing plant processes only one ton per year, and the measurement accuracy is also in the order of percent, then it is crucial for the useful applicability of material balance principles to remember how toxic plutonium is and also to remember that only a few kilograms are needed for manufacturing a nuclear bomb.

A third aspect is the verification of the source data necessary for the establishment of a material balance. It is clear that this aspect plays a role only in those cases where material accountability is a means of detecting losses or diversion, where the source data are generated by one group (e.g., representatives of the industry), and where these data are verified by another group (e.g., representatives of a control authority). Since the possibility of data falsification cannot be excluded in such cases – otherwise verification measures would be meaningless – the problem of evaluation of the various existing falsification strategies arises again. Furthermore, as in the problem of appropriate relations between throughputs, measurement accuracies, and importance of the material, the problem of the appropriate verification effort in relation to the objectives must be considered.

If one studies applications of the material accountability principle, existing as well as theoretical ones, one realizes that only in a few cases is the principle in the narrow sense – establishment of a material balance by comparison of book and physical inventories – used. This is the case in all situations in science and technique where continuity equations are established or where mass balances like that of the earth's carbon dioxide or oxygen and nitrogen[10] are established; it is also the case for the nuclear material example. In many other cases material accountability means simply a measurement of all material flows as input, product, and waste; in other words, it means only the establishment of book inventories. This is even the case when the diversion possibility cannot be excluded, e.g., in the noble-metal-processing industries; here, in fact, the additional containment measures replace the taking of inventory and thus play a key role in the whole security system. Another case is material balances where the interest is centered on some waste streams that contain materials that cannot be balanced in a natural way, because they are generated in the plant. An example of this is the waste water chemical oxygen demand in the bottle industry, which has been discussed extensively by Ayres *et al.*[11] In the discussion of concrete applications in Chapters 6 and 7 of this volume, an attempt is made to clarify when and why material balances in the narrow sense are established and when and why only a more general material accountability is performed.

In accordance with the ideas on material accountability outlined so far – principle, verification, and applications – this monograph is structured as follows: In Chapter 2 the material balance principle will be described for one inventory period, and the problem of the statistical significance of the difference between book and closing real inventories will be discussed. Next, consideration will be generalized to a sequence of inventory periods. This is the first time that the difference between the two situations (where diversion of material need not be considered and where it cannot be excluded) becomes important: in the latter case, diversion strategies are taken into account. Analysis of the data verification problem is the subject of Chapter 3. This problem was intensively investigated in the course of the analysis of the nuclear material safeguards problem because of its importance in that field. In this monograph, a survey is given of the techniques that have been proposed for the comparison of measurement data and that may become important in various fields. In Chapter 4 a two-part analysis of the systems aspects of a control system is presented: data verification and establishment of a material balance. A conflict situation representation of the control system is given, and the question of the "global parameters" of the problem is raised, a problem that is intimately connected with the question of the unavoidable degree of subjectivity in the system.

In Chapter 5 the ideas developed in previous chapters are applied to the nuclear materials safeguards system, and the specific case of an irradiated-fuel-reprocessing plant is analyzed. Special emphasis is given to practical formulas and to limitations of application. The sixth chapter gives an idea of the extent of applications of the material accountability principle in the technological and economic fields; these applications range from very old and special ones, like mint material accountability, to the very broad use in the planning of socialist economies. Environmental accountability is the subject of Chapter 7; here the applications that have been mentioned in the preceding chapters are treated at some length. Finally, in the eighth chapter some possible applications in the arms control area are presented; they seem worthy of discussion here because of the vital importance of the subject and, furthermore, because this area offers an opportunity for particularly useful application of the ideas developed throughout this book.

(During the final stages of work on this monograph, a paper was published by Marsden *et al.*[12] who attempt to conceive a kind of "unified theory of mass balances." Three different examples (analysis of the mass flow in a chemical reactor, a river pollution analysis with the help of the Streeter–Phelps equations, and the analysis of the Solow economic growth model) demonstrate the close similarities between completely different problem areas in the form of a one-to-one correspondence of the basic quantities used in the three models. Furthermore, they offer some ideas on how to generalize the mass balance concept to a tool for any kind of interdisciplinary modeling. In this monograph, I have not gone so far, as the emphasis was more on the demonstration of the applicability of the material accountability principle to practical problems; however, the ideas of Marsden *et al.* seem quite appealing and could be considered as a straightforward theoretical extension of the ideas presented here.)

What is the intended audience for the material to be presented in the following chapters? The monograph speaks primarily to practitioners in the technical and economic area who have to establish material accountability systems for any material and who want information about the ways and means, the problems involved, and the benefits that may be realized by such systems. In addition, the monograph applies to physicists, chemists, biologists and environmental scientists who are investigating material balances in the course of their research and who may acquire some idea of the power of the material balance tool. Finally, it is intended for theoreticians in the economic and political area; in this regard, this monograph may be viewed as a further elaboration of some ideas expressed by Kneese *et al.*[6] and Göran.[9]

One final remark should be made about the mathematical methods used. Some may feel that a much more sophisticated formalism has been used than is necessary for the treatment of the practical examples presented here, and it could be argued that the presentation of some good computer programs would be of more help to the practitioners than the sometimes complicated formulae given here. The argument for the use of analytical solutions is that they provide a unique way of *explaining* the structure of the models and their achievements as well as their limitations. This is also the reason special attention has been given to the discussion of special cases and approximations: they provide an illustration of and an intuitive insight into the problems and their solutions. Furthermore, the intent was to present the material in such a form that it could be understood by a reader with a knowledge of mathematics and statistics that can be acquired through elementary textbooks like those of Brownlee[13] and of Bennett and Franklin.[14] Two sections perhaps require somewhat more knowledge (sections 3.3.1 and 4.4); however, a complete understanding of these sections is not necessary to the understanding of the rest of the book.

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2 The Material Balance Concept

In this chapter, the basic idea of the establishment of a material balance is presented. For this purpose a material balance area, through which material passes in a given interval of time, is defined. After the definition of the book inventory – initial physical inventory plus throughput (receipts minus shipments) – the material principle is formulated; this principle holds that if no material has been lost or diverted, then the book and physical inventories at any given time should be equal. This is simply a consequence of the law of conservation of matter.

This principle does not hold precisely on actual data because of random measurement errors and other uncertainties. Therefore, a decision problem arises, a problem that is complicated, if one considers a sequence of material balances, by the common occurrence of correlations between different elements of the sequence.

Two situations are considered in this chapter. The first is one in which the aspect of diversion of material is irrelevant – for instance, in the case of balances in nature (the carbon dioxide cycle of the earth will be considered in Chapter 7), as well as in the case of balances established for process control or for management purposes in various areas of technology and economics (some examples will be discussed in Chapter 6). In the second situation, the possibility of diversion of material will be taken into account; in fact, we will consider examples where the material balance principle is a tool for detecting the diversion of material. Here, "strategies" enter the scene.

Before going into *quantitative* considerations, we will demonstrate the usefulness of the material balance concept even in *qualitative* terms with the help of a little quiz:

Consider two bottles each containing the same amount of wine; red wine in one and white wine in the other. Now one spoonful of red wine is put into the white wine bottle and after some mixing a spoonful is taken from the white and red wine

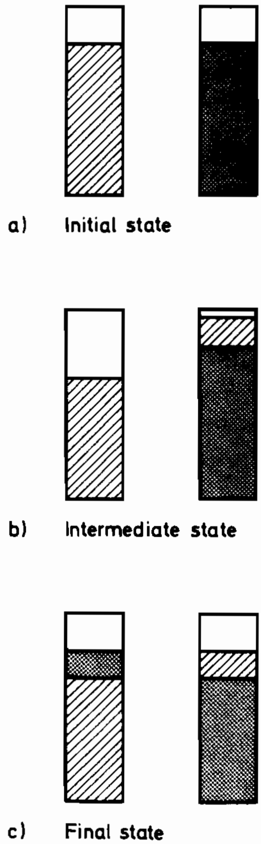

FIGURE 2.1 Mixing of wine (see text).

mixture and put back into the red wine bottle. The question is whether there is more red wine in the white wine or vice versa.

One can calculate with some effort that both bottles must contain the same amounts of foreign wines, but one can also show this directly with a material balance argument: if the red wine bottle contained more white wine than the white wine bottle contained red wine, or inversely, then the total amounts of white and red wine could not add up to the amounts originally given! The problem and its solution are illustrated in Figure 2.1.

2.1 THE MATERIAL BALANCE PRINCIPLE FOR ONE INVENTORY PERIOD

Let us consider a well-defined closed box that contains at a given time t_0 some material into which material enters and from which material leaves during a given interval of time (t_0, t_1). This box, which in the following discussion is called the material balance area, may represent, for example, an industrial material processing plant or the air above a given land area that contains some pollutants.

The material contained in the material balance area at time t_0 is called the *physical inventory* I_0. The algebraic sum of the amounts of material that enter and leave the material balance area in the interval of time (t_0, t_1) – which in the case of an industrial plant are called receipts and shipments – is called the *throughput D*. The physical inventory at t_0 plus the throughput in (t_0, t_1) gives the *book inventory B* at t_1, i.e., the amount of material that should be contained in the material balance area at time t_1:

$$B: = I_0 + D. \tag{2.1}$$

The amount of material actually contained in the material balance area at time t_1 is called the physical inventory I_1.* If all material contained in, and passing through, the material balance area in the interval of time (t_0, t_1) is carefully accounted for, and if no material has disappeared or has been diverted, then the difference between the book inventory B at t_1 and the physical inventory I_1 should be zero. This is simply a consequence of the law of conservation of matter. However, as not all of these conditions must be satisfied, the difference between these two quantities at the end of one inventory period, which for historical reasons has been called material unaccounted for *(MUF)*†

$$MUF: = B - I_1 = I_0 + D - I_1, \tag{2.2}$$

is not always zero. Thus arises the problem of finding out the various causes of this difference being nonzero and, furthermore, of trying to separate them.

This decision theoretical problem will be outlined for the simplest case in the next section. The definitions given in this section are summarized in Figure 2.2; other figures that represent applications of the material accountability principle appear in later chapters (see, e.g., Figures 5.2, 5.3, 6.1, 6.3, 7.1, and 7.3).

* A nice illustration of the difference between book and physical inventory was given by the Bavarian humorist Carl Valentin (1882–1948) in the form of a little sketch: a delivery man comes to a landlady and gives her a birdcage saying: "Here I deliver to you the birdcage with the canary inside that you bought some days ago." The landlady takes the cage, looks at it, and says: "But there is no canary inside!" Now an endless dispute arises in which the man explains in great detail that he himself had put the canary into the cage, that he carefully closed the door, that he continuously watched it until now, that there was no hole in the cage, and so on, and that therefore the canary *must be* in the cage (book inventory!) whereas the landlady continuously repeats and tries to convince the man that there is *no* canary in the cage (physical inventory!).

† It would be better to call this quantity "book-physical inventory difference," as in most cases the material is accounted for, but with measurement errors. In fact, this term has been used for some time (see, e.g., Stewart[1]), but MUF had already been used in so many papers and even in agreements and treaties that there was no chance of changing this somewhat misleading terminology.

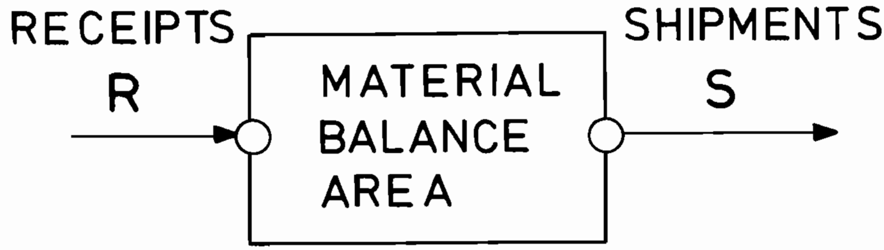

FIGURE 2.2 The material balance concept. Time t_0: beginning physical inventory I_0. Time interval (t_0, t_1): throughput D = receipts R − shipments S. Time t_1: book inventory $B = I_0 + D$; ending physical inventory I_1; MUF $= B - I_1$.

2.2 MEASUREMENT ERRORS, RANDOM LOSSES, AND DIVERSION: FORMULATION OF THE TEST PROBLEM

Let us assume that the measurement of the physical inventories I as well as that of the material throughput cannot be done without committing measurement errors and, furthermore, that there will always be some unmeasured losses – material flows into or out of the area that for any reason are not measured at all. Then, as already mentioned, the problem arises of deciding whether the difference between the book and the physical inventories at the end of the inventory period can be explained by these two sources of uncertainty or if there is another reason, for example, the diversion of material in cases where this is possible.

The solution of this decision theoretical problem requires some statistical formalism. More precisely, it requires the use of *test theory* methods. As the kinds of formulae and arguments that are used below will recur throughout this monograph, they are described here in some detail.

In order to outline the procedure, we will assume that the *probability distributions* of the measurement errors and of the random losses are known. We write the results of the measurements of the physical inventories I_0 and I_1 and of the book inventory D, which we now treat as *random variables*, in the following form

$$\begin{aligned} I_0 &= E(I_0) + e_0 \\ I_1 &= E(I_1) + e_1 \\ D &= E(D) + e_D \end{aligned} \tag{2.3}$$

where $E(I_0)$, $E(I_1)$, and $E(D)$ are the expected values of the random variables I_0, I_1, and D – that is, the true values of the inventories I_0, I_1, and D in those cases in which there are no *persistent systematic errors* – and where e_0, e_1, and e_D are the random errors of the measurements of I_0, I_1, and D. The expected values and the variances (i.e., the *first two moments* of the probability distributions of e_0, e_1, and e_D) are given by the following expressions:

$$
\begin{aligned}
E(e_0) &= 0, \qquad \operatorname{var}(e_0) = \sigma_{I_0}^2 \\
E(e_1) &= 0, \qquad \operatorname{var}(e_1) = \sigma_{I_0}^2 \\
E(e_D) &= 0, \qquad \operatorname{var}(e_D) = \sigma_{D}^2
\end{aligned}
\tag{2.4}
$$

Furthermore, we assume that during the inventory period a *random loss l* occurs. This loss we also conceive of as a random variable whose probability distribution is known and whose expected value and variance are given by

$$E(l) = 0,^{\dagger} \qquad \operatorname{var}(l) = \sigma_l^2. \tag{2.5}$$

The measurement errors and the random losses may cause a nonzero book–physical inventory difference, as already explained. In order to understand this, we write Eq. (2.2) with the help of the expressions (2.3) in the following form:

$$MUF = E(I_0) + e_0 + E(D) + e_D - E(I_1) - e_1 - l. \tag{2.6}$$

Because of the conservation of matter we have

$$E(I_0) + E(D) - E(I_1) - l = 0. \tag{2.7}$$

Therefore, from Eqs. (2.4) and (2.5) we obtain

$$E(MUF) = 0. \tag{2.8}$$

We call this relation, which is a consequence of the assumption that there are only measurement errors and random losses with the properties (2.4) and (2.5) the *null hypothesis H_0*.

We can now formulate our problem, which is to find out whether the non-vanishing book–physical inventory difference is caused only by measurement errors and random losses. In statistical terms: we have to *test the null hypothesis H_0*. We achieve this by choosing a *significance threshold s* for the sample value (realized value) of the book–physical inventory difference $M\hat{U}F$ and deciding

$$H_0 \text{ correct if } M\hat{U}F \leqslant s. \tag{2.9}$$

The value of the significance threshold s is fixed with the help of the *probability of error of the first kind* α, which is defined by

$$\alpha := \operatorname{prob}\{MUF > s \mid H_0\}. \tag{2.10}$$

In words, α is the probability that "H_0 not correct" will be stated if, in fact, H_0 is true. The problem of the appropriate choice of the value of α is discussed in Chapter 4.

If the result of the measurement is

$$M\hat{U}F > s, \tag{2.11}$$

† We consider here internal plant losses under stable process conditions. For new plants or if one has to consider external losses, the expected value of l will be greater than zero.

we conclude that "the null hypothesis H_0 is not correct" or "the alternative hypothesis H_1 is correct." The nature of the physical problem determines whether we want to formulate the alternative hypothesis H_1 explicitly. Let us assume that it is reasonable to formulate H_1 in the following way:

$$H_1 : E(MUF) = M, \tag{2.12}$$

where M is a quantity greater than zero. (The choice of an appropriate value for M will also be discussed in Chapter 4.) In this case, we can characterize the test by the *probability of error of the second kind* β, which is defined by

$$\beta : = \text{prob}\ \{MUF \leqslant s \mid H_1\}. \tag{2.13}$$

In words, β is the probability that "H_1 not correct" will be stated if, in fact, H_1 is true (or, in line with standard statistical terminology, it is the probability that the statement "H_0 not correct" will not be made).

The probabilities of errors of the first and second kind for normally and independently distributed measurement errors and random losses are given by

$$1 - \alpha = \Phi\left(\frac{s}{\sigma}\right) \tag{2.14}$$

and

$$1 - \beta = \Phi\left(\frac{M}{\sigma} - U_{1-\alpha}\right), \tag{2.15}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt \tag{2.16}$$

is the normal (Gaussian) distribution function, U is inverse, and

$$\sigma^2 = 2\sigma_I^2 + \sigma_D^2 + \sigma_l^2 \tag{2.17}$$

is the variance of the material unaccounted for.

Generally, in this monograph we will not present formal proofs of mathematical statements or theorems. As in this case, however, we will occasionally give informal proofs in order to convey an idea of the mathematics involved.

The relations (2.14) to (2.17) can be derived as follows: from the definition of α (Eq. 2.10), we get

$$1 - \alpha = \text{prob}\ \{MUF \leqslant s \mid H_0\},$$

or, with Eqs. (2.4) and (2.7),

$$1 - \alpha = \text{prob}\left\{\frac{e_0 + e_D - e_1 - l}{\sigma} \leqslant \frac{s}{\sigma} \mid H_0\right\},$$

where σ is given by (2.17). As we assumed e_0, e_D, e_1, and l to be normally and independently distributed, the random variable $(e_0 + e_D - e_1 - l)/\sigma$ is normally distributed with expected value zero and variance 1, and so Eq. (2.14) holds.

From the definition of the alternative hypothesis H_1 (Eq. 2.12), we take the random variable $(e_0 + e_D - e_1 - l - M)/\sigma$ to be normally distributed with expected value zero and variance σ; therefore we get from the definition of β

$$\beta = \text{prob}\left\{\frac{MUF - M}{\sigma} \leqslant \frac{s - M}{\sigma}\right\},$$

or

$$\beta = \Phi\left(\frac{s}{\sigma} - \frac{M}{\sigma}\right).$$

Elimination of s with Eq. (2.14) gives Eq. (2.15).

Since the purpose of the test procedure described so far is to detect unusual losses (i.e., losses that cannot be explained by experience) or diversion, for obvious reasons we call the probability of the error of the first kind, α, the *false alarm probability*, and we call one minus the probability of the error of the second kind, $1 - \beta$, the *probability of detection.* Because of the central importance throughout this monograph of Eq. (2.15), which establishes a relation between false alarm probability α, variance of measurements and random losses σ^2, amount M assumed to be missing (or diverted), and probability of detection, we will discuss it here in some detail:

1. The probability of detection increases with increasing amount M assumed to be missing (or diverted). This property is a natural requirement in any detection system.
2. The probability of detection increases with decreasing standard deviation σ. This is reasonable, too. If one remembers that the standard deviation ordinarily decreases with increasing effort (money or man-hours), this property means that the probability of detection increases with increasing effort.
3. The probability of detection increases with increasing false alarm probability. This is a well-known property of any detection system (e.g., fire alarm system): the more sensitive the system is, the higher is its false alarm rate.

Up to now, Eq. (2.15) has been used in order to determine the value of the probability of detection $1 - \beta$. We can use it, however, to determine the value of any of the four quantities involved. For this reason, it is sometimes written in a symmetric form:

$$U_{1-\alpha} + U_{1-\beta} = \frac{M}{\sigma}.$$

A numerical example that can easily be obtained with the help of any table for the normal (Gaussian) distribution function may be formulated as follows: if an amount of material is missing (or diverted) that is 3.3 times as large as the standard deviation σ, and if the false alarm probability α is 0.05, then the probability of detection is 0.95.

Figure 2.3 is a graphic representation of Eq. (2.15) in the form of a nomograph. Since Eq. (2.15) will be used to determine the value of any of the four quantities,

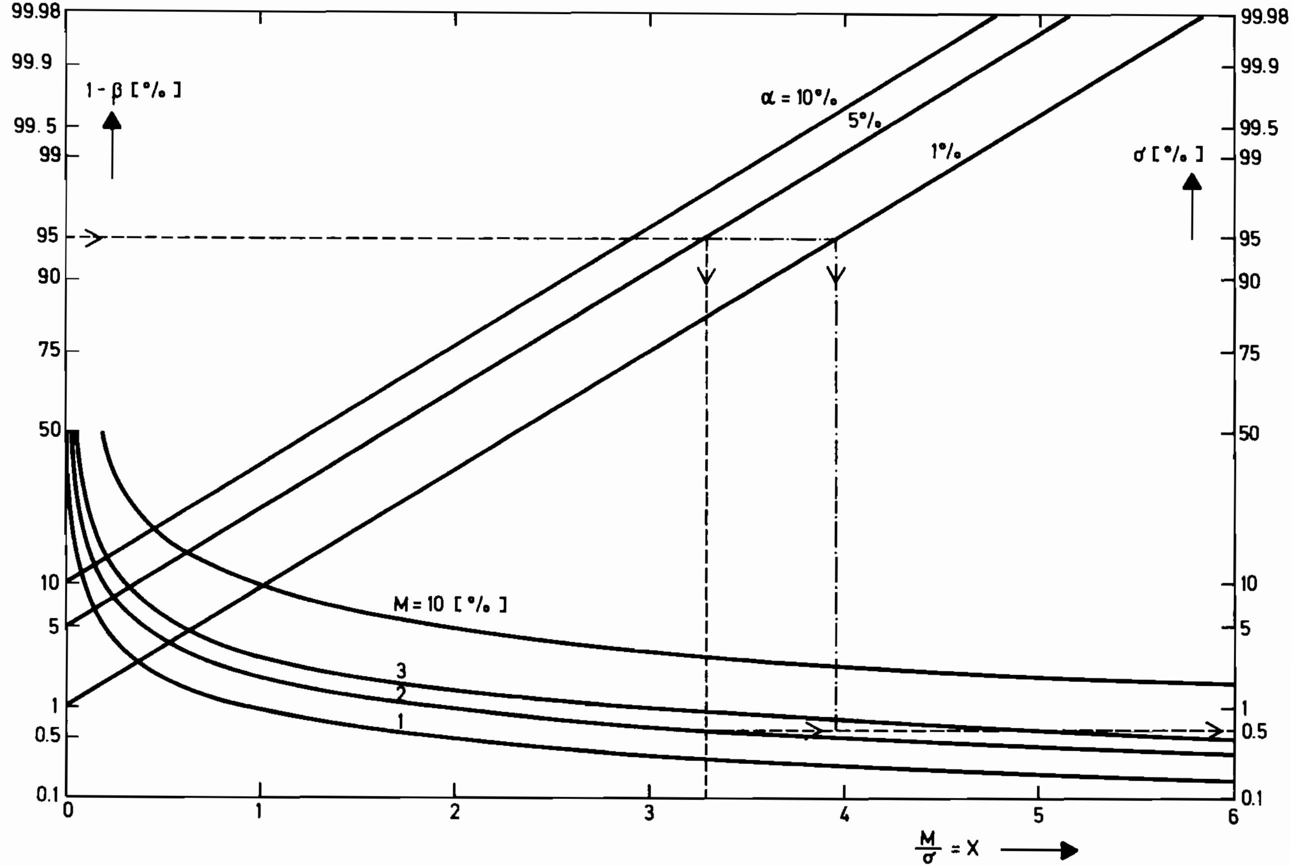

FIGURE 2.3 Nomograph for the equation $1 - \beta = \phi(M/\sigma - U_{1-\alpha})$.

we can, with the help of this nomograph, give the values of three of the four parameters and determine the value of the fourth as shown by the example. It should be noted once more that we have assumed that the variances of measurement errors and random losses are known. The estimation of measurement error variances may require considerable experimental effort,[2] and the estimation of variances of random losses can be achieved only with the help of long-term historical MUF data.[3–5] Although these problems will not be treated in further detail, a remark about the importance of *systematic measurement errors* in this specific connection is in order. For simplicity we consider one part of the throughput measurement D:

Let us assume that the receipts in the interval (t_0, t_1) consist of n different batches and that each single batch i is measured with a random error e_i, $i = 1, \ldots, n$, and a systematic error d that is common to all errors of the sequence of n batches.

As an illustration of this we can imagine that the measurements are performed with the help of a calibration curve and that in the establishment of this calibration curve random errors are committed. As long as one uses the same calibration curve, the error of this curve remains the same – i.e., it is systematic in the usual sense of the word. However, it should become clear from this example that this systematic error is of random origin and varies randomly from calibration to calibration. In other words, random and systematic errors are not logical alternatives but have to be defined in connection with the specific experiment under consideration. The important difference between these two types of errors is their mode of propagation, as will be shown.

The resulting variance of the sum of all receipts can then be determined as follows: let R_i be the measurement result of one batch:

$$R_i = E(R) + e_i + d, \qquad i = 1, \ldots, n, \tag{2.18}$$

with

$$\begin{aligned} E(e_i) &= 0, \quad \mathrm{var}(e_i) = \sigma_r^2 = E^2(R) \cdot \delta_r^2 \\ E(d) &= 0, \quad \mathrm{var}(d) = \sigma_s^2 = E^2(R) \cdot \delta_s^2 \end{aligned} \tag{2.19}$$

Then the relative standard deviation δ_{Sum} of the sum

$$\mathrm{Sum} := \sum_{i=1}^{n} R_i \tag{2.20}$$

is given by

$$\delta_{\mathrm{Sum}}^2 = \frac{1}{n} \cdot \delta_r^2 + \delta_s^2 . \tag{2.21}$$

This formula can be derived in the following way: from Eqs. (2.18), (2.19), and (2.20) we have

$$\mathrm{var}\,(\mathrm{Sum}) = \mathrm{var}\left(n \cdot E(R) + \sum_{i=1}^{n} e_i + n \cdot d \right) = n \cdot \sigma_r^2 + n^2 \cdot \sigma_s^2 .$$

Therefore, with

$$\text{var (Sum)} = E^2(\text{Sum})\cdot\delta^2_{\text{Sum}}$$

we obtain

$$\delta^2_{\text{Sum}} = \frac{n\cdot\sigma_r^2}{n^2\cdot E^2(R)} + \frac{n^2\cdot\sigma_s^2}{n^2\cdot E^2(R)},$$

which, with Eqs. (2.19), is equivalent to Eq. (2.21). This result shows that for very large batch numbers the dominating part of the variance is the variance of the systematic error even if for the single measurement this is small compared with that of the random error.

We have already mentioned the problem of estimating error variances. It is clear that in all cases where the total measurement error is composed of random and systematic errors according to Eq. (2.18), this problem is even more complicated; here, *analysis-of-variance* types of experiments have to be performed.[6]

2.3 SEQUENCE OF INVENTORY PERIODS

In this section, we consider a reference time interval $(0,T)$, say 1 year, and assume that during this time $n + 1$ physical inventories will be taken (the value of n will be discussed in section 2.5). A number of new problems now arise. The first one to be discussed is the problem of the starting inventory for each of the inventory periods.

Let us assume that at the end of one inventory period the result of the test is "the null hypothesis is correct," in other words, that no material has disappeared. Then, in principle, one would assume that the ending physical inventory of the preceding inventory period should be taken as the starting inventory for the next inventory period, as it represents the amount of material actually found in the material balance area. However, the uncertainty of this value could be very high compared with that of the book inventory. In such a case it would be better to take the ending book inventory.

Stewart[7] has proposed estimating the starting inventory for the period (t_{i-1}, t_i) with the help of a linear combination of the ending book and physical inventories of the preceding period (t_{i-2}, t_{i-1}) in such a way that the estimate is unbiased and has a minimum variance. This unbiased minimum variance estimate S_{i-1} for the starting inventory of the ith inventory period is given by the following formula:

$$S_{i-1} = a_{i-1}\cdot B_{i-1} + (1 - a_{i-1})\cdot I_{i-1}, \tag{2.22a}$$

where

$$a_{i-1} = \frac{\text{var}(I_{i-1})}{\text{var}(B_{i-1}) + \text{var}(I_{i-1})}. \tag{2.22b}$$

The variance of this estimate is given by

$$\frac{1}{\text{var}(S_{i-1})} = \frac{1}{\text{var}(B_{i-1})} + \frac{1}{\text{var}(I_{i-1})}. \tag{2.23}$$

How can we understand these formulae? An unbiased estimate for the starting inventory is given by

$$\widetilde{S}_{i-1} = \widetilde{a}_{i-1} \cdot B_{i-1} + (1 - \widetilde{a}_{i-1}) \cdot I_{i-1}\,; 0 \leqslant \widetilde{a}_{i-1} \leqslant 1.$$

The variance of this estimate is

$$\text{var}(\widetilde{S}_{i-1}) = \widetilde{a}_{i-1}^2 \cdot \text{var}(B_{i-1}) + (1 - \widetilde{a}_{i-1})^2 \cdot \text{var}(I_{i-1}).$$

The value of $\widetilde{a}_{i-1}$ that minimizes var $(\widetilde{S}_{i-1})$ is determined by the following equation:

$$a_{i-1} \cdot \text{var}(B_{i-1}) - (1 - a_{i-1}) \cdot \text{var}(I_{i-1}) = 0. \tag{2.24}$$

Clearly, the variance of this estimate is smaller than each of the variances var(B_{i-1}) and var(I_{i-1}). Thus, even when one of the two ending inventories is much more precise than the other, it is useful to take the less precise inventory into account as well.

The estimate as given above has another property that is important for the following reason: if for every inventory period the starting inventory is chosen as described by formula (2.22), then the book–physical inventory differences of different inventory periods are uncorrelated. The proof of this statement can be understood immediately. The book–physical inventory difference for the jth involves only constants. Then the covarience between MUF_1 and MUF_j is

$$MUF_j = a_{j-1} \cdot B_{j-1} + (1 - a_{j-1}) \cdot I_{j-1} + D_j - I_j.$$

Let $j > i$. Then S_{j-1} can be written as

$$S_{j-1} = c_{j-1} + b_{j-1} \cdot S_i,$$

where the c_{j-1} term involves I's and B's with subscripts larger than i, and b_{j-1} involves only constants. Then the covariance between MUF_i and MUF_j is

$$\begin{aligned} \text{cov}(MUF_j, MUF_i) &= \text{cov}(S_{j-1} + D_j - I_j, S_{i-1} + D_i - I_i) \\ &= b_{j-1} \cdot \text{cov}(S_i, S_{i-1} + D_i - I_i) \\ &= b_{j-1} \cdot \text{cov}(a_i \cdot (S_{i-1} + D_i) + (1 - a_i) \cdot I_i, S_{i-1} + D_i - I_i). \end{aligned}$$

As $\text{cov}(S_{i-1} + D_i, I_i) = 0$, and as $\text{cov}(X, X) = \text{var}(X)$, we get

$$\begin{aligned} \text{cov}(MUF_j, MUF_i) &= -b_{j-1} \cdot [(1 - a_i) \cdot \text{var}(I_i) - a_i \cdot \text{var}(B_i)] \\ &= 0, \end{aligned}$$

if we use Eq. (2.24) by putting i instead of $i - 1$.

The fact that the book–physical inventory differences are *uncorrelated* if the starting inventories are chosen according to Eq. (2.22) has a further consequence whose importance will become evident. As it has been assumed initially that all measurement errors are normally distributed, it follows that the book–physical inventory differences of different inventory periods are *independent*. This is important in the following considerations.

The second problem in the treatment of a sequence of inventory periods is the appropriate extension of the test procedure outlined in section 2.2 to a sequence of inventory periods. The null hypothesis in this case is, clearly,

$$H_0: E(MUF_1) = \ldots = E(MUF_n) = 0. \tag{2.25}$$

The direct translation of the method described in section 2.2 would be to perform n tests, one after each single inventory period. However, there are reasons to control the test as a whole – in other words, to control the resulting probability of error of the first kind α, which is given by

$$1-\alpha = \text{prob}\,\{MUF_1 \leqslant s_1 \wedge MUF_2 \leqslant s_2 \wedge, \ldots, \wedge MUF_n \leqslant s_n \,|\, H_0\},$$

where s_i is the significance threshold of the ith test, and which gives with the zero-correlation property

$$1-\alpha = \text{prob}\,\{MUF_1 \leqslant s_1 \,|\, H_0\} \ldots \text{prob}\,\{MUF_n \leqslant s_n \,|\, H_0\}, \tag{2.26}$$

or, with Eq. (2.10)

$$1-\alpha = \prod_{i=1}^{n} (1-\alpha_i), \tag{2.27}$$

where a_i is the false-alarm probability for the ith inventory period. Another possibility would be to test the null hypothesis as a whole, e.g., on the basis of a multivariate statistic like that described by Anderson.[8] The disadvantage of this is that one gets no information at all about the single inventory periods.

In the following section we will proceed by performing n tests in the sense of section 2.2, where we choose the values of the single significance thresholds in such a way that the boundary condition (Eq. 2.27) is met for a given value of the overall probability of an error of the first kind.

2.4 MATERIAL ACCOUNTABILITY AS A TOOL FOR DETECTING THE DIVERSION OF MATERIAL

In cases where we know that diversion of material is not possible, the alternative hypothesis H_1 need not be formulated explicitly, as mentioned in section 2.3. According to the test procedure described above, there is a certain degree of freedom in the choice of the n significance thresholds for the single-inventory periods of the reference interval of time; this is so because we have only one boundary condition (Eq. 2.27). This freedom can be used, for example, to simplify the procedure by choosing the same values for the probability of an error of the first kind α_i for all periods:

$$\alpha_1 = \ldots = \alpha_n = :1 - \sqrt[n]{1-\alpha}. \tag{2.28}$$

On the other hand, in all cases where one has to take into account the possibility of the diversion of material, the alternative hypothesis H_1 plays a major role. This was indicated in section 2.2, where we introduced a probability of detection $1-\beta$, which was a function of the amount M assumed to be diverted. In the following, we assume that material accountability is performed primarily in order to detect any diversion of material – we will discuss this point once more at the end of this section. Therefore, we formulate the alternative hypothesis H_1 in such a way that we assume that in the ith inventory period the amount M_i of material is diverted:

$$H_1 : E(I_{i-1} + D_i - I_i) = M_i, \qquad i = 1, \ldots, n. \tag{2.29}$$

According to the choice of the starting inventory, the expected value of the book–physical inventory difference at the end of an inventory period is not given simply by the amounts of material disappearing in these inventory periods.

Under the assumption that in the jth inventory period ($j = 1, \ldots, n$) the amount M_j disappears (alternative hypothesis H_1) the expected value of the book–physical inventory difference of the ith inventory period is determined by the recursive relation

$$E(MUF_i|H_1) = a_{i-1} \cdot E(MUF_{i-1}|H_1) + M_i ; E(MUF_1|H_1) = M_1. \tag{2.30}$$

These relations can be proven as follows: if one defines

$$E(I_j) = : E_j, \qquad j = 0, 1, \ldots, n$$

one has, according to this assumption,

$$E_{j-1} + E(D_j) - E_j = M_j.$$

Therefore,

$$\begin{aligned} E(MUF_i|H_1) &= E(S_{i-1}) + E(D_i) - E_i \\ &= a_{i-1}[E(S_{i-2}) + E(D_{i-1})] + (1-a_{i-1}) \cdot E_{i-1} + E(D_i) - E_i \\ &= a_{i-1} \cdot [E(MUF_{i-1}|H_1) - E(D_{i-1}) + E_{i-1} + M_{i-1} - E_{i-2} + E_{i-1}] \\ &\quad + (1-a_{i-1}) \cdot E_{i-1} + E(D_i) - E_i \\ &= a_{i-1} \cdot E(MUF_{i-1}|H_1) + M_i. \end{aligned}$$

Let us return to the question of the appropriate determination of the significance thresholds s_i for the n inventory periods during the reference interval of time $(0, T)$ in a case in which a diversion of material cannot be excluded. We assume that a "critical amount M of material" exists in such a way that a diversion of this amount should be detected with a guaranteed probability, where

$$M = \sum_{i=1}^{n} M_i \tag{2.31}$$

and where M_i is the amount of material disappearing in the ith inventory period.

The optimal choice of the significance thresholds s_i (or the probabilities of errors of the first kind α_i) for the ith inventory period may then be formulated as that choice that minimizes the probability of error of the second kind for *any* alternative hypothesis H_1 given by Eq. (2.29) subject to the boundary conditions (2.27) and (2.31).

Because of the relations (2.30) the expected value y_i of the ith book–physical inventory difference is given by

$$y_i = a_{i-1} \cdot y_{i-1} + M_i \tag{2.32}$$

and thus the probability of error of the second kind for the ith inventory period, according to Eq. (2.15), is given by

$$\beta_i = 1 - \Phi\left(\frac{y_i}{\sigma_i} - U_{1-\alpha_i}\right), \tag{2.33}$$

where

$$\sigma_i^2 := \operatorname{var}(S_{i-1}) + \operatorname{var}(D) + \operatorname{var}(I_i)$$

is the variance of the ith material unaccounted for.

On this basis, we can formulate the problem of the determination of the optimal set of probabilities of errors of the first kind $(\alpha_1^*, \ldots, \alpha_n^*)$ in the following way:

The optimal set $(\alpha_1^*, \ldots, \alpha_n^*)$ of the probabilities of errors of the first kind is that set that determines the total optimal guaranteed probability of an error of the second kind,

$$\beta^{**} = \prod_{i=1}^{n} \phi\left(U_{1-\alpha_i}^* - \frac{y_i^*}{\sigma_i}\right). \tag{2.34}$$

The total optimal guaranteed probability of an error of the second kind is defined by

$$\beta^{**} = \min_{\substack{\alpha_1 \ldots \alpha_n: \\ 1-\alpha = \prod_i (1-\alpha_i)}} \max_{\substack{M_1 \ldots M_n: \\ M = \sum_i M_i}} \prod_{i=1}^{n} \Phi\left(U_{1-\alpha_i} - \frac{y_i}{\sigma_i}\right). \tag{2.35}$$

Intuitively, a person or a group who wants to divert material would do the inverse, i.e., choose its strategy $x \in X$, where

$$X = \{(M_1, \ldots, M_n) \colon \sum_i M_i = M > 0\}, \tag{2.36}$$

in such a way that the probability of an error of the second kind is maximized for any strategy $y \in Y$, where

$$Y = \{(\alpha_1, \ldots, \alpha_n) \colon \prod_{i=1}^{n} (1 - \alpha_i) = 1 - \alpha\}. \tag{2.37}$$

Thus we may formulate the optimization problem as a two-person zero-sum game (X, Y, β) where the sets of pure strategies are given by the sets (2.36) and (2.37) and where the payoff to the first player is given by the probability of error of the second kind.

It can be shown that the optimal strategies $(M_1^*, \ldots, M_n^*)$ and $(\alpha_1^*, \ldots, \alpha_n^*)$ of the optimization problem (2.34) exist and are unique. We will not give the derivation of the rather complicated analytical expressions for the optimal strategies, which are obtained with the help of Lagrange multiplier technique; instead, the reader is referred to theorem 3.10 in the original work of Avenhaus and Frick.[9] We will, however, present the analytical expressions for the optimal strategies: they are given as solutions of the following two systems of equations:

$$\frac{e^{-\left(x_i + \frac{U^2(e^{x_i})}{2}\right)}}{\sigma_i(1-a_i)} - \frac{e^{-\left(x_{i-1} + \frac{U^2(e^{x_{i-1}})}{2}\right)}}{\sigma_{i-1}\cdot(1-a_{i-1})} = 0, \qquad i = 2, \ldots, n-1,$$

$$\frac{e^{-\left(x_n + \frac{U^2(e^{x_n})}{2}\right)}}{\sigma_n} - \frac{e^{-\left(x_{n-1} + \frac{U^2(e^{x_{n-1}})}{2}\right)}}{\sigma_{n-1}\cdot(1-a_{n-1})} = 0,$$

$$\sum_{j=1}^{n} x_j = \ln(1-\alpha); \quad x_j = \ln(1-\alpha_j); \tag{2.38}$$

$$e^{x_i + \frac{U^2(e^{x_i})}{2}} \cdot Q\left(U(e^{x_i}) - \frac{y_i}{\sigma_i}\right) - e^{x_{i-1} + \frac{U^2(e^{x_{i-1}})}{2}}$$

$$\cdot Q\left(U(e^{x_{i-1}}) - \frac{y_{i-1}}{\sigma_{i-1}}\right) = 0, \qquad i = 1, \ldots, n,$$

$$y_n + \sum_{j=1}^{n-1}(1-a_j)\cdot y_j = M, \tag{2.39}$$

y_j and a_j are given by Eqs. (2.22b) and (2.32).

For the purpose of illustration, we give the optimal strategies for two inventory periods, (M_1^*, M_2^*) and (α_1^*, α_2^*). According to Eqs. (2.38) and (2.39), they are solutions of the following equations:

$$(1-\alpha_1)\cdot\sigma_1\cdot(1-a)\cdot\exp(\tfrac{1}{2}U^2_{1-\alpha_1}) - (1-\alpha_2)\cdot\sigma_2\cdot\exp(\tfrac{1}{2}U^2_{1-\alpha_2}) = 0$$

$$(1-\alpha_1)\cdot(1-\alpha_2) = 1-\alpha$$

$$-\frac{1}{\sigma_1}\cdot\Phi'\left(U_{1-\alpha_1} - \frac{M_1}{\sigma_1}\right)\cdot\Phi\left(U_{1-\alpha_2} - \frac{M}{\sigma_2} + \frac{(1-a)\cdot M_1}{\sigma_2}\right)$$

$$+\frac{1-a}{\sigma_2}\cdot\Phi\left(U_{1-\alpha_1} - \frac{M_1}{\sigma_1}\right)\cdot\Phi'\left(U_{1-\alpha_2} - \frac{M}{\sigma_2} + \frac{(1-a)M_1}{\sigma_2}\right) = 0$$

$$M_1 + M_2 = M,$$

where $\Phi'(x)$ is the first derivative of the Gaussian distribution function $\Phi(x)$ and where a is given by Eq. (2.22b).

Two important properties of these solutions must be mentioned here; they can be deduced from the general solution, but it is easier to do so from the special solution given above:

- For $a_i = 0$ – that is, for the case where the physical inventories are measured precisely and therefore are taken as the starting inventories for the next inventory periods – one can show with the help of some algebra[10] that the smaller the variances σ_i^2 of the book–physical inventory differences (with appropriate starting inventory) are, the smaller and corresponding optimal significance thresholds

$$s_i^* = \sigma_i \cdot U_{i-\alpha_i^*}$$

are and the smaller the amounts of material M_i^* to be diverted in the ith inventory period are. This can be interpreted in such a way that in inventory periods during which the technical possibility (expressed by the variance of the measurement errors) of detecting a diversion is good, the operator will divert only small amounts of material, if any, and the inspection team will check the material balance only in a rather loose way. We will come back to this point in Chapter 3, where we will find the same situation again.

- One can conclude immediately from the set of equations (2.38) that the optimal strategy $(\alpha_1^*, \ldots, \alpha_n^*)$ of the inspection team does not depend on the value of the total amount M to be diverted in the reference time. This is very important because in both cases, i.e., for accidental losses as well as for intentional diversion of material, the plant operator who wants to detect losses and the control authority that wants to detect diversion of material can at best have an idea of the *order of magnitude* of the amount of missing material, but not of its precise value. The fact that the optimal inspection strategy does not depend on the value of M tells us that it is, in fact, not necessary to know the precise value of M in order to establish an optimal control scheme.

We have stated before that the optimization problem would be intuitively formulated as a two-person zero-sum game with the error of the second kind as the payoff. We will discuss this assumption in greater detail in the fourth chapter. Here, we will mention only that from the point of view of optimizing the probability of error of the second kind the way proposed in section 2.2 is not necessarily the best way to estimate the starting inventory. In fact, other estimates can be given.[11] However, as these estimates lead to very difficult correlation problems, they are not very useful from the practical point of view, and therefore we will use only the estimate given by Eq. (2.22).

Let us conclude this section with a remark about the purpose of the optimization procedure just described. The idea was to maximize the probability of detection of *any* diversion strategy for a given total amount M to be diverted. Therefore, we had to determine the *best* diversion strategy from the diverter's point of view. As a result, we obtained the optimal guaranteed probability of detection. This procedure is also reasonable in cases where intended diversion of material need not be considered, but only accidental losses: it gives the optimal control strategy for the detection of losses of total size M (which need not be known) that occur in any mode, including the worst mode from the detection point of view. In other words, for this case, too, we have determined the optimal control scheme and the optimal guaranteed probability of detection. Such a procedure is commonly used in the game theoretical treatment of statistical problems, where one talks about "games against malevolent nature."

2.5 FREQUENCY OF TAKING INVENTORY

In the two foregoing sections we have assumed that there are n inventory periods per reference time interval $(0, T)$. The question of the existence of an "optimum number n" of inventory periods now arises.

In order to tackle this problem we have to formulate the appropriate optimization criterion. We will proceed in two steps. First, we assume that the crtiterion is again the overall *probability of detection*. This is in line with the formulation of the optimization problem as given by Eq. (2.35). Second, and this is a new aspect, we take the *detection time* into consideration.

The analysis of the first problem, i.e., the optimization of the overall probability of detection with respect to the number n of inventory periods leads to the following somewhat surprising result, which will be reported here without the rather lengthy proof[10,12]: the total optimal guaranteed probability of error of the second kind β^{**} as given by Eq. (2.34) is smaller for one inventory period per reference time than for n inventory periods per reference time for any $n > 1$.

This result is surprising, as one would expect that additional physical inventories would increase the quality of the statement. It is to be noted that it was not proven that n inventory periods are better than m inventory periods for $m > n$ – i.e., strict monotony was not proven. However, numerical calculations indicate that, in fact, β^{**} is a strictly increasing function of the number n of inventory periods; an example is given in Figure 2.4.

From these results one could conclude that in any case it is best to have only one inventory period for the reference interval of time. However, when one has to take into account diversion of material, the detection time also must be considered. It is clear that this detection time is determined by the number of inventory periods, as a statement about detection can be made only after the book–physical inventory comparison.

From what has been said up to now, one could make two assumptions about the detection time. First, one would assume that the shorter an inventory period is, the shorter the detection time is. Second, according to the result stated above, with an increasing number of inventory periods per reference time, the probability of detection decreases. Therefore, detection may depend on the values of the

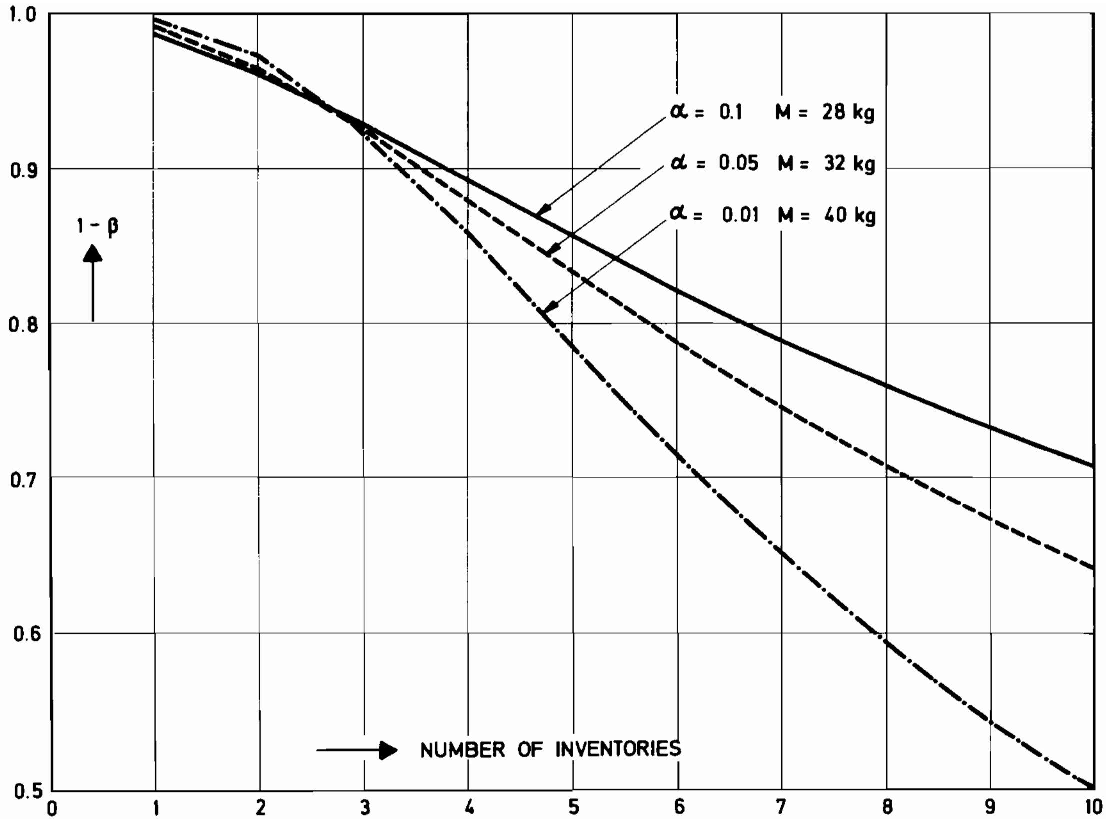

FIGURE 2.4 Probability of detection $1-\beta$ as a function of the number of inventory periods per year with α and M as parameters. Data from Avenhaus and Frick.[12] $\mathrm{var}(I_i) = 0.013\ \mathrm{kg}^2$ for $i = 0.1 \ldots$; $\sum_i \mathrm{var}(D_i) = 63.26\ \mathrm{kg}^2$.

parameters of the stronger of the two effects. From these considerations one concludes that the expected detection time T is the appropriate optimization criterion for the detection time point of view, because it takes into account both aspects, the actual time at which detection may occur and the probability for the detection at that time.

The expected detection time, say T_2, which shall be measured in units of inventory periods, is the sum of the products of detection times $i (i = 1, \ldots, n)$ and probabilities for first detection at time i; these probabilities are given by

$$P_i \cdot \prod_{j=1}^{i-1} (1 - P_j), \qquad i = 1, \ldots, n \tag{2.40a}$$

(this has to be understood as P_1 for $i = 1$), where P_i is given by

$$P_i := \text{prob} \{MUF_i > s_i \mid H_1\}. \tag{2.40b}$$

We note in passing that (2.40a) can be written in such a simple form only because the different MUF_i's are independent.

One is now faced with the difficulty of taking into account the possibility that no detection at all occurs during the reference time; the probability of this is given by

$$\prod_{i=1}^{n} (1 - P_i),$$

where P_i is again given by Eq. (2.40b). If we call a the detection time for the case in which detection occurs only after the end of the reference time, then the expected detection time is given by the following formula:

$$T_2 := \sum_{i=1}^{n} i \cdot P_i \cdot \prod_{j=1}^{i-1} (1 - P_j) + a \cdot \prod_{i=1}^{n} (1 - P_i), \tag{2.41}$$

where P_i again is given by Eq. (2.40b). The difficulty with this formula is, as already stated, that there exists no natural numerical value for a; for optimization purposes we may arbitrarily put $a = n + 1$.

A more satisfying optimization criterion is the expected detection time T_1 under the *condition* that detection will take place during the reference time. This criterion again takes into account both aspects of detection time – actual time and probability of detection at that time. It is given by the following formula:

$$T_1 := \frac{\sum_{i=1}^{n} i \cdot P_i \cdot \prod_{j=1}^{i-1} (1 - P_j)}{1 - \prod_{i=1}^{n} (1 - P_i)}, \tag{2.42}$$

where P_i is once again given by (2.40b). Comparison with formula (2.41) shows that a kind of normalization has been performed based on the following formula:

$$\sum_{i=1}^{n} P_i \cdot \prod_{j=1}^{i-1} (1 - P_j) + \prod_{i=1}^{n} (1 - P_i) = 1.$$

Formula (2.42) can be derived as follows: let U and T be random variables with a finite number of realizations $u_1, \ldots, u_n$ and $t_1, \ldots, t_m$. Then the probability for the realization $u_j, j = 1, \ldots, n$, under the condition that $t_i, i = 1, \ldots, m$ holds, is given by

$$\text{prob}\,\{U = u_j \mid T = t_i\} := \frac{\text{prob}\,\{U = u_j \wedge T = t_i\}}{\text{prob}\,\{T = t_i\}}.$$

Therefore, the expected value of U under the condition that t_i holds is given by

$$E_{t_i}(U) = \sum_{j=1}^{n} u_j \cdot \frac{\text{prob}\,\{U = u_j \wedge T = t_i\}}{\text{prob}\,\{T = t_i\}}.$$

Now we define $n \to n + 1$, $m = 2$; $u_i = i$ for $1 \leqslant i \leqslant n$, $u_{n+1} > n$; and $t_1 = 1, t_2 = 0$. This means that we define t_1 as "detection happens during the reference time" and t_0 as "detection does not happen during the reference time" and, furthermore, that we define $U = i$ as "detection happens first at the ith inventory taking." Therefore we have, with (2.40b),

$$\text{prob}\,\{U = i\} = P_i \cdot \prod_{j=1}^{i-1} (1 - P_j) \quad \text{for} \quad 1 \leqslant i \leqslant n$$

$$\text{prob}\,\{U = u_{n+1}\} = 1 - \prod_{i=1}^{n} (1 - P_i)$$

$$\text{prob}\,\{T = 1\} = 1 - \prod_{i=1}^{n} (1 - P_i)$$

$$\text{prob}\,\{T = 0\} = \prod_{i=1}^{n} (1 - P_i).$$

Now we have

$$\{T = 1\} \wedge \{U = i\} = \{U = i\} \quad \text{for} \quad 1 \leqslant i \leqslant n,$$

$$\{T = 1\} \wedge \{U = u_{n+1}\} = \emptyset \text{ (empty set)}.$$

From this we get

$$T_1 := E_{t_1}(U) = \sum_{j=1}^{n} u_j \cdot \frac{\text{prob}\,\{U = u_j\}}{\text{prob}\,\{T = t_1\}},$$

which leads immediately to the formula to be proven.

For the purpose of illustration we will give some special cases of the formulae (2.41) and (2.42). For $P_1 = \ldots = P_n =: P$ we get (for $a = n + 1$)

$$T_1 = \sum_{i=1}^{n} \frac{i \cdot P \cdot (1 - P)^{i-1}}{1 - (1 - P)^n} \tag{2.43}$$

$$T_2 = \sum_{i=1}^{n} i \cdot P \cdot (1 - P)^{i-1} + (n + 1) \cdot (1 - P)^n, \tag{2.44}$$

which gives, with the help of some algebra,

$$T_1 = \frac{1}{P} + n \cdot \left(\frac{1}{1-(1-P)^n} - 1 \right)$$

$$T_2 = \frac{1}{P} + \left(2n - \frac{1-P}{P} \right) \cdot (1-P)^n .$$

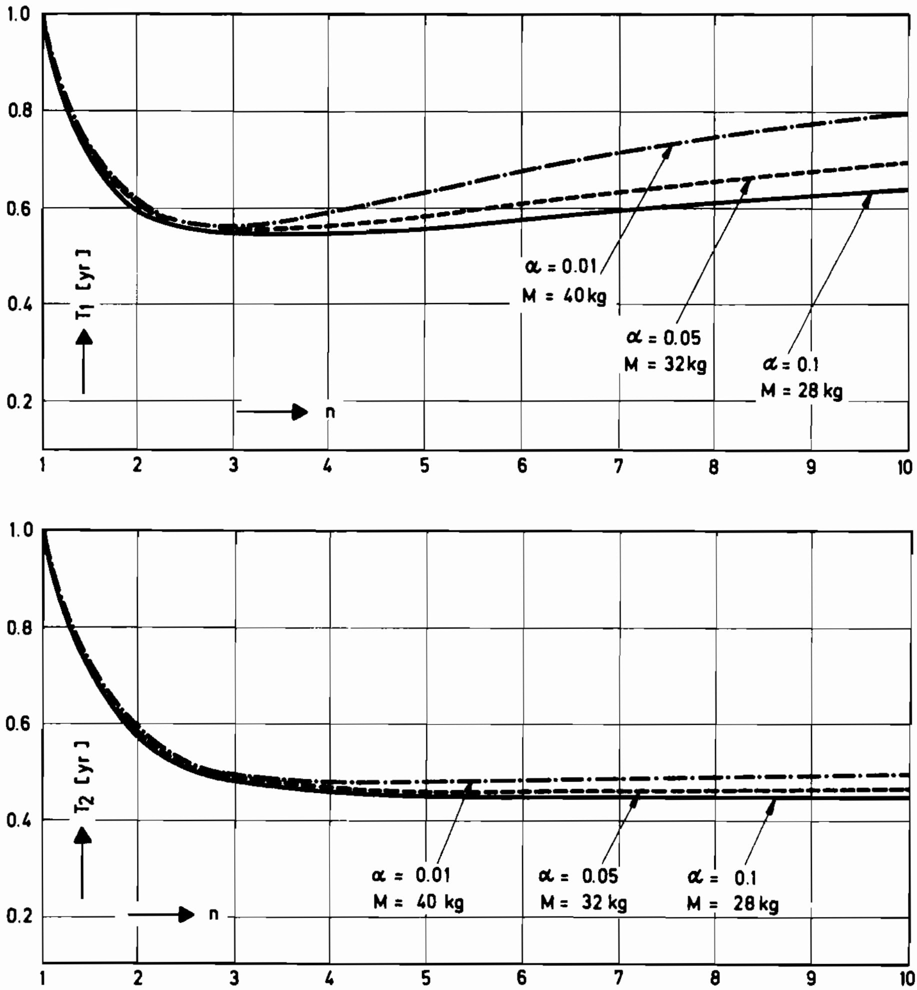

FIGURE 2.5 Expected detection times T_1 (Eq. 2.42) and T_2 (Eq. 2.41) as a function of the number of inventory periods per year with α and M as parameters. Data from Avenhaus and Frick.[12]

For $1-P \ll 1$ and $1-P \ll 2nP$ we get

$$T_1 = \frac{1}{P} + n \cdot (1-P)^n$$

$$T_2 = \frac{1}{P} + 2n \cdot (1-P)^n .$$

Both forms look quite similar in this case.

The first derivative of T_1 with respect to n is

$$\frac{\partial T_1}{\partial n} = \frac{(1-P)^n}{1-(1-P)^n} \cdot \left(1 + \frac{n \cdot \ln(1-P)}{1-(1-P)^n}\right) \qquad (2.45)$$

which is for $n = 1$ greater (or smaller) than zero if $P <$ (or $>$) $-\ln(1-P)$, i.e., the existence of a minimum of T_1 with respect to n depends on the value of P. It should be noted, however, that formula (2.45) is of only limited use for the determination of the optimum number n of inventory periods even in cases where the single probabilities of detection are approximately the same: As the number n of inventories has to be varied for *fixed* reference times, i.e., for fixed throughput, the single probabilities of detection depend themselves on n (except for cases where the variances of the book–physical inventory differences are determined primarily by the variances of the physical inventories).

Figure 2.5 presents the results of the calculations, based on the data used in Figure 2.4, for both the detection times (measured in fractions of the reference time, $a = n + 1$ in the case of T_2). In both cases one sees that there exists a minimum that may be explained with the arguments given above.

The question of how to choose between or combine the conflicting optimization criteria – total probability of error of the second kind β and expected detection time T – together with the question of the appropriate values of critical amount M of material and overall probability of error of the first kind α, will be discussed in Chapter 4.

2.6 SUBDIVISION OF MATERIAL BALANCE AREAS

Let us assume that we have a very large material balance area, and let us further assume that the errors of inventory and throughput measurements are so large that the resulting uncertainty of the established material balance is intolerable. Under these circumstances, one could subdivide the material balance area into n subareas as illustrated in Figure 2.6. For simplicity we assume that the material flows from the left to the right; in other words, we exclude the possibility of recycling material.

According to Eq. (2.2) the book–physical inventory difference for the ith subarea, $i = 1, \ldots, n$, is given by

$$MUF_i = I_{0i} + R_i - S_i - I_{1i}, \qquad i = 1, \ldots, n, \qquad (2.46)$$

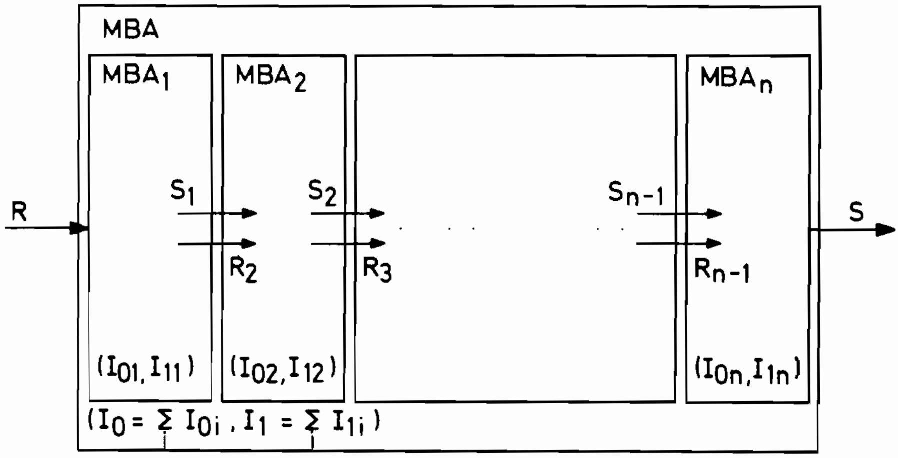

FIGURE 2.6 Subdivision of the original material balance area into n subareas.

where $R_1 = R$ are the receipts and $S_n = S$ the shipments of the original area, and where the shipments of the ith subarea are the receipts of the $i+1$st subarea. We assume that these transfers between the subareas are measured independently as receipts and shipments and that these measurements are unbiased, i.e.,

$$E(S_i) = E(R_{i+1}), \qquad i = 1, \ldots, n-1. \tag{2.47}$$

Addition of all n equations (2.46) gives

$$\begin{aligned} \sum_{i=1}^{n} MUF_i &= \sum_{i=1}^{n} I_{0i} + R - S - \sum_{i=1}^{n} I_{1i} \\ &=: I_0 + R - S - I_1 \\ &=: MUF_{\text{tot}}. \end{aligned} \tag{2.48}$$

In other words, addition of all n equations (2.46) gives the material balance equation for the original material balance area.

Let us assume now that for each of the n material balance subareas significance tests are performed as described in section 2.1. Then, when the amount M_i of material is diverted in the ith subarea, with

$$\sum_{i=1}^{n} M_i = M, \tag{2.49}$$

the following expression for the total probability of detection

$$1 - \beta_n = 1 - \prod_{i=1}^{n} \Phi\left(U_{1-\alpha_i} - \frac{M_i}{\sigma_i}\right) \tag{2.50a}$$

is obtained where

$$\sigma_i^2 = \text{var}\,(I_{0i} + R_i - S_i - I_{1i}) \tag{2.50b}$$

is the variance of the ith book–physical inventory difference MUF_i, $i = 1, \ldots, n$. This is true because of the independence of the different material balances. Therefore, the question is whether this probability of detection is larger or smaller than the probability of detection for the original area, which is given by the following well-known expression:

$$1 - \beta = \Phi\left(U_{1-\alpha} - \frac{M}{\sigma}\right), \tag{2.51a}$$

where M is given by Eq. (2.49) and

$$\sigma^2 = \text{var}\,(I_0 + R - S - I_1) \tag{2.51b}$$

is the variance of the book–physical inventory difference of the original material balance area, and where, for the purpose of comparison,

$$\alpha = 1 - \prod_{i=1}^{n} (1 - \alpha_i) \tag{2.52}$$

is the overall false alarm probability, which will be the same for the one large and for the n small material balance areas.

With the same kinds of argument as in the sequence of inventories case, one now can show that for any subdivision the probability of detection $1 - \beta$ as given by Eq. (2.50a) is smaller than that given by Eq. (2.51a) when both "players" use optimal strategies in the sense of section 2.2. In other words, from the point of view of probability of detection it is better not to subdivide the original area.

There is, however, still another argument: as in the foregoing section, where frequent inventories were performed in order to have a short *detection time*, one might want to subdivide the original area in order to better *localize* the loss or the diversion. In fact, there could be reasons that would make it desirable to know *where* a loss or a diversion had happened if it had happened. The analogy between the sequence of inventories case and this case can be carried even further. If we want to determine the expected detection location (measured in numbers of subareas), we get the same expressions as earlier: the expected detection location, under the condition that detection takes place, is given by the same expression as (2.42):

$$E(L) = \frac{\sum_{i=1}^{n} i \cdot P_i \prod_{j=1}^{i-1} (1 - P_j)}{1 - \prod_{i=1}^{n} (1 - P_i)}. \tag{2.53}$$

Here the analogy ends. Whereas in the foregoing section we could use this expression in order to determine the optimal number of inventory periods (i.e., that number that minimizes the expected detection time), such an approach does not have any meaning here.

Generally, one may conclude that from the point of view of probability of detection it is best to have one large material balance area and only one inventory period in the reference time under consideration. There might, however, be good reasons to better localize the detection in time or place that would justify the introduction of additional physical inventories or the subdivision of the large material balance area into a number of small ones.

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3 Data Verification

In Chapter 2 it was assumed that the data necessary for the establishment of a material balance are correct except for measurement errors – in other words, that these data are not falsified intentionally. There are cases where there is no reason to take into account the possibility of falsification, e.g., all cases of balances of mass flows existing in nature. However, there are also cases where such data falsification cannot be excluded; these are all cases where the material balance principle is used as a control or safeguards tool.

In this chapter, we will describe data verification procedures. This means that in this chapter we always have the control function of the establishment of the material balance in mind. In addition, it should be mentioned that the verification techniques described in this chapter are useful not only for verifying data that are necessary for the establishment of a material balance; they also may be used for more general data verification purposes as well.

In the following sections we will first treat the verification of physical inventory data, and then the verification of material flow data. The reason for this procedure is that in the second case, in contrast to the first, we have to take into account the sequential aspects of the problem, which makes the treatment more complicated.

The techniques and procedures discussed have been developed only in the last few years. At the end of the chapter, a short remark will be made about the relation between the material presented here and the material contained in the literature on business accountancy.

3.1 INVENTORY VERIFICATION: ONE CLASS OF MATERIAL

In order to be specific, let us assume that there is an industrial plant that processes important or rare or dangerous material that has to be safeguarded and, furthermore, that the safeguards are performed by accounting for the material in the way described in Chapter 2. As we have seen, this procedure requires regular physical inventories. We assume that the physical inventory is taken in such a way that the production of the plant is stopped* and that all the material in the plant is available in the form of a unique class of measurement units that we will call "batches."

In this specific case the safeguards procedures are as follows: the operator of the plant measures the material contents of these batches and reports the data to the inspection authority. An inspection team verifies the reported data with the help of independent measurements on a random sampling basis. If there are no significant differences, the inspection team will take *all* the operator's data as correct. If there are significant differences, a "second action level" of the inspection authority will be the consequence.

This second action level is necessary because there is always the possibility of a false alarm, i.e., the possibility that significant differences appear even though the data are correct. We expect that in a second action level it will become clear whether or not the alarm was justified. We will not go into the problems of second action levels here; these questions are treated in a general framework in the next chapter.

Let x_j, $j = 1, \ldots, N$, be the measurement result obtained by the operator for the material content of the jth batch. Furthermore, let T_j be the true value of the material content of the jth batch, and let e_{Oj} be the random and d_O the systematic measurement error common to all measurements, i.e.,

$$x_j = T_j + d_O + e_{Oj}, \qquad j = 1, \ldots, N. \tag{3.1}$$

The variances of the errors are assumed to be known:

$$\operatorname{var}(d_O) =: \sigma_{Os}^2, \operatorname{var}(e_{Oj}) =: \sigma_{Or}^2, \qquad j = 1, \ldots, N. \tag{3.2}$$

If we assume that both the random and systematic errors are independent and normally distributed, we have

$$x_j - T_j = d_O + e_{Oj} \sim N(0, \sigma_{Os}^2 + \sigma_{Or}^2). \tag{3.3}$$

On the basis of a random sampling plan the inspection team chooses n batches and measures their material contents. Let y_i, $j = 1, \ldots, n$, be the result of the inspection team's measurement of the material content of the jth batch, and let e_{Ij} and d_I be the random and systematic errors. The variances of these errors are

$$\operatorname{var}(d_I) =: \sigma_{Is}^2, \operatorname{var}(e_{Ij}) =: \sigma_{Ir}^2, \qquad j = 1, \ldots, n. \tag{3.4}$$

They need not be the same as those of the operator's measurement errors because, for example, the inspection team may use different instruments.

* It should be mentioned here that it is an oversimplification to assume that a plant is shut down in order to take a physical inventory only for safeguards purposes because neither the plant management nor the safeguards authority would pay the costs connected with such a measure. In reality, a plant will be shut down only if there are also some internal plant reasons for doing so; otherwise, "partial inventories" or "running inventories" or simply crude estimates without any process interference will be performed.

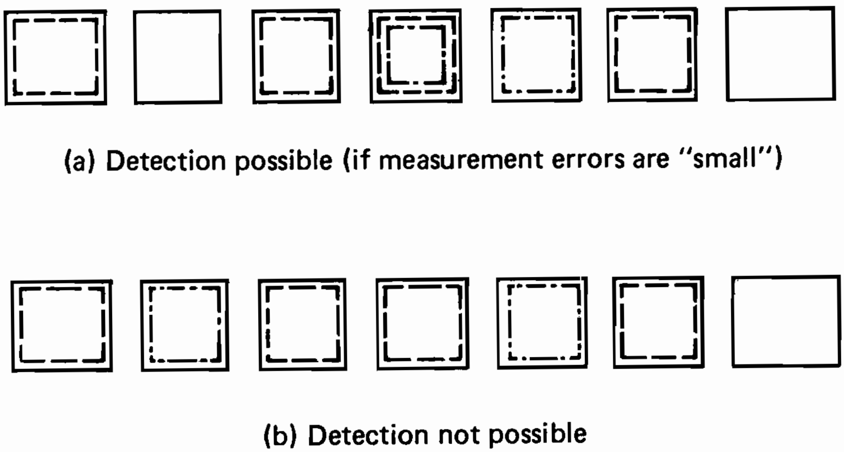

FIGURE 3.1 The random sampling problem. —— Batches available ($N = 7$), data of which are reported to the inspector. – – – Batches, data of which are verified by the inspector ($n = 4$). –·–·– Batches, data of which are falsified by the operator ($r = 2$).

If the operator does not divert any material from batch j after he has reported the data, we have, by analogy to (3.3)

$$y_j - T_j = d_I + e_{Ij} \sim N(0, \sigma_{Is}^2 + \sigma_{Ir}^2), \qquad j = 1, \ldots, n. \qquad (3.5)$$

If the operator diverts the amount μ from r batches after he has reported the data but before the inspection team's measurements, we have

$$y_j - T_j = d_I + d_{Ij} - \mu \sim N(-\mu, \sigma_{Is}^2 + \sigma_{Ir}^2), \qquad j = 1, \ldots, r. \qquad (3.6)$$

For the purposes of illustration, a numerical example for the sample sizes n and r is given in Figure 3.1. It should be noted that the expected value of the difference $y_j - T_j$ represents the amount of material μ that has been diverted only in those cases where the material content of the jth batch is represented by one datum y_j. If the material content of one batch is given by several data (e.g., total weight and concentration) and if, furthermore, these data are verified separately, μ has a different meaning. An example of this is given in Chapter 5.

We assume here that the amount diverted by the operator is the same for all batches from which he diverts material. This assumption as well as the value of this amount will be discussed later. Clearly, it could be assumed, too, that the operator diverts the material only after the verification procedure of the inspection team has been finished. This would mean that there would be a chance that such a diversion would be detected through the establishment of the material balance. The data falsification discussed here is assumed to be performed in such a way that the material balance does not show a significant difference. The interplay of the two diversion strategies, data falsification and use of the uncertainty of the material balance, will also be discussed in the next chapter.

There are several statistical procedures available for the comparison of the operator's data with those of the inspection team. Here, we will concentrate on the so-called D-statistic, introduced into this field by K. B. Stewart,[1] but we will mention another statistic at the end of this section.

The idea of the D-statistic is to form, with the help of the inspectors' data, an unbiased estimate I of the whole material content of all batches of the class

$$I: = \frac{N}{n} \sum_{j=1}^{n} y_j \tag{3.7}$$

and to compare this value with the corresponding estimate B formed with the help of the operator's data

$$B: = \frac{N}{n} \sum_{j=1}^{n} x_j. \tag{3.8}$$

It is to be noted that only those operator's data for which corresponding inspection data exist are used in the form (3.8), the reason being that in forming the difference D,

$$D: = B - I, \tag{3.9}$$

the variations of the true values are eliminated.

If the operator does not falsify any data – in accordance with Chapter 2 we call this assumption the null hypothesis H_0 – the expected value of the difference is zero:

$$E(D/H_0) = 0. \tag{3.10}$$

If the operator falsifies r batches in such a way that he takes from every falsified batch the amount μ – we call this assumption the alternative hypothesis H_1 – the expected value of the difference is

$$E(D/H_1) = \mu \cdot r = : M, \tag{3.11}$$

which can be derived easily by using the properties of the hypergeometric distribution. Therefore, just as in the case of the material balance test described in Chapter 2, a test will be performed with null and alternative hypotheses given by Eqs. (3.10) and (3.11).

Let us again define probabilities of errors of the first and second kind by

$$\begin{aligned} \alpha: &= \text{prob}\{D > s/H_0\} \\ \beta: &= \text{prob}\{D \leqslant s/H_1\} \end{aligned} \tag{3.12}$$

where s is the significance threshold of the test. In the following discussion we will call $1 - \beta$ the probability of detection. Instead of the more general case considered in Chapter 2, we deal here with situations where the objective of the verification measures is the detection of missing material. In the same sense we call α the false alarm probability.

The probability of detection $1-\beta$ as a function of the false alarm probability α is given by the following formula:

$$\beta = \sum_{l=\max(0,n+r-N)}^{\min(n,r)} \Phi\left(U_{1-\alpha} - \frac{\mu \cdot l}{\sqrt{n \cdot \sigma_r^2 + n^2 \cdot \sigma_s^2}}\right) \cdot \frac{\binom{r}{l} \cdot \binom{N-r}{n-l}}{\binom{N}{n}}, \qquad (3.13)$$

where Φ is the normal (Gaussian) distribution function, U its inverse, and

$$\begin{aligned} \sigma_r^2: &= \sigma_{Or}^2 + \sigma_{Ir}^2 \\ \sigma_s^2: &= \sigma_{Os}^2 + \sigma_{Is}^2. \end{aligned} \qquad (3.14)$$

One recognizes the characteristic mixture of the measurement uncertainty aspect, expressed by the Φ-function, and the random sampling aspect, expressed by the hypergeometric coefficients. The derivation of this formula, which has been given by Avenhaus,[2] is as follows:

The distribution function of the random variable D under the alternative hypothesis H_1 is given by

$$F_{D/H_1}(x): = \text{prob}\,\{D < x/H_1\},$$

which can be written in the following form:

$$F_{D/H_1}(x): = \text{prob}\left\{\frac{N}{n} \cdot \sum_j d_j + \frac{N}{n} \cdot \mu \cdot Z < x\right\},$$

where d_j is the sum of the measurement errors of the operator's and inspection team's measurements of the jth batch and Z is the number of falsified batch data in the inspection team's sample. The random variable Z can take every integer value l between max $(0, n+r-N)$ and min (n,r). Therefore, with

$$x_2: = \frac{N}{n} \cdot \mu \cdot l; \qquad x_1: = x - \frac{N}{n} \cdot \mu \cdot l$$

and because of the independence of the random variables $\sum_j d_j$ and Z, we get

$$F_{D/H_1}(x) = \sum_l \text{prob}\left\{\frac{N}{n} \cdot \sum_j d_j < x - \frac{N}{n} \cdot \mu \cdot l\right\} \cdot \text{prob}\,\{Z < l\}.$$

According to the foregoing consideration the random variables d_j are normally distributed with expected values and variances

$$E(d_j) = 0; \qquad \text{var}(d_j) = \sigma_r^2 + \sigma_s^2$$

where σ_r^2 and σ_s^2 are given by (3.14). In addition, the random variable Z is hypergeometrically distributed, so we get

$$F_{D/H_1}(x) = \sum_l \Phi\left(\frac{x - \mu \cdot l}{\sqrt{n \cdot \sigma_r^2 + n^2 \cdot \sigma_s^2}}\right) \cdot \frac{\binom{r}{l} \cdot \binom{N-r}{n-l}}{\binom{N}{n}}.$$

Because of the definitions (3.12), the probability of no detection β and the false alarm probability α are given by

$$\beta = F_{D/H_1}(s), \qquad 1 - \alpha = F_{D/H_0}(s),$$

where $F_{D/H_0}(x)$ is given by

$$F_{D/H_0}(x) = \Phi\left(\frac{x}{\sqrt{n \cdot \sigma_r^2 + n^2 \cdot \sigma_s^2}}\right).$$

Elimination of the significance threshold s with the help of the false alarm probability α finally gives formula (3.13).

Formula (3.13) can be used, for example, for the determination of the sample size n, if all other parameters are fixed. Because of its complicated structure, however, it is difficult to use it in practice. Therefore, approximations will be discussed in the next section. For illustrative purposes, five limiting cases are given:

1. For exact measurements, i.e., for $\sigma^2 = 0$, we get

$$\beta = \frac{\binom{N}{0} \cdot \binom{N-r}{n-0}}{\binom{N}{n}} = \frac{\binom{N-r}{n}}{\binom{N}{n}},$$

or, after some reordering,

$$\beta = \prod_{i=0}^{r-1} \left(1 - \frac{n}{N-i}\right).$$

For $r \ll N$ we get from this formula the following simple formula

$$\beta = \left(1 - \frac{n}{N}\right)^r. \qquad (3.15)$$

It should be noted that for exact measurements and "drawing with replacement" instead of "drawing without replacement," as has been assumed until now, we get instead of (3.15)

$$\beta = \left(1 - \frac{r}{N}\right)^n, \qquad (3.16)$$

which can be understood immediately.

2. For the sample size $n = 1$ we get

$$\beta = (1 - \alpha) \cdot \left(1 - \frac{r}{N}\right) + \Phi\left(U_{1-\alpha} - \frac{\mu}{\sqrt{\sigma_r^2 + \sigma_s^2}}\right) \cdot \frac{r}{N},$$

which can be understood easily: if there are r falsifications and the sample size is 1, then the probability of choosing a falsified batch is r/N, and the probability of choosing a correct one is $1 - r/N$. If one chooses a correct batch, the probability of not detecting a falsification is $1 - \alpha$. If one chooses a falsified batch, the probability of not detecting the falsification is

$$\Phi\left(U_{1-\alpha} - \frac{\mu}{\sqrt{\sigma_r^2 + \sigma_s^2}}\right),$$

in accordance with formula (2.15).

3. For one falsified batch, $r = 1$, and $N > n + 1$ we get

$$\beta = (1-\alpha) \cdot \left(1 - \frac{n}{N}\right) + \phi\left(U_{1-\alpha} - \frac{\mu}{\sqrt{n\sigma_r^2 + n^2 \cdot \sigma_s^2}}\right) \cdot \frac{n}{N},$$

which can be explained in the same way as above.

4. If all batch data are verified, i.e., $n = N$, we get

$$\beta = \Phi\left(U_{1-\alpha} - \frac{\mu \cdot N}{\sqrt{N \cdot \sigma_r^2 + N^2 \cdot \sigma_s^2}}\right).$$

This case corresponds exactly to the test described in the second chapter: we compare two quantities that have the same expected value under the null hypothesis but whose expected values under the alternative hypothesis differ by the amount $M = \mu \cdot N$. Therefore, we obtain the same formula as given by (2.15).

5. If all batch data are falsified, i.e., for $r = N$, we get

$$\beta = \Phi\left(U_{1-\alpha} - \frac{\mu \cdot n}{\sqrt{n \cdot \sigma_r^2 + n^2 \cdot \sigma_s^2}}\right);$$

this expression is again of the same structure as (2.15).

If the inspection team wants to use formula (3.13) – or an appropriate approximation – for the determination of the necessary sample size, the question of the value of the parameter μ arises, since, unlike the other parameters, it is not known to the inspection team. A reasonable procedure would be to determine that value that maximizes the probability of error of the second kind β, because one would then have a guaranteed probability of detection. Because of the complicated structure of (3.13), this optimization cannot be carried through analytically; one has to resort to numerical calculations. Some examples are given in Figure 3.2. From these calculations, one may draw the following conclusions.

Let us assume that the operator wants to divert the amount

$$M = \mu \cdot r. \tag{3.17}$$

For an amount M that is far larger than the total measurement variance, it is optimal for the operator to falsify as *few* batch data as possible, consistent with the

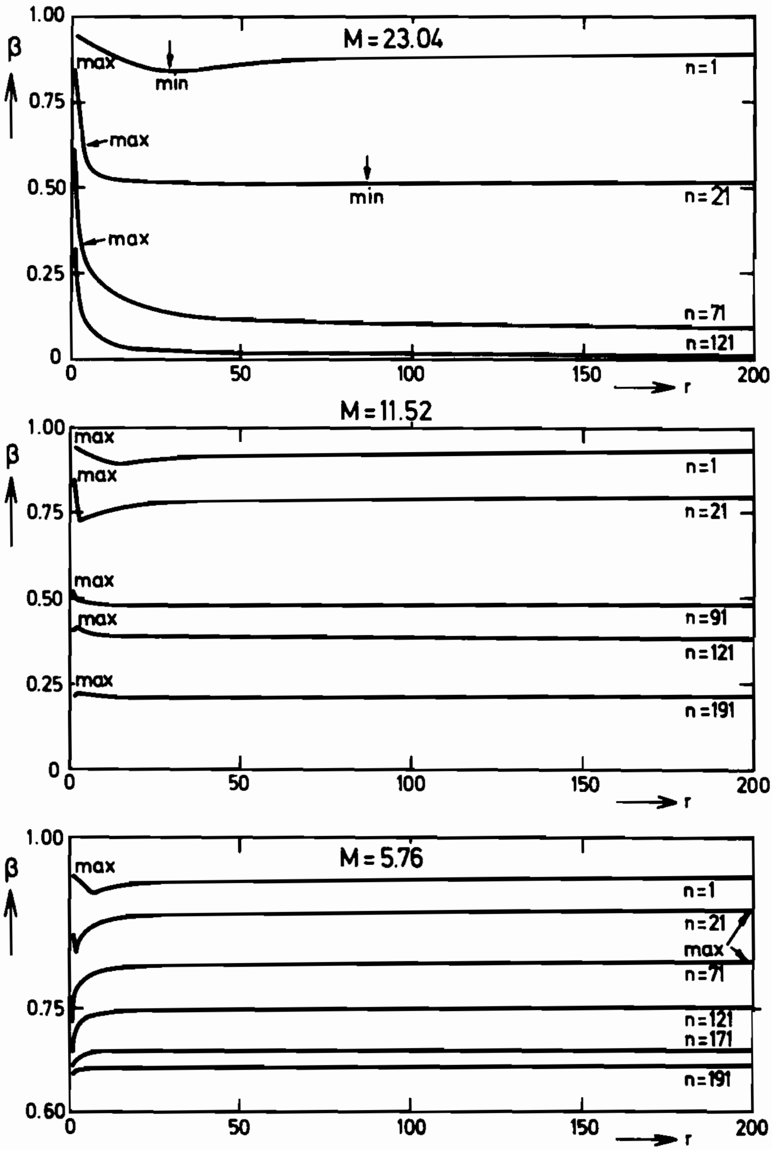

FIGURE 3.2 Probability of no detection

$$\beta = \sum_{l} \phi\left(U_{1-\alpha} - \frac{l}{\sqrt{n}} \cdot \frac{\mu}{\sigma}\right) \frac{\binom{r}{l}\binom{N-r}{n-l}}{\binom{N}{n}}$$

for given $M = \mu \cdot r$ as a function of r, with n as parameter, $N = 200$, $\sigma = 0.3271$, $\alpha = 0.05$. (From Avenhaus.[2])

boundary condition (3.17). In the opposite case, i.e., for an amount M smaller than the measurement variance, the operator should falsify *all* batch data, again consistent with the boundary condition (3.17).

The meaning of "to falsify as few data as possible" depends on the technical situation. An upper limit for the amount μ to be falsified per batch is the material content. In some cases the inspection team may use, in addition, a rough and cheap measurement device with whose help *all* batches are checked before the precise measurement instrument is used for a limited number of batches. In these cases a falsification is possible only within the limits of the accuracy of the rough instrument.

In the derivation of formula (3.13) we assumed that the operator would falsify the batch data in such a way that the difference between reported and real value is the same for all falsified batches. One can give examples where it would be better for the operator to use a different strategy, i.e., to vary the differences. However, as this would require a large computational and organizational effort, we do not consider it to be a practical strategy.

As already mentioned, there are further possibilities for comparing the operator's and the inspection team's measurement data: the so-called P-statistic. Here, the operator's data on a single batch are compared with the inspection team's data on the same batch such that the total false alarm probability is not greater than a given value. Before going into the mathematical details it should be mentioned that, unlike the D-statistic, this statistic is not capable of handling systematic errors – this is the main reason that the D-statistic is generally preferred. If systematic errors can be neglected, then the values of the parameters of the problem determine which of the two statistics gives the lower probability of detection for a fixed false alarm probability. In addition, the P-statistic has an important property in view of the sequential problems that are discussed in the last section of this chapter.

The resulting probability of no detection as a function of the false alarm probability for the comparison of data for one class of material is given by the following formula:

$$\beta = \sum_{l=\max(0,n+r-N)}^{\min(n,r)} \phi\left(U_{\sqrt[n]{1-\alpha}} - \frac{\mu}{\sigma}\right)^l \cdot (1-\alpha)^{1-l/n} \cdot \frac{\binom{r}{l}\binom{N-r}{n-l}}{\binom{N}{n}} \qquad (3.18)$$

where

$$\sigma^2 = \sigma_{Ir}^2 + \sigma_{Or}^2, \qquad (3.19)$$

where α is the resulting false alarm probability, which is related to the single test false alarm probability α' by the relation $1-\alpha = (1-\alpha')^n$, and where all the other parameters are defined as before.

The derivation of this formula is as follows.[2] The inspection team compares each of its measurement results y_j with the corresponding result x_j reported by the operator. This comparison consists of a test of significance on the basis of the difference $d_j := x_j - y_j$; the null and the alternative hypotheses H_0 and H_1 are given by

$$E(d_j/H_0) = 0; \qquad E(d_j/H_1) = -\mu$$

for all verified batches.

According to our assumptions the differences d_j are normally distributed random variables, the distributions of which are given by (3.3), (3.5), and (3.6). Therefore, if s is the significance threshold of the single test, the single probabilities of detection and the false alarm probabilities α' are given by

$$\alpha' := \text{prob}\,\{d_j > s/H_0\} = \Phi\left(\frac{s}{\sigma}\right)$$

$$p_j := \text{prob}\,\{d_j > s/H_1\} = \Phi\left(\frac{\mu}{\sigma} - \frac{s}{\sigma}\right).$$

Elimination of the significance threshold s with the help of the false alarm probability α gives

$$p_j = \Phi\left(\frac{\mu}{\sigma} - U_{1-\alpha'}\right).$$

Now the probability of not detecting a falsified batch is composed of all probabilities of finding among the n verified batches l falsified ones *without* recognizing them as such, and $n - l$ unfalsified ones that *are* recognized as such. As the probability of finding among n batches l falsified batches, if a total of r batches are falsified, is determined by the hypergeometric distribution, the probability of not finding any falsified batch is given by

$$\beta = \sum_{l=\max(0,n+r-N)}^{\min(n,r)} \left[\Phi\left(U_{1-\alpha'} - \frac{\mu}{\sigma}\right)\right]^l \cdot (1-\alpha')^{n-l} \cdot \frac{\binom{r}{l}\cdot\binom{N-r}{n-l}}{\binom{N}{n}}.$$

The resulting false alarm probability α for the whole class is obtained if one puts $r = \mu = 0$; this gives

$$\alpha = 1 - (1-\alpha')^n.$$

Elimination of the single false alarm probability α' with the help of the resulting false alarm probability α finally gives formula (3.18).

As in the case of the probability of detection based on the D-statistic procedure, we will discuss for illustrative purposes several limiting cases:

1. For exact measurements, i.e., for $\sigma^2 = 0$, we get, as in the case of the D-statistic,

$$\beta = \prod_{i=0}^{r-1}\left(1 - \frac{n}{N-i}\right).$$

Clearly, in both cases the probability of detection is reduced to the probability of selecting at least one falsified batch among the n batches selected for verification purposes.

2. For sample size $n = 1$ and $r < N - 1$ we get

$$\beta = (1-\alpha)\cdot\left(1 - \frac{r}{N}\right) + \Phi\left(U_{1-\alpha} - \frac{\mu}{\sigma}\right)\cdot\frac{r}{N}.$$

That this is exactly the same expression we got for the D-statistic can be understood easily: as there exists only one pair of data to be compared, both verification procedures are the same.

3. For one falsified batch, $r = 1$, and $n < N - 1$, we get

$$\beta = (1-\alpha)\cdot\left(1-\frac{n}{N}\right) + \Phi\left(U_{\sqrt[n]{1-\alpha}} - \frac{\mu}{\sigma}\right)\cdot(1-\alpha)^{1-1/n}\cdot\frac{n}{N}$$

$$= (1-\alpha')^n\cdot\left(1-\frac{n}{N}\right) + \phi\left(U_{1-\alpha'} - \frac{\mu}{\sigma}\right)\cdot(1-\alpha')^{n-1}\cdot\frac{n}{N},$$

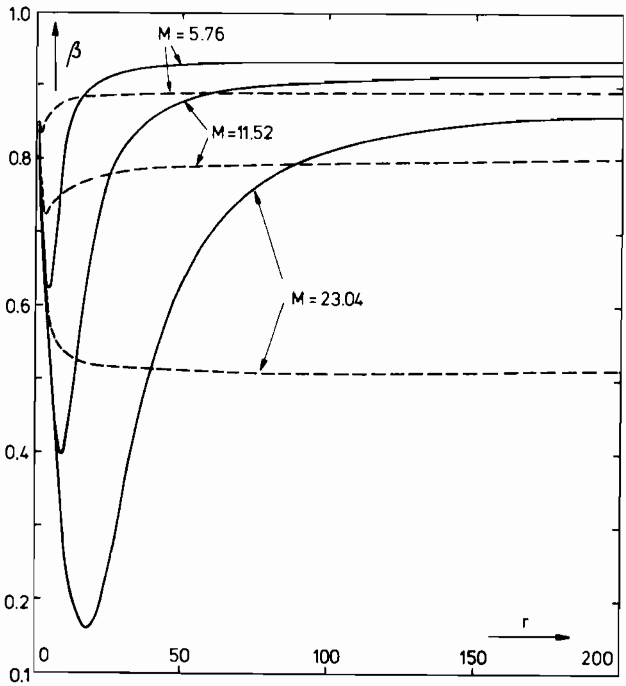

FIGURE 3.3 Probabilities of no detection

$$\beta_D = \sum_l \phi\left(U_{1-\alpha} - \frac{l}{\sqrt{n}}\cdot\frac{\mu}{\sigma}\right)\cdot\frac{\binom{r}{l}\binom{N-r}{n-l}}{\binom{N}{n}} \quad \text{(dashed line)}$$

$$\beta_P = \sum_l \phi\left(U_{\sqrt[n]{1-\alpha}} - \frac{\mu}{\sigma}\right)^l\cdot(1-\alpha)^{1-l/n}\cdot\frac{\binom{r}{l}\binom{N-r}{n-l}}{\binom{N}{n}}$$

for given $M = \mu\cdot r$ as a function of r; $N = 200$, $\sigma = 0.327$, $\alpha = 0.05$, $n = 21$. (From Avenhaus.[2])

which can be understood in the same way as in the corresponding case of the D-statistic.

4. If all batch data are verified, i.e., for sample size $n = N$, we get

$$\beta = \Phi\left(U_{1-\alpha'} - \frac{\mu}{\sigma}\right)^r \cdot (1-\alpha')^{N-r},$$

which means that the probability of no detection is the product of r probabilities of no detection and of $N-r$ probabilities of committing no false alarms.

5. If all batch data are falsified, i.e., for $r = N$, we get

$$\beta = \Phi\left(U_{1-\alpha'} - \frac{\mu}{\sigma}\right)^n,$$

which means that the probability of no detection is the product of n probabilities of no detection.

For $n > 1$, Figure 3.3 gives an indication of the relation between P- and D-statistics. The maxima of the probability of detection as a function of number r of falsified batches for a given total amount M to be falsified are more pronounced in case of the P-statistic. However, as they are not interesting for the operator because he wants to minimize the probability of detection, and as the inspection team cannot influence the choice of r, these maxima are without any importance in the framework of the safeguards problem.

3.2 INVENTORY VERIFICATION: SEVERAL CLASSES OF MATERIAL

In this section we assume that the inventory of the plant consists of R different classes of batches, with completely different values of such parameters as total batch numbers, material contents, and measurement variances. One can get an idea of such a situation if one imagines a chemical plant where the material to be safeguarded enters the plant in a given physical and chemical form (e.g., liquid or solid, pure or as a compound) and leaves the plant in a different form. Therefore, if production is stopped so that a physical inventory can be taken, the material will be available in classes of batches with different characteristics.

According to the terminology of the foregoing section, let X_{ij}, $j = 1, \ldots, N_i$, $i = 1, \ldots, R$, be the measurement result obtained by the operator for the material content of the jth batch of the ith class. Furthermore, let T_{ij} be the corresponding true values of the material contents and let e_{Oij} be the random and d_{Oi} be the systematic measurement errors common to all measurements of the operator in the ith class, i.e.,

$$x_{ij} = T_{ij} + d_{Oi} + e_{Oij}, \qquad j = 1, \ldots, N_i, i = 1, \ldots, R. \tag{3.20}$$

The class-specific variances are again assumed to be known:

$$\text{var}(d_{Oi}) = :\sigma^2_{Osi}, \qquad \text{var}(e_{Oij}) = :\sigma^2_{Ori}, \qquad j = 1, \ldots, N_i, i = 1, \ldots, R. \tag{3.21}$$

If we assume, in addition, that both the random and systematic errors are independent and normally distributed, we have

$$x_{ij} - T_{ij} = d_{Oi} + e_{Oij} \sim N(0, \sigma^2_{Osi} + \sigma^2_{Ori}), \qquad j = 1, \ldots, N_i, i = 1, \ldots, R. \tag{3.22}$$

We assume again that the operator reports all his data to the safeguards authority and that on the basis of a sampling plan the inspection team chooses n_i, $i = 1, \ldots, R$, batches from the ith class in order to determine their material contents with the help of independent measurements. Let y_{ij}, $j = 1, \ldots, n_i$, $i = 1, \ldots, R$, be the measurement result of the inspection team for the material content of the jth batch of the ith class, and let e_{Iij} and d_{Ii} be the random and systematic errors of these measurements, respectively. The variances of these errors are

$$\text{var}(d_{Ii}) = :\sigma^2_{Isi}, \qquad \text{var}(e_{Iij}) = :\sigma^2_{Iri}, j = 1, \ldots, n_i, i = 1, \ldots, R. \tag{3.23}$$

As already mentioned in section 3.1, these variances need not be the same as the corresponding operator's variances given by Eqs. (3.21), as the inspection team may use different instruments.

For the safeguards authority the key problem now arises of how to choose the sample sizes n_i, $i = 1, \ldots, R$, for the different classes of material. We tackle this problem in the following way: let us assume that there is only a finite amount C of inspection effort available for the verification of the operator's reported inventory data and, furthermore, that the effort required for the verification of one batch datum of the ith class is ϵ_i (given in terms of money or man-hours). Then, the sample sizes n_i have to be selected in such a way that the boundary condition

$$C > \sum_{i=1}^{R} \epsilon_i \cdot n_i \tag{3.24}$$

is met and the overall probability of detection is maximized.

On the other hand, one has to assume that the operator who wants to divert the amount M of material by means of data falsification will do this in the way that is most efficient from his point of view. This means that one has to assume that he chooses his sample sizes r_i, $i = 1, \ldots, R$, in such a way that the boundary condition

$$M \leqslant \sum_{i=1}^{R} \mu_i \cdot r_i \tag{3.25}$$

is met where μ_i is the amount of material diverted by means of falsification of one batch of the ith class such that the overall probability of detection is minimized.

3.2.1 EXACT SOLUTION FOR A SPECIAL CASE

In this section, we first consider a special case that offers us the advantage of giving analytical solutions. We then treat the general problem with the help of some approximations. Finally, we give a general discussion that will show how the solutions of the special case can be applied to practical situations.

We assume in this section that both "players," i.e., the plant operator and the inspection team, decide independently and without knowing about the intentions of the other that they will limit their actions to only one class of material, not necessarily the same one. One may imagine that the selection of the class will be done with the help of a random number generator. This means that the selection of the batches whose data have to be verified will be performed in two steps:

1. Selection of the class (say, class i)
2. Selection of the n_i batches of the class selected such that n_i is the largest integer smaller than C/ϵ_i according to (3.24), i.e.,

$$n_i = \left[\frac{C}{\epsilon_i}\right] \tag{3.26}$$

The same procedure will be used by the operator in case he wants to divert the amount M of material by means of data falsification: he will first select the class and then the r_j batches within the class such that r_j is the smallest integer larger than M/μ_j according to (3.25), i.e.,

$$r_j = \left\langle\frac{M}{\mu_j}\right\rangle. \tag{3.27}$$

It should be noted that there may be a difference in the ways the parties determine their sample sizes: the values of $\epsilon_i, i = 1, \ldots, R$, are fixed if the measurement instruments are given. The operator, however, may choose any value of μ_j within the limits discussed earlier; he may choose r_j and μ_j such that the class probability of detection is minimized. For the following considerations it is important only that the values of r_j and μ_j be determined for all classes $j = 1, \ldots, R$ before a specific class is selected.

One important point has to be made here: in the forthcoming considerations we do not include the possibility of the operator's behaving legally; that is, we assume that in any case the operator will divert the amount M of material. In brief, the reason for this is that the whole analysis serves the purpose of optimizing the safeguards effort, which has as its aim the detection of the diversion of the amount M of material. A more detailed discussion of this assumption is given in Chapter 4.

The problem outlined so far may be formulated as a two-person zero-sum game where the payoff to the operator as player 1 is one minus the probability of detection and the payoff to the inspection team as player 2 is the negative payoff to the operator. The probability of detection is $1-\beta_i$ if both players choose the ith class for their action, where β_i is given either by Eq. (3.13) or by Eq. (3.18); and the probability of detection is the false alarm probability α_i if the inspection team chooses the ith class and the operator chooses a different class.

Therefore, we have an $R \times R$ matrix game where the payoff to the operator is

β_i if both parties choose the same class and the payoff to the operator is $1-\alpha_i$ if the inspection team chooses the ith class and the operator chooses a different class.

$$\begin{array}{ll} & \text{Inspection team: verification in class:} \\ & \begin{array}{ccccc} 1 & 2 & 3 & \dots & R \end{array} \\ \begin{array}{lc} & 1 \\ \text{Operator:} & 2 \\ \text{diversion} & 3 \\ \text{from} & . \\ \text{class} & . \\ & . \\ & R \end{array} & \begin{pmatrix} \beta_1 & 1-\alpha_2 & 1-\alpha_3 & \dots & 1-\alpha_R \\ 1-\alpha_1 & \beta_2 & 1-\alpha_3 & \dots & 1-\alpha_R \\ 1-\alpha_1 & 1-\alpha_2 & \beta_3 & \dots & 1-\alpha_R \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ 1-\alpha_1 & 1-\alpha_2 & 1-\alpha_3 & \dots & \beta_R \end{pmatrix} \end{array} \tag{3.28}$$

All elements of the ith column have the value $1-\alpha_i$ with the exception of the element on the main diagonal, which has the value β_i.

It should be noted that a special case of this $R \times R$ matrix game $[\beta_i = p \cdot (1-\alpha_i)$ for $i = 1, \dots, R]$ has been described by Karlin[3] where it has been used for the analysis of the distribution of election funds in R different election areas. Generally, this game may be used for the analysis of conflict or competition problems in R different areas where, because of limited resources, both players are forced to concentrate their actions on only one of the R areas.

In the following, we consider only the case

$$\max_i \alpha_i < \min_i (1-\beta_i), \sum_{i=1}^{R} \frac{\max_j \alpha_j - \alpha_i}{1-\beta_i-\alpha_i} < 1 \tag{3.29}$$

because only for cases of this type can analytical formulae be given that are of interest in practical situations. The first assumption is reasonable in any case; the second assumption is fulfilled for $\alpha_1 = \alpha_2 = \dots = \alpha_R$ and also for $R = 2$ and $\alpha_1 \neq \alpha_2$, but it may not be fulfilled if the class false alarm probabilities vary strongly and if, furthermore, the number of classes is large.

Let q_i, $i = 1, \dots, R$, be the probability that the inspection team will select the ith class, and let p_i, $i = 1, \dots, R$, be the probability that the operator will select the ith class. Then the optimal strategies $q^* := (q_1^*, \dots, q_R^*)$ and $p^* := (p_1^*, \dots, p_R^*)$ of the inspection team and of the operator; the resulting "false alarm" probability α^*, which is defined as

$$\alpha^* = \sum_{i=1}^{R} \alpha_i \cdot q_i^* ; \tag{3.30}$$

and the value of the game β^*, i.e., one minus the guaranteed probability of detection, are given by the following relations:

$$\sum_{i=1}^{R} \frac{\alpha^* - \alpha_i}{1 - \beta_i - \alpha_i} = 0 \tag{3.31a}$$

$$\frac{1}{1 - \beta^* - \alpha^*} = \sum_{i=1}^{R} \frac{1}{1 - \beta_i - \alpha_i} \tag{3.31b}$$

$$q_i^* = \frac{1 - \beta^* - \alpha^*}{1 - \beta_i - \alpha_i}, \; p_i^* = \frac{1 - \beta^* - \alpha_i}{1 - \beta_i - \alpha_i}, \qquad i = 1, \ldots, R. \tag{3.31c}$$

For the value β^* of the game, we have the following limits:

$$\max_i \alpha_i \leqslant 1 - \beta^* \leqslant \min_i (1 - \beta_i). \tag{3.32}$$

As can be checked easily, the optimal strategies q^* and p^* fulfill the necessary conditions

$$\sum_{i=1}^{R} q_i^* = 1, \; \sum_{i=1}^{R} p_i^* = 1.$$

The proof of the relations (3.29) can be performed by using the saddle-point criterion,[2] which in our case reads as follows:

$$\beta(p, q^*) \leqslant \beta(p^*, q^*) \leqslant \beta(p^*, q), \tag{3.33}$$

where $\beta^* = \beta(p^*, q^*)$. This criterion can be understood easily: the strategy q^* of the inspection team is the best against any of the operator's strategies – in other words, against the best of the operator's strategies. The same holds for the strategy p^* of the operator. That the solutions (3.31) fulfill the relations (3.33) can easily be checked; in our case, in fact, equality holds among the three quantities in (3.33).

It must be noted that the relations (3.31), which represent the solution of the matrix game (3.28) (i.e., the second stage of the two stages of the batch selection procedure mentioned on page 46), also represent a solution of the complete two-stage game.

Before clarifying the meaning of the false alarm probability α^*, we will discuss some features of the solution (3.31) of the matrix game given by the matrix (3.28):

1. If all class false alarm probabilities are chosen to be the same, i.e., if $\alpha_1 = \ldots = \alpha_R =: \alpha$, we get

$$\alpha^* = \alpha,$$

and, furthermore,

$$q_i^* = \frac{1 - \beta^* - \alpha}{1 - \beta_i - \alpha} = p_i^*, \qquad i = 1, \ldots, R,$$

which means that both players' probabilities q_i^* and p_i^*, with which they decide on action in the ith class, are the same. In addition we see that the probability with which the operator chooses a class for action is larger, the smaller the class's probability of detection is. This is perfectly reasonable, and it is also reasonable that the strategy of the inspection team accordingly is the same, i.e., to choose that class with high probability that is chosen with high probability by the operator. This result corresponds exactly to the result we obtained for the sequence of inventory periods, where both the inspection team and the operator concentrated their actions on periods in which the detection probability was low.

2. In the limiting case $\alpha_1 = \ldots = \alpha_R = 0$, we get

$$\frac{1}{1-\beta^*} = \sum_{i=1}^{R} \frac{1}{1-\beta_i} > R,$$

or

$$1-\beta^* < \frac{1}{R},$$

which is clear because only one class is selected; this, however, leads to low probabilities of detection if R is large.

Let us come back once more to the resulting "false alarm" probability α^*, which was defined by

$$\alpha^* = \sum_{i=1}^{R} \alpha_i \cdot q_i^* .$$

We used quotation marks because this probability α^* has a somewhat different meaning from the false alarm probability we had defined earlier. In fact, as we made the assumption at the beginning of the treatment of this special problem that the operator will divert the amount $M > 0$ of material, any alarm is justified in the sense that material has been diverted. On the other hand, α^* is the expected probability of getting an alarm in one class while the diversion takes place in another class. Therefore, it is indeed a false alarm in the literal sense of the term.

Whether the term false alarm is appropriate in this connection depends upon the entire inspection procedure, including "second action levels" (see section 4.2). If, in case of an alarm, only measurements of batches *of the same class* are repeated, then this alarm will be recognized as false. If, on the other hand, measurements of batches *of all classes* are repeated in case of an alarm, then one probably will detect the falsification of data, and the alarm really leads to the detection of an illegal action, which means that in such a scheme there is no such thing as a false alarm. In the following discussion we will assume for simplicity that the first alternative is in effect.

In the inspection procedure that has been described so far, the class false alarm probabilities α_i were free parameters and, consequently, the resulting false alarm probability depended on the set of false alarm probabilities chosen by the inspection team. One could now take a different position and postulate that the resulting false alarm probability (in other words, the rate of false accusations) is such an important quantity that the value of it has to be fixed *a priori*. This would mean that the inspection team can then choose the set $(\alpha_1, \ldots, \alpha_R)$ of class false alarm probabilities within the limits posed by the boundary condition

$$\alpha = \sum_i \alpha_i \cdot q_i,$$

where α has a given value. It is clear that the inspection team will choose the set $(\alpha_1, \ldots, \alpha_R)$ such that the guaranteed probability of detection $1-\beta^*$ is maximized.

Unfortunately, this optimization procedure cannot be carried through analytically because of the complicated dependence of the class detection probabilities $1-\beta_i$ on the class false alarm probabilities α_i (Eq. 3.13 or Eq. 3.18). However, one can give an approximate solution if one assumes that the class detection probabilities $1-\beta_i$ do not depend on the class false alarm probabilities α_i. Since this still leads to complicated analytical formulae, and since we will not make further use of these results, the interested reader is referred to the original work.[2]

3.2.2 APPROXIMATE SOLUTION FOR THE GENERAL CASE

In this section we consider the case in which both "parties," i.e. plant operator and inspection team, do not limit their activities to one class but rather spread them over all R classes of material. In the special case treated before, the question of the appropriate distribution of inspection effort on the different classes did not appear explicitly (one class was selected and the whole effort was spent on this class); in the situation considered now, the effort distribution question is the central problem to be considered. Here, we will base our treatment on the D-statistic; however, we will comment on the use of the P-statistic at the end of the section.

We begin our considerations by generalizing the D-statistic introduced in section 3.1 to the case of R classes. We obtain the following expression:

$$D: = \sum_{i=1}^{R} \frac{N_i}{n_i} \sum_{j=1}^{n_i} (x_{ij} - y_{ij}). \tag{3.34}$$

The expected value of this random variable under the null hypothesis (no data falsification at all) is zero:

$$E(D/H_0) = 0. \tag{3.35}$$

The expected value under the alternative hypothesis (falsification of r_i batch data by the amounts μ, $i = 1, \ldots, R$) is given by

$$E(D/H_1) = \sum_i \mu_i \cdot r_i = : M. \tag{3.36}$$

The variances of D under the two hypotheses are given by the following expressions:

$$\mathrm{var}(D/H_0) = :\sigma^2_{D/H_0} = \sum_{i=1}^{R} N_i^2 \cdot \left(\frac{\sigma^2_{ri}}{n_i} + \sigma^2_{si}\right) \tag{3.37}$$

$$\mathrm{var}(D/H_1) = :\sigma^2_{D/H_1} = \sum_{i=1}^{R} N_i^2 \left[\frac{\sigma^2_{ri}}{n_i} + \sigma^2_{si} + \mu_i^2 \cdot \frac{r_i}{N_i} \cdot \frac{N_i - r_i}{N_i} \cdot \left(\frac{1}{n_i} \cdot \frac{N_i}{N_i - 1} - \frac{1}{N_i - 1}\right)\right] \tag{3.38}$$

where

$$\sigma^2_{ri} = \sigma^2_{Ori} + \sigma^2_{Iri}, \sigma^2_{si} = \sigma^2_{Osi} + \sigma^2_{Isi}, \qquad i = 1, \ldots, R. \tag{3.39}$$

These variances are the sums of the measurement error variances (σ^2_{D/H_0}) and of the random sampling variances, which are derived from the hypergeometric

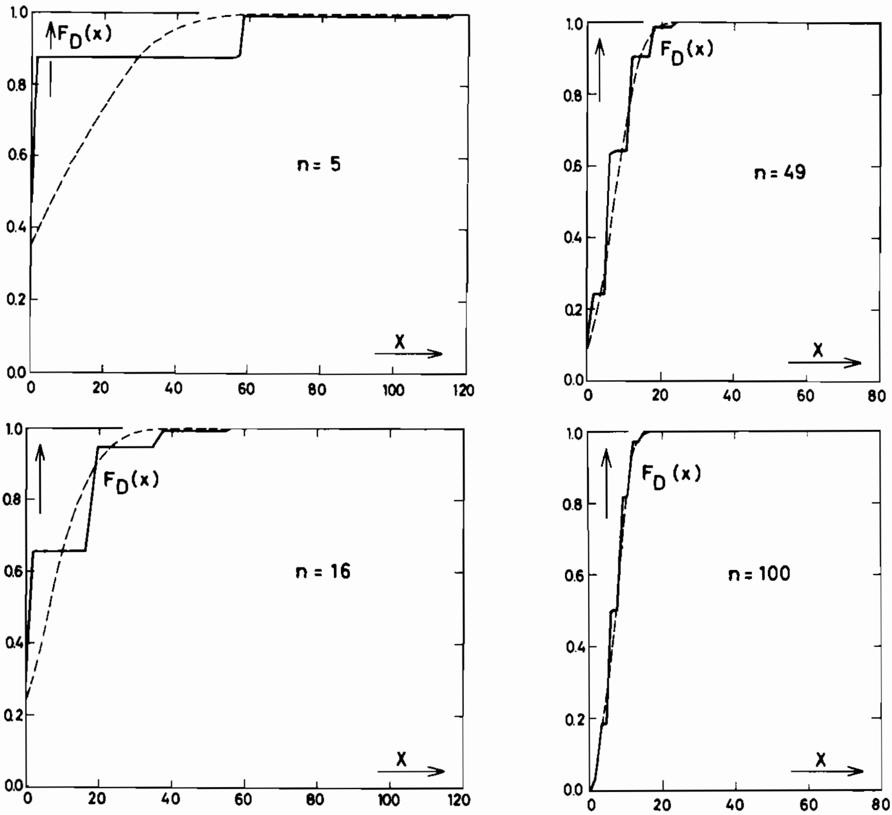

FIGURE 3.4 Distribution function $F_D(x)$ of the D-statistic

$$F_D(x) = \sum_{l=\max(0,n+r-N)}^{\min(n,r)} \phi\left(\frac{x - \mu \cdot l}{\sqrt{n \cdot \sigma_r^2 + n^2 \cdot \sigma_s^2}}\right) \cdot \frac{\binom{r}{l}\binom{N-r}{n-l}}{\binom{N}{n}}$$

and its approximation by the normal (Gaussian) distribution (dashed line) for $N = 200$, $\mu = 1.44$, $\sigma_r = 0.002$, $\sigma_s^2 = 0$, $r = 5$ and various values of n. (From Avenhaus.[2])

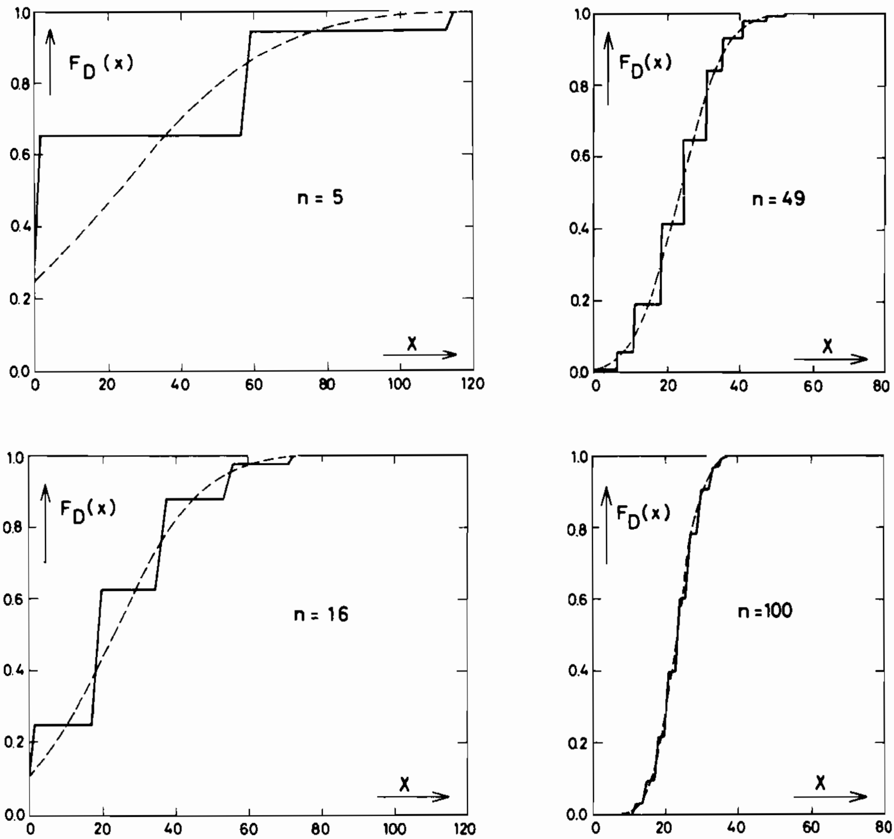

FIGURE 3.5 Distribution function $F_D(x)$ of the D-statistic and its approximation by the normal (Gaussian) distribution (dashed line) for $N = 200$, $\mu = 1.44$, $\sigma_r = 0.002$, $\sigma_s^2 = 0$, $r = 16$ and various values of n (From Avenhaus.[2])

distribution. For $N_i \gg 1$, formula (3.38) simplifies to

$$\sigma^2_{D/H_1} = \sum_i N_i^2 \cdot \left[\frac{\sigma^2_{ri}}{n_i} + \sigma^2_{si} + \mu_i^2 \cdot \frac{r_i}{N_i} \cdot \frac{N_i - r_i}{N_i}\left(\frac{1}{n_i} - \frac{1}{N_i}\right)\right].$$

We now assume that the random variable D is approximately normally distributed, with expected values and variances given by Eqs. (3.35) to (3.39):

$$\begin{aligned} D/H_0 &\sim N(0, \sigma^2_{D/H_0}) \\ D/H_1 &\sim N(M, \sigma^2_{D/H_1}). \end{aligned} \tag{3.40}$$

Figures 3.4 to 3.6 offer some graphic representations of numerical calculations to suggest the parameter combinations for which this approximation is valid.

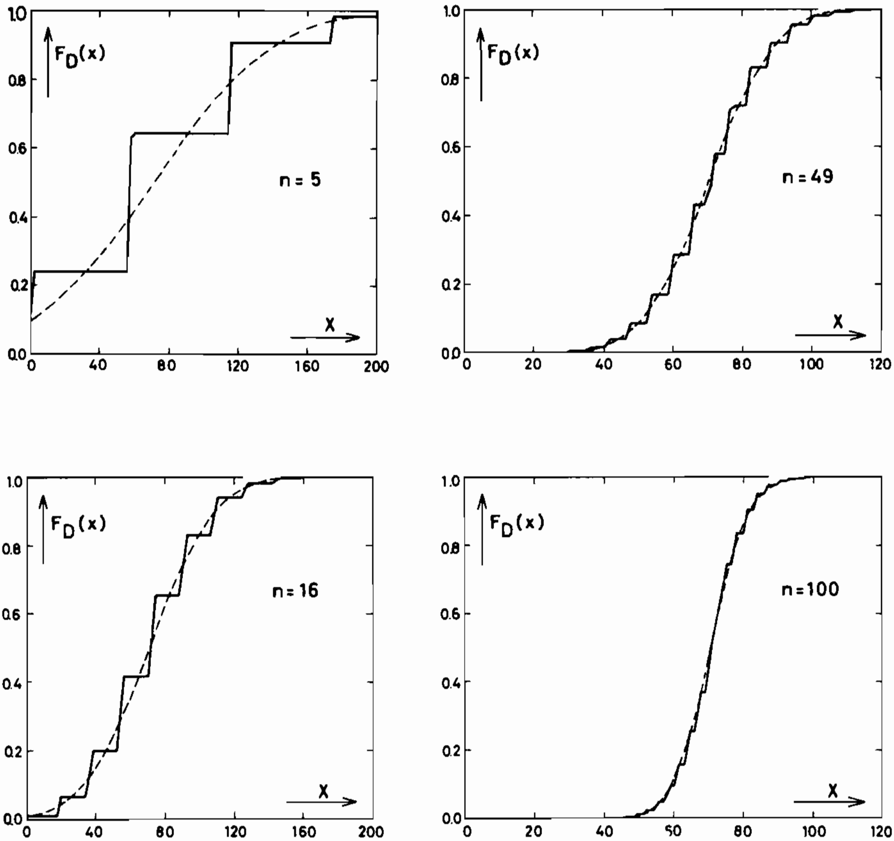

FIGURE 3.6 Distribution function $F_D(x)$ of the D-statistic and its approximation by the normal (Gaussian) distribution (dashed line) for $N = 200$, $\mu = 1.44$, $\sigma_r = 0.002$, $\sigma_s^2 = 0$, $r = 49$ and various values of n. (From Avenhaus.[2])

Now, if a test is performed in the same way as in the case of only one class of material, the relation between probability of detection, false alarm probability, and critical mass is given by the following relation:

$$1 - \beta = \Phi\left(\frac{M}{\sigma_{D/H_1}} - U_{1-\alpha} \cdot \frac{\sigma_{D/H_0}}{\sigma_{D/H_1}}\right), \tag{3.41}$$

which differs from relation (2.15) only because the variances under the different hypotheses are different.

In order to find the optimal distribution of a given inspection effort C, we again consider a two-person zero-sum game where the set of pure strategies of the inspection team is given by all possibilities of choosing the sample sizes n_i, $i = 1, \ldots, R$, of the different classes consistent with the boundary condition (3.24), where the set of pure strategies of the operator is given by all possibilities of choosing the sample sizes r_i, $i = 1, \ldots, R$, of the different classes consistent with the boundary condition (3.25), and where the payoff to the operator is given by one minus the probability of detection β, as given by Eq. (3.41). It is assumed, furthermore, that the values of the parameters M, C, μ_i, ϵ_i, $i = 1, \ldots, R$, are given. At the end of this section we will see that, within the framework of our assumptions, it is best for the operator to choose the largest values of the parameters μ_i that are consistent with the technical boundary conditions.

As the normal distribution function $\Phi(x)$ is a monotone function of its argument x, we may take the negative argument of the function in Eq. (3.41) as the payoff to the operator. In the following, we will consider only the case $M \gg U_{1-\alpha} \cdot \sigma_{D/H_0}$. In this case, the dependence of σ_{D/H_0} on the variables n_i will have no influence on the result, so we may take the variance σ^2_{D/H_1} as the payoff to the operator.

The optimization of the variance σ^2_{D/H_1} was proposed first by Stewart,[1] who, however, optimized the variance only with respect to the inspector's variables n_i for given sets $(r_1, \ldots, r_R)$ and $(\mu_1, \ldots, \mu_R)$ by using the method of Lagrangian multipliers (i.e., he considered the sample sizes n_i to be continuous variables). Bouchey *et al.*[4] treated the same problem by using the method of dynamical programming (i.e., they were able to keep the variables n_i discrete).

An approximate solution of the two-person zero-sum game described above can be given as follows. Under the assumptions

$N_i \gg 1$, $r_i \ll N_i$, n_i and r_i are continuous variables for $i = 1, \ldots, R$
$\sigma^2_{ri}/n_i \ll \sigma^2_{si}$ for $i = 1, \ldots, R$

the optimal strategies $(n_1^0, \ldots, n_R^0)$ and $(r_1^0, \ldots, r_R^0)$ and the value of the game are given by the following formulae:

$$n_i^0 = \frac{C}{\sum_j N_j \cdot \epsilon_j \cdot \mu_j} \cdot N_i \mu_i, \qquad i = 1, \ldots, R \tag{3.42a}$$

$$r_i^0 = \frac{M}{\sum_j N_j \cdot \epsilon_j \cdot \mu_j} \cdot N_i \epsilon_i, \qquad i = 1, \ldots, R \tag{3.42b}$$

$$\sigma^{02}_{D/H_1} = \sum_{i=1}^{R} \left(N_i^2 \cdot \sigma^2_{si} + \frac{M}{C} \cdot N_i \cdot \epsilon_i \cdot \mu_i \right). \tag{3.42c}$$

In order to give an idea of the structure of these solutions, we sketch their derivation. From the assumptions, it follows that the variance σ^2_{D/H_1} can be written in the following form:

$$\sigma^2_{D/H_1} = \sum_i \left(N_i^2 \cdot \sigma_{si}^2 + N_i \cdot \mu_i^2 \cdot \frac{r_i}{n_i} \right). \tag{3.43}$$

As the first term is independent of the strategies, the saddle-point criterion may be written in the following form:

$$\sum_i N_i \cdot \mu_i^2 \cdot \frac{r_i}{n_i^0} \leqslant \sum_i N_i \cdot \mu_i^2 \cdot \frac{r_i^0}{n_i^0} \leqslant \sum_i N_i \cdot \mu_i^2 \cdot \frac{r_i^0}{n_i}. \tag{3.44}$$

The validity of the left-hand side can be seen immediately if one uses the boundary condition (3.24). The right-hand side is equivalent to the inequality

$$\sum_i \epsilon_i \cdot n_i \cdot \sum_i \frac{N_i^2 \cdot \mu_i^2 \cdot \epsilon_i}{n_i} \geqslant \left(\sum_i N_i \cdot \mu_i \cdot \epsilon_i \right)^2,$$

which represents a special form of the Cauchy–Schwarz inequality.

It should be noted that the solutions (3.42) may not satisfy the boundary conditions n_i, $r_i \leqslant N_i$, $i = 1, \ldots, R$, as these boundary conditions are not taken into account in the formalism. If this is the case, analytical solutions cannot be given, and one has to look for numerical solutions.

From the solutions (3.42) we can conclude that it is best for the operator to choose the μ_i, $i = 1, \ldots, R$, as large as possible (i.e., as large as is compatible with the technical possibilities; see page 41). This is in accordance with the results described in section 3.2 and with our assumption that the critical amount M has to be large compared to the measurement uncertainties ($M \gg U_{1-\alpha} \cdot \sigma_{D/H_0}$).

The quantity $N_i \cdot \epsilon_i$ describes the effort necessary for the verification of *all* batches in the ith class. The quantity $N_i \cdot \mu_i$ describes the amount of material that can be diverted by the operator if he diverts from all batches of the ith class the amount μ_i. Therefore, the solutions (3.42) can be interpreted in the following way:

- The ratios of the inspection team's optimal sample sizes for the different classes must be equal to the ratios of the maximal amounts of material that can be diverted by the operator from the different classes. The ratios of the operator's optimal sample sizes for the different classes must be equal to the ratios of the efforts necessary to verify all batches of the different classes.

An approximate solution similar to the one given by Eqs. (3.42) can be derived if the P-statistic is used as the basis for the data comparison. Under the assumptions

$r_i \ll N_i$, $n_i \ll N_i$, n_i and r_i are continuous variables for $i = 1, \ldots, R$
$\mu_i \ll \sigma_i \cdot U_{n_i\sqrt{1-\alpha_i}}$ for $i = 1, \ldots, R$

an approximate solution of the two-person zero-sum game described above is given by the following formulae:

$$n_i^0 = \frac{C}{\sum_i \epsilon_i \mu_i N_i/\pi_i} \cdot \frac{\mu_i \cdot N_i}{\pi_i} \tag{3.45a}$$

$$r_i^0 = \frac{M}{\sum_i \epsilon_i \mu_i N_i/\pi_i} \cdot \frac{\epsilon_i \cdot N_i}{\pi_i}, \tag{3.45b}$$

where

$$\pi_i = \Phi\left(\frac{\mu}{\sigma_i} - U_{n_i\sqrt{1-\alpha_i}}\right), \qquad i = 1, \dots, R \tag{3.45c}$$

is the probability of detecting a falsification if the data of one falsified batch of the ith class are verified.

The derivation of this solution starts with the following approximation of the class probability of detection for the P-statistic, Eq. (3.18).

$$1 - \beta_i = \frac{r_i \cdot n_i}{N_i} \cdot \pi_i, \qquad i = 1, \dots, R. \tag{3.46}$$

The overall probability of detection is therefore given by the following relation:

$$1 - \beta = 1 - \prod_{i=1}^{R}\left(1 - \frac{r_i \cdot n_i}{N_i} \cdot \pi_i\right)$$

and can be approximated by

$$1 - \beta = \sum_{i=1}^{R} \frac{r_i \cdot n_i}{N_i} \cdot \pi_i.$$

Therefore, the saddle-point criterion reads as follows:

$$\sum_i \frac{r_i \cdot n_i}{N_i} \cdot \pi_i \leqslant \sum_i \frac{r_i^0 \cdot n_i^0}{N_i} \cdot \pi_i \leqslant \sum_i \frac{r_i^0 \cdot n_i^0}{N_i} \cdot \pi_i.$$

It can be proven immediately that the formulae (3.45) fulfill these relations if one takes into account the boundary conditions (3.24) and (3.25).

On the basis of this approximation an interesting relation can be established between the results described in this section and those described in section 3.2.1. In section 3.2.1 the activities of the two players were limited to one class; however, that class was selected with the help of a random experiment. Therefore, the sample sizes are random variables, and one can determine the expected values of the sample sizes, both of the inspection team and of the operator. If one now takes the approximation used above and determines the ratios of the expected sample sizes, the results are the same as those given by Eqs. (3.45). This result may be interpreted in the following way:

- The exact solution of the approximate problem and the approximate solution of the exact problem give the same result for the approximate probability of detection. In other words, we may state that we have not committed Kimball's error of the third kind.[5]

This result may be used for another approximation of the optimal distribution of the verification effort in case the assumptions made in this section are not justified: we calculate the *expected* sample sizes for the general case (action in only one class) on the basis of the solutions (3.31) – i.e., we calculate

$$\bar{n}_i := n_i \cdot q_i^*, \qquad i = 1, \ldots, R,$$

where q_i^* is the optimized probability of selecting the ith class for the purpose of data verification (optimal inspection strategy), and take these expected sample sizes as *real* sample sizes for the general case (action in all classes). We do the same with the operator's sample sizes r_i. Then, with the help of Eqs. (3.35) to (3.41), we can calculate the guaranteed probability of detection.

Let us make a final remark about the usefulness of the solutions given above. There is an interesting generalization of the formulae (3.42) that is based on the exact measurement approximation and "drawing with replacement" formula (3.16), for R classes:

$$\beta = \prod_{i=1}^{R} \left(1 - \frac{r_i}{N_i}\right)^{n_i}. \qquad (3.47)$$

Without giving the proof, we just report that for

$$\sum_i \mu_i \cdot N_i \gg M$$

(i.e., for an assumed diversion that is smaller than the total amount that *can* be diverted), the only saddle-point of the function (3.47) in the domain

$$\left\{ n_1, \ldots, n_R, r_1, \ldots, r_R : \sum_{i=1}^{R} \epsilon_i \cdot n_i = C, \sum_{i=1}^{R} \mu_i \cdot r_i = M \right\}$$

is determined by the following relations:

$$n_i^0 = \frac{C}{\sum_j \mu_j \cdot \epsilon_j \cdot N_j \cdot \exp(-\kappa \cdot \epsilon_j)} \cdot \mu_i \cdot N_i \cdot \exp(-\kappa \cdot \epsilon_i) \qquad (3.48a)$$

$$r_i^0 = N_i \cdot [1 - \exp(-\kappa \cdot \epsilon_i)], \qquad i = 1, \ldots, R, \qquad (3.48b)$$

where the parameter κ is uniquely determined by the following equation:

$$\sum_{i=1}^{R} \mu_i \cdot N_i \cdot \exp(-\kappa \cdot \epsilon_i) = \sum_{i=1}^{R} \mu_i \cdot N_i - M. \qquad (3.48c)$$

The value $\beta^0 = \beta(n_1^0, \ldots, n_R^0, r_1^0, \ldots, r_R^0)$ at the saddle-point – in other words, the optimal guaranteed probability of detection $1 - \beta^0$ – is given by the following simple relation

$$1 - \beta^0 = 1 - \exp(-\kappa \cdot C). \qquad (3.48d)$$

This solution has the following interesting properties:

1. As κ is greater than zero, one sees immediately that

$$0 < r_i^0 < N_i$$
$$0 < n_i^0, \qquad i = 1, \ldots, R,$$

in contrast to the situation for the solutions (3.42), where these properties were not necessarily given. A necessary but not sufficient condition for $r_i < N_i, i = 1, \ldots, R$, is

$$\sum_{i=1}^{R} \epsilon_i \cdot N_i > C.$$

It is intuitive that there is no sufficient condition because the drawing-with-replacement model permits the spending of *any* effort.

2. For $\kappa \cdot \epsilon_i \ll 1$ for $i = 1, \ldots, R$, i.e., when

$$C \ll \sum_{i=1}^{R} \epsilon_i \cdot N_i$$

we obtain the old solutions (3.42).

3. For $\epsilon_1 = \ldots = \epsilon_R = : \epsilon$ we obtain $\exp(-\kappa \cdot \epsilon) = 1 - M/\sum_i \mu_i \cdot N_i$, and therefore,

$$n_i^0 = \frac{C}{\sum_j \mu_j \cdot \epsilon \cdot N_j} \cdot \mu_i \cdot N_i$$
$$r_i^0 = \frac{M}{\sum_j \mu_j \cdot \epsilon \cdot N_j} \cdot \epsilon \cdot N_i, \qquad i = 1, \ldots, R,$$

which resembles the solution (3.42) quite strongly.

4. As κ does not depend on the value of C, the optimal sample sizes r_i^0 *do not* depend on the value of C. Unfortunately, the optimal sample sizes n_i^0 *do* depend on the value of M in general; they *do not* depend on the value of M only for small values of M and for small variations of the single batch inspection efforts ϵ_i, as we have seen above. Furthermore, as κ is a monotonically increasing function of M, as can be seen by implicit differentiation of Eq. (3.48c), we get from (3.48d) that the optimal guaranteed probability of detection β^0 is a monotonically increasing function of M and a monotonically increasing function of C, which is quite reasonable.

These properties illustrate the usefulness and the stability of the solutions (3.48) as well as of the solutions (3.42) and (3.45). Therefore, and in view of the complexity of the general problem and the effort necessary for obtaining better solutions, we consider these solutions to be sufficient for practical problems.

3.3 FLOW MEASUREMENT VERIFICATION

So far, it has been assumed that the batches whose data must be verified are all available at the same time and, therefore, that the inspection team can take random samples and proceed in the way described. The verification of the physical inventory of a plant at a given time was taken as a representative situation for the application of this scheme. In this section we shall see the conditions under which we can apply the scheme to the verification of flow measurements as well.

Let us consider the material entering an industrial plant during a given interval of time. We assume that the material arrives in the form of batches that are available for some time and that then go into the production process. There they lose their identity and thus are no longer available for measurement and data verification purposes. This means that the verification can be done only before the batches go into the production process and that the methods described in the preceding section are not generally applicable. However, there are other situations.

In order to illustrate these other possibilities, we consider batches where a complete determination of the material content includes a net weight determination and a chemical concentration analysis. We assume that the chemical concentration determination will be performed in such a way that a sample is drawn and analyzed: these samples are assumed to be stored until the end of the inventory period, when a verification procedure can be applied as described in section 3.2. The different classes of batches may be given by the classes of input, product, waste, and physical inventory batches. In the case of the net weight determinations these procedures cannot be applied: once the batches have entered the process stream, the weight data can no longer be verified.

In the following, we will consider this situation. Two different cases will be treated:

- The operator measures the parameter in question (e.g., the net weight of the batch), reports it to the inspection authority, and the batch is still available for some time after this so that the inspection team can decide whether the data of this batch shall be verified. We shall call this procedure independent sequential verification.
- The operator measures the parameter in question and reports it to the inspection authority. However, there is no time for a later verification of these data because the batch immediately enters the production process. In this case the inspection team can perform its own measurements only at the same time as the plant operator; in a real situation the inspection team will simply observe the operator when he takes his measurements. We will call this procedure dependent sequential verification.

3.3.1 INDEPENDENT SEQUENTIAL VERIFICATION*

If one analyzes this case, a fundamental difference from the cases treated so far can be observed: at the beginning of the sequence of batches the inspection team does

* This section was written in collaboration with Jean-Pierre Ponssard, École Polytechnique, Paris.

not know whether the operator intends to divert any material by means of data falsification. It might happen that the operator decides only from batch to batch if he will falsify the data of the batch, with the result that at the end of the sequence he has not falsified any data. This situation was analyzed by Dresher[6] and later by Höpfinger;[7,8] these investigations, however, dealt with only one class of material and did not take measurement errors into account. Because this situation is less important than the case in which the operator decides at a given point in time either to divert some material within a certain time or not to divert any material we will pursue it no further. We consider once again the case in which there is only one class of material. There are a total of N batches, n batch data will be verified (this is known to the operator), and r batch data will not be falsified. The comparison will be made pairwise; the probability that a correct batch datum will be recognized as correct is α' and the probability that a falsified batch datum will be recognized as correct is β. The overall probability of detection can be determined with the help of a formalism developed by Dresher[6]: let one minus the overall probability of detection $P(n, r, N)$ again be the payoff to the operator. The optimal probability of detection $P(n, r, N)$ can then be determined easily by solving the following 2×2 matrix game:

Inspection team \ Operator	Falsification	No falsification
Verification	$-(1-\beta)+\beta \cdot P(n-1, r-1, N-1)$	$-\alpha' + (1-\alpha') \cdot P(n-1, r, N-1)$
No verification	$-P(n, r-1, N-1)$	$-P(n, r, N-1)$

(3.49)

The meaning of the different matrix elements can be understood easily. Consider the upper left element: if in the first stage the operator falsifies his data and the inspection team verifies these data, then this falsification is either detected (with probability $1-\beta$) or not detected (with probability β). If it is not detected, then the "game" goes on to the next stage, i.e., a game has to be considered with $N-1$ stages and sample sizes $n-1$ and $r-1$. The meaning of the other matrix elements can be understood in a similar way.

The boundaries for the solution are as follows:

1. For $r = N$ and $n \geqslant 1$, we have

$$P(n, N, N) = 1 - \beta^N, \tag{3.50a}$$

2. For $n = N$ and $r \geqslant 1$, we have

$$P(N, r, N) = 1 - \beta^r \cdot (1 - \alpha')^{N-r}, \tag{3.50b}$$

3. For $n = 0$ we have

$$P(O, r, N) = 0. \tag{3.50c}$$

It should be noted that the game in which the operator has not decided *a priori* whether to divert any material leads to the same payoff matrix as given by (3.49). The only difference consists in the boundary conditions (3.50).

As can be shown easily, the guaranteed overall probability of detection is given by the following formula:

$$P = \sum_l \Phi \left(U_{\sqrt[n]{1-\alpha}} - \frac{\mu}{\sigma} \right)^l \cdot (1-\alpha)^{1-l/n} \cdot \frac{\binom{r}{l} \cdot \binom{N-r}{n-l}}{\binom{N}{n}}, \tag{3.51}$$

where

$$1 - \alpha = (1 - \alpha')^n.$$

The optimal strategy of the inspection team is to verify the ith batch with probability

$$P_{ij} = \frac{n-j}{N-(i-1)}, \qquad j = 0, \ldots, n; \qquad i = 1, \ldots, N, \tag{3.52}$$

if j of the foregoing batches have been verified. The optimal strategy of the operator is to falsify the data of the ith batch with probability

$$q_{ij} = \frac{r-k}{N-(i-1)}, \qquad k = 0, \ldots, r; \qquad i = 1, \ldots, N, \tag{3.53}$$

if k of the foregoing batches have been verified.

The result (3.51) is especially interesting because it corresponds exactly to the nonsequential case if one uses the P-statistic, formula (3.18). This means that under the assumption that the operator has decided before the beginning of the sequence to falsify r batches, the situation is the same as in the nonsequential case. This may be explained by the fact that the operator has no chance to learn from the past (i.e., how many batches had already been verified by the inspection team).

There are great difficulties in extending this consideration to the case of several classes. One way of doing so might be to approach the problem in the same way as in the special case described in section 3.2.2; that is, to assume that both parties limit their actions to one class and that they decide before the start of the sequence with the help of a random experiment which of the classes they will select. In this case, formulae (3.31) can be applied again, and the class probabilities of detection are given by Eq. (3.51). However, there is no indication that it is possible to use the expected sample sizes calculated with these formulae for the general case, as had been proposed at the end of the section; numerical calculations for simple cases show that these cases are much more complicated than the corresponding non-sequential cases. Thus, the expected sample sizes calculated from the special case may give an indication of the distribution of effort among different sequences of batches in the form of a lower limit for the probability of detection. However, the problem must be considered unsolved, at least from the analytical point of view; simulation studies for special numerical cases seem to provide the only means of obtaining solutions.

3.3.2 DEPENDENT SEQUENTIAL VERIFICATION

In the case where the inspection team generates its data together with the operator, it is clear that those data of the operator that are verified will not be falsified, since the operator knows before he reports his data which data will be verified by the inspection team. If any data are falsified, it will be those that are not verified. However, even in these cases, it is possible for the inspection authority to make a statement about the correctness of all data; this possibility is outlined in the following paragraphs. As we have done up to now, we will again consider first the case of only one class of material with N batches, n of which are verified by the inspection team.

Let T be the long-term average true value of the material content (or more generally, of the batch parameter considered) of the batches of the sequence. It is supposed that this long-term average is well known because of long experience with production practices. For a sequence of N batches let f_j be the difference between the true value of the jth batch and T. Furthermore, let d_O and d_I be the systematic errors of the operator and the inspection team, respectively, and let e_{Oj} and e_{Ij} be their random errors.

If no data are falsified (null hypothesis H_0), we have for the data x_j reported by the operator

$$x_j = T + f_j + d_O + e_{Oj}, \qquad j = 1, \ldots, N \tag{3.54}$$

and for the inspection team's data y_j

$$y_j = T + f_j + d_I + e_{Ij}, \qquad j = 1, \ldots, n. \tag{3.55}$$

If r batch data are falsified by the amount μ of material (alternative hypothesis H_1), we have for the r falsified data reported by the operator

$$x_j = T - \mu + f_j + d_O + e_{Oj}, \qquad j = 1, \ldots, r. \tag{3.56}$$

The variances σ_r^2, σ_{Or}^2, σ_{Ir}^2, σ_{Os}^2, σ_{Is}^2 of the variation of the true values ("batch-to-batch variation") and of the measurement errors of operator and inspection team are assumed to be known:

$$\operatorname{var} f_j =: \sigma_v^2, \qquad j = 1, \ldots, N$$

$$\operatorname{var} d_{O,I} =: \sigma_{O,Is}^2, \operatorname{var} e_{O,I,j} =: \sigma_{O,Ir}^2, \quad j = 1, \ldots, N. \tag{3.57}$$

The test procedure now is the following: the inspection team compares the estimate of the material content of all batches of the sequence that has been formed on the basis of the inspection team's independent measurements with the estimate formed on the basis of the operator's measurements; in other words, the inspection team evaluates the difference D.

$$D: = \frac{N}{n} \sum_{j=1}^{n} y_j - \sum_{j=1}^{N} x_j. \tag{3.58}$$

It should be noted that in contrast to the D-statistic, described in section 3.1, *all* operator's data are used; this is the reason the variation of the true values has to be taken into account.

If we assume that all measurement errors, as well as the deviations f_j, are normally distributed, then the random variable D is normally distributed under the null as well as under the alternative hypotheses. It has the following expected values and variances:

$$E(D/H_0) = 0 \qquad E(D/H_1) = \mu \cdot r = :M \tag{3.59}$$

$$\operatorname{var}(D/H_0) = \operatorname{var}(D/H_1) = :\sigma^2$$

where

$$\sigma^2 = N^2 \left(\frac{1}{N} \cdot \sigma_{Or}^2 + \frac{1}{n} \cdot \sigma_{Ir}^2 + \sigma_{Os}^2 + \sigma_{Is}^2 + \left(\frac{1}{n} - \frac{1}{N} \right) \cdot \sigma_v^2 \right).$$

According to earlier considerations, the probability of detection $1 - \beta$ is given by

$$1 - \beta = \Phi \left(\frac{M}{\sigma} - U_{1-\alpha} \right), \tag{3.60}$$

where α is the false alarm probability.

If we want to extend the treatment to the case of R classes of material, it is not always correct to determine the difference D between the inspection team's data and the operator's data, summed over all classes. In order to understand this, we consider the two classes of input and output batches. In case of a falsification, the operator will report too small input data and too large output data, which means that the difference may cancel partly or completely if they are added. In this case it would be reasonable to consider the difference of the differences. However, there are also cases where the operator could report too small values for both classes (e.g., if he intends to cover up a falsification in a second plant for which the output of the first plant is the input). Therefore, it would be reasonable to consider all possible sums and differences of the class differences between the inspection team's data and the operator's data. The variance of the D-statistic is the same in all these cases.

According to earlier considerations, the problem again consists in minimizing the variance

$$\operatorname{var}(D/H_1) = \sum_{j=1}^{R} \left(N_i(\sigma_{Osi}^2 - \sigma_{vi}^2) + N_i^2 (\sigma_{Osi}^2 + \sigma_{Isi}^2) + \frac{N_i^2}{n_i} (\sigma_{Iri}^2 + \sigma_{vi}^2) \right) \tag{3.61}$$

with respect to the boundary condition

$$\sum_{i=1}^{R} \epsilon_i \cdot n_i \leqslant C.$$

If one considers the sample sizes n_i to be approximately continuous variables, then the problem can be solved with the help of the method of Lagrangian parameters. The resulting optimized sample sizes n_i^0 are given by

$$n_i^0 = \frac{1}{\sum_j N_j \cdot \epsilon_j(\sigma_{vj}^2 + \sigma_{Irj}^2)} \cdot N_i \cdot \sqrt{\frac{\sigma_{vi}^2 + \sigma_{Iri}^2}{\epsilon_i}}, \qquad i = 1, \ldots, R. \qquad (3.62)$$

Formulae of this type have been given by Cochran.[9]

It should be noted that the strategy of the operator (i.e., the way he determines the sample sizes r_i under the boundary condition $\sum_i \mu_i \cdot r_i = M$ for a given value of M) does not have any influence on the probability of detection. This is true as long as we assume that the batch-to-batch variation is known to the inspection team and that this information will not be improved on the basis of the measurement results. If, on the contrary, this were the case, then the operator's strategy (e.g., taking large amounts from a few batches of one class) would definitely have an influence on the inspection team's strategy and, therefore, on the resulting probability of detection.

3.4 CONCLUDING REMARK

As stated in the introduction to this chapter, we have considered only situations in which the data verification procedures are meant to detect falsified data and in which data are falsified for the purpose of diverting material in such a way that the procedure for establishment of material balance cannot detect such a diversion. This is the reason we could not use the techniques and procedures developed for business accountancy, as described, for example, by W.E. Deming.[10] In Deming's book, various sampling schemes for the evaluation of inventories of materials are described and the practical problems connected with the implementation of these schemes are discussed, but always under the implicit assumption that nobody will falsify data on purpose. Nevertheless, as these methods are based on long practical experience, one should try to extend their use to the situations considered here.

On the other hand, it is useful to repeat here what was said at the end of the fourth section of Chapter 2. If one develops stratified sampling schemes in order to detect accidental losses or in order to minimize errors, one should try to find the best scheme – that scheme that is the best against *all* loss or error "strategies." In this sense, the methods presented here are also useful when data falsification need not be taken into account.

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4 Systems Aspects

The subject of this chapter is, once again, a system for the control of material flows that is based on the principles of material accountability and data verification described in the two preceding chapters. In both subsystems a game theoretical treatment was used for the determination of the optimal strategies of the control authority, with the probability of nondetection as the payoff to the operator. However, neither the zero-sum assumption, i.e., the assumption that the gain of one party is the loss of the other, nor the choice of the payoff function was justified by first-principle arguments. In addition, the terms "false alarm probability," which includes the idea that there exists a mechanism that can recognize an alarm as false, and "probability of detection" were introduced heuristically and without any control scheme that justified these terms.

In this chapter, we will analyze the conflict situation between the two parties – control authority and plant operator – and show that the zero-sum game with the probability of detection as the payoff to the control authority does in fact describe the situation appropriately. In addition, we will discuss the necessity of different action levels for the control authority in order to clarify whether or not an alarm was justified. Furthermore, we will discuss what parameters of the total system can be determined, and how many "global" parameters have to be fixed *a priori.* For this purpose, we have to consider the control problem as a whole, which is complicated because the two subsystems are not independent. In the last section of this chapter an alternative control scheme will be discussed that combines the two aspects of the problem from the very beginning in the sense that the inspection team uses only its own data to establish the material balance.

4.1 THE CONTROL PROBLEM AS CONFLICT SITUATION

Here, we consider a well-defined control problem for one material balance area during an interval of time (t_0, t_1). We choose as an example the control of the material flow of a single industrial plant. We assume that an inspection effort C (man-hours or money or both) is available for this control task and that the operator of the plant decides at the beginning of the interval of time whether he will divert any material.

If one tries to estimate the value of the amount M of material that the operator will divert in case he plans to do so, one has to ask for the motivation of the operator. Probably he is not interested in diverting amounts below a certain threshold amount M_0. Furthermore, he will not divert more than a certain amount, say M_1, of material because if he did, the probability of detection would be high. On the other hand, the control authority is not interested in searching for micrograms of material, first because there would be no chance of detection and second because, with few (but important) exceptions, the diversion of such amounts would do no harm to anybody. Thus, the control authority will assume that the plant operator will either not divert any material at all during the interval of time (t_0, t_1) or that he will divert the amount M of material, with $M_0 < M < M_1$.

If the plant operator diverts the amount $M > 0$, then the interests of the control authority and those of the plant operator are opposed to each other: the control authority wants to detect the diversion, whereas the plant operator wants to camouflage it. If the plant operator does not intend to divert any material, then the interests of both parties are parallel; both parties consider it to be a gain if the legal behavior is recognized as legal behavior, and they consider it to be a loss if an alarm is raised even if the operator behaves legally.

This partial conflict situation may be described by a noncooperative two-person game, in which the sets of strategies X_M and Y_C of both players are limited by the finite available effort C and by the total amount M of material assumed to be diverted, and in which the payoffs to the plant operator (as player 1) and to the control authority (as player 2) are defined as follows:

$(-a, b)$	for diversion and detection	
$(c, -d)$	for diversion and no detection	(4.1)
$(-e, -f)$	for no diversion and detection (i.e., for a false alarm)	
$(0, 0)$	for no diversion and no detection	

In making these definitions, we assume that the inspection effort C is small compared to the ideal or real loss for the control authority in the various possible situations and that it can be neglected in the payoff to the control authority. The inspection effort C is treated as a parameter of the set of strategies of the control authority whose value is determined either *a priori* (as will be assumed here) or by means of a special criterion (we will come back to this point on page 68). This separation of payoff parameters and inspection effort, which was first proposed by Bierlein,[1,2] allows us to normalize the payoff parameters such that the payoff to both players is zero for no diversion and no detection. We assume, further, that the values of the parameters a, b, c, d, and f are greater than zero. About the value of e we do not make any assumptions: in case of a false alarm (which is assumed to be clarified in a second action level – see the next section) there might remain a net loss for the plant operator because of unavoidable process disturbances; there might, however, also be a net gain because the control authority had to pay a penalty for unjustified accusation.

In the following discussion we first consider the case in which false alarms cannot happen (i.e., $\alpha = 0$), and then the more realistic and more complicated case in which false alarms have to be taken into account.

4.1.1 FALSE ALARMS ARE NOT POSSIBLE

According to the definitions (4.1), the expected gains (or losses) of the plant operator and of the control authority

$$\begin{aligned}&(-a\cdot[1-\beta(x,y)]+c\cdot\beta(x,y), b\cdot[1-\beta(x,y)]\\&-d\cdot\beta(x,y)) = :(B(x,y),I(x,y)) \qquad \text{for diversion}\\&\qquad\qquad = (0,0) \qquad\qquad \text{for no diversion}\end{aligned} \tag{4.2}$$

where $1-\beta(x, y)$ is the probability of detection if the plant operator plays a strategy $x \in X_M$ and the control authority plays $y \in Y_C$.

Let us consider for the moment only the "illegal" game, i.e., the game in which the plant operator will divert material. To solve this game means to find the equilibrium points (x^*y^*) of the game that are defined by

$$\begin{aligned}&-a\cdot[1-\beta(x^*,y^*)]+c\cdot\beta(x^*,y^*)\geqslant -a\cdot[1-\beta(x,y^*)]\\&+c\cdot\beta(x,y^*) \text{ for all } x\in X_M,\\&+b\cdot[1-\beta(x^*,y^*)]-d\cdot\beta(x^*,y^*)\geqslant b\cdot[1-\beta(x^*,y)]\\&-d\cdot\beta(x^*,y) \text{ for all } y\in Y_C.\end{aligned} \tag{4.3}$$

As we have assumed that the values of the parameters a, b, c, and d are greater than zero, these two inequalities are equivalent to the following inequalities:

$$\beta(x^*,y)\geqslant\beta(x^*,y^*)\geqslant\beta(x,y^*) \qquad \text{for all } x\in X_M, y\in Y_C. \tag{4.4}$$

These inequalities, however, can be interpreted as the *saddle-point criterion of a two-person zero-sum game* with "no probability of detection" as the payoff to player 1. In other words, the equilibrium points (x^*, y^*) in the noncooperative non-zero-sum game (X_M, Y_C, B, I) are the same as the saddle points in the zero-sum game (X_M, Y_C, β).

Up to now we have assumed that the value of the inspection effort C was given *a priori*. But how can we determine this value if we are asked to do so? For this purpose, we need a criterion for *sufficient* inspection effort. The criterion first given by Bierlein[1] is defined as follows:

• The inspection effort has to be determined such that the plant operator is induced to behave legally. This will be achieved if his expected gain in case of legal behavior is larger than his expected gain in case of illegal behavior.

From (4.2) we therefore get the following sufficient condition for the inspection effort:

$$-a \cdot [1-\beta(x^*,y^*)] + c \cdot \beta(x^*,y^*) \leqslant 0, \tag{4.5}$$

or, equivalently,

$$1-\beta(x^*,y^*) \geqslant \frac{1}{1+\frac{a}{c}}.$$

If we define

$$\frac{1}{1+\frac{a}{c}} = :1-\beta_0,$$

we can formulate the criterion as follows:

• The inspection effort has to be determined such that in case of the diversion of the amount M of material the probability $1-\beta_0$ of detection is guaranteed.

This criterion may be helpful when it is not possible to agree on numerical values of the payoff parameters b and d, but when one instead prefers to think in terms of detection probabilities.

Let us summarize and discuss the results obtained so far. The optimal inspection strategy $y^* \in Y_C$ for a *given* inspection effort C can be determined as the saddlepoint of an "illegal" two-person zero-sum game with the probability of no detection as the payoff to the control authority. This means that the optimal inspection strategy *does not depend* upon the values of the payoff parameters of the players. This result is of eminent practical importance because it is impossible in many cases to give numerical values to the payoff parameters; this is, by the way, one reason the practical usefulness of game theoretical considerations has frequently been doubted.

One could argue that the exclusive consideration of the "illegal" game has to be interpreted in such a way that it is assumed *a priori* that the plant operator will behave illegally and that, therefore, control is no longer necessary. Here, the answer is that an illegal action must still be *detected* and that the consideration of the illegal game serves the purpose of determining the optimal detection scheme.

The determination of the optimal inspection *strategy* for a given effort is the systems analyst's most important task because the question of the appropriate inspection *effort* is usually handled by administrators or by politicians, at least in cases of major importance. Nevertheless, we have formulated a mathematical technical criterion for the "appropriate" effort, which, however, requires knowledge of the values of the payoff parameters or, equivalently, of the appropriate probability of detection.

In the following, it is always assumed that the value of the inspection effort is fixed *a priori*.

4.1.2 FALSE ALARMS ARE POSSIBLE

When false alarms cannot be ruled out, the scheme for the determination of optimal inspection strategies that we have developed on the preceding pages does not work. First, it does not seem possible to reduce the general noncooperative game to a zero-sum game because the equivalence of the inequalities (4.3) and the saddle-point criterion (4.4) fails. Second, the false alarm probability α enters the scene as a new quantity whose value must be specified. We will see, nevertheless, that we can handle the new situation with a procedure that is quite similar to the one developed before.

We start with an auxiliary consideration. Let us assume that there exists an inspection problem for which the gains and losses in the different situations are given by (4.1). Let us assume furthermore that the set of strategies X_M of the plant operator is the set of possible probabilities p of diverting the amount M of material, and that the set of strategies Y of the control authority is the set of possibilities of choosing the value of the false alarm probability α:

$$X_M = \{p : 0 \leqslant p \leqslant 1\}, \qquad Y = \{\alpha : 0 \leqslant \alpha \leqslant 1\}. \tag{4.6}$$

Then, with the help of (4.2), we can derive the expected gains of the two players: We get for the plant operator

$$\widetilde{B}(\alpha, p) = (-a \cdot (1 - \beta(\alpha)) + c \cdot \beta(\alpha)) \cdot p - e \cdot \alpha \cdot (1 - p) \tag{4.7a}$$

and for the control authority

$$\widetilde{I}(\alpha, p) = (b \cdot [1 - \beta(\alpha)] - d \cdot \beta(\alpha)) \cdot p - f \cdot \alpha \cdot (1 - p), \tag{4.7b}$$

where $1 - \beta(\alpha)$ is the probability of detection as a function of the false alarm probability α. Thus, we have a noncooperative two-person game $(X_M, Y, \widetilde{B}, \widetilde{I})$ whose solution is again given by equilibrium points (p^*, α^*), which are defined by analogy to the inequalities (4.3).

The details of the analysis of this game can be found in Frick and Avenhaus,[3] but the more important results are as follows:

1. For $a \leqslant e$ there exists one equilibrium point, which is given by $(p^*, \alpha^*) = (1, 1)$. This result can be understood as follows: If the plant operator's loss is greater in case of detected diversion than in case of a false alarm (such a situation is, of course, most unlikely), the control authority will always state a diversion, i.e., choose $\alpha^* = 1$. Inversely, in such a case the plant operator will divert material with certainty, as this causes a smaller loss for him.

2. For $a > e$ one can show that

If there exists an equilibrium point (p^*, α^*), it has the properties $0 < p^* < 1$, $0 < \alpha^* < 1$.

If β is convex, there exists an equilibrium point.

If β is convex and differentiable, there exists exactly one equilibrium point, which is given by the following equations

$$-a + e \cdot \alpha + (a + c) \cdot \beta(\alpha) = 0 \tag{4.8a}$$

$$[f - (b + d) \cdot \beta'(\alpha)] \cdot p - f = 0 \tag{4.8b}$$

where $\beta'(\alpha)$ is the first derivative of β. (Note that the optimal value of α depends only upon the payoff parameter values of the plant operator.)

The fact that in the reasonable case $a > e$ the plant operator will divert material with a probability greater than zero leads to the question of how the control authority should choose the value of α so that the plant operator will behave legally – that is, so that he will choose $p = 0$. In the spirit of the criterion (4.5), the set of strategies Y' of the control authority that induces the plant operator to legal behavior is given by

$$Y' = \{\alpha : 0 \leqslant \alpha \leqslant 1, \tilde{B}(p, \alpha) < \tilde{B}(0, \alpha)\}.$$

Now one can show that the game $(X_M, Y', \tilde{B}, \tilde{I})$ has no equilibrium point. Therefore one has to argue as follows: the optimal counterstrategy of the plant operator against any inspection strategy $y \in Y'$ is $p = 0$, which, according to (4.7a), gives an expected payoff

$$\tilde{I}(0, \alpha) = -f \cdot \alpha$$

to the control authority. This means that the control authority will choose the smallest possible value of $\alpha \in Y'$. Let us denote $\bar{\alpha} := \inf\{\alpha \in Y'\}$. Then we can show that for differentiable β we have $\bar{\alpha} = \alpha^*$ where α^* is the solution of (4.8a). This means that the control authority has to choose an "ϵ–good" strategy $\bar{\alpha} + \epsilon$, where ϵ is a small positive number.

Let us come back to our original problem, which we illustrate with the help of the material accountability example. Our task is to determine the optimal inspection strategy; in our example, this means determining the optimal set $(\alpha_1^*, \ldots, \alpha_n^*)$ of false alarm probabilities (i.e., significance thresholds) for the n inventory periods. It can be shown[4] that the noncooperative game $(\tilde{X}_M, \tilde{Y}, \tilde{B}, \tilde{I})$, where

$$\begin{aligned}
\tilde{X}_M &= X_{M1} \cup X_2 \cup X_{M3}, \\
X_{M1} &= \{(M_1, \ldots, M_n) : \sum_i M_i = M > 0\} \\
X_2 &= \{(0, \ldots, 0)\} \text{ (legal behavior)} \\
X_{M3} &= \{p : 0 \leqslant p \leqslant 1\} \\
\tilde{Y} &= \{(\alpha_1, \ldots, \alpha_n) : 0 \leqslant \alpha_i \leqslant 1 \text{ for } i = 1, \ldots, n\}
\end{aligned}$$

and where $\tilde{B}$ and $\tilde{I}$ are given by (4.7), has a uniquely determined equilibrium point

$[(M_1^*, \ldots, M_n^*), p^*; (\alpha_1^*, \ldots, \alpha_n^*)]$ that can also be calculated by the following two-step procedure:

1. Determine the saddle-point $[(M_1^*, \ldots, M_n^*), (\alpha_1^*, \ldots, \alpha_n^*)]$ of the zero-sum game $(\hat{X}_M, Y, \beta)$ where $\hat{X}_M$ and $\tilde{Y}_C$ are given by the sets (2.36) and (2.37) and where β is the n-fold product of (2.33). In other words, solve the problem outlined in section 2.4.
2. Determine the equilibrium point (p^*, α^*) of the noncooperative game $(X_M, \tilde{Y}, \tilde{B}, \tilde{I})$ where X_M and Y are given by the sets (4.6) and where $\beta(\alpha)$ is the value of the game $(\hat{X}_M, Y_\alpha, \beta)$.

This important result, which also holds for other control schemes (e.g., the combined material balance and data verification schemes discussed in Chapter 5), allows us to handle the complex problem of optimizing inspection strategies in control systems where false alarms cannot be excluded. The following procedure may be used in dealing with this complex problem.

If it is possible to estimate the values of the payoff parameters (4.1), then one can solve the problem either as a whole or with the help of the two-step procedure outlined above. If, on the contrary, such estimates cannot be given, then one has to fix the value of the resulting false alarm probability *a priori* and solve the "illegal" two-person zero-sum game, with the probability of no detection as the plant operator's payoff.

Thus, the analysis of this problem is similar to that of the less complicated case in which false alarms need not be taken into account. The only substantial difference is that the resulting false alarm probability represents a new parameter whose value must be determined *a priori* if the payoff parameter values cannot be estimated. We will come back to this point in section 4.3.

4.2 ACTION LEVELS

In the foregoing section we introduced the terms "false alarm probability" and "probability of detection," and we remarked that in a simple inspection scheme that has only one action level this nomenclature has no meaning. When data are verified with the help of the D-statistic, then, if the difference D is significant, an alarm will be raised. However, nothing has been said about whether any further action is taken to find out if the alarm was justified.

In this section we will discuss possible control procedures with more than one action level (as well as those with only one) and see what the meaning of α and β is in this framework. Since no such procedures have actually been worked out, these considerations are hypothetical; they are important, however, in clarifying the relationship of the schemes described in the foregoing chapters.

4.2.1 ONE ACTION LEVEL ONLY

Let us assume that the control procedure consists of only one action level: the inspection team generates its own data and compares these data with those reported by the plant operator. If these two sets of data show no significant differences, then the material balance is closed with the data of the operator. If the material balance establishment shows no significant difference between book and physical inventory, it is stated that the operator behaved legally. If either one of the two tests – data verification or material balance – shows significant differences, it is stated that material has been lost or diverted.

The subsystems of this control system have been described at length in Chapters 2 and 3. The effort C introduced in section 3.2 is in fact the total inspection effort, as no other activities are foreseen. The disadvantage of this system is that it offers no possibility of correcting an erroneous accusation. The control authority knows that, according to the chosen significance levels and the measurement variances, there is a probability α that an erroneous accusation will be made, but in practice they cannot decide whether the accusation is justified. This means that in this model there exists no "false alarm" but only an "alarm." One can still call α the false alarm probability as it is an important quantity for the design of a control system, but in this situation it is irrelevant.

It is clear that in realistic control systems – especially in international systems like the nuclear materials safeguards system to be described in the next chapter – more action levels are required, since one cannot afford too many false alarms.

4.2.2 SEVERAL ACTION LEVELS

We now assume that at the end of the first action level it is decided either that the difference between the inspection team's data and the operator's data is not significant and therefore that the operator behaved legally, or that the difference is significant. In the latter case, a second action level follows.

At this second action level, all measuring instruments, those of the operator and those of the inspection team, are recalibrated in order to eliminate systematic measurement errors; furthermore, all data are checked for transcription errors. On this basis, it is determined whether a false alarm was raised. If the reexamination leads to the same conclusions, it is stated that in fact material was lost or diverted, and the inspection ends with a report to the control authority.

This cannot be considered a true two-action-level procedure because at the second action level no new measurements are performed. We assume that the additional effort C_2 for the second action level as well as the false alarm probability α (in other words, the expected effort $\alpha \cdot C_2$ for the second action level) is given. This means that the optimal procedure for the control of material as described in the two preceding chapters is not changed by the introduction of a second action level of this kind.

At the second action level, additional measurements could be performed. One could, for example, repeat the old measurements with higher accuracy, or one could measure all batches of all classes. However, there are objections to such procedures. One objection is that such additional measures still do not make it absolutely clear whether material was lost or diverted, as the random measurement errors cannot be eliminated; a repetition of the operator's measurements would be useless, as he could explain the falsification of his first data as transcription or measurement errors. Another objection is that the total inspection effort must be limited somehow, as inspections disturb the operation of the plant (at least one plant representative must be available to accompany the inspectors). In other words, the plant operator must be protected against overambitious inspection teams.

If one wants to analyze an inspection procedure that truly consists of two action levels, one must be aware that the inspection mode at the second action level may influence the strategy of the operator – and thus of the inspection team – at the first level. If the same batches that had been verified at the first level were verified at the second level, the operator would limit his falsification at the first level to only a few batches. If, however, on the second action level all batches were verified, the operator would falsify as many batch data as possible by as small an amount as possible. Therefore, a mathematical treatment of a procedure with more than one action level would have to include the second action level from the outset. In order to do this, one would have to know the details of the second action level procedures. As already mentioned, there is no known case where such procedures are formally established, so further detail is irrelevant.

4.3 PARAMETERS OF THE TOTAL CONTROL SYSTEM

In the discussion of the two subsystems of a material flow control system in Chapters 2 and 3, some parameters had to be fixed without further justification. In the material balance establishment subsystem these parameters were length of the inventory period (t_0, t_1); probability of detection $1-\beta$; false alarm probability α; and critical mass M. In the data verification subsystem we had these same parameters, as well as inspection effort C. The values of these parameters could not be chosen independently because for each subsystem there is one relation between these parameters (see, e.g., Eq. (2.15) and Eq. (3.41)). This means that in total the values of $3 + 3 = 6$ parameters have to be fixed *a priori*, if we choose for both subsystems the same reference time.

It is necessary that each inspection system be as formalized as possible; it was already mentioned that the plant operator must be protected against the inspection authority. The next question to be dealt with is the possibility of reducing the number of parameters whose values must be subjectively fixed *a priori*. In order to tackle this question, let us consider the simplified case in which the verification of the data of the plant operator can be performed with the help of the D-statistic described in Chapter 3 (a realistic example of such a case will be given in the next chapter). The whole inspection procedure at the end of one inventory period consists of the performance of two tests of significance: the D-test (for the data verification) and the MUF-test (for the material balance establishment).

The whole inspection problem at the end of an inventory period may be viewed as a composite test-of-significance problem in the sense of multivariate statistical analysis (see, e.g., Miller[4]), where the null hypothesis H_0 is given by

$$E(D/H_0) = 0, \qquad E(MUF/H_0) = 0, \tag{4.9}$$

and where the alternative hypothesis is given by

$$E(D/H_1) = M_1 > 0, \qquad E(MUF/H_1) = M_2 > 0, \tag{4.10}$$

where the values of M_1 and M_2 are still to be discussed (see Eq. 4.26).

Because the inspection problem, which consists of the two subsystems, is treated as a whole, one does not fix the significance thresholds s_1 and s_2 of the two tests by fixing the two false alarm probabilities α_1 and α_2 independently. Instead, one fixes a single false alarm probability α, which is defined by

$$1-\alpha := \text{prob}\,\{D \leqslant s_1 \wedge MUF \leqslant s_2/H_0\}. \tag{4.11}$$

As this equation is of crucial importance for further considerations, we will discuss it in greater detail.

According to the results described in Chapter 2, the random variable MUF under the null hypothesis H_0 is normally distributed, with expected value 0 and variance σ^2_{MUF} (given by Eq. 2.17). According to Chapter 3, the random variable D under the null hypothesis H_0 is at least approximately normally distributed with the expected value 0 and variance σ^2_{D/H_0} (given by Eq. 3.35):

$$D \sim N(0, \sigma^2_{D/H_0}), \qquad MUF \sim N(0, \sigma^2_{MUF}). \tag{4.12}$$

The two random variables, however, are not independent because the data of the plant operator occur in both D and MUF; we call ρ_{H_0}, defined by

$$\rho_{H_0} = \frac{\text{cov}\,(D, MUF)}{\sigma_{D/H_0} \cdot \sigma_{MUF}} = \frac{E(D \cdot MUF/H_0)}{\sigma_{D/H_0} \cdot \sigma_{MUF}}, \tag{4.13}$$

the correlation between these two random variables under the null hypothesis. Therefore, the two random variables $D/\sigma_{D/H_0}$ and MUF/σ_{MUF} are distributed according to a bivariate normal distribution whose density is defined by

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2xy\rho + y^2}{2(1-\rho^2)}\right). \tag{4.14}$$

The false alarm equation (4.11) is given by the following expression:

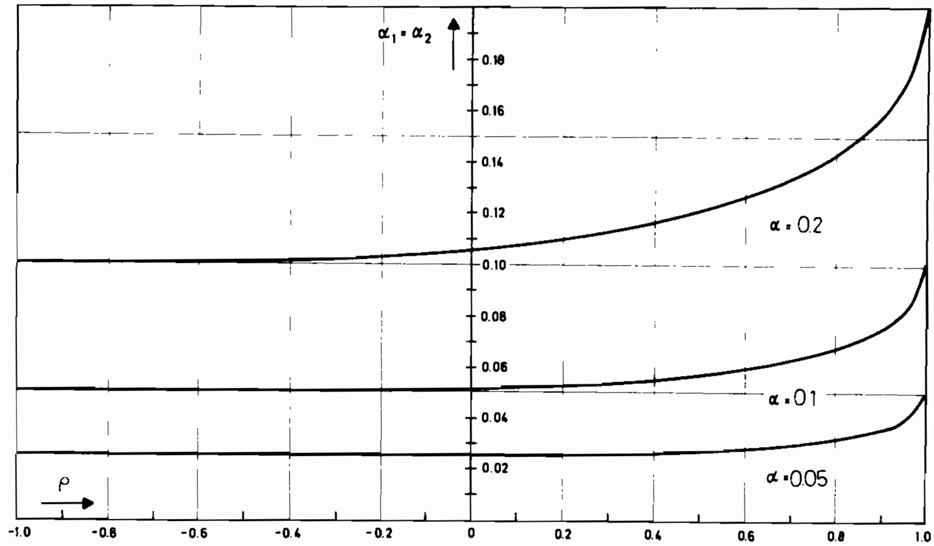

FIGURE 4.1 Single test false alarm probabilities α_1 and α_2 for $\alpha_1 = \alpha_2$ as a function of correlation ρ with total false alarm probability α as parameter. (From Avenhaus and Nakicenovic.[6])

$$1 - \alpha = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\frac{s_1}{\sigma_{D/H_0}}} dx \int_{-\infty}^{\frac{s_2}{\sigma_{MUF}}} dy \ \exp\left(-\frac{x^2 - 2xy\rho + y^2}{2(1-\rho^2)}\right). \quad (4.15)$$

If we remember the relations between the significance thresholds s_1 and s_2 and the single false alarm probabilities α_1 and α_2 (Eqs. 3.39 and 2.14)

$$1 - \alpha_1 = \Phi\left(\frac{s_1}{\sigma_{D/H_0}}\right), \qquad 1 - \alpha_2 = \Phi\left(\frac{s_2}{\sigma_{MUF}}\right), \quad (4.16)$$

then we can eliminate the significance thresholds in Eq. (4.15) and get a relation between the total false alarm probability $1 - \alpha$ and the single false alarm probabilities α_1 and α_2:

$$1 - \alpha = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{U_{1-\alpha_1}} dx \int_{-\infty}^{U_{1-\alpha_2}} dy \ \exp\left(-\frac{x^2 - 2xy\rho + y^2}{2(1-\rho^2)}\right). \quad (4.17)$$

Some limiting cases of this equation can be established very easily if one uses the properties of the bivariate normal distribution[5]:

- For $\rho = 0$ we get

$$1 - \alpha = (1 - \alpha_1) \cdot (1 - \alpha_2), \quad (4.18)$$

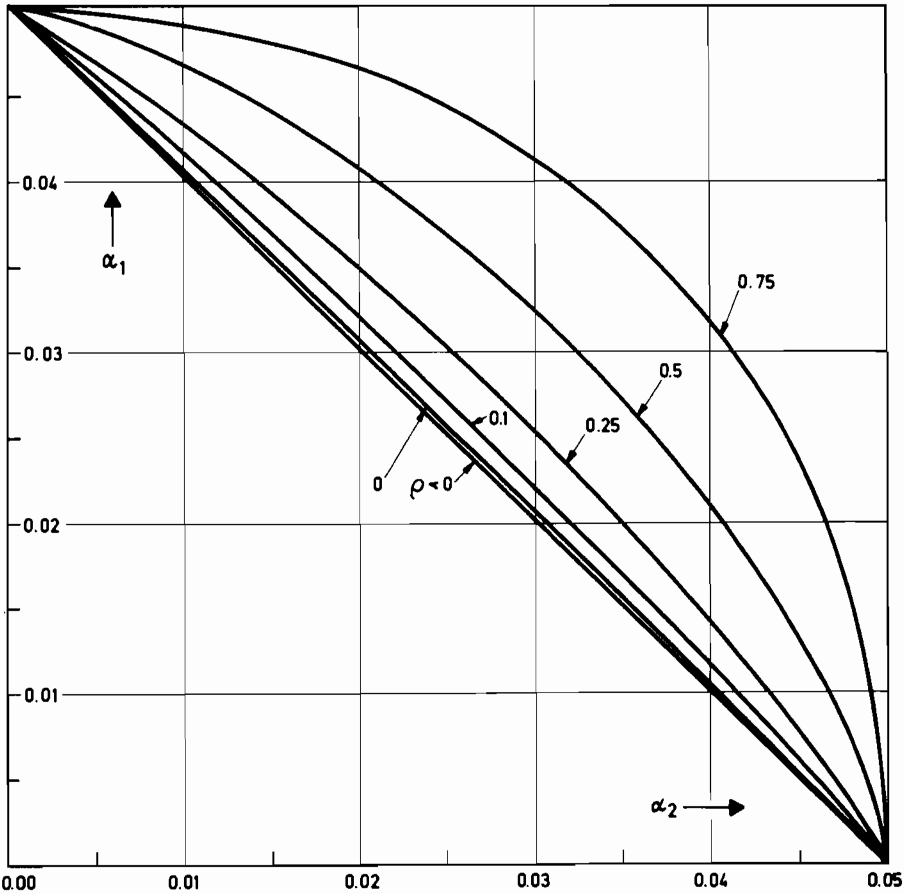

FIGURE 4.2 Mutual dependence of the single test false alarm probabilities α_1 and α_2 with correlation ρ as parameter for total false alarm probability $\alpha = 0.05$. (From Avenhaus and Nakicenovic.[6])

which is a well-known relation in multivariate statistical analysis.

- For $\rho = 1$ we get

$$\alpha = \begin{cases} \alpha_1 & \text{for} \quad \alpha_2 \leqslant \alpha_1 \\ \alpha_2 & \phantom{\text{for}} \quad \alpha_2 \geqslant \alpha_1 \end{cases}. \tag{4.19}$$

- For $\rho = -1$ we get

$$\alpha = \alpha_1 + \alpha_2 . \tag{4.20}$$

- For $-1 \leqslant \rho \leqslant 1$ we get

$$\alpha = \begin{cases} \alpha_1 & \text{for} \quad \alpha_2 = 0 \\ \alpha_2 & \qquad\quad \alpha_1 = 0 \end{cases}. \tag{4.21}$$

In addition, we can deduce from Bonferroni's inequality[4] that for $-1 \leqslant \rho \leqslant 1$ we have

$$\alpha \leqslant \alpha_1 + \alpha_2. \tag{4.22}$$

Figures 4.1 and 4.2 illustrate equation (4.17). Figure 4.1 shows for $\alpha_1 = \alpha_2$ the dependence of α_1 on ρ_1 with α as parameter; Figure 4.2 shows the dependence of α_1 on α_2, with ρ as parameter and for fixed $\alpha = 0.05$. The main result is that for $\rho < 0$ (which is the case in the example given in Chapter 5) Eq. (4.17) can be approximated very well by Eq. (4.18).

The composite test of significance (D, MUF) is not yet fully characterized by a given value of the total false alarm probability α, because the two significance thresholds are not yet determined completely by Eq. (4.15). In order to find these thresholds we determine the total probability of an error of the second kind β of the test, or, in other words, the total probability of detection $1-\beta$. This is defined by

$$\beta := \text{prob}\,\{D \leqslant s_1 \wedge MUF \leqslant s_2 / H_1\}. \tag{4.23}$$

Using Eqs. (4.10) and taking into consideration that for the D-statistic the variance under the null hypothesis is different from the variance under the alternative hypothesis, we get for the total probability of detection

$$\beta = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\frac{\sigma_{D/H_0} \cdot U_{1-\alpha_1} - M_1}{\sigma_{D/H_1}}} dx \int_{-\infty}^{U_{1-\alpha_2} - \frac{M_2}{\sigma_{MUF}}} dy$$

$$\exp\left(-\frac{x^2 - 2xy\rho + y^2}{2(1-\rho^2)}\right). \tag{4.24}$$

For the vanishing correlation ($\rho = 0$) this formula reduces to

$$\beta = \Phi\left(\frac{\sigma_{D/H_0} \cdot U_{1-\alpha_1} - M_1}{\sigma_{D/H_0}}\right) \cdot \Phi\left(U_{1-\alpha_2} - \frac{M_2}{\sigma_{MUF}}\right) = \beta_1 \cdot \beta_2. \tag{4.25}$$

Now we can solve the problem of choosing appropriate values for the single false alarm probabilities α_1 and α_2 as well as for the single critical masses M_1 and M_2 in Eqs. (4.10). We suppose that the control authority wants to determine the values of α_1 and α_2 such that the guaranteed probability of detection is optimized if the operator diverts the amount M of material. This means that the maximum (with respect to the inspection strategies $(\alpha_1, \alpha_2 : (1-\alpha_1)(1-\alpha_2) = 1-\alpha)$) of the minimum (with respect to the diversion strategies $(M_1, M_2 : M_1 + M_2 = M)$) of the probability of detection has to be determined. The optimal values of the single false alarm probabilities α_1 and α_2 are therefore determined with the help of the following optimization problem:

$$\max_{\substack{\alpha_1,\alpha_2:\\(1-\alpha_1)(1-\alpha_2)=1-\alpha}} \quad \min_{\substack{M_1,M_2:\\M_1+M_2=M}} \quad 1-\beta(\alpha_1,\alpha_2;M_1,M_2) \tag{4.26}$$

where $\beta(\alpha_1, \alpha_2; M_1, M_2)$ is given by Eq. (4.24). For practical purposes it would be of advantage to put $\alpha_1 = \alpha_2$; in fact, it will be shown in the next chapter that this choice does not cause any significant deviation from the optimal probability of detection.

Before we go on, it should be noted that there is another conceivable way of performing the composite test of significance for D and MUF. One could take linear combinations of D and MUF as test statistics so that the new statistics are independent and are thus much simpler to treat. In fact, such a scheme has been discussed recently by Hough *et al.*[7] The more complicated scheme outlined here seems preferable, however, because it is important for a control authority to test the "physical" statistics D and MUF and to be able to see immediately where the significant difference, if any, arises.

Let us now come back to our original question of how many inspection system parameter values have to be fixed *a priori* and how many can be determined later on. We may imagine the following procedure.

First, the inspection budget (man-hours or money) for a given plant during a given reference time (e.g., 1 year) has to be fixed. As was discussed in the first section of this chapter, one could also imagine that the value of the budget is determined according to a criterion of the kind given on page 69; however, we assume here that such an important parameter is discussed in the political arena, especially because such long-term considerations as number of available inspectors must be taken into account. Second, the length of an inventory period is determined. This question was discussed at the end of the second chapter: if the probability of detection were the only criterion, one would have as few inventories as possible (e.g., once a year). On the other hand, the detection time, which is another criterion, would call for as short an inventory period as possible. A third aspect is that physical inventory taking usually means a disturbance of the plant operations – which should be as small as possible. Therefore, a compromise should be made among these different points of view; the actual length of an inventory period will in most cases be determined on a pragmatic basis.

The next parameter to be discussed is the total false alarm rate. While a high false alarm rate increases the sensitivity of the inspection system, the inspection authority cannot afford too many false alarms, for obvious reasons. The actual value of the total false alarm rate will depend strongly on the second action level procedure. If there are efficient methods for checking the validity of an "alarm" at the first action level, one will not be so sensitive, while an alarm on the second level would immediately throw some doubt on the legality of the plant operator's behavior.

The two remaining parameters, critical mass and total probability of detection $1-\beta$, should be considered together: if the value of M is fixed, that of $1-\beta$ is determined by Eq. (4.26) and vice versa. Whether the probability of detection

$1-\beta$ in case of a diversion of an amount M will be considered sufficient depends on the nature of the inspection problem. Eventually, the value of one of the other parameters, e.g., the budget, must be changed. Clearly, the efficiency that can be reached also depends on the accuracy of the measurements.

We have now reduced the number of parameters whose values have to be fixed *a priori* from the six mentioned at the beginning of this section to the four outlined above. Is a further reduction possible?

In principle, further reduction can be achieved in the following way: we consider several plants and fix the values of the parameters – length of inventory period, budget, false alarm probability, and critical mass – for all the plants together. Then the values of the plant parameters are determined with the help of an optimization procedure as discussed above. This corresponds to our procedure for deciding on a verification effort and determining the critical masses for the single classes given in the third chapter, where only the values of the parameters C and M were given for all classes and where the single-class parameter values were determined by an optimization procedure. If several plants are considered at once, there would be more difficulties, technical as well as fundamental: there would be new correlations (e.g., shipper–receiver correlations) that could not be treated as conveniently as in the case analyzed above, and it is doubtful if such a "lumping" of different plants would be acceptable to the plant operators. It remains to be seen, therefore, how far such a formalization can go in a real case. It is clear, however, that, no matter what, the values of some parameters have to be fixed subjectively *a priori*.

4.4 MATERIAL ACCOUNTABILITY VERIFICATION SYSTEM NOT USING OPERATOR'S DATA*

So far we have considered one possible material accountability verification scheme: the plant operator generates all data necessary for the establishment of a material balance and reports these data to the control authority. The control authority verifies these data with the help of independent measurements. If there are no significant differences between the operator's reported data and the findings of the control authority, then the material balance is closed with the operator's data.

In this section, we consider an alternative material accountability verification scheme, which does not make use of the data reported by the operator but rather is based exclusively on the data generated by the control authority. Such a system could be of use in situations where there is no reason for a plant operator to maintain a complicated measurement system or where, for some reason, the records are not available.

It is clear that if the inspection team cannot measure the data of all material batches processed in the plant under consideration (e.g., if the inspection budget or

* This section was written in collaboration with W.S. Jewell, University of California, Berkeley.

time is limited), then some *prior information* about the average material contents of the different batches as well as the batch-to-batch variation has to be used. Therefore, a ***Bayesian approach*** seems natural for the treatment of problems of this kind. On the other hand, this prior information will not be very detailed, and so we will use the principles of *credibility theory*,[8,9] where it is necessary to know only the first two moments of the prior distribution.

In the following, we first consider only one class of material, and then R different classes (inputs, outputs, and so on) with the problem of material balance closure. Finally, we discuss the problem of optimization of a given inspection effort. Since the batch-to-batch variation of the true material contents within one class is normally much larger than the measurement variance, we will neglect the measurement errors here; they could easily be taken into account, however, if necessary.

4.4.1 ONE CLASS OF MATERIAL

Let us consider one class of material consisting of N batches. An inspection team measures the material content of n of these N batches precisely and wants to estimate the total material content of the class with the help of the n data. The true values of the material content of the batches vary from batch to batch; because of long-term experience, however, the inspection team has prior information about the average value and the batch-to-batch variation of the true material content.

This prior information may be specified in the following way: the true material content x_j of the jth batch is a random variable with a likelihood density $p(\tilde{x}_j|\theta)$, where θ is the parameter (possibly a vector), representing the unknown variation that has occurred in this production run. In Bayesian analysis, the parameter θ itself is considered as a random variable with a prior density $p(\tilde{\theta})$. We do not assume that the complete expressions of $p(\tilde{x}_j|\theta)$ and $p(\tilde{\theta})$ are known to the inspection team, but only the expected value m,

$$m: = E\{\tilde{x}_j\} = EE\{\tilde{x}_j|\tilde{\theta}\} = \int d\theta \cdot \theta \cdot p(\theta) \cdot \int dx_j \cdot x_j \cdot p(x_j|\theta),$$
$$j = 1, \ldots, N \tag{4.27}$$

and the two components of the variance

$$\mathrm{var}\,\{\tilde{x}\} = E\{\tilde{x}_j^2\} - (E\{\tilde{x}_j\})^2 = :E + D, \tag{4.28}$$

which are given by

$$E = E\,\mathrm{var}\,\{\tilde{x}_j|\tilde{\theta}\} = E_\theta\{E_x\{\tilde{x}_j^2|\tilde{\theta}\} - (E_x\{\tilde{x}_j|\tilde{\theta}\})^2\}$$
$$D = \mathrm{var}\,E\{\tilde{x}_j|\tilde{\theta}\} = E_\theta\{(E_x\{\tilde{x}_j|\tilde{\theta}\})^2\} - (E_\theta E_x\{\tilde{x}_j|\tilde{\theta}\})^2 \qquad j = 1, \ldots, N.$$

For later purposes we note that

$$\mathrm{cov}\,\{\tilde{x}_i, \tilde{x}_{j\neq i}\} = E\{\tilde{x}_i\tilde{x}_{j\neq i}\} - E\{\tilde{x}_i\}\cdot E\{\tilde{x}_{j\neq i}\} = D. \tag{4.29}$$

As we must differentiate carefully between random variables and their actual values, we indicate random variables by a tilde. In addition, we will put the random variables whose moments have to be determined into curled brackets. Notice that, even though the random variables $\tilde{x}_j$ are independent, given θ they are, *a priori*, dependent random variables; in other words, it is possible to make inferences about future values of the random variables $\tilde{x}_j$ from observed values because they have the same (unknown) value of θ.

Assume that the inspection team has measured the material contents of $n \leqslant N$ batches (for simplicity we relabel the batches so that these are the first n batches); let $\underline{x} = (x_l, \ldots, x_n)$ be the result of these measurements. The problem is to estimate the total material content of the class using these data and their prior information (4.27 and 4.28). Since we know x, we must estimate $(\tilde{x}_{n+1}, \ldots, \tilde{x}_N)$.

The idea of the credibility approach is to take an estimate $f_n(x)$ for the material content x_{n+1} of the n + 1st batch, which is *linear in the data* and which minimizes *the preposterior variance* of the forecast error defined by

$$H_x = E\{[\tilde{x}_{n+1} - f_n(\underline{\tilde{x}})]^2\}. \tag{4.30}$$

(H_x is, in fact, a variance since $f_n(\underline{\tilde{x}})$ will be an unbiased estimate; i.e., $E\{\tilde{x}_{n+1} - f_n(\underline{\tilde{x}})\} = 0$.)
As a linear form, we take

$$f_n(\underline{\tilde{x}}) = z_0 + z_1 \cdot \frac{1}{n} \cdot \sum_{j=1}^{n} \tilde{x}_j \tag{4.31}$$

since there is no reason to use a different weighting factor for each x_j. Then H_x is given by

$$\begin{aligned} H_x = {} & E\{\tilde{x}_{n+1}^2\} + z_0^2 + \frac{z_1^2}{n^2} \cdot E\left\{\left(\sum_{j=1}^{n} \tilde{x}_j\right)^2\right\} - 2z_0 \cdot E\{\tilde{x}_{n+1}\} \\ & - 2\frac{z_1}{n} \cdot E\left\{\tilde{x}_{n+1} \sum_{j=1}^{n} \tilde{x}_j\right\} + 2z_0 \cdot \frac{z_1}{n} \cdot E\left\{\sum_{j=1}^{n} \tilde{x}_j\right\} \end{aligned} \tag{4.32}$$

and we get, with Eqs. (4.27), (4.28), and (4.29)

$$E\{\tilde{x}_{n+1}^2\} = \text{var}\{\tilde{x}_i\} + (E\{\tilde{x}_i\})^2 = D + E + m^2$$

$$\begin{aligned} E\left\{\left(\sum_{j=1}^{n} \tilde{x}_j\right)^2\right\} = {} & n \cdot (\text{var}\{\tilde{x}_i\} + (E\{\tilde{x}_i\})^2) + n \cdot (n-1) \cdot (\text{cov}\{x_j', x_j\} \\ & + (E\{x_i\})^2) = n \cdot (D + E + m^2) + n \cdot (n-1) \cdot (D + m^2), \end{aligned}$$

$$E\left\{\tilde{x}_{n+1} \sum_{j=1}^{n} x_j\right\} = n \cdot (\text{cov}\{\tilde{x}_j, \tilde{x}_j'\} + (E\{\tilde{x}_i\})^2) = n \cdot (D + m^2)$$

the following expression for the preposterior variance H_x:

$$H_x = D + E + m^2 + z_0^2 + \frac{z_1^2}{n^2} \cdot [n \cdot (D + E + m^2)$$
$$+ n \cdot (n-1) \cdot (D + m^2)] - 2z_0 m - 2z_1 \cdot (D + m^2) + 2 \cdot z_0 \cdot z_1 \cdot m. \qquad (4.33)$$

The optimal values of z_0 and z_1 are determined by

$$\frac{\partial H_x}{\partial z_0} = 0 \qquad \text{and} \qquad \frac{\partial H_x}{\partial z_1} = 0,$$

which finally gives

$$z_1 = \frac{n}{n + \dfrac{E}{D}} \qquad \text{and} \qquad z_0 = m \cdot (1 - z_1). \qquad (4.34)$$

Notice that (4.31) and (4.34) can, in fact, be used to estimate the future realization of any random variable $\tilde{x}_j$, $j = n + 1, \ldots, N$. The minimum of the preposterior variance of H_x is given by

$$\min H_x = E + \frac{E}{n + \dfrac{E}{D}} \qquad (4.35)$$

These results have an intuitive interpretation: for $nD \gg E$ we obtain $z_1 \approx 1$, $z_0 \approx 0$, and therefore

$$f_n(\underline{x}) \approx \frac{1}{n} \sum_{j=1}^{n} x_j.$$

That is, we use primarily the information contained in the data. Note that this could happen either because the number of examples is very large or because D, the variance for our prior information, is large. For $nD \ll E$ we obtain $z \ll 1$, and therefore

$$f_n(\underline{x}) \approx m;$$

i.e., we use primarily the prior information m.

We now estimate the sum S of all material in the class

$$S = \sum_{j=1}^{N} x_j \qquad (4.36)$$

by the true values of the material contents in the first n batches, $\sum_{j=1}^{n} x_j$, plus the sum of the estimates of the remaining $N - n$ material contents:

$$\sum_{j=1}^{n} x_j + (N - n) \cdot f_n(x). \qquad (4.37)$$

Using (4.27), we obtain the following estimate $F^n(x)$ of the sum S:

$$F^n(x) = (N - n) \cdot (1 - z_1) \cdot m + \left(\frac{N - n}{n} \cdot z_1 + 1 \right) \cdot \sum_{j=1}^{n} x_j. \qquad (4.38)$$

The preposterior variance of the forecast error of this estimate, which is defined by

$$H_s = E\{[\tilde{S} - F^n(\tilde{x})]^2\}, \tag{4.39}$$

is not just the sum of $(N-n)$ terms H_x in (4.35), because the same value of θ applies throughout, and thus the error terms are correlated. However, it can be written in simplified form as

$$H_s = \operatorname{var}\left\{\alpha \sum_{j=1}^{n} \tilde{x}_j + \sum_{j=n+1}^{N} \tilde{x}_j\right\}, \qquad \alpha = -\frac{N-n}{n} \cdot z_1. \tag{4.40}$$

Therefore, we get

$$H_s = (\alpha^2 \cdot n + (N-n)) \cdot \operatorname{var}\{\tilde{x}_i\} + (\alpha^2 \cdot n \cdot (n-1) + (N-n) \cdot (N-n-1) + 2 \cdot \alpha \cdot n \cdot (N-n)) \cdot \operatorname{cov}\{\tilde{x}_i, \tilde{x}_{i \neq i'}\}$$

which gives, with (4.28) and (4.29), the final result:

$$H_s = N \cdot (N-n) \cdot E + (N-n)^2 \cdot \frac{E}{n + \frac{E}{D}}. \tag{4.41}$$

It should be noted that we would have obtained the same result if we had estimated S in the general form

$$\sum_{j=1}^{n} x_j + F_2^n(x); \quad F_2^n(x) = z_0 + \frac{z_1}{n} \cdot \sum_{j=1}^{n} x_j$$

where the constants z_0 and z_1 had to be determined by minimizing the preposterior variance

$$H_{s_2} = E\{[\tilde{S}_2 - F_2^n(\tilde{x})]^2\}.$$

The reason for this is that we limited ourselves to choosing an unbiased *linear* estimate for S.

Let us consider again two special cases. For $n = N$ we get, from (4.38) and (4.41),

$$F^N(x) = \sum_{j=1}^{N} x_j; \qquad H_s = 0.$$

That is, the true value of the total material content is known. For $n = 0$ we get

$$F^N(x) = N \cdot m; \qquad H_s = N \cdot E + N^2 \cdot D.$$

That is, the total material content is estimated only on the basis of the prior information. One also sees that D behaves like the variance of a *systematic error* that persists in all estimates because θ remains the same.

4.4.2 SEVERAL CLASSES OF MATERIALS: OPTIMIZATION OF VERIFICATION EFFORT

Let us consider now one inventory period and assume for simplicity that the physical inventories at the beginning and at the end of the inventory period are zero. The material flowing through the plant during this inventory period may be classified into R classes of material: R_1 input classes and $R-R_1$ output classes. Let x_{ij} be the true material content of the jth batch of the ith class, which will be measured by the inspection team in case this batch is selected for measurement. If the ith class is an input class, x_{ij} is positive; otherwise, it is negative.

We assume that the random sampling scheme of the inspection team is to select n_i out of the N_i batches of each class at the end of the inventory period. For example, one may imagine a chemical plant, where samples from all batches are drawn and stored and where only a fraction of these samples are analyzed at the end of the inventory period.

In the following we assume that for the inspection of the material flow during the inventory period under consideration there is only the amount C of inspection effort (given in man-hours or in monetary terms) available. Furthermore, it is assumed that the observation of one batch datum of the ith class requires the effort ϵ_i. Therefore, the question of how to distribute the effort among the different classes arises: in other words, how to choose the class sample sizes n_i such that the boundary condition

$$C \geqslant \sum_{i=1}^{R} \epsilon_i \cdot n_i \tag{4.42}$$

is met.

Before beginning the statistical analysis that provides us with the basis necessary for solving this optimization problem, we have to formulate the two hypotheses of the control authority:

- If no material has been lost or diverted (null hypothesis H_0) the material balance principle postulates that at the end of the inventory period the algebraic sum of all throughputs must be zero; in other words:

$$\sum_{i=1}^{R} \sum_{j=1}^{N_i} x_{ij} = 0. \tag{4.43}$$

- If the plant operator wants to divert material during the inventory period under consideration (alternative hypothesis H_1), Eq. (4.43) no longer holds.

Let us assume that the operator does not change the number of batches in each class simply by taking away some of the batches; rather, he diverts from r_i batches of the ith class the amount μ_i of material. Let us assume, furthermore, that the operator decides at the beginning of the inventory period whether he will divert any material. Finally, let us assume that the diversion takes place in the first R_1 classes *after* the inspection team's measurements and in the remaining $R-R_1$ classes *before*

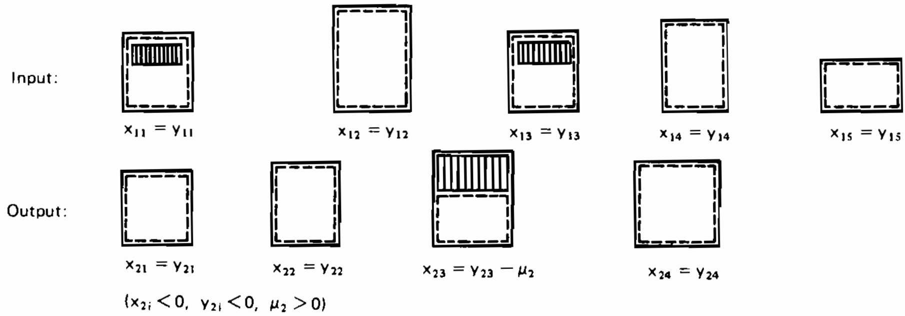

FIGURE 4.3 Material balance (zero opening and closing inventories). Null hypothesis (no diversion): $\sum_{i=1}^{5} x_{1i} + \sum_{i=1}^{4} x_{2i} = 0$. Alternative hypothesis (diversion: $\sum_{i=1}^{5} y_{1i} + \sum_{i=1}^{4} y_{2i} = 1 \cdot \mu_1 + 1 \cdot \mu_2$. *Solid box*: material contents x_{ij} measured by the inspection team in case of no diversion. *Dashed box*: material contents y_{ij} measured by the inspection team in case of diversion. *Shaded area:* diverted material.

the inspection team's measurements (the reason being that input batches are measured immediately after their arrival, and output batches immediately before their shipment). If A_i are the sets of batches from the ith class from which the operator diverts the amount μ_i of material, then we have, instead of Eq. (4.43), the following relation for the true material contents y_{ij} of the batches to be measured by the inspection team:

$$\sum_{i=1}^{R} \sum_{j=1}^{N_i} y_{ij} = \sum_{i=1}^{R} \mu_i \cdot r_i = :M \tag{4.44}$$

where

$$y_{ij} = \begin{cases} x_{ij} + \mu_i & \text{for all batches from } A_i, i = R_1 + 1, \ldots, R \\ x_{ij} & \text{otherwise} \end{cases}$$

and where x_{ij} is the material content of the jth batch of the ith class to be measured by the inspection team in case of no diversion. An illustration of this relation is given in Figure 4.3 for $R = 2, N_1 = 5, N_2 = 4$, and $r_1 = 1$.

As in the case of one class of material we now assume that the prior information available to the inspection team is the knowledge of the values of the following parameters:

$$m_i := E\{\tilde{x}_{ij}\} = E_\theta\{m_i(\tilde{\theta})\}, \quad j = 1, \ldots, N_i, \quad i = 1, \ldots, R \tag{4.45}$$

$$E_{ii'} := \operatorname{cov}\{\tilde{x}_{ij}, \tilde{x}_{i'j'}\} = \begin{cases} :E_i & \text{for } i = i' \\ 0 & \phantom{\text{for }} i \neq i' \end{cases} \tag{4.46}$$

$$D_{ii'}: = \operatorname{cov}\{m_i(\tilde{\theta}), m_{i'}(\tilde{\theta})\} = \begin{cases} :D_i & \text{for} \quad i = i' \\ 0 & \quad\quad\; i \neq i' \end{cases} \tag{4.47}$$

We have assumed here that the batch-to-batch variations of *different* classes do not depend on each other. It should be noted that this assumption seems to contradict Eq. (4.43), where such a dependence is given explicitly. However, this equation is a material balance equation that may be interpreted in such a way that the last output batch can contain only the amount of material that has been left (and that may be excluded from the random sampling procedure). This means that only the last batch depends on the foregoing batches; it does not imply a nonzero correlation between all batches of the R classes under consideration.

The statistical analysis of the control scheme outlined so far has to solve two problems. First, an estimate of the algebraic sum of all batches of all classes, which we again call material unaccounted for (*MUF*), must be made. Second, the significance test with respect to the two hypotheses formulated above must be constructed, and its efficiency determined. In the following discussion, we will limit the analysis to the first problem because the solution of the second question would require a mathematical effort far beyond the scope of this monograph. Instead, we refer the interested reader to the original literature.[10]

Under the assumption that no material is diverted or is missing, the *MUF* is estimated according to the considerations of the foregoing section with the help of the following formula

$$F_{MUF}(x_1, \ldots, x_R) = \sum_{i=1}^{R} F_i^n(x_i) = \sum_{i=1}^{R} \left((N_i - n_i) \cdot m_i \cdot (1 - z_{1i}) + \left[(N_i - n_i) \cdot \frac{z_{1i}}{n_i} + 1 \right] \cdot \sum_{j=1}^{n_i} x_{ij} \right), \tag{4.48}$$

where z_{1i} is, according to (4.34), given by

$$z_{1i} = \frac{n_i}{n_i + \dfrac{E_i}{D_i}} .$$

It is clear that this estimate is simply the sum of the class estimates (4.38) because we have neglected the correlation between batches of different classes, given θ. One can, however, derive the corresponding formula for nonvanishing correlations without major difficulties.

The preposterior variance of this estimate is given by

$$\begin{aligned} H_{MUF} &= E\{[M\tilde{U}F - F_{MUF}(\tilde{x}_1, \ldots, \tilde{x}_R)]^2\} \\ &= \sum_{i=1}^{R} \left[(N_i - n_i) \cdot E_i + (N_i - n_i)^2 \cdot \frac{E_i}{n_i + E_i D_i} \right] . \end{aligned} \tag{4.49}$$

This variance may be used as a criterion for the optimization of the distribution of a given control effort C; thus, the optimization problem reads, together with (4.43):

$$\min_{\substack{n_1,\ldots,n_R:\\ C \geq \sum_i \epsilon_i \cdot n_i}} \sum_{i=1}^{R} \left[(N_i - n_i) \cdot E_i + (N_i - n_i)^2 \cdot \frac{E_i}{n_i + \dfrac{E_i}{D_i}} \right].$$

This procedure may be considered an analogue of the one we used for the determination of the "best" estimate for the opening inventories in a sequence of inventory periods. This estimate was also formed on the basis of the null hypothesis that no material is missing or diverted. It was stated in Chapter 2 (p. 23) that in principle the opening inventory must be estimated in such a way that the probability of detection is optimized. In the same sense, one could argue here that the efficiency of the test should first be calculated, and then the estimate of the material unaccounted for determined such that the efficiency of the test is maximized. We did not proceed this way for the same reasons we gave in the opening inventory case: first, because of the complexity of the mathematical problem and, second, because it is difficult to formulate an alternative hypothesis when only random losses, and not diversion of material, have to be detected.

We will conclude the presentation of this credibility approach here without giving a numerical example, mainly because in the nuclear material case the control procedure has been established in a different way, as already indicated. However, in the last chapter we will come back to the scheme just described.

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5 International Nuclear Material Safeguards

As already indicated in the introduction, there is one example where material accountability has become of paramount importance: the international nuclear material safeguards system, set up in partial fulfillment of the nonproliferation treaty for nuclear weapons. This system went into operation some years ago, and the experience gained since that time could be of use in establishing other international control systems.

In the following sections, the development of the nuclear material safeguards system is reviewed in order to demonstrate why material accountability became the system's primary tool. Next, the nuclear fuel cycle and the various material measurement problems inherent in it will be described. Finally, a numerical example is presented.

5.1 NONPROLIFERATION TREATY AND IAEA NUCLEAR MATERIAL SAFEGUARDS SYSTEM

Measures for safeguarding the nuclear material in the nuclear fuel cycle industry are of special importance for three reasons:

- The material is very valuable (plutonium and highly enriched uranium are worth twice as much as an equivalent amount of gold).
- Furthermore, the material is extremely toxic (the lethal body burden of plutonium is about 10^{-6} g).
- Finally, and most important, the material can be used for the construction of nuclear weapons.

It was for all of these reasons that, from the very outset, safeguards were more important for nuclear material than for other materials (e.g., gold), where only one of the reasons given is valid. However, it was the last of the three reasons that caused the international interest in this special safeguards problem; this interest finally led to the Treaty on the Non-Proliferation of Nuclear Weapons.[1]

On March 5, 1970, the nonproliferation treaty took effect after ratification by 43 member states of the United Nations. This treaty, which had been drafted by the United Kingdom, the Soviet Union, and the United States, and signed by these three states on July 1, 1968, was meant to prevent the acquisition of nuclear weapons by states other than those that already possessed them. The treaty thus implies international safeguards that guarantee the timely detection of any diversion of nuclear material from the peaceful nuclear fuel cycle. The authority responsible for the implementation of these safeguards measures is the International Atomic Energy Agency (IAEA) in Vienna.

Although at the time the treaty was ratified, the United States, for example, had at least 25 years of experience with nuclear material safeguards, it soon turned out that the *international* control of *national* fuel cycles would cause completely new problems. For this reason, research and development work was initiated in several nations with the purpose of developing a practicable and acceptable international safeguards system. This work was started in the signatory states (even though their own nuclear fuel cycles were not subject to these safeguards) as well as in non-nuclear-weapons states with large peaceful nuclear fuel cycles (e.g., Japan and the Federal Republic of Germany). The international effort was coordinated and stimulated by the IAEA through consultants,[2-4] workshops, panels, and symposia.[5-13] At the Geneva Conference in 1971[14] an evaluation and overview of the work done so far was given, and it is illuminating to look at the structure of the whole system as it was conceived at that time (Figure 5.1): even though the detailed tools and procedures had just been fixed and therefore no rigorous analysis of the system was possible, all relevant aspects of the problem had been recognized and described very clearly.

An important step toward a safeguards system that could be accepted by all states was made when the Safeguards Committee, which had been created by the Board of Governors of the IAEA and which negotiated from July 1970 until February 1971, succeeded in establishing a model agreement[15] that was meant to serve as an example for all the safeguards agreements between IAEA and the individual states.

In the model agreement, it was laid down that *material accountability* is used as the safeguards measure of primary importance, with *containment* and *surveillance* as complementary measures. Here, material accountability has to be understood exactly as it has been defined in Chapter 2. For this purpose it is necessary to have well-defined areas for the physical inventory and channels for the flow of the nuclear material; otherwise, a measurement of the material would be impossible. This was the basis of the introduction of the concept of *strategic points* into the Non-Proliferation Treaty (see IAEA,[15] article 116): the safeguards measures must be

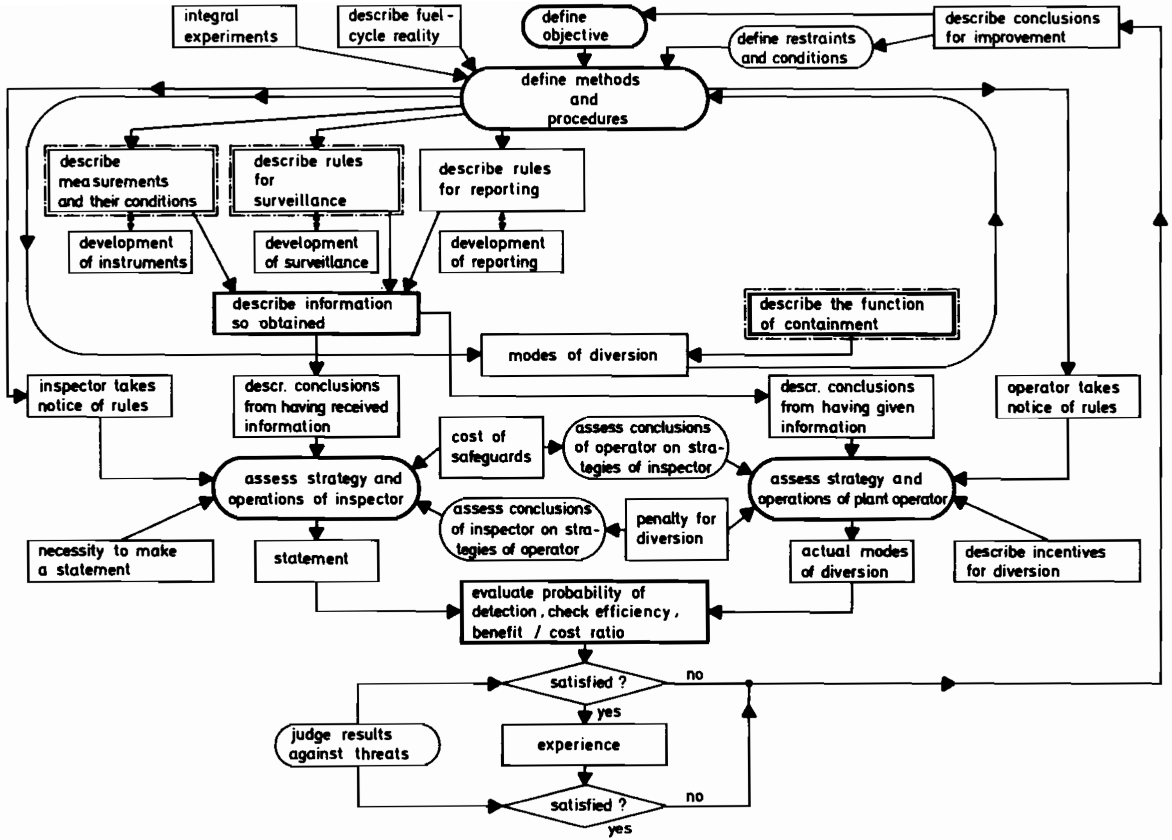

FIGURE 5.1 Structure of the model for design of safeguards. (After Häfele.[14])

concentrated at as few places as possible. Containment measures are understood to be physical security measures such as concrete walls, special devices for the transport of the material (birdcages), seals, and alarm installations. Surveillance measures are understood to be those measures that are taken when, because of the special situation, direct inspection is the only possibility (e.g., during reactor core loading).

It should be emphasized that these measures as well as their relative importance were agreed upon because of the special boundary conditions of the international safeguards problem. The major boundary conditions that allowed the system to be accepted on an international level may be summarized by the key words *rational, objective*, and *formalized.* This means that not the subjective impression of an individual inspector but only a formalized system that provides quantitative statements is accepted as a basis for a judgment of whether a state behaved legally in the sense of the nonproliferation treaty. It was for this special purpose that material accountability became the most important measure, because unlike either containment or surveillance measures, it provides quantitative statements by its very nature.

Furthermore, the procedures for the performance of nuclear material safeguards were laid down in the model agreement: the operator of a nuclear plant generates all source data for the establishment of a material balance and reports these data to a national (or regional) authority, which reports them to the international safeguards authority; the international authority verifies these reported data with the help of independent measurements made on a random sampling basis. If there are no significant differences between the operator's reported data and the international authority's own findings, then all the operator's data are accepted and the material balance is closed on the basis of the operator's data alone.

Finally, it should be mentioned that there are model agreement statements about the *maximum routine inspection effort.* In order to make meaningful statements about this effort, the importance of the nuclear material (in the sense of the nonproliferation treaty) processed in the different plants of the nuclear fuel cycle had to be defined. Thus, the concept of *effective kilogram* was introduced (in fact, this concept was in use before the nonproliferation treaty, in connection with bilateral safeguards agreements made under the auspices of the IAEA). According to this concept, 1 kg of plutonium corresponds to one effective kilogram, whereas 1 kg of uranium with an enrichment of 0.01 and above corresponds to a quantity in effective kilograms that is obtained by taking the square of the enrichment. The maximum routine inspection effort, given in inspection man-hours spent in a nuclear plant, is determined on the basis of the annual throughput or inventory of nuclear material expressed in effective kilograms.

Before concluding the description of the IAEA safeguards system, some words should be said about the development of international safeguards since 1971. The first problem to be solved was that of the relation between the national and regional safeguards authorities on one hand and the international authority on the other. Here, EURATOM, the safeguards authority of the European Community, posed a special problem: as it is itself an international authority, the question of the relation of two international authorities had to be treated.[16] Second, more attention has been paid to the *physical protection* of nuclear material and nuclear plants; even though this problem is the responsibility of the individual states, the IAEA developed some recommendations. We will come back to this point at the end of the chapter. Finally, the question of stopping further proliferation of nuclear weapons by means other than international safeguards has received attention. The ideas developed in this connection center on "limitation of the export of sensitive nuclear plants" and establishment of international "regional nuclear centers," where all nuclear activities related to power generation are concentrated.[17] However, as these ideas are still under discussion, and as they are not directly connected with our subject, we will not go into detail here.

Let us come back to the IAEA international safeguards system as established in 1971. The efficiency of this system was estimated only very roughly when the model agreement was negotiated. Clearly, such an estimate could be begun only after the principles, the tools, and the procedures had been laid down precisely. The first detailed analyses were finished several years later.[18] Before we discuss them, however, some ideas about the nuclear fuel cycle and the problem of the measurement of nuclear material should be clarified.

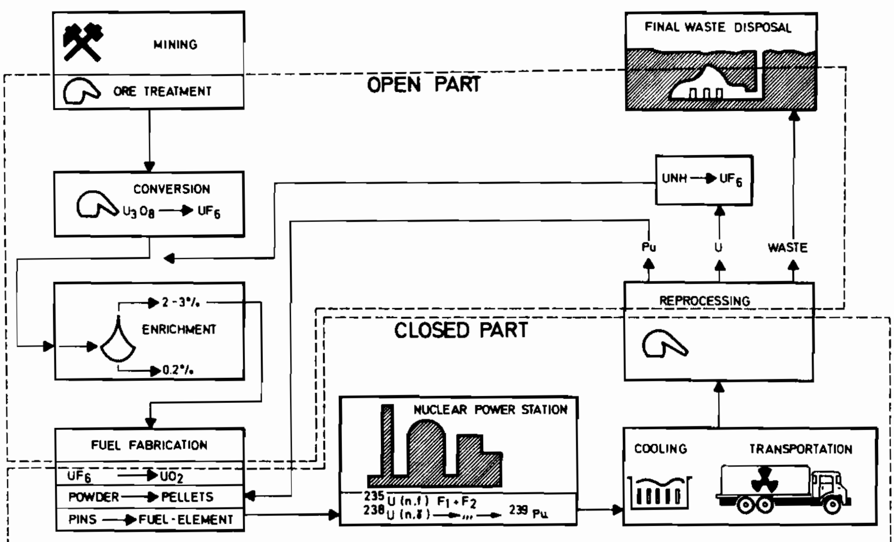

FIGURE 5.2 The nuclear fuel cycle, based on the light water reactor.

5.2 NUCLEAR FUEL CYCLE AND NUCLEAR MATERIAL MEASUREMENT

The nuclear fuel cycle industry is composed of all those industrial activities that are connected with the production of energy through nuclear fission processes (other activities based on nuclear fission, like radiomedicine, are not relevant here). The main activities of a nuclear fuel cycle based on the light water reactor are represented in Figure 5.2.

In the uranium mine, natural uranium ore is extracted; 0.7% of the uranium in this ore is found as ^{235}U and 99.3% as ^{238}U. Preliminary processing produces uranate (yellow cake), which is transported to the conversion plant. Here, it is converted into gaseous uranium hexafluoride (UF_6), which is sent to the enrichment plant, where the uranium is enriched, raising the ^{235}U concentration to 2–3%. The uranium hexafluoride containing the enriched uranium is sent to the fuel fabrication plant, where the uranium fluoride is first converted into uranium oxide (UO_2). The uranium oxide powder is pressed into pellets; these pellets are sintered and loaded into fuel pins that are finally fitted to fuel elements. The fuel elements are brought to the nuclear power station, where they constitute the reactor core. The fuel elements remain at the station for approximately 3 years, during which the fuel is consumed, i.e., the ^{235}U isotope is split by neutron-induced nuclear fission, yielding fission products and heat. In addition, transuranium elements are produced, the most important among these being *plutonium* (mainly ^{239}Pu, but also other isotopes).

The spent fuel elements that are removed from the reactor core are stored for some months at the reactor site to allow the short-lived fission products to decay, and then they are transported in heavily shielded transport casks to the chemical reprocessing plant. There, the remaining uranium is separated from the plutonium, the other transuranium elements, and the fission products. The uranium is sent back to the enrichment plant, and the plutonium is either stored (for later use in fast breeder reactors) or sent to the fuel fabrication plant where it is added to the uranium, thus serving as additional fuel (this procedure is called thermal recycling of plutonium). The remaining transuranium elements and the fission products constitute the highly radioactive waste, which after some treatment is stored in the final waste disposal (i.e., deposited in such suitable geological formations as salt caverns).

As indicated in Figure 5.2, the nuclear fuel cycle is divided into an *open part*, where the nuclear material is handled in an accessible form, and a *closed part*, where the nuclear material is handled in a form that is not directly accessible. This division is important in view of the various methods for measurement of the nuclear material: in the open part, so-called direct methods may be used, where the material is measured in the usual way by determining gross and net volumes or weights, sampling, and performing chemical analyses, but this is not possible in the closed part. Here, indirect or nondestructive methods have to be used to measure the nuclear material contained in the fuel elements, which means that the emitted neutrons, rays, or the heat of decay are used for the measurement.

It should be noted that both types of measurements, direct and indirect, pose difficult technical problems. Therefore, it has become crucially important that the safeguards material accountability evaluate the errors connected with these measurements. In recent years there has been an ongoing effort devoted to estimating – with the help of interlaboratory tests – the variances of all random and systematic error components of the measurements; an indication of the size of this effort may be obtained from Kraemer and Beyrich[19] and Beyrich and Drosselmeyer.[20] An account of all measurement problems in the nuclear industry is given by Jones[21] and, from the statistical point of view, by Jaech.[22]

Without going into all the details of the measurement of the flow of nuclear material, one specific case should be considered: the entry of material into the chemical reprocessing plant. Here, the uranium and plutonium contents of the spent fuel elements have to be measured. This is done, after the separation of the fuel from the cladding material of the fuel pins and the dissolution of the fuel, in the "accountability tank." The measurement at this place plays a unique role in the whole fuel cycle for two reasons: it is the first time that the plutonium generated in the reactor is measured and, furthermore, the measurement itself is the most difficult of all measurements in the whole fuel cycle because of the presence of all the highly radioactive fission products. For these reasons, special attention has been given to this measurement problem.[23,24] Thus, the entry into the chemical reprocessing plant deserves the name "strategic point" in the sense of the nonproliferation treaty.

The measurement of the physical inventory poses a different problem in each plant of the nuclear fuel cycle. In the conversion and fabrication plants, physical inventories are taken simply by stopping the production process and measuring all the material available. If the plants are large, then one will not stop production in the whole plant, but only in parts. In any case, one will collect the material in some parts of the plant in order to facilitate the taking of inventory. In the case of the chemical reprocessing plant such a procedure is not possible since the material is not accessible because of its radioactivity. Here, either a washout is performed (i.e., all of the material is taken out of the plant so that the physical inventory is zero), or the material contained in the plant is collected in several tanks and estimated very roughly. Another method that is still being developed[25] and that does not require any interruption of the reprocessing procedure is the in-process inventory determination method, which is based on the differences of the isotopic composition of the entering irradiated fuel. Finally, in the enrichment plants, none of the methods listed can be applied directly because the enrichment process cannot be stopped, nor can isotopic differences be used, if the entering material is natural uranium. Thus, one can only *estimate* the amount of material in the process itself; however, this amount is only a very small fraction of the material circulating in the whole plant.

It should be noted that all the production and processing procedures in the nuclear fuel cycle, as well as the measurement techniques (especially the indirect measurement methods), are still being developed. The data presented in the papers cited, as well as in the following numerical example, are data as of today – tomorrow they may have to be revised.

5.3 NUMERICAL EXAMPLE

Among the many case studies that have been done in recent years[26] is an analysis of the accountability problem in a chemical reprocessing plant; it has all the characteristic features of the general material accountability problem as well as those of the specific safeguards problem. The data used here are from Avenhaus *et al.*[27] and Avenhaus and Nakicenovic.[28]

5.3.1 BASIC DATA

The West Valley plant of Nuclear Fuel Services (NFS) is located in the United States about 30 miles south of Buffalo, New York. The plant reprocesses irradiated nuclear fuel elements by means of the PUREX solvent extraction process. The capacity of the plant is 300 tons of low-enriched uranium per year (in the form of low-enriched UO_2 or uranium metal). This means 1,000 kg of low-enriched uranium per day, assuming 300 working days per year. The base-line process is designed for low-enriched UO_2 in stainless-steel–zirconium alloy tubes; with modification of the front-end treatment only, however, natural uranium fuel clad in aluminum can be processed.* The processing procedure (Figure 5.3) will not be described in detail here, but some remarks will be made about input, product, losses, and inventory.

The element to be processed is first removed from the storage pool, placed in a fixture on the inspection table, and marked for sawing. It is then transferred to the saw table; the scrap metal cut off is taken in scrap buckets to the general purpose cell (GPC) for eventual burial. The fuel bundle is pushed by a ram out of its casing into a shear feed magazine, and the magazine is transferred to the shear. The chopped fuel is discharged through a chute into baskets in the GPC, and these are loaded into the dissolver in the GPC by a crane. The amounts of fuel and acid charged to the dissolver are adjusted to yield a ^{235}U concentration that is approximately 50% of the critical concentration. Finally, the dissolver solution and rinses are collected in the accountability and feed-adjustment tank (3D–1), which is equipped with heating and cooling coils, condenser, air sparger, and liquid-level and specific-gravity instruments.

The low-enriched uranium product is loaded into a tank trailer and shipped in quantities of about 4.2 metric tons of uranium per shipment. The recovered plutonium product is stored in geometrically safe tanks from which it is loaded into 10-liter bottles that are then packed in birdcages. Each 10-liter bottle contains 2–3 kg of plutonium. Shipments of plutonium are scheduled either when there are 20 bottles of packaged product in storage or at the end of each campaign.

* The plant is under reconstruction; these data will no longer be valid when operations resume.

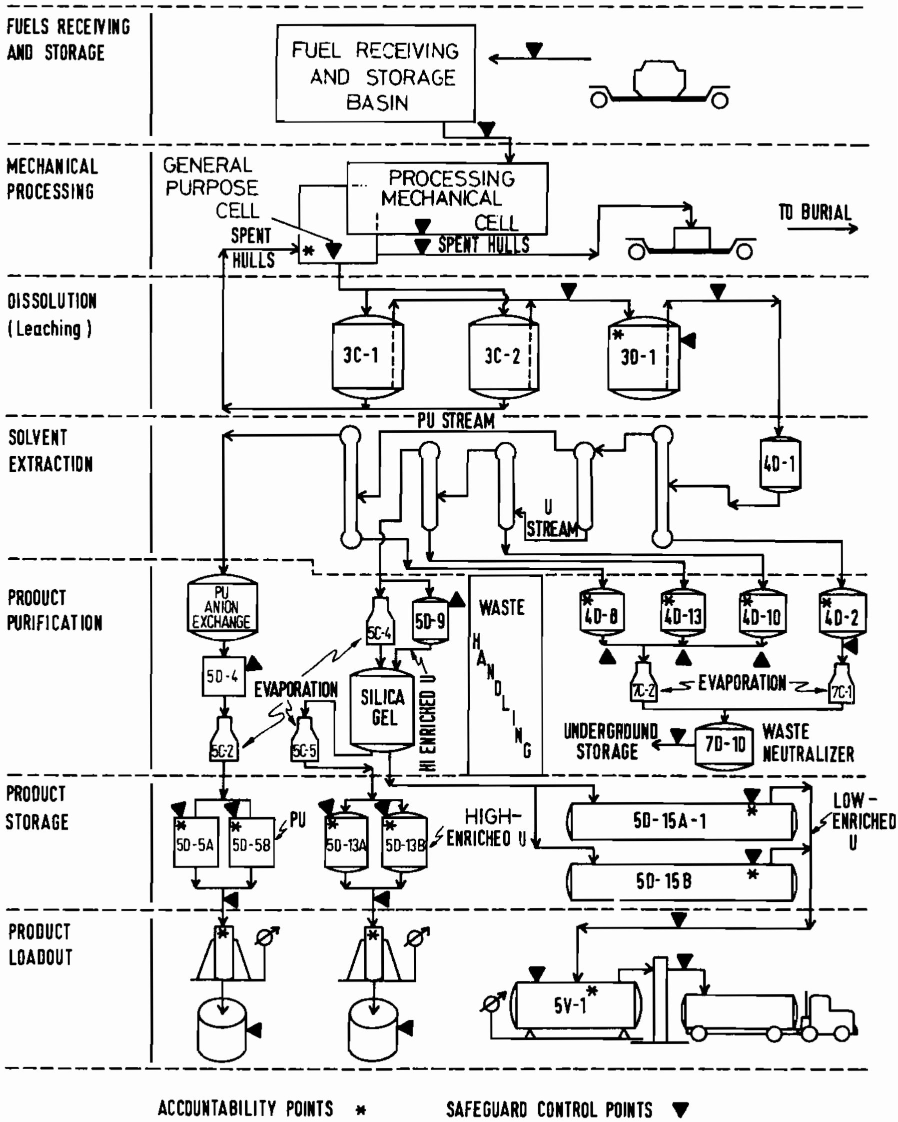

FIGURE 5.3 Reprocessing procedure at NFS, major streams. See text for explanation. (Adapted from Avenhaus *et al.*[27])

Two kinds of loss are considered: liquid waste and hull losses. Losses of solid waste from the laboratories are not considered to be important, and the amount of material that is recycled in the course of acid recycling can also be neglected.

The plutonium inventory of the plant during a campaign can be estimated with an accuracy of about 10% by means of rough estimates of the plutonium content

TABLE 5.1 NFS Campaign and Batch Data for the Reference Time T (6 months) for the Plutonium Throughput

Pu throughput (kg)	875
Liquid waste (% of input)	0.9
Hull losses (% of input)	0.1
Number of campaigns	5
Number of working days	125
Input	
Input/campaign (kg)	175
Number of batches/campaign	25
Batch volume (liters)	4,000
Pu content/batch (kg)	7
Batch-to-batch variation (%)	10
Product	
Number of batches/campaign	76
Weight of batch (kg)	15
Pu content/batch (kg)	2.28
Batch-to-batch variation (%)	10
Liquid waste	
Number of batches/campaign	90
Batch volume (liters)	5,000
Pu content/batch (kg)	0.019
Batch-to-batch variation (%)	10

SOURCE: Avenhaus *et al.*[27]

of the different tanks. The in-process inventory determination method does not work very well in the NFS plant because the plutonium product tanks are too large. The only accurate method of determining the inventory of the plant is to perform a flushout after the end of a campaign. Representative data from actual NFS campaigns are given in Table 5.1.

Plutonium is the only material in the process that is of strategic value; for this reason, we restrict our considerations to this element. The measurement system needed for complete plutonium accountability may be described as follows: input is measured in the input accountability tank (3D–1) by determining the volume (by the dip-tube system) and the concentration (through isotopic dilution analysis) of a sample. The samples for plutonium product specification and plutonium concentration analysis (amperometric titration and isotopic analysis) are drawn from the product storage tank (5D–5A or 5D–5B). The product loadout quantities are based on the net product solution weight and the reported assay values. Liquid waste is collected in the central waste tank (7D–10). Measurement is based on volume determination (level indicator) and analysis of a sample (TTA-extraction α counting) drawn from the central waste tank. The plutonium content in the hulls is determined by weighing the baskets (gross and tare), taking samples of end and middle pieces, and analyzing the samples.

TABLE 5.2 Pu Measurement System for NFS Plant

		Standard Deviation per Single Measurement (%)		Effort per Single Measurement	
Class	Measurement	Random	Systematic	Man-hours	Cost ($ U.S.)
Input	Volume determination (dip-tube system)	0.35	0.1	0.7	–
	Sampling	1	–	1.5	–
	Concentration determination (isotopic dilution analysis)	0.6	0.3	–	400
Product	Weighing	0.02	–	3	–
	Sampling	0.5	–	2.25	–
	Concentration determination (amperometric titration and isotopic analysis)	0.4	0.3	–	200
Liquid waste	Volume determination (level indicator)	5	5	0.1	–
	Sampling	50	–	0.5	–
	Concentration determination (TTA-extraction α counting)	15	10	2	40
Physical inventory	Washout	0.577 [kg]		–	

SOURCE: Avenhaus *et al.*[27]

In Table 5.2, the plutonium measurement system is summarized, and the effort required to perform the measurements, expressed in man-hours and money, is given. Obviously much more effort is devoted to safeguards in the plant than is indicated in Table 5.2. However, as this additional effort represents a kind of baseload effort that cannot be reduced and that is not subject to any optimization procedure, we will not report on it here; for further information, the reader should refer to the original work.[27]

5.3.2 MATERIAL ACCOUNTABILITY PROCEDURES

According to Chapter 2, the establishment of the material balance includes the establishment of the

Initial physical inventory I_0
Book inventory $B(I_0$ + input − product − waste)
Closing physical inventory I_1

In the following, we will discuss the measurement of these quantities as well as the verification of the relevant data in detail. We will then determine the efficiency of the combined system for *one inventory period* – we do not consider the problems connected with a sequence of inventory periods, which have been analyzed in Chapter 2.

Physical Inventories

We assume that the true values (in kilograms) of the physical inventories I_0 and I_1 after a flushout are

$$E(I_0) = E(I_1) = 1. \tag{5.1a}$$

We assume further that the variation of these inventories is of the same order of magnitude:

$$0 \leqslant I_{0,1} \leqslant 2. \tag{5.1b}$$

If we assume, in addition, that the physical inventories are equally distributed random variables with a range given by (5.1b), we obtain the following variances of the physical inventories after a flushout:

$$\mathrm{var}(I_0) = \mathrm{var}(I_1) = 0.333. \tag{5.1c}$$

Input

One measurement of the plutonium content g_{1j} (in grams) of the jth input batch consists of

- Volume determination v_{1j} (in liters)
- Drawing of a sample
- Concentration measurement c_{1j} of the sample (in grams of Pu per liter)

Therefore, *in the case of no data falsification* the operator reports the total plutonium input during one inventory period; this input has been calculated on the basis of the following input batch data:

$$g_{1j} = v_{1j} \cdot c_{1j}, \qquad j = 1, \ldots, N_1. \tag{5.2a}$$

The results of the single measurements can be written in the following form:

$$v_{1j} = E(v_1) + e_{1j}^{vr} + e_1^{vs} \tag{5.2b}$$

$$c_{1j} = E(c_1) + e_{1j}^{cr} + e_1^{cs} + d_{1j}^{0c} \tag{5.2c}$$

where $E(v_1)$ and $E(c_1)$ are the true values of the quantities to be measured; e_{1j}^{vr} and e_1^{vs} are the random and systematic errors of the volume determination; e_{1j}^{cr} and e_1^{cs} are the random and systematic errors of the concentration determination; and d_{1j}^{0c} is the sampling error in the operator's sample. Here, it has been assumed for simplicity that the true values for all batches are the same. The generalized expressions can be given easily.

The variances of these errors are abbreviated as follows:

$$\begin{aligned} \operatorname{var}(e_{1j}^{vr}) &= :\sigma_{vr1}^2 \\ \operatorname{var}(e_{1j}^{vs}) &= :\sigma_{vs1}^2 \\ \operatorname{var}(e_{1j}^{cr}) &= :\sigma_{cr1}^2 \\ \operatorname{var}(e_{1j}^{cs}) &= :\sigma_{cs}^2 \\ \operatorname{var}(d_{1j}^{0c}) &= :\sigma_{s1}^2 . \end{aligned} \tag{5.3}$$

If one assumes that one calibration per inventory period is performed both for the volume and for the concentration measurement, and if one neglects error terms of the second order, then the total input reported by the operator is given by

$$\begin{aligned} \text{Input} = N_1 \cdot E(v_1) \cdot E(c_1) + \sum_{j=1}^{N_1} [E(v_1) \cdot (e_{1j}^{cr} + e_1^{cs} + d_{1j}^{0c}) \\ + E(c_1) \cdot (e_{1j}^{vr} + e_{1j}^{vs})] \end{aligned} \tag{5.4}$$

and the variance of the total reported input is given by the following formula, if one takes into account that random and systematic errors are propagated differently:

$$\begin{aligned} \operatorname{var}(\text{Input}) &= E^2(v_1) \cdot (N_1 \cdot \sigma_{cr1}^2 + N_1^2 \cdot \sigma_{s1}^2 + N_1 \cdot \sigma_{cs1}^2) \\ &\quad + E^2(c_1) \cdot (N_1 \cdot \sigma_{vr1}^2 + N_1^2 \cdot \sigma_{vs1}^2). \end{aligned} \tag{5.5}$$

Waste

For waste the situation is the same as in the case of input except that all the characteristic quantities have different values. Thus, for the variance of the total reported waste during the inventory period – characterized by the index 3 – we get the following formula:

$$\begin{aligned} \operatorname{var}(\text{Waste}) &= E^2(v_3) \cdot (N_3 \cdot \sigma_{cr3}^2 + N_3^2 \cdot \sigma_{s3}^2 + N_3 \cdot \sigma_{cs3}^2) \\ &\quad + E^2(c_3) \cdot (N_3 \cdot \sigma_{vr3}^2 + N_3^2 \cdot \sigma_{vs3}^2), \end{aligned} \tag{5.6}$$

which needs no further explanation.

Product

The situation in the case of the product is different, since not the volume but the total weight of the batch is determined by taking the gross and the tare weight of the batch and its container; thus, the systematic errors of these measurements are cancelled. Therefore, one has for the material content g_{2j} (in grams of plutonium) of the jth product batch

$$g_{2j} = v_{2j} \cdot c_{2j}, \qquad j = 1, \ldots, N_2 \tag{5.7a}$$

$$v_{2j} = E(v_2) + e_{2j}^{vg} + e_{2j}^{vt} \qquad \text{[kg solution]} \tag{5.7b}$$

$$c_{2j} = E(c_2) + e_{2j}^{cr} + e_2^{cs} + d_{2j}^{0c} \left[\frac{\text{g Pu}}{\text{kg sol}}\right] \tag{5.7c}$$

where $E(v_2)$ and $E(c_2)$ are the true values of the measurements, e_{2j}^{vg} and e_2^{vt} are the random errors of the gross and tare weights of the weighing procedure, e_{2j}^{cr} and e_2^{cs} are the random and systematic errors of the concentration measurement, and d_{2j}^{0c} is the sampling error of the concentration measurement.

The variances of these errors are abbreviated in the same way as in the case of the input:

$$\begin{aligned} \text{var}(e_{2j}^{vg}) &= \text{var}(e_{2j}^{vt}) = :\sigma_{v2}^2 \\ \text{var}(e_{2j}^{cr}) &= :\sigma_{cr2}^2 \\ \text{var}(e_{2j}^{cs}) &= :\sigma_{cs2}^2 \\ \text{var}(d_{2j}^{0c}) &= :\sigma_{s2}^2 \end{aligned} \tag{5.8}$$

Therefore, the variance of the total product reported during the inventory period is given by the following formula:

$$\begin{aligned} \text{var (Product)} &= \text{var}\left(\sum_{j=1}^{N_2} [E(v_2) \cdot (e_{2j}^{cr} + e_2^{cs} + d_{2j}^{0c}) + E(c_2) \cdot (e_{2j}^{vg} + e_{2j}^{vt})]\right) \\ &= E^2(v_2) \cdot (N_2 \cdot \sigma_{cr2}^2 + N_2^2 \cdot \sigma_{s2}^2 + N_2 \cdot \sigma_{cs2}^2) \\ &\quad + E^2(c_2) \cdot 2N_2 \cdot \sigma_{v2}^2 . \end{aligned} \tag{5.9}$$

Material Unaccounted For

According to Eq. (2.2), material unaccounted for is defined as

$$MUF: = I_0 + \text{Input} - \text{Product} - \text{Waste} - I_1 . \tag{5.10}$$

If the operator does not divert any material (null hypothesis H_0), the expected value of MUF is zero (see Eq. 2.8); in case of diversion of the amount M, the expected value of MUF is M (see Eq. 2.12). The variance of MUF is, in both cases, given by

$$\text{var}(MUF) = :\sigma_{MUF}^2 = 2\,\text{var}(I_0) + \text{var(Input)} + \text{var(Product)} + \text{var(Waste)}, \tag{5.11}$$

where the single expressions are given by Eqs. (5.1c), (5.5), (5.6), and (5.9). Numerical values for the square roots of all variances (relative standard deviations) are listed in Table 5.3. The variance of the material unaccounted for is given in Table 5.3, according to which the standard deviation of MUF amounts to 4.1 kg of plutonium in one inventory period (half a year). Using Eq. (2.15) we conclude that for a false alarm rate of $\alpha = 0.05$ about 13.5 kg of plutonium must be lost or diverted to ensure a probability of detection of $1 - \beta = 0.95$.

TABLE 5.3 Variance of the Material Unaccounted for (*MUF*) for One Inventory Period

Class	Variance (kg^2)	Standard Deviation (kg)
Input	8.564 (Eq. 5.5)	2.926
Product	6.837 (Eq. 5.9)	2.615
Waste	0.958 (Eq. 5.6)	0.979
Inventory	0.333 (Eq. 5.1c)	0.577
MUF	17.026 (Eq. (5.11)	4.126

SOURCE: Avenhaus *et al.*[27]

5.3.3 VERIFICATION PROCEDURES

It is assumed that the inspector watches all the measurements necessary for taking the physical inventory and that he need not verify the volume and weight determinations or the sampling procedures, as they are automated and therefore, tamper-proof. It is further assumed that the inspector verifies the concentration determinations on the basis of a random sampling scheme and that both the operator and the inspector use the same measurement methods.

If the operator wants to divert material by means of data falsification, he proceeds as follows. He dilutes r_1 of his input samples in order to simulate a smaller amount of input. In this way he gains material that he can divert. Therefore, instead of (5.2c) we have

$$\begin{aligned} c_{1j} &= E(c_1) + e_{1j}^{cr} + e_1^{cs} + d_{1j}^{0c} - \mu_1^c \qquad \text{for } j = 1, \ldots, r_1 \\ c_{1j} &= E(c_1) + e_{1j}^{cr} + e_1^{cs} + d_{1j}^{0c} \qquad \text{for } j = 1, \ldots, N_1 - r_1. \qquad (5.12) \end{aligned}$$

The operator reports, however, $c_{ij} + \mu_1$, for $j = 1, \ldots, r_1$ in order to maintain the material balance.

He proceeds in the same way for the product and the waste, except that in these two cases he concentrates the samples. (Clearly, the effects will be the same if the operator does not dilute or concentrate samples but simply reports wrong data). Therefore, if $c_{ij}^{0,I}$, $i = 1, 2, 3$, $j = 1, \ldots, n_i$, are the results of the concentration measurements *reported* by the operator and those of the inspection team, then the D-statistic according to Eq. (3.9) is given by

$$D = \frac{N_1}{n_1} \cdot \sum_{j=1}^{n_1} (c_{1j}^I - c_{1j}^0) + \frac{N_2}{n_2} \cdot \sum_{j=1}^{n_2} (c_{2j}^0 - c_{2j}^I) + \frac{N_3}{n_3} \cdot \sum_{j=1}^{n_3} (c_{3j}^0 - c_{3j}^I). \quad (5.13)$$

The reason for this special choice of signs has already been explained; it will be important for the correlation considerations below.

The expected values of the D-statistic under the null and alternative hypotheses are given by

$$E(D/H_0) = 0 \qquad (5.14a)$$

$$E(D/H_1) = \sum_i \mu_i^c \cdot r_i, \tag{5.14b}$$

where μ_i^c is the amount by which the concentration of a falsified batch of class i is falsified. The amount of material that can be diverted in this way is not given simply by the amount $\sum_i \mu_i^c \cdot r_i$, but rather by

$$M_2 = \sum_{i=1}^{3} E(v_i) \cdot \mu_i^c \cdot r_i = \sum_{i=1}^{3} \mu_i \cdot r_i, \tag{5.15}$$

where μ_i is defined by

$$\mu_i := E(v_i) \cdot \mu_i^c. \tag{5.16}$$

As one can see from Eqs. (5.14) and (5.15), M_2 and $E(D/H_1)$ are not identical. Therefore, the optimization procedure sketched in Chapter 3 must be modified, as a different boundary condition, Eq. (5.15), has to be used; instead of Eqs. (3.42) we now get the following optimal sample sizes n_i^0 of the inspection team and r_i^0 of the operator [for the reasons given at the end of section 3.2 (p. 58) we limit our consideration to these simple formulae] :

$$n_i^0 = \frac{C}{\sum_j \frac{N_j}{E^2(v_j)} \cdot \mu_j \cdot \epsilon_j} \cdot \mu_i \cdot \frac{N_i}{E^2(v_i)}, \tag{5.17a}$$

$$r_i^0 = \frac{M_2}{\sum_j \frac{N_j}{E^2(v_j)} \cdot \mu_j \cdot \epsilon_j} \cdot \epsilon_i \cdot \frac{N_i}{E^2(v_i)}, \qquad i = 1, 2, 3. \tag{5.17b}$$

Under these conditions we get for the expected value of the D-statistic under the alternative hypothesis H_1

$$E(D/H_1) = \sum_i \frac{\mu_i}{E(v_i)} \cdot r_i^0,$$

or, if we use the expression for the optimum sample size r_i^0, Eq. (5.17b),

TABLE 5.4 Input Data for the Concentration Measurement Verification

	Class i	Total Number of Batches N_i	Batch Size $E(v_i)$	Pu Content per Batch (kg)	Effort ϵ_i per Verification ($ U.S.)	Amount μ_i per Batch to be Diverted (kg)
Input	1	125	4,000 liters	7	400	0.1470
Product	2	380	15 kg	2.28	200	0.0342
Waste	3	450	5,000 liters	0.019	40	0.0114

SOURCE: Avenhaus *et al.*[27]

TABLE 5.5 Inspector's Optimal Sample Sizes

	C (% of Max. Effort)								
Sample	100	80	60	50	30	20	10	5	1
n_1^0	125	96	26	1	1	1	1	1	1
n_2^0	380	380	380	358	214	142	70	34	5
n_3^0	450	17	5	1	1	1	1	1	1

SOURCE: Avenhaus *et al.*[27]

TABLE 5.6 Operator's Optimal Sample Sizes

	Amount M_2 To Be Diverted (kg)											
Sample	0.1	0.5	1	2	3	4	5	6	7	8	9	10
r_1^0	0	1	1	1	1	1	1	1	1	1	1	1
r_2^0	3	10	25	54	83	112	142	171	200	229	258	288
r_3^0	1	1	1	1	1	1	1	1	1	1	1	1

SOURCE: Avenhaus *et al.*[27]

$$E(D/H_1) = \frac{\sum_i \frac{N_i}{E^2(v_i)} \cdot \epsilon_i \cdot \mu_i}{\sum_i \frac{N_i}{E^3(v_i)} \cdot \epsilon_i \cdot \mu_i} \cdot M_2. \tag{5.18}$$

The basic data for the verification scheme are collected in Table 5.4. Because of the large difference of the amounts μ_i^c by which the data have to be falsified, practically all of the effort must go to the product stream. This does not mean, however, that the input and waste stream data need not be verified at all. The following procedure is proposed. If only a small effort can be expended, only one batch is verified in the input and one in the waste stream; the rest goes to the product stream. If resources available are more than enough for verification of all product batches, then the remaining effort must be distributed between input and waste according to formula (5.17).

The optimal sample sizes n_i^0 are given in Table 5.5 as a function of the total effort C. The optimal numbers of falsified batches r_i^0 are given in Table 5.6 as a function of the total amount M_2 assumed to be diverted.

5.3.4 DETERMINING THE CORRELATION BETWEEN DATA VERIFICATION AND MATERIAL BALANCE ESTABLISHMENT

It was previously mentioned that the random variables *MUF* and *D* are stochastically dependent because the data of the operator are used both for data verification and for material balance establishment. In case of the null hypothesis H_0, we get, because $E(MUF) = E(D/H_0) = 0$ and by using Eqs. (5.4), (5.5), (5.6), and (5.9), the following expression for the covariance between MUF and D under the null hypothesis H_0

$$
\begin{aligned}
\operatorname{cov}(MUF, D/H_0) &= E([MUF - E(MUF)] \cdot [D - E(D)]/H_0) \\
&= E(MUF \cdot D/H_0) \\
&= E\Bigg(\Bigg(N_1 \cdot E(v_1) \cdot E(c_1) - N_2 \cdot E(v_2) \cdot E(c_2) - N_3 \cdot E(v_3) \cdot E(c_3) \\
&\quad + \sum_{j=1}^{N_1} [E(v_1) \cdot [e_{1j}^{cr} + e_1^{cs} + d_{1j}^{0c} \\
&\quad + E(c_1) \cdot [e_{1j}^{vr} + e_{1j}^{vs}]] - \sum_{j=1}^{N_2} [E(v_2) \cdot [e_{2j}^{cr} + e_2^{cs} + d_{2j}^{0c}] \\
&\quad + E(c_2) \cdot [e_{2j}^{vs} + e_{2j}^{vt}]] - \sum_{j=1}^{N_3} [E(v_3) \cdot [e_{3j}^{cr} + e_{3j}^{cs} + d_{3j}^{0c}] \\
&\quad + E(c_3) \cdot [e_{3j}^{vr} + e_{3j}^{vs}]]\Bigg) \cdot \\
&\quad \Bigg(\frac{N_1}{n_1} \cdot \sum_{j=1}^{n_1} [f_{1j}^{cr} + f_1^{cs} + d_{1j}^{Ic} - e_{1j}^{cr} - e_1^{cs} - d_{1j}^{0c}] \\
&\quad + \frac{N_2}{n_2} \cdot \sum_{j=1}^{n_2} [e_{2j}^{cr} + e_2^{cs} + d_{2j}^{0c} - f_{2j}^{cr} - f_2^{cs} - d_{2j}^{Ic}] \\
&\quad + \frac{N_3}{n_3} \cdot \sum_{j=1}^{n_3} [e_{3j}^{cr} + e_3^{cs} + d_{3j}^{0c} - f_{3j}^{cr} - f_3^{cs} - d_{3j}^{Ic}]\Bigg)\Bigg)
\end{aligned} \tag{5.19}
$$

where f and d^I are the errors of the inspector corresponding to those of the operator. This formula looks rather horrible. It should be noted that its structure is not particularly complicated, but the detail of the technical problem, i.e., the measurement effort for establishing the material balance leads to such a lengthy expression.

If we omit the vanishing terms in Eq. (5.19), we obtain a much simpler expression, namely

$$\operatorname{cov}(MUF, D/H_0) = -E\Bigg[E(v_1)\cdot\frac{N_1}{n_1}\cdot\left(\sum_{j=1}^{n_1}[(e_{1j}^{cr})^2+(d_{1j}^{0c})^2]+n_1^2\cdot e_1^{cs}\right)$$
$$+E(v_2)\cdot\frac{N_2}{n_2}\cdot\left(\sum_{j=1}^{n_2}[(e_{2j}^{cr})^2+(d_{2j}^{0c})^2]+n_2^2\cdot e_2^{cs}\right)$$
$$+E(v_3)\cdot\frac{N_3}{n_3}\cdot\left(\sum_{j=1}^{n_3}[(e_{3j}^{cr})^2+(d_{3j}^{0c})^2]+n_3^2\cdot e_3^{cs}\right)\Bigg],$$

which can finally be reduced to the following formula:

$$\operatorname{cov}(MUF, D/H_0) = -\sum_{i=1}^{3}[E(v_i)\cdot N_i\cdot(\sigma_{cri}^2+\sigma_{di}^2+n_i\cdot\sigma_{csi}^2)]. \quad (5.20)$$

This means that *MUF* and *D* are *negatively* correlated.

From Eq. (5.20) we obtain the *correlation coefficient* for the null hypothesis H_0, which is defined by

$$\rho_{H_0} := \frac{\operatorname{cov}(MUF, D/H_0)}{\sqrt{\operatorname{var}(MUF)}\cdot\sqrt{\operatorname{var}(D/H_0)}}. \quad (5.21)$$

For the alternative hypothesis H_1 (diversion of the amounts M_1 and M_2 by means of the two strategies) we have, instead of Eq. (5.17),

$$\operatorname{cov}(MUF, D/H_1) = E((MUF-M_1)\cdot[D-E(D)]/H_1), \quad (5.22)$$

where the expected value of the D-statistic under the alternative hypothesis H_1 according to (5.14b) is given by

$$E(D) = \sum_v \mu_v^c\cdot r_v.$$

The covariance between *MUF* and *D* under the alternative hypothesis H_1 is calculated by using Eqs. (5.4), (5.5), (5.6), (5.9), and (5.12) and leads to the following expression:

$$E\Big((MUF-M_1)\cdot(D-E(D))\Big) =$$

$$E\Bigg(\bigg(N_1\cdot E(v_1)\cdot E(c_1)-N_2\cdot E(v_2)\cdot E(c_2)-N_3\cdot E(v_3)\cdot E(c_3)$$

$$+\sum_{j=1}^{N_1}[E(v_1)\cdot[e_{1j}^{cr}+e_1^{cs}+d_{1j}^{0c}]+E(c_1)\cdot[e_{1j}^{vr}+e_{1j}^{vs}]]$$

$$-\sum_{j=1}^{N_2}[E(v_2)\cdot[e_{2j}^{cr}+e_2^{cs}+d_{2j}^{0c}]+E(c_2)\cdot[e_{2j}^{vs}+e_{2j}^{vt}]]$$

$$-\sum_{j=1}^{N_3}[E(v_3)\cdot[e_{3j}^{cr}+e_3^{cs}+d_{3j}^{0c}]+E(c_3)\cdot[e_{3j}^{vr}+e_{3j}^{vs}]]-M_1\bigg)$$

$$+\left(\frac{N_1}{n_1}\sum_{j=1}^{n_1}[f_{1j}^{cr}+f_1^{cs}+d_{1j}^{Ic}-e_{1j}^{cr}-e_1^{cs}-d_{1j}^{0c}]+\frac{N_1}{n_1}\cdot h_1\cdot\mu_1^c\right.$$

$$+\frac{N_2}{n_2}\cdot\sum_{j=1}^{n_2}[e_{2j}^{cr}+e_2^{cs}+d_{2j}^{0c}-f_{2j}^{cr}-f_2^{cs}-d_{2j}^{Ic}]+\frac{N_2}{n_2}\cdot h_2\cdot\mu_2^c$$

$$\left.+\frac{N_3}{n_3}\cdot\sum_{j=1}^{n_3}[e_{3j}^{cr}+e_3^{cs}+d_{3j}^{0c}-f_{3j}^{cr}-f_3^{cs}-d_{3j}^{Ic}]+\frac{N_3}{n_3}\cdot h_3\cdot\mu_3^c-\sum_j\mu_j^c\cdot r_j\right)\Bigg), \quad (5.23)$$

where h_v, $v = 1, 2, 3$ are the numbers of batch data falsified by the operator and contained in the inspection team's samples.

With the help of the material balance relation

$$N_1\cdot E(v_1)\cdot E(c_1)-N_2\cdot E(v_2)\cdot E(c_2)-N_3\cdot E(v_3)\cdot E(c_3) = M_1$$

and because of the independence of the *e, d,* and *f* on one hand and h_v on the other hand, we obtain the same expression for the covariance between *MUF* and *D* under the alternative hypothesis as under the null hypothesis:

$$\mathrm{cov}(MUF, D/H_1) = \mathrm{cov}(MUF, D/H_0), \quad (5.24)$$

which also means that in this case we have $\rho < 0$. However, because of the difference of the variance of the *D*-statistic for H_0 and H_1 we have, instead of (5.21),

$$\rho_{H_1} = \frac{\mathrm{cov}(MUF, D/H_0)}{\sqrt{\mathrm{var}(MUF)}\cdot\sqrt{\mathrm{var}(D/H_1)}}. \quad (5.25)$$

5.3.5 OVERALL PROBABILITY OF DETECTION

In Figure 5.4, the results of the numerical calculations for the overall probability of detection $1-\beta$ according to Eqs. (4.18) and (4.11) are presented for one inventory period (6 months) for the parameters $M = M_1 + M_2 = 10$ kg Pu, $\alpha = 0.05$, $\alpha_1 = \alpha_2$, and for varying M_1 (and M_2) and effort C. The corresponding probabilities of detection for $\rho = 0$, which have been calculated according to Eqs. (4.19) and (4.12) are almost the same as those for $\rho < 0$, which is not surprising, in view of the discussion in section 4.3. As can be checked numerically, the minimum of the probability of detection is given approximately for those values of M_1 and M_2 for which the following relation holds:

$$\frac{M_1}{\sigma_{MUF}} = \frac{E(D/H_1)}{\sigma_{D/H_1}}, \quad (5.26)$$

which is intuitive because of the symmetry of the formulae, at least for $\rho = 0$.

At first sight it seems strange that for a certain range of the M_1 (and M_2) values, $1-\beta$ *decreases* with increasing effort C. The explanation is as follows: the variance σ^2_{D/H_1} decreases monotonically with increasing C (this is intuitive). This means that

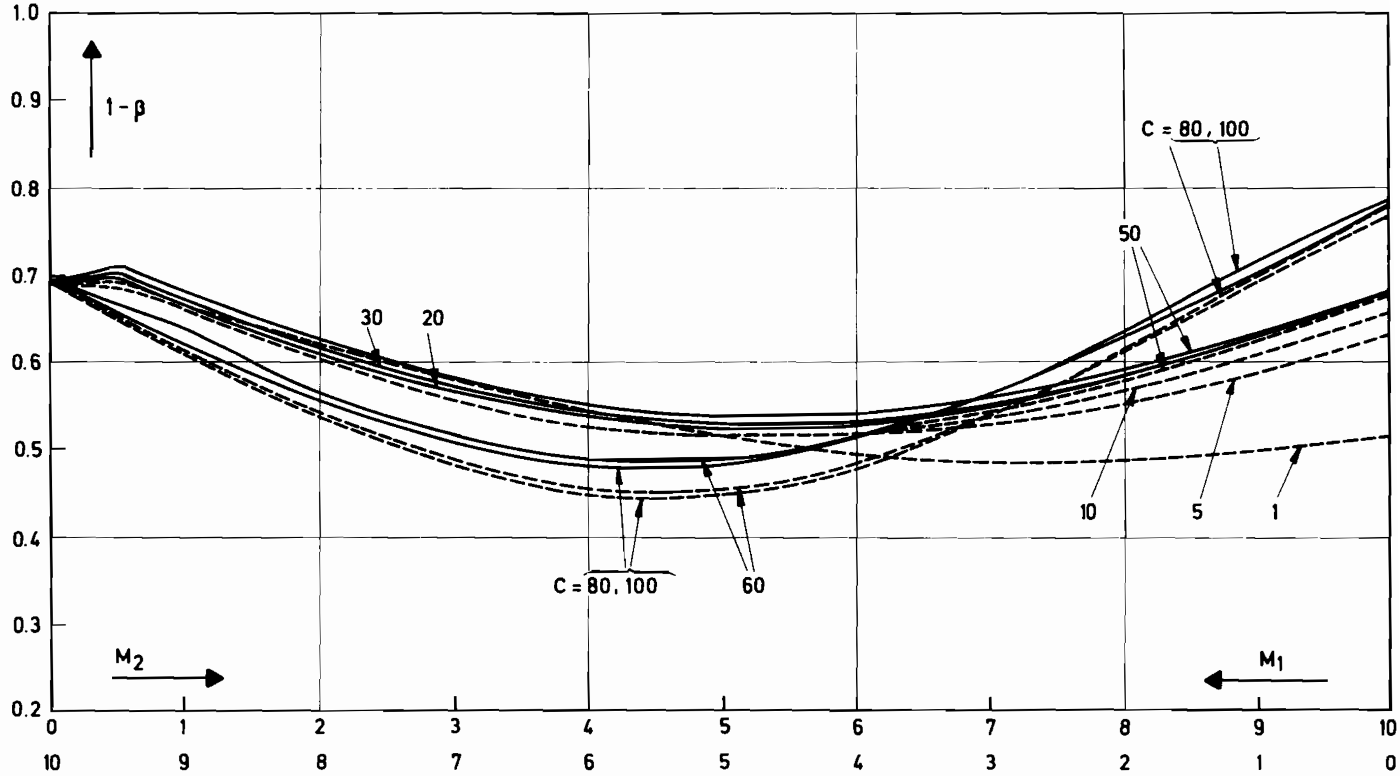

FIGURE 5.4 Total probability of detection $1-\beta$ as a function of amount M_1 and M_2 of material diverted, with effort C (percent of maximum effort) as parameter, and $M_1 + M_2 = 10$ (kg), $\alpha_1 = \alpha_2$, $\alpha = 0.05$. Dashed lines: $\rho = 0$. Solid lines: $\rho \neq 0$. For $C = 10$, 5, and 1 dashed and solid lines coincide. (Adapted from Avenhaus and Nakicenovic.[28])

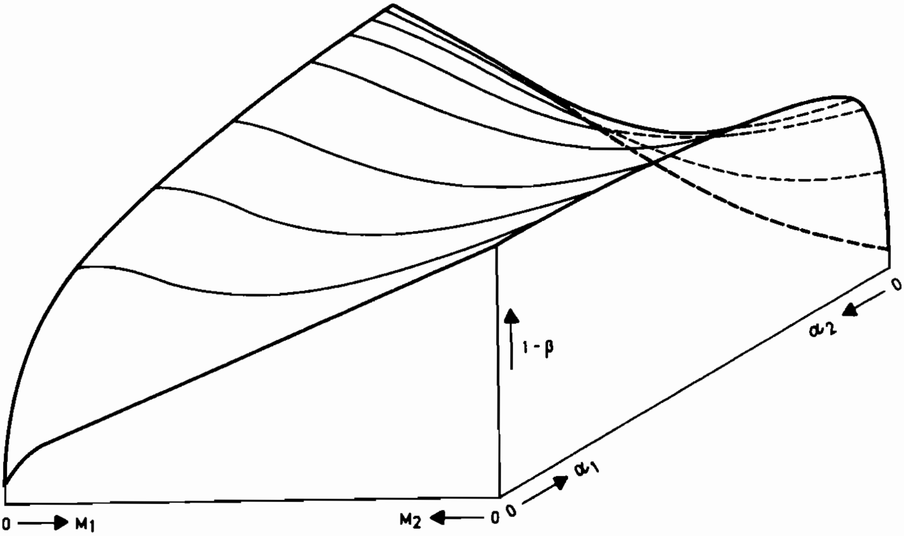

FIGURE 5.5 Total probability of detection $1-\beta$ as function of amount M_1 of material diverted and single false alarm probability α_1, with $M_1 + M_2 = 10$ (kg), $\alpha = 0.05$, and $C = 50\%$ of maximum effort.

$1-\beta_2$ as given by Eq. (3.39) increases with increasing C if the argument of the Φ-function is positive and decreases if the argument is negative.

In Figure 5.5, for a fixed effort C (50% of maximum effort), $1-\beta$ is represented as a function of M_1 (and M_2) and α_1 (and α_2) for $M = M_1 + M_2 = 10$ kg Pu, and $\alpha = 0.05$. The saddle surface is easily recognizable, which illustrates the minimax problem posed in section 4.3, Eq. (4.26). The maximum of $1-\beta$ is approximately given for $\alpha_1 = \alpha_2$ if Eq. (5.26) holds.

The numerical calculations may be summarized by stating that the overall guaranteed probability of detection for a given effort C and a total amount M of diverted material for one inventory period is calculated according to formulae (4.25) and (4.18) for $\alpha_1 = \alpha_2$; M_1 and M_2 are chosen according to (5.26).

5.4 CONCLUSION

In light of the example presented in the last section, the relevance of material accountability for the nuclear material safeguards problem should be examined once again. In fact, one could argue that the figures shown in Figure 5.4 indicate that too much material has to be diverted before a reasonable probability of detection is reached and, therefore, that the material accountability tool is not the most appropriate one. There are two answers to this objection.

First, it is very unlikely that a *state* will divert very small amounts of material in order to accumulate them and construct a nuclear weapon, and prevention of the diversion of very small amounts of material by a *private group*, so-called nuclear theft (see, e.g., Willrich and Taylor[29] and Krieger[30]), is the responsibility of the *national* authorities. These national authorities might use a safeguards system completely different from the one described above in order to meet their objectives; one could imagine, for example, that the containment principle might play a much more important role.

Second, as was pointed out in section 5.1, international acceptance depended heavily on the system's being rational, formalized, and objective. It is clear that the accountability system that has become opperational is highly formalized by its very nature. In addition, it has a high degree of objectivity: according to the analysis presented, subjectivity is limited to the choice of the values of only a few parameters, like total inspection effort, total false alarm probability, and frequency of inventory periods. It was for this reason that these complicated analyses have been performed; even if, in practice, much less sophisticated formulae can be used, it is necessary to understand the structure of the system and to be able to determine the degree of subjectivity remaining in it.

In summary, it has been shown that, of the different possible tools for an international nuclear material safeguards system – material accountability, containment, and surveillance – material accountability best meets the various requirements and boundary conditions imposed on such a system.

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6 Material Accountability in Technology and Economics

In Chapter 5, one specific and rather recent application of the principle of material accountability was presented. It is, in fact, this new application that has stimulated research in the field – research into both theory (i.e., statistical and game theoretical) and practice (i.e., measurement). In this chapter, we will present some applications in the areas of technology and economics that differ from the nuclear safeguards application in several aspects. First, some of them have a very long tradition: mint material accountability, for example, goes back to the ancient Greeks. Second, the statistical treatment, along the lines of that described in Chapter 2, is of minor importance in most cases; the reasons for this will be given later. Finally, the purpose of the establishment of material balances varies from case to case; it cannot always be formulated so clearly and so mathematically as in the nuclear material safeguards case, and, indeed, this was the reason for discussing this case first.

First, two applications in the chemical industry are described – distillation and isotope separation – where material accountability is used extensively for process control purposes. Applications from the metal industry are discussed next; in particular, mint material accountability will be given as an example – an example that has surprisingly strong similarities to the nuclear material case. The second half of the chapter is concerned with more general applications. First, we will look at the Materials–Process–Product Model (MPPM) developed by Ayres and his co-workers; this is an attempt to treat economic problems with the help of material accountability considerations. The broad use of material balances as a management tool in socialist economies is discussed next (the discussion, however, is a brief and general one, because a detailed description is certainly beyond the scope of this monograph).

6.1 THE CHEMICAL INDUSTRY

Material balance methods have a long history in the chemical industry. They have been used mainly for process control purposes, which means that the problem of detecting loss or diversion of material to be processed does not play a major role under normal conditions. The material presented in this section can give only a very rough idea of the use of material balances. Much more complete are the descriptions in textbooks on process control (see, for example, Buckley,[1] Treyball,[2] or Skinskey[3]). Here, only a demonstration of applications in this very important field will be given.

In the following sections, two examples are given that are considered to be representative of many others: fractionated distillation and isotope separation.

6.1.1 DISTILLATION

According to Buckley,[1] the control of distillation columns is one of the most intriguing and challenging branches of process control. Hundreds of papers have been written on this subject, and the number of proposed control schemes probably runs into the thousands. Here, we will limit ourselves to only a very few aspects.

Process control of distillation columns has two primary facets, product quality control and material balance control. Material balance control must

- Cause the average sum of the distillate and tails streams to be exactly equal to the average feed rate
- Cause the resulting adjustment in process flows to be smooth and gradual to avoid upsetting either the column or the downstream process equipment fed by the column
- Maintain the column holdup and the overhead and bottom inventories between upper and lower limits

These control objectives must be satisfied in the face of any possible disturbances in feed flow rate, composition, thermal condition, and so on.

For simplicity, we limit our analysis to an ideal and binary distillation. In Figure 6.1, a sketch of such a column is given, consisting of an enriching and condenser section and a stripping and reboiler section; each section consists of many "trays," or stages, or units. Let us assume that F(moles/min) of a binary system are fed into the column, x_F being the mole fraction of the more volatile component. Furthermore, let P and W(moles/min) be the distillate product and tails waste streams removed from the column, and x_P and x_W be the corresponding mole fractions of the more volatile component. Then the mass balances for the whole system as well as for the more volatile component read as follows:

$$F = P + W \tag{6.1}$$

$$F \cdot x_F = P \cdot x_P + W \cdot x_W \tag{6.2}$$

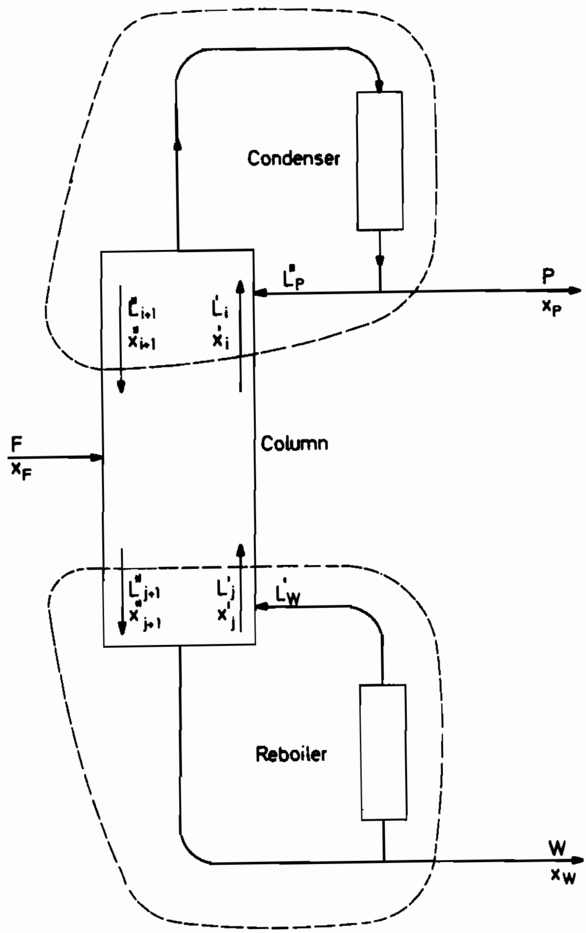

FIGURE 6.1 Material balance of a simple fractionating column. (After Buckley.[1])

Elimination of W gives

$$P = \frac{x_F - x_W}{x_P - x_W} \cdot F, \tag{6.3}$$

which describes the removal rate of distillate P as a function of the feed rate F. Normally, the values of x_P and x_W are specified, while x_F may vary because of changes in the feed composition.

According to Figure 6.1 we also define for the enriching and stripping sections

L'_i = total vapor flow from ith tray (moles/min)
L''_{i+1} = total liquid overflow from $(i + 1)$th tray (moles/min)
x'_i = mole fraction of more volatile component in L'_i
x''_{i+1} = mole fraction of more volatile component in L''_{i+1}

Then the mass balance equations for the enriching section

$$L_i' = P + L_{i+1}'' \tag{6.4}$$

$$L_i' \cdot x_i' = P \cdot x_P + L_{i+1}'' \cdot x_{i+1}'', \tag{6.5}$$

and for the stripping section accordingly ($i \to j$)

$$L_{j+1}'' = W + L_j' \tag{6.6}$$

$$L_{j+1}'' \cdot x_{j+1} = W \cdot x_W + L_j' \cdot x_j'. \tag{6.7}$$

From these equations we get for the enriching section

$$x_i' = \frac{L_{i+1}''/F}{L_i'/F} \cdot x_{i+1}'' + \frac{(L_i' - L_{i+1}'')/F}{L_i'/F} \cdot x_P \tag{6.8}$$

and for the stripping section

$$x_j' = \frac{L_{j+1}''/F}{L_j'/F} \cdot x_{j+1}'' - \frac{(L_{j+1}'' - L_j')/F}{L_j'/F} \cdot x_W, \tag{6.9}$$

or, if distillate and tails streams are essentially pure – i.e., if $P \simeq x_F \cdot F$, $W \simeq (1 - x_F) \cdot F$ – then we get for the enriching section

$$x_i' \simeq \frac{L_{i+1}''/F}{L_i'/F} \cdot x_{i+1}'' + \frac{x_F}{L_i'/F} \cdot x_P \tag{6.10}$$

and for the stripping section

$$x_j' \simeq \frac{L_{j+1}''/F}{L_j'/F} \cdot x_{j+1}'' - \frac{1 - x_F}{L_j'/F} \cdot x_W. \tag{6.11}$$

From equations (6.4) and (6.5) we get a relation for the difference of the mole fractions x_i' of the total vapor flow of one tray and x_{i+1}'' of the total liquid overflow of the next higher tray:

$$x_{i+1}'' - x_i' = \frac{x_i' - x_P}{L_{i+1}''/P}. \tag{6.12}$$

This means that x_{i+1}'' is smaller than x_i' by an amount that decreases if the backflow ratio L_{i+1}''/P increases. In case of total backflow ($L_{i+1}''/P \to \infty$), x_{i+1}'' is equal to x_i'.

Equation (6.8) or (6.10) defines the "operating line" of the enriching section (E-line), and Eq. (6.9) or (6.11) the operating line of the stripping section (S-line). These lines are represented graphically in McCabe–Thiele diagrams (Figure 6.2). The point to be made here is that these diagrams represent nothing but material balance relations for material flows in distillation (or absorption or extraction) columns.

We will not go into the details of the analysis of more complicated systems, like multicomponent systems (for information on these see, e.g., Chien[5]) or recycling systems (see, e.g., Buckley[6] and De Armas[7]; the latter describes their use in sugar

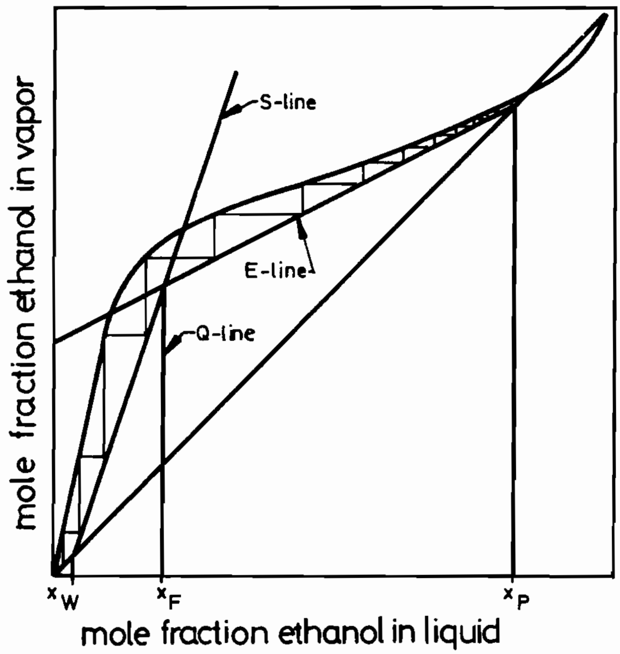

FIGURE 6.2 Distillation diagram for an ethanol purification column. Most of the trays in the column (the "steps" in the diagram) are rectifying or enriching (E) trays rather than stripping (S) trays. The Q-line is determined by the feed composition x_F and temperature. The two operating lines are determined by overhead and bottoms composition specifications and by the position of the Q-lines. (Adapted from Ackley.[4])

factories); rather, we will close this section with some general conclusions that can be drawn from the relations above.

First, let us specify that x_P and x_W are to be held constant. Then, if the feed tray location is to be held constant, we must always hold the number of trays in the enriching section constant in the McCabe–Thiele diagram, as well as the number of trays in the stripping section.

If feed composition (and thermal condition) is constant, then we want the operating lines to remain the same as feed flow changes. As shown by equations (6.8) and (6.9), the operating lines will not change as long as the ratios of distillate to feed, reflux to feed, boilup to feed, and tails to feed are held constant. This may be achieved by fixing any one of three pairs: (a) reflux-to-feed and boilup-to-feed ratio, (b) reflux-to-feed and tails-to-feed ratio, or (c) boilup-to-feed and distillate-to-feed ratio.

If the feed thermal condition is constant, if feed composition is not necessarily constant, and if the distillate and tails streams are essentially pure, then the operating lines are given by equations (6.10) and (6.11). One sees that as the feed composition changes the slopes of the operating lines must be changed so that on the McCabe–Thiele diagram the number of trays in the enriching and stripping sections remain the same.

Finally, let us determine the minimum number of stages necessary to reach prescribed distillate and tails specifications for a given feed specification. The number of trays takes a minimum for total backflow, i.e., for $L'_{i+1}/P \to \infty$; this means, according to (6.12), that

$$x''_{i+1} = x'_i. \tag{6.13}$$

Now let the efficiency of one tray be described by the separation factor α, which is defined by

$$\alpha := \frac{x'}{1-x'} \bigg/ \frac{x''}{1-x''} = \frac{\xi'_x}{\xi''_x}, \tag{6.14}$$

where ξ'_x and ξ''_x are called atomic fractions. From (6.13) and (6.14) we then get the following relation for the total number n of trays:

$$\frac{x_P}{1-x_P} = \alpha^n \cdot \frac{x_W}{1-x_W}.$$

If this equation is solved for n, we obtain the *Fenske equation*[8] for the minimum number of trays:

$$n = \frac{1}{\ln\alpha} \cdot \ln \frac{x_P}{x_W} \cdot \frac{1-x_W}{1-x_P}. \tag{6.15}$$

6.1.2 ISOTOPE SEPARATION

In principle, isotope separation (the most important separations being the separation of uranium-238 and uranium-235 and of hydrogen and deuterium) may be considered a special case of distillation. In fact, the basic material balance equations (6.1) and (6.2) also apply here. However, at least in the case of uranium, special problems arise, as the mass differences between the two isotopes are very small; in other words, one must solve the problem of "close separation." Therefore, one needs special techniques with respect to the basic principles (diffusion, centrifuges, or separation nozzles) as well as with respect to the technology of separation columns, which are here called "cascades." In the following, our considerations are limited to the second aspect, which, at least theoretically, can be handled independently of the separation principle.

An ideal cascade does not mix material flows of a high degree of separation with those of a lower degree; in other words, one wants to retain the "separation work" once it has been achieved. This idea is represented graphically in Figure 6.3. Let us assume that the head separation factor β that is defined by the atomic fractions of the lighter element of the head streams of two succeeding separation units is independent of the position of the unit:

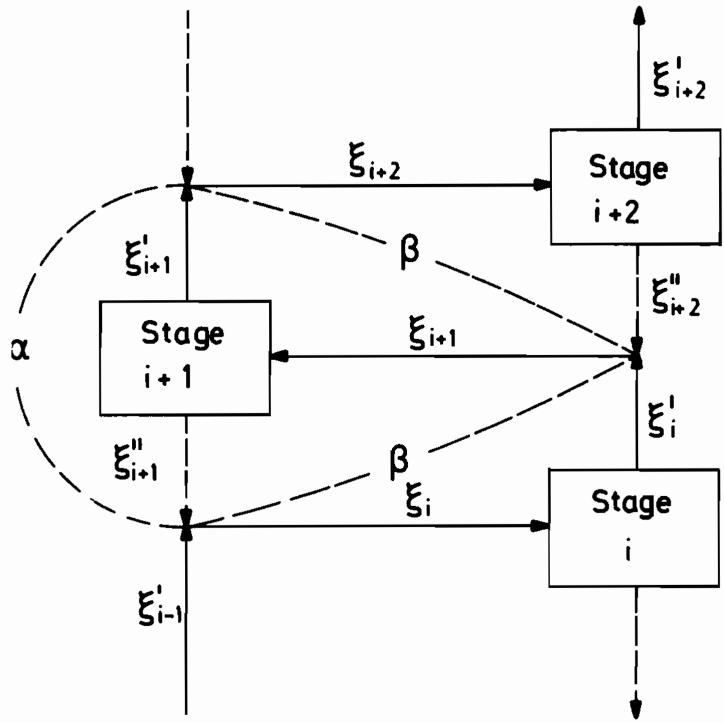

FIGURE 6.3 Principle of the ideal cascade.

$$\xi'_i = \beta \cdot \xi'_{i-1} . \tag{6.16}$$

For an ideal cascade, it must be guaranteed that

$$\xi''_{i+1} = \xi'_{i-1} = \xi_i , \tag{6.17}$$

which, together with (6.16), leads to

$$\xi'_i = \beta \cdot \xi''_{i+1} ,$$

or, for $i + 1$ instead of i, again using (6.16), to

$$\xi'_{i+1} = \beta^2 \cdot \xi''_{i+1} . \tag{6.18}$$

Now, from the definition of the separation factor α we get

$$\xi'_{i+1} = \alpha \cdot \xi''_{i+1} . \tag{6.19}$$

Therefore, comparing (6.18) and (6.19), we get

$$\beta = \sqrt{\alpha} . \tag{6.20}$$

For the close-separation cascade (i.e., when $\beta - 1 \ll 1, \alpha \ll 1$) we get

$$\beta - 1 = \frac{\alpha - 1}{2}. \tag{6.21}$$

The total number of stages in an ideal cascade can be determined in a way similar to that used for the determination of (6.12):

$$n = \frac{1}{\ln \beta} \cdot \ln \frac{x_P}{x_W} \cdot \frac{1 - x_W}{1 - x_P} - 1,$$

or, with (6.20),

$$n = \frac{2}{\ln \alpha} \cdot \ln \frac{x_P}{x_W} \cdot \frac{1 - x_W}{1 - x_P} - 1. \tag{6.22}$$

This means that the number of stages in an ideal cascade for a given separation (x_P, x_W) is twice the minimum number needed for total backflow minus one. Similarly, the number of stages in the stripping section is given by

$$n_W = \frac{1}{\ln \beta} \cdot \ln \frac{x_F}{x_W} \cdot \frac{1 - x_W}{1 - x_F}. \tag{6.23}$$

The backflow ratio for an ideal cascade can be formed as follows: From (6.12) we get

$$\frac{L''_{i+1}}{P} = \frac{x_P - x'_i}{x'_i - x''_{i+1}}. \tag{6.24}$$

From (6.17), it can be seen that for the ideal cascade $x'_i = x_{i+1}$. If we express x_{i+1} by x''_{i+1}, we get, with (6.20),

$$\frac{L''_{i+1}}{P} = \frac{1}{\beta - 1} \cdot \left(\frac{x_P}{x''_{i+1}} - \frac{\beta(1 - x_P)}{1 - x''_{i+1}} \right). \tag{6.25}$$

In the stripping section, the backflow ratio is accordingly given by

$$\frac{L'_i}{W} = \frac{1}{\beta - 1} \cdot \left(\frac{1 - x_W}{1 - x'_i} - \frac{\beta \cdot x_W}{x'_i} \right). \tag{6.26}$$

If we eliminate the mole fractions x''_i, x'_i by using the following relations for the enriching section

$$x'_i = x_{i+1} = x''_{i+2} = \frac{\beta^i \cdot x_P}{\beta^i \cdot x_P + \beta^n \cdot (1 - x_P)}$$

and for the stripping section ($i \to j$)

$$x''_j = x_{j-1} = x'_{j-2} = \frac{\beta^{j-1} \cdot x_W}{\beta^{j-1} \cdot x_W + (1 - x_W)},$$

we get for the enriching section

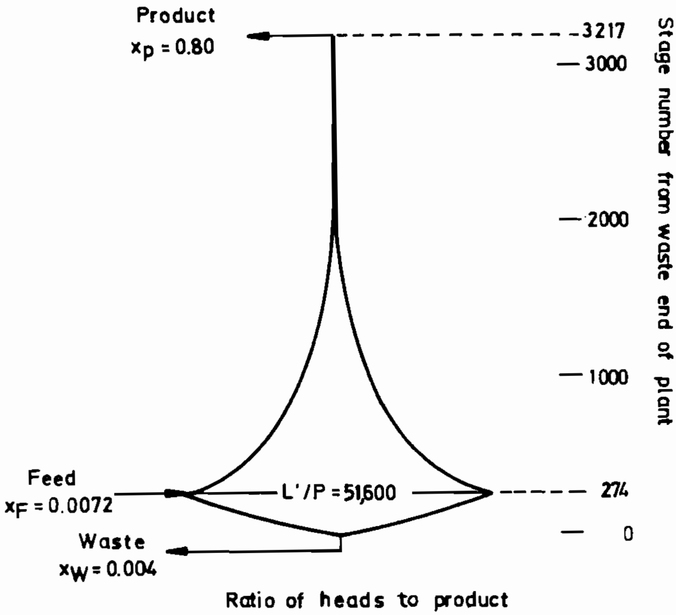

FIGURE 6.4 Heads flow rate *vs.* stage number in ideal cascade; $\beta = 1.00215$. (Adapted from Benedict and Pigford.[9])

$$\frac{L''_{i+1}}{P} = \frac{1}{\beta - 1} \cdot [x_P \cdot (1 - \beta^{i-n}) + (1 - x_P) \cdot \beta \cdot (\beta^{n-i} - 1)] \qquad (6.27)$$

and for the stripping section

$$\frac{L'_j}{W} = \frac{1}{\beta - 1} \cdot [x_W \cdot \beta \cdot (\beta^j - 1) + (1 - x_W) \cdot (1 - \beta^{-j})]. \qquad (6.28)$$

These equations can be used to illustrate the shape of an ideal cascade. In Figure 6.4 the variation of the interstage flow with the number of stages is given for an ideal cascade, which separates natural uranium ($x_F = 0.0072$) into enriched uranium ($x_P = 0.8$) and depleted uranium ($x_W = 0.004$) by means of gas diffusion ($\alpha = 1.0043$). For one mole product of the form of Eqs. (6.1) and (6.2), $F = 249$ mole, $W = 248$ mole; the total number of stages is $n = 3{,}217$ according to (6.22), and the number of stages in the stripping section is $n_W = 274$ according to (6.23). The heads flow rate in the enriching section is given, using (6.17), by

$$L'_i = L''_{i+1} = \frac{1}{0.00215} [0.8(1 - 1.00215^{i-3{,}217})$$

$$+ 0.2 \cdot 1.00215 \cdot (1.00215^{3{,}217-i} - 1)], \qquad 274 < i < 3{,}217.$$

The heads flow rate in the stripping section is given, using (6.28), by

$$L_j' = \frac{248}{0.00215} \cdot [1.00215 \cdot (1.00215^j - 1) \cdot 0.004 + 0.996 \cdot (1 - 1.00215^{-j})], \qquad 0 < j < 274.$$

It is worth noting that the graph in Figure 6.4, which is nothing more than a special consequence of material balance relations, has become the symbol for isotope separation.

Before concluding this section, a remark should be made about the statistical analysis of the problems discussed so far. It is obvious that diversion or loss of material plays no role here. Therefore, the considerations in Chapters 1 and 2 are inapplicable. Still, it is important to analyze the effect of errors or disturbances on the stability of the system. Although some work has been done in this field,[3] the impression remains that more needs to be done.

Only recently, a methodologically oriented paper[10] was published that discusses the problems just mentioned. In this paper, the establishment of multicomponent material balances for complex chemical processes is sketched, and the following discussion concentrates on the measurement aspects of the problem, emphasizing

Design of measuring places and accuracy of methods of measurement (including random and systematic errors)

Determination of balance periods for quasistationary continuous processes

Description of computer program packages

Even though the paper does not treat any of these three points in any detail because of its survey character, it does indicate the direction to be followed in further inquiry.

6.2 THE METAL INDUSTRY

In the preceding section we demonstrated that in the chemical industry material accountability is used primarily for process control purposes. In this section we will show that in the metal processing industry material accountability is used for a wide variety of purposes: for process control, for internal plant economy, and also for detection of losses and diversion of material. This wide spectrum of applications reflects the great variations in the value of the metal to be processed. In steel plants large amounts of relatively cheap metals are processed and, therefore, the only purpose of material accountability is to help run the plant economically; in gold and silver processing plants, however, the purpose of material accountability is to detect any loss or diversion of even small amounts of the very expensive material.

In this section, we describe briefly the establishment of so-called metal balances in plants processing lead, zinc, and similar metals. We will not go into greater detail because the accounting techniques used are well known and straightforward. We then discuss silver in federal mints, which provides an illuminating example of the use of material accountability.

6.2.1 METAL BALANCES

Let us consider the processing of metals like copper, lead, or zinc. The ore is taken from a mine, and after some preliminary treatment (e.g., milling) it is brought to a plant specializing in the metal under consideration. Here, the concentrated ore is processed by means of various techniques (preroasting, sintering, reduction) until the metal is in the various forms in which it is sold (in the case of zinc, for example, these forms are sheet zinc and coating zinc for coating or galvanization).

A rough comparison of inputs and outputs of typical plants shows that about 17 percent of the ingoing material is lost in the form of various wastes and hidden inventories. This high figure may be considered the reason for the establishment of metal balances; among other things, these balances serve to keep track of the losses at different production steps and thus to keep them to a minimum. It is for this reason that metal balances are somewhat different from, for example, nuclear material balances. Whereas in the nuclear case the entire plant represents a single material balance area, here the plant is subdivided into many material balance areas for which separate material balances are established daily. These material balance areas correspond to the main processing units of the plant. Thus, at the end of a day the plant management knows the size of the losses in each unit and can decide whether they are compatible with normal operating experience.

There is another difference in the methodology of establishing material balances in nuclear material and metals. As described in Chapter 5, the statistical evaluation of measurement errors plays a central role in the whole nuclear materials safeguards system. With metals, the measurement errors (in weight determination and chemical analyses, for instance) are negligible, compared to the plant internal losses and the losses to the environment. On the other hand, as indicated, there is long-term experience in the form of statistical data about the values of various loss categories; this means that significance tests of the form described in Chapter 2 can be performed.

There is little in the scientific literature about these metal balances – perhaps because they seem too prosaic or straightforward to the practitioner. Indeed, they are not sophisticated mathematically, unlike the chemical balances described in the preceding section. One reference[11] has been found that reports the performance of detailed analyses of concrete cases. However, we will not go into the details of that work, since the data given there may well be outdated. More recent publications about metal balances treat special aspects such as optimization methods[12] or the use of computerized techniques.[13]

Finally, it should be kept in mind that, like the material balances in the chemical

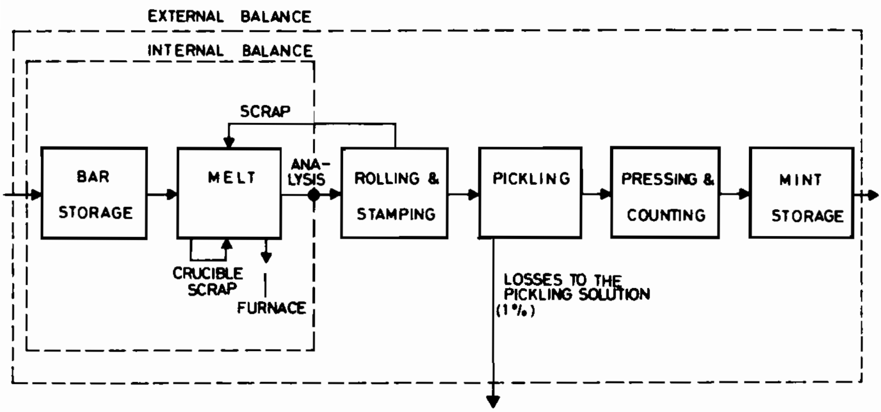

FIGURE 6.5 Flow diagram of the silver processed in a federal mint in the Federal Republic of Germany.

industry, traditional metal balances are used for process control; however, unlike balances in the chemical industry, their emphasis is on the control of losses.

6.2.2 MINT MATERIAL ACCOUNTABILITY

It is not surprising that there are great similarities between nuclear accountability and mint material accountability. In both cases, the material is valuable and must be safeguarded against losses or diversion; it seems reasonable, therefore, that the same safeguards principle be applied in both cases.

Before looking into mint material accountability, we must differentiate between two different kinds of coins produced in a federal mint in the Federal Republic of Germany: ordinary coins to be used as currency, and silver or gold coins that are issued only on special occasions and that are bought by collectors. In the framework of our considerations, the difference between these two kinds of coins is the following. While the intrinsic value of the material for regular coins is not so high, safeguards against diversion are necessary at every stage of the production process – not only the final product (money) but also the input in terms of blanks that can be used, for example, in all coin-operated machines. With silver or gold coins, on the other hand, it is the material that has to be safeguarded because of its value.

In the production of ordinary coins federal mints perform *digital accountability*: input and output measurements by means of counting and of weight determinations of batches with an accuracy of a single coin. Although complicated and refined measures are taken in order to prevent illegal recycling of material that results in double counting – which represents the only possibility for undetected diversion of material – this case is not so interesting from a methodological point of view; our attention will, therefore, focus on the second case.

The flow of silver in a federal mint is represented graphically in Figure 6.5 (for

TABLE 6.1 Silver Throughput Data in a Federal Mint[a]

Silver input (tons/yr)	120
Amount of material recycled (mainly scrap from rolling and stamping) (tons/yr)	15
Number of inventory periods/yr	4
Input	
Number of silver bars/yr	4,000
Weight of 1 silver bar (kg)	30
Batch-to-batch variation (%)	< 10
Product	
Number of silver coins/yr	1.7×10^6
Weight of 1 coin (g)	11.2
Silver content of 1 coin (g)	7
Batch-to-batch variation of silver content of 1 coin (%)	1
Losses in the pickling solution (% of throughput)	1
Hidden inventory	(see text)

[a] Data from Avenhaus and Hartmann.[15]

more details see the publication of the FRG federal mint[14]); gross data from a major federal mint in the Federal Republic of Germany are given in Table 6.1. The melt consists of silver bars received from the state, together with copper and recycled scrap. The molten material is poured into permanent moulds; producing billets (ingots) of an average weight of 3 kg. After a heat and pickling procedure the billets are rolled into strip from which the rounds or blanks are stamped. These blanks are repickled once and finally stamped into coins. They are counted and put into jute sacks that are stored in the safe deposits of the mint until they are shipped. Scrap generated during rolling and stamping is recycled along with the crucible residue; the pickling solution containing the silver is removed to a special plant where the silver is recovered.

From this description of the silver flow it is clear that two kinds of material balance are important. The first can be called the external balance; it is simply the comparison of the book inventory and the physical inventory of the plant, and it is important for knowing whether material has disappeared accidentally or been diverted deliberately. The second kind of material balance, which we will call the internal material balance, refers only to that part of the plant where the silver undergoes chemical transformations; this balance is important because in this part of the production process major uncertainties and even losses cannot be avoided.

Let us first consider the internal balance, assuming the following: Such a balance is established for a production campaign that lasts 3 months; the physical inventory in the internal material balance area (see Figure 6.5) is assumed to be zero before the start and after the end of the campaign. The establishment of the internal material balance is represented in Table 6.2. The input consists of silver bars and recycled scrap. The output is the coins taken from the furnace. The measurement methods, their error variances, and the variance of the material unaccounted for are given in Table 6.3. As can be seen, the standard deviation of the material unaccounted for amounts to 0.12% of the input.

TABLE 6.2 Internal Silver Balance in a Federal Mint

Length of inventory period (mo)	3
Input (I)	
Silver bar	
Number of n_{I_1} of batches/inventory period	1,000
Ag weight G_{I_1} of 1 batch (kg)	30
Batch-to-batch variation (%)	< 10
Scrap	
Number of n_{I_2} of batches/inventory period	360
Total weight g_{I_2} of 1 batch (kg)	21
Ag concentration c_{I_2} of 1 batch (kg metal/kg silver)	625
Batch-to-batch variation (%)	< 10
Output (O)	
Number of n_O of batches/inventory period	360
Total weight g_O of 1 batch (kg)	154
Silver concentration c_O of 1 batch (kg metal/kg silver)	.625
Batch-to-batch variation (%)	< 10
Expected Material Unaccounted For [$E(MUF)$]	
$E(MUF) = \text{Initial inventory}^a + n_{I_1} \cdot G_{I_1} + n_{I_2} \cdot g_{I_2} \cdot c_{I_2}$ $- n_0 \cdot g_0 \cdot c_0 - \text{Ending inventory}^a$ $= 1{,}000 \cdot 30 + 360 \cdot 21 \cdot 0.625 - 360 \cdot 154 \cdot 625$ $= 0 \text{ (kg)}$	

[a] Not taken into account.

So far we have not considered losses or hidden inventories. There are no statistical data that could be taken into account in the way described in Chapter 2. However, one case was reported of an old furnace that, on being decommissioned, was found to contain more than 80 kg of silver as hidden inventory. Considering the uncertainty of the material balance established for one campaign, it is not surprising that these losses, accumulated over years, were not detected. Given the price of silver, however – 1 kg of silver is worth about $140 – some effort to increase the overall accuracy seems to be justified.

The input of the external material balance is the same as that of the internal balance. The product consists of silver coins that are weighed in lots of 300; the silver concentration measurement is the same as for the internal balance. In addition, there is an output from the silver contained in the pickling solution. The external material balance establishment is not discussed here as it is analogous to that of the internal balance. One interesting point is that in this connection a verification procedure is in effect: there are specifications for the coins with respect to their total weight and silver content that are verified by destructive methods on a random sampling basis.

In sum, one may say that although the silver coin verification procedure serves a different purpose from the verification procedure in the nuclear material case, the example of mint material accountability contains all the essential features analyzed in the preceding chapters.

TABLE 6.3 Accuracy of Internal Silver Balance in a Federal Mint

Input	
Silver bar (I1)	
Standard deviation $\sigma_{I_1 Gr}$ of random error of weight determination of 1 batch (kg)	0.001
Standard deviation $\sigma_{I_1 Gs}$ of systematic error of weight determination of 1 batch (kg)	0.001
Scrap (I2)	
Standard deviation $\sigma_{I_2 Gr}$ of random error of weight determination of 1 batch (kg)	0.001
Standard deviation $\sigma_{I_2 Gs}$ of systematic error of weight determination of 1 batch (kg)	0.001
Standard deviation $\sigma_{I_2 Gr}$ of random error of concentration measurement of 1 batch (%)	0.3
Standard deviation $\sigma_{I_2 G_s}$ of systematic error of concentration measurement of 1 batch (%)	0.3
Output (O)	
Same as for scrap	

Variance of Material Unaccounted For[a]

$$
\begin{aligned}
\text{var}(MUF) &= n_{I_1} \cdot \sigma^2_{I_1 Gr} + \frac{n_{I_1}}{100} \cdot (100 \cdot \sigma_{I_1 Gs})^2 \\
&\quad + n_{I_2} \cdot \sigma^2_{I_2 gr} \cdot E^2(c_{I_2}) + \frac{n_{I_2}}{100} \cdot (100 \cdot \sigma_{I_2 gs})^2 \cdot E^2(G_{I_2}) \\
&\quad + E^2(g_{I_2}) \cdot n_{I_2} \cdot \sigma^2_{I_2 cr} + E^2(g_{I_2}) \cdot \frac{n_{I_2}}{100} \cdot (100 \cdot \sigma_{I_2 c})^2 \\
&\quad + n_O \cdot \sigma^2_{Ogr} \cdot E^2(c_0) + \frac{n_O}{100} \cdot (100 \cdot \sigma_{Ogs})^2 \cdot E^2(c_O) \\
&\quad + E^2(g_O) \cdot n_O \cdot \sigma^2_{Or} + E^2(g_O) \cdot \frac{n_O}{100} \cdot (100 \cdot \sigma_{Ocs})^2 \\
&= 1{,}000 \cdot 10^{-6} + \frac{1000}{10} \cdot (100 \cdot 10^{-3})^2 \\
&\quad + 360 \cdot 10^{-6} \cdot 0.625 + \frac{360}{100} \cdot (100 \cdot 10^{-3})^2 \cdot 0.625^2 \\
&\quad + 21^2 \cdot 360 \cdot (0.625 \cdot 3 \cdot 10^{-3})^2 + 21^2 \cdot \frac{360}{100} \cdot (100 \cdot 0.625 \cdot 3 \cdot 10^{-3})^2 \\
&\quad + 360 \cdot 10^{-6} \cdot 0.625^2 + \frac{360}{100} \cdot (100 \cdot 10^{-3})^2 \cdot 0.625^2 \\
&\quad + 154^2 \cdot 360 \cdot (0.625 \cdot 3 \cdot 10^{-3})^2 + 154 \cdot \frac{360}{10} \cdot (100 \cdot 0.625 \cdot 3 \cdot 10^{-3})^2 \\
&= 1{,}270\ (\text{kg}^2) \\
\sqrt{\text{var}(MUF)} &= 35.6\ \text{kg} = 0.12\%\ \text{of input}
\end{aligned}
$$

[a] Weight determination and concentration measurement calibration every 100th measurement.

6.3 THE MATERIALS–PROCESS–PRODUCT MODEL

So far, we have discussed cases where material accountability considerations are used in special industries for specific technological reasons: for process control and for the detection of diversion or loss of material. This section and the next report on more general applications in the area of economics. As a detailed description would take too much space here, only a survey is given and the more important literature cited.

In 1972, Ayres[16] published his ideas on the applicability of material accountability to economic problems under the title "Materials–Process–Product Model." Two years later, a major "Feasibility Demonstration Based on the Bottle Manufacturing Industry" was reported by Ayres and his co-workers.[17] As this work complements the ideas that have been developed and illustrated so far, it is briefly described here.

The MPPM principle is represented graphically in Figure 6.6. Boxes on the far left represent inputs required for the model components to the right of them.

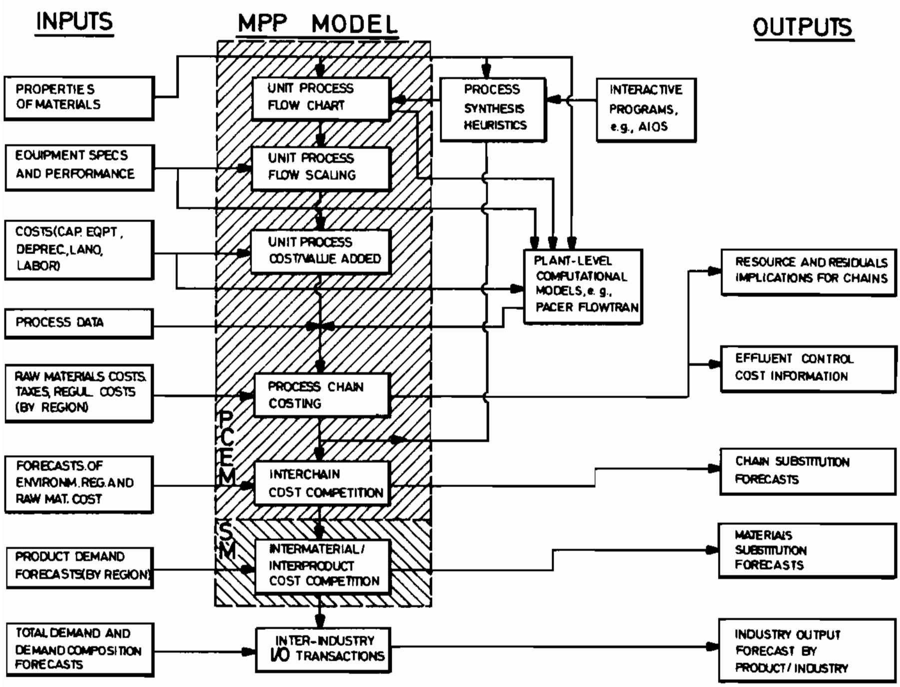

FIGURE 6.6 Materials–Process–Product Model. See text for explanation. (From Ayres *et al.*[17])

Boxes on the far right indicate outputs from the model components to the left of them. The boxes at the upper right represent several computational options for augmenting the amount of detail used in process description. The component at the bottom of the diagram is a future model extension. The model in its present and more limited form is shown by the highlighted region in the center.

Mathematically, the model may be visualized as a collection of nodes, representing processes that are interconnected by directional lines, each of which corresponds to a specific material input or output. The overall economic flow moves from left (raw materials) to right (final products), but the flows between intermediate processes may go in either direction, provided that materials are conserved so that inputs and outputs always balance. The processes on the left tend to be concerned with the separation of components (e.g., metals from ore) and chemical recombination in desired forms (e.g., alloys, basic chemicals). The processes in the center tend to be more concerned with physical transformations (e.g., spinning, rolling, stamping, casting, drawing) and fabrication (weaving, lamination, welding, fastening, plating, painting, etc.). The processes on the right tend to be concerned with systems integration, packaging, distribution, sales, maintenance, and the like.

As the implementation of this theoretical framework would clearly represent a major effort (there are at least 500 important industrial materials) the MPPM authors proceed to demonstrate the practical value of the model in a much smaller effort. They built the Process Chain Evaluation Model (PCEM), which is simply a collection of computer programs for constructing "chains" or "networks" of processes; these programs compute consolidated material and energy input/output data and value-added data for each such chain or network. With the help of PCEM and an appropriate data bank one can analyze several specific kinds of problems at the level of aggregation of an industry, a firm, or a material.

Ayres and co-workers[17] describe an application of these ideas to the bottle industry. In this industry, several raw materials compete, the more important ones being glass, polyethylene, and polyvinyl chloride. Ayres and his associates showed that the PCEM makes possible the assessment of the outcome of competition between alternative processes and raw material choices, as a function of alternative assumptions regarding available technology, raw or intermediate material prices, market prices for the final product, market size for the final product, tax rates, or regulatory constraints (on waste emission, for example).

The starting point of the analysis is the specification of all inputs and outputs (Table 6.4); according to the principle of conservation of mass, the sum of the material inputs and outputs must be zero. The second step is the establishment of an economic balance for the process, profit being the difference between the sales revenue and the costs of raw materials, utilities, process operation, annualized investment, and taxes. The third step is a detailed analysis of the process chains, leading from the basic raw materials to the end product from both a material flow and production cost point of view. These details are not important here, but what can be achieved with this analysis again demonstrates the broad applicability of material accountability considerations.

TABLE 6.4 Soda-Lime Glass Bottle Manufacturing Materials – Input/Output[a]

Utilities	
Electrical energy	0.13000 kWh
Fuel energy	5.62600 M Btu
Cooling water or equivalent	0.71000 gal
Primary Inputs	
Sodium carbonate	0.27580 lb
Calcium carbonate	0.24820 lb
Silicon dioxide	0.54080 lb
Calcium silicate	0.01790 lb
Potassium aluminum silicate	0.03220 lb
Recycled glass	0.10200 lb
Secondary Inputs	
Water	0.63000 lb
Sodium sulfate	0.00040 lb
Potassium nitrate	0.00280 lb
Arsenious oxide	0.00050 lb
Manganese dioxide	0.00040 lb
Product	
Glass bottles	1.00000 lb
Gaseous Wastes	
Steam	0.63000 lb
Carbon monoxide	0.01000 lb
Carbon dioxide	0.21000 lb
Silicon dioxide	0.00100 lb

SOURCE: Ayres *et al.*[17]
[a] Normalized per unit pound product.

Ayres *et al.*[17] emphasize the impact of alternative forms of taxation – for instance,

"Effluent charges" on specific pollutants, such as ethylene dichloride
Taxes levied on materials purchased but not accounted for in final products
Taxes levied on value added by processing
Taxes levied on undesirable components of final products, e.g., toxic materials

Furthermore, different forms of effluent charges are discussed, among them:

Charges that are linear or nonlinear to the amount emitted
Threshold values, or emission levels below which no charge is levied

PCEM can also be used in addressing the question of technological substitution, which means here the substitution of one basic material for another. Which material is the most promising one under which conditions – for example, under raw material scarcity or changing tax policies?

Unfortunately, space limitations prevent thorough treatment of this issue, but one final remark should be made about the statistical analysis of the problem. Ayres and his co-workers did not perform calculations of this kind; they do note, however, that process accounting has not attempted to trace the origins of small material losses from the process but that increasing environmental concern will lead to much closer process monitoring although some losses will always be hard to identify. Here a statistical analysis of the type discussed in Chapter 2 is in order.

6.4 USE OF MATERIAL BALANCES IN SOCIALIST ECONOMIES

In the foregoing section we presented new ideas on how and why to use material balances in industrial plants. This section is devoted to the use of material balances in socialist economies. Of course, only an idea of this extremely important field of applications can be given; in fact, this sketch is meant to put the main ideas described in this monograph into perspective by describing those developed and applied by socialist economists.

An indication of the role of the material balance concept in the overall socialist economic system is found, for example, in the official Soviet description of the economy of the USSR.[18] The outline to be given here is based on an article by Kossov and Baranov.[19] A detailed description is given by Aganbegyan and Granberg.[20,21] As far as known to the author, the best description in English literature is given by Montias.[22]

Intersectoral balances, as they are called in the Soviet literature, are used as an instrument for analyzing and planning the production and distribution process in a national economy. The principles of intersectoral balancing were used for the first time in the USSR in the national economic plan for 1923–1924. The intensive development of these methods was carried through in the second half of the fifties in several institutions, including the Economic-Mathematical Institute of the Academy of Sciences of the USSR.

These intersectoral balances include different kinds of balances: balances expressed in monetary terms, balances expressed in physical terms, and also balances expressed in mixed terms. All these balances may refer to the whole national economy, to regional and inter-regional economies, and to intersectoral and inter-industrial links, as well as to the production and distribution of given commodities. They can differ with respect to time horizon (dynamic balances), or they can be static. They are used for a variety of purposes – for reporting activities for different time schedules, for planning, for investigation, and others.

Monetary balances usually consist of four divisions. The first division gives the intersectoral flow of materials for current production; the main outcome of this division is intermediate products. The second division deals with the structure of final products. The third division shows the amortization and newly created values.

The fourth division reflects the structure of partial distribution of newly created values.

Balances in physical terms usually consist of two divisions. The first division reflects resources and intersectoral capacities for the production. The second division characterizes the distribution of the current production and the final consumption. Because of the general scope of this monograph we are interested here only in the second division of the balances in physical terms.

Let x_i, $i = 1, \ldots, n$, be the gross output of the ith commodity in a given interval of time, let a_{ij} be the technological coefficient showing the amount of the jth commodity required to produce the unit amount of the ith commodity, and, furthermore, let y_i be the final demand for the ith commodity in the interval of time under consideration. Then the material balances for all n commodities read as follows:

$$x_i = \sum_{j=1}^{n} a_{ij} \cdot x_j + y_i, \qquad i = 1, \ldots, n. \tag{6.29}$$

If we define vectors X and Y and matrix A by

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \tag{6.30}$$

then we can write the n equations (6.29) in the following concise form:

$$X = A * X + Y, \tag{6.31}$$

or, introducing the unity matrix E,

$$(E - A) * X = Y \tag{6.32}$$

These relations are mainly established to determine the gross output X as a function of the final demand Y. In other words, the problem is to solve the system of equations (6.32) for X:

$$X = (E - A)^{-1} * Y \tag{6.33}$$

which means the inversion of matrix $E - A$.

It is clear that the determination of the gross output X for a given interval of time cannot be determined simply by inverting matrix $E - A$. Among other substantial reasons, this is not possible because of the large number of commodities, which run into thousands. Several approximation procedures have been used instead, the most intuitively appealing being an iterative approach.

One starts out with a (somewhat unrealistic) assumption that the first set of output commodities y_i, $i = 1, \ldots, n$, circulated to all industries only consists of final demands. With the help of input–output coefficients a_{ij} (established by the individual industrial sectors as well as by higher planning agencies) first estimates of gross outputs $x_i^{(1)}$, $i = 1, \ldots, n$, according to

$$x_i^{(1)} = \sum_{j=1}^{n} a_{ij} \cdot y_j + y_i, \qquad i = 1, \ldots, n, \tag{6.34}$$

are calculated. These estimates make up a second set of output commodities. The previous summation procedure is repeated to yield second estimates $x_i^{(2)}$, $i = 1, \ldots, n$, according to

$$x_i^{(2)} = \sum_{j=1}^{n} a_{ij} \cdot x_j^{(1)}, \qquad i = 1, \ldots, n. \tag{6.35}$$

It can be shown that every new set of gross output commodities obtained in this manner would come close to the perfectly consistent set of commodities that could be calculated by direct inversion of the matrix $E - A$.

There are further possibilities for solving the problem of inverting matrix $E - A$; however, we will not discuss these now but go back to the underlying assumption that gross output commodities can be systematically derived from a bundle of final demands. It is clear that certain intermediate products have a higher value to the economy than the end-uses they can generate. Take, for example, sectors of the economy that are given top priority in order to widen bottlenecks. These kinds of problems are hard to solve with the help of physical balances alone; therefore, the interaction between the physical and the monetary balances is of great importance.

Let us try to compare the balances treated here with those previously considered in this monograph: As one may deduce from the definitions of the commodities x_i and y_i and from the set of equations (6.29), in the second division of the balances in physical terms *flows of material* are balanced. This means that in any given time interval the intermediate and final demands must be equal to their gross production. This is true if no changes in the inventory of commodities are considered. Theoretically, this can be understood if it is remembered that one purpose of these balances is the planning and the control of the entire production and distribution process. In practice, of course, inventory changes and depreciation are taken into account.

In earlier sections of this monograph we introduced the notion of material unaccounted for. If one uses the iterative procedure for inversion of the matrix $E - A$, after a finite number of steps a difference remains between two consecutive sets of gross output commodities. If we want to use this difference as a measure of compliance with final demands and gross output commodities, then we might similarly call it material unaccounted for.

6.5 COMPUTER-AIDED MATERIAL ACCOUNTABILITY PROCEDURES

The examples of applications of the principle of material accountability recounted above are heterogeneous – they vary from very specialized (mints) to very general (socialist economic planning). In concluding this chapter, it seems reasonable to remark on the use of computers and computer programs in this connection. It is clear that the use of computers of any size is common wherever numerical calculations have to be handled (e.g., when complicated measurements that generate large amounts of data are necessary) or if complicated mathematical formulae have to be used that cannot be solved numerically by hand calculations [as is the case for Eqs. (2.38) and (2.39), for example]. We will not discuss, however, these two types of use but will concentrate on some recently published work that aims directly at developing material balance programs for specific purposes. In fact the highest percentage of publications on material accountability matters in the last 2 years deals with computer software problems. Here, we will report on three classes of papers, treating

Techniques and programs facilitating routine material accountability procedures
On-line techniques for process control purposes
Planning and design techniques in a general sense

The first class of papers deals with all the situations in which material enters and leaves a facility, or a storage area of a facility, and in which the material can be accounted for without any measurement errors. This is the case, for example, if items or batches are *counted* but not *measured* for material content. Here, the emphasis is on the development of appropriate forms, on maximizing simplicity, and on minimizing the time to be spent by the personnel responsible for these matters. Powley,[23] Jacoby and Kabel,[24] and Blankenburg[25] have written papers of this kind; clearly, however, a whole literature exists for these problems.

In the second class of papers, industrial processes are considered; here, material accountability (or mass balance) measures are used for process control purposes, much as described in the first section of this chapter. The main reason for the development of computer programs here is to have the results immediately available in order to allow timely adjustment of the process. Examples of this type of work are given by Cutten *et al.*[13] and McCracken.[26] Naturally, there are other reasons for developing computer programs – for example, optimization of process input[12] or real-time methods for the control of the material flows in nuclear facilities in order to detect losses or diversion of particular nuclear materials;[27,28] however, as they may be categorized under the heading of "on-line" or "real-time" techniques, and here again an enormous literature exists, we will not go into detail.

In the last class of papers there are three subclasses. First, there are papers that deal with the design and planning of production processes in which the establishment of material balances plays an important role in the determination of yield accounting, in allocating costs, in calculating production levels and raw material requirements, and so on. A survey of programs for questions of this sort is given by Harris *et al.*[28] Second, there are papers that describe programs for the design of industrial plants and even for the design of a complex of interlinked plants; for example, Sparrow *et al.*[30] describe a computer package for the design of multiproduct batch plants, and Walsham[31] offers a simulation model for the design of a petrochemical complex. Finally, there are papers that handle software problems in connection with planned economies; it is clear that because of the magnitude of the problems the question of developing appropriate computer models is crucial to the success of the whole approach. About recent developments in that field not much is known to the author; one example, however, is the paper of Rudner.[32]

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7 Environmental Accountability

In earlier times man had an impact on his natural environment that was, at most, of regional importance. Usually, either all materials taken from the environment and used were given back again without further consequences – this was the case, for example, in a purely agricultural society – or the amounts taken were so small that they could be disregarded. For example, up to 1860 practically the only primary energy source was the combustion of wood. With the large-scale use of coal after 1870, the amounts of primary energy needed increased in such a way that there was no longer complete recycling of material; however, the amounts in question still were small. Of course, in earlier times there were examples of human activities that caused irreversible changes (e.g., the erosion of complete landscapes as a consequence of uncontrolled deforestation, or the exhaustion of mines). However, in these cases the consequences were, at most, of regional extent. Only in recent times have human activities developed in such a way that their influence on the environment may cause irreversible changes of global extent.

The general problem of possible irreversible global changes in the environment, as well as the question of what could be achieved in preventing such changes, is detailed here for examples of material balances existing in nature that man can disturb, namely the oxygen and carbon dioxide balance and the radiation balance of the earth–atmosphere system. These problems are well known. The point to be made here is that they have grown in such a way that their interrelations with other problems have become most important, as have effects that were earlier considered minor. As a result, they can no longer be treated in an isolated, discipline-oriented way – the application of systems analysis becomes necessary.

One of the tools that could prove useful is material accountability in a very general form, and several groups have already proposed its use. Kneese and coworkers[1] promoted the idea mainly in connection with the problem of waste disposal in industrialized societies. Drawing on his experience with a global control system based mainly on material accountability (the IAEA nuclear materials safeguards system), Häfele[2] proposes the analysis and establishment of environmental balances.

In the following sections, we will discuss some natural material balances and the influence of man on these balances. The carbon dioxide cycle of the earth will be treated quantitatively, and some policy decision problems connected with the combustion of fossil fuels will be sketched. Finally, examples of local and regional environmental accountability problems will be given.

7.1 MAN'S IMPACT ON NATURAL MATERIAL BALANCES

The oxygen and carbon dioxide cycle in nature uninfluenced by man has the following form. During the day a high rate of photosynthesis leads to the net production of O_2 and net consumption of CO_2. At night, with no photosynthesis, a net consumption of O_2 and production of CO_2 takes place. However, there is an overall net input of CO_2 and net output of O_2. The O_2 and CO_2 balances are closed if one takes into account the decomposition of dead plants. In case of complete mineralization of the plant mass, the total O_2 produced is consumed again and the total CO_2 consumed is put back into the atmosphere. Some people assume that the O_2 inventory of the atmosphere as a whole results from the formation of fossil fuels, i.e., from plant masses not completely rotted.

In a modern agricultural society there still is a net production of oxygen. On agricultural land in the Federal Republic of Germany, an average of 10 tons of O_2 per hectare per year is produced and 3.5 tons of O_2 is consumed (by animals, man, and fuels). Thus, there is a net output of 6.5 tons of O_2 per hectare per year.[3] However, if one considers the F.R.G. as a whole, one arrives at an O_2 production of 200 million tons per year and a consumption of 700 million tons (of which 600 million tons is used for the combustion of fossil fuels). Therefore, a deficit of 500 million tons of O_2 per year remains. Even if this holds only for a highly industrialized country, one may ask whether we are consuming the atmosphere's oxygen. We will come back to this point later.

The combustion of fossil fuel results in a carbon dioxide production of enormous magnitude. In 1960, 10.8 billion tons of CO_2 was released into the atmosphere all over the world. This has already resulted in a measurable increase in the atmosphere's CO_2 content. Measurements in Hawaii (i.e., at a place that is far from local CO_2 sources) indicate that the CO_2 has been increasing throughout the world by about 0.2 percent per year: to the 320-ppm current world average, 0.7 ppm is added each year.[4] To ask what consequences this may have, one must consider the CO_2 cycle in nature.[5]

Consider Figure 7.1. On land CO_2 is taken up by vegetation and stored in plants and humus. The magnitude of this reservoir is similar to that of the atmosphere, and the exchange time is probably of the order of 30 to 40 years. The ocean provides a much larger reservoir and has the potential of storing some 60 times as much

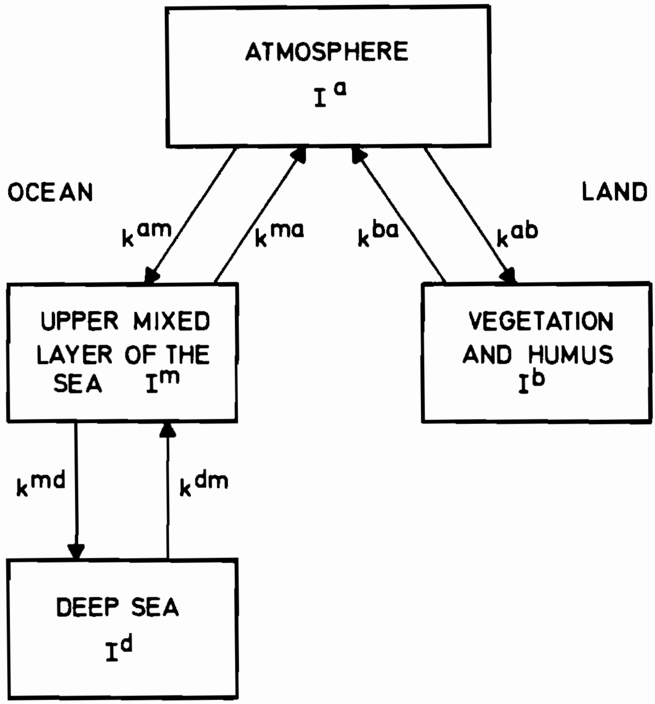

FIGURE 7.1 Natural reservoirs of carbon dioxide. I^x is the content of reservoir x: k^{xy} describes the transition from reservoir x to reservoir y. (Adapted from Sawyer.[6])

CO_2 as the atmosphere. The upper layers of the sea (above the thermocline) must, however, be distinguished from the deeper layers of the ocean. The upper layers are well mixed and are in contact with the atmosphere, but they can hold only about as much CO_2 as exists in the atmosphere. Studies of the concentration of ^{14}C, which is produced by cosmic rays in the atmosphere and subsequently decays to ^{12}C, suggest that the rate of transfer of CO_2 from the atmosphere to the upper layers of the ocean requires some 5 to 10 years for the transfer of a quantity equivalent to that in the atmosphere. Transfer to the deep ocean from the upper layers is a slower process, and as a result it would probably be a matter of centuries before the deep ocean reached equilibrium with any new level of concentration in the atmosphere.

It is estimated that at present about half the CO_2 released to the atmosphere by burning fossil fuels is kept in the atmosphere. Thus the question arises of where the rest is and what the consequences of this storage may be, and, additionally, what the consequences of the increase of the CO_2 content of the atmosphere may be. For this purpose models must be developed (and have been developed) and tested by global monitoring systems. Models for the CO_2 balance of the earth are being developed – by Fairhall[7] and by Zimen and Altenhein,[8] for example. They state

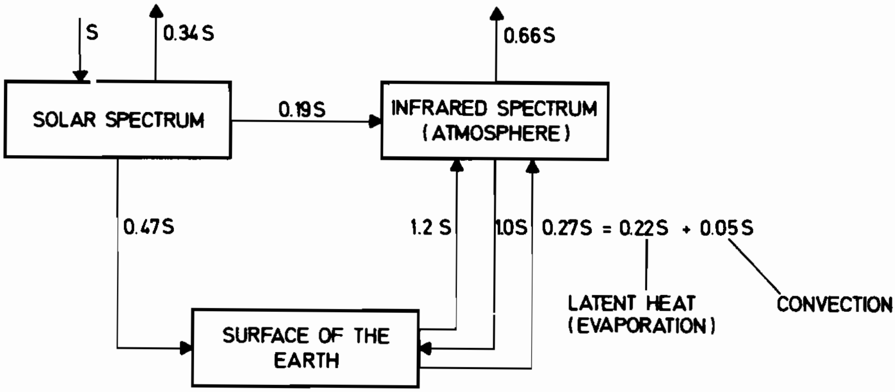

FIGURE 7.2 Distribution of solar power input. Solar input above the atmosphere $S = 340\,W/m^2$. (From Häfele.[9])

that the CO_2 will be stored in the sea, where it could have a serious effect on all calcareous organisms and on the food chains of which they are a part. Other models state that the increased CO_2 content of the atmosphere changes the radiation balance of the earth, resulting in an increase in the global average temperature (the so-called greenhouse effect[6]).

To explain this greenhouse effect, in Figure 7.2 the radiation balance of the earth is shown.[9] The solar input to the atmosphere, averaged over day and night and all zones of the globe, is $340\,W/m^2$. Roughly 47% of this energy ($160\,W/m^2$) reaches the surface of the earth; the rest is absorbed within the atmosphere or is reflected immediately. Solar radiation is absorbed within the atmosphere by the various trace gases, principally water vapor, CO_2, O_3, and molecular oxygen, and by dust and haze. Additional radiation is absorbed by liquid water droplets and ice crystals within clouds. The right-hand side of Figure 7.2 shows the radiation balance between atmosphere and earth: infrared radiation from the earth to the atmosphere and vice versa and heat losses of the earth through evaporation and convection. As early as 1863 Tyndall suggested that the blanketing effect of increased CO_2 would cause climatic changes by changing the surface temperature. Increased CO_2, because of its strong absorption (and later emission) of infrared radiation at 12 to 18 μm, would reradiate energy downward to the earth's surface and further inhibit radiative cooling at the ground surface. Radiative equilibrium models, including a convective adjustment, suggest[4] that the projected 18 percent increase in CO_2 concentration in the atmosphere by the year 2000 would result in an increase of the surface temperature of about one-half degree and a stratospheric cooling of 0.5° to 1°C at 20 to 25 km. However, these models neglect the important interacting dynamics and thermodynamics of the atmosphere, as well as the ocean–atmosphere interaction. This neglect makes the computed temperature changes very uncertain.

The possibility of verifying (or proving false) the conclusions and models described above with the help of appropriate measurements is worth consideration. In the case of oxygen, measurements have been performed since 1910. Intensive measurement from 1967 to 1970 did not show[10] that the oxygen content of the atmosphere had changed, within the accuracy of the measurements. These measurements had been proposed because it was feared that, in addition to the oxygen consumption caused by the combustion of fossil fuels, the herbicides and pesticides concentrated by the basic photosynthetic organisms could affect their population, thereby modifying the equilibrium concentration of oxygen in the earth's atmosphere. As stated before, this could not be verified by the measurements. There are some investigators who hold that the oxygen balance is regulated in completely different ways, for instance, by large forest fires.[11] In any case, in view of the risk involved, the Study Group for Critical Environmental Problems (SCEP)[4] has proposed performing measurements, with increasing accuracy, at least every 10 years.

In the case of the CO_2 system of the earth, proposals have been made to measure the three interacting systems of source, route, and reservoir. As already mentioned, sources are the combustion of fossil fuel and the release and take-up of CO_2 by the oceans. In the latter case, observations and aerial measurements have already been made in the three areas where there is a strong exchange between deep and surface water: the northern North Atlantic, the far northwest Pacific, and the Weddell Sea. Global averages are measured, e.g., in Hawaii. It has already been proposed to have at least four observation stations on the earth, far from local sources. Cost estimates have been made for a "sufficiently" dense (in time and space) global monitoring system.[4,10]

In the study of the radiation balance of the earth, satellite observations make a unique contribution to the understanding of atmospheric energetics (this is not the case for oxygen or carbon dioxide). Apart from their use as important atmospheric probes, satellites provide us with direct observations, on a real-time scale, of the distribution of various radiation parameters at the upper boundary of the atmosphere. Accurate satellite measurements can provide global distributions of the albedo as well as the absorption, emission, and net radiation balance of the overall earth – atmosphere system.[12]

7.2 A GLOBAL EXAMPLE: THE CARBON DIOXIDE CYCLE OF THE EARTH

The CO_2 cycle of the earth has already been described in connection with other natural balances. One may summarize the problems outlined there by quoting Sawyer[6]: "There is little doubt that in assessing the future level of CO_2 in the atmosphere it is important to understand fully the balance between the CO_2 in the atmosphere and the ocean."

Quantitative models for the global CO_2 cycle have been developed by several groups, with varying degrees of refinement.[7,8,13] The following analysis, which stresses the material accountability point of view along the lines of Chapter 2, follows the argumentation developed by Avenhaus and Hartmann.[14]

7.2.1 THE FOUR BOXES MODEL

The four boxes model for the CO_2 cycle of the earth may be described as follows (see Figure 7.1). There are four boxes: atmosphere (a), biosphere (b), upper mixed layer of the sea (m), and deep sea (d). At time t_i these boxes contain the CO_2 inventories I_i^a, I_i^b, I_i^m and I_i^d (measured in moles). In the time interval (t_i, t_{i+1}) parts of the inventories are exchanged; the transition from box x to box y is determined by the exchange coefficient k^{xy}. In the following we will consider 1 year time intervals, so the exchange coefficients are measured in reciprocal years (the problems associated with the choice of the appropriate time steps are treated elsewhere[14] and will not be discussed here). In addition, we have during (t_i, t_{i+1}) the CO_2 input $n_{i,i+1}$ into the atmosphere that results from the burning of fossil fuels. Therefore, according to Figure 7.1, we have the following relations for the CO_2 inventories in the different boxes at time t_{i+1}:

$$
\begin{aligned}
I_{i+1}^a &= I_i^a - k^{ab} \cdot I_i^a - k^{am} \cdot I_i^a + k^{ba} \cdot I_i^b + k_i^{ma} \cdot I_i^m + n_{i,i+1} \\
I_{i+1}^b &= I_i^b + k^{ab} \cdot I_i^a - k^{ba} \cdot I_i^b \\
I_{i+1}^m &= I_i^m + k^{am} \cdot I_i^a + k^{dm} \cdot I_i^m - k^{ma} \cdot I_i^m - k^{md} \cdot I_i^m \\
I_{i+1}^d &= I_i^d + k^{md} \cdot I_i^m - k^{dm} \cdot I_i^d
\end{aligned}
\qquad (7.1)
$$

These relations describe a transition from the state $I_i = (I_i^a, I_i^b, I_i^m, I_i^d)$ to the state I_{i+1}^t, which can be written*

$$
I_{i+1}^t = A * I_i^t + N_{i,i+1}^t \qquad (7.2)
$$

where $N_{i,i+1} = (n_{i,i+1}, 0, 0, 0)$, and where the matrix A is given by

$$
A = \begin{pmatrix}
1 - k^{ab} - k^{am} & k^{ba} & k^{ma} & 0 \\
k^{ab} & 1 - k^{ba} & 0 & 0 \\
k^{am} & 0 & 1 - k^{ma} - k^{md} & k^{dm} \\
0 & 0 & k^{md} & 1 - k^{dm}
\end{pmatrix} \qquad (7.3)
$$

Before going on we give for completeness the time-continuous version of the set (7.1) of equations, which represents a set of linear differential equations:

$$
\frac{dI^t(t)}{dt} = A' * I^t(t) + N^t(t) \qquad (7.1')
$$

where $N(t) = [n(t), 0, 0, 0]$, and where $A' := A - E$, E being the unity matrix and A being the matrix (7.3).

* The vector X is defined as a row vector; therefore, the transposed vector X^t is a column vector. Matrix multiplication is denoted by an $*$.

Let us now consider first the "preindustrial" situation (until 1860), which is characterized by

$$N_{i,i+1} = 0 \qquad \text{for all } i = 0, 1, 2, \ldots . \tag{7.4}$$

During this period the CO_2 cycle was in equilibrium, which is described in the framework of our model by

$$I_{i+1}^t = A * I_i^t = I_i^t, \tag{7.5}$$

where I_i^t is the equilibrium state and A is the matrix (7.3). This means that the equilibrium state is an eigenstate of the matrix A with the eigenvalue 1.

It can be shown, in fact, that the matrix A has the eigenvalue 1; even more, it can be shown[15] that any nonnegative square matrix in which the sums of the column vectors are 1 has the eigenvalue 1 with an associated positive eigenvector that is determined up to a positive factor. Furthermore, one sees immediately that any matrix of this form describes a *material-conserving transition*, which means that if $\Sigma(I)$ is the sum of all inventories in the boxes under consideration, then this sum remains the same after the transition $A * I$: i.e., we have $\Sigma(I) = \Sigma(A * I)$. These general properties are important because we will consider models with more than 4 boxes.

A special eigenstate of the matrix A with the eigenvalue 1 is given by

$$I = (1, k^{ab}/k^{ba}, k^{am}/k^{ma}, k^{am}/k^{ma} \cdot k^{md}/k^{dm}). \tag{7.6}$$

This means we consider a state where the atmospheric CO_2 inventory is normalized to one and where the other inventories are expressed as multiples of the atmospheric CO_2 inventory. This formula will be especially useful for the applications.

From Eq. (7.5) we find immediately that the equilibrium state I_{i+n} is obtained from the state I_i by an n-fold multiplication of this state with the matrix A, i.e.,

$$I_{i+n}^t = A^n * I_i^t, \quad n = 1, 2, \ldots . \tag{7.5'}$$

It can be understood intuitively that A^n fulfills the material conservation condition $\Sigma(I_{i+n}) = \Sigma(I_i)$, but formal proof is easy. In addition, one can show that for $n \to \infty$ the matrix A^n approaches the limiting matrix A^∞,

$$A^\infty := \lim_{n \to \infty} A^n = (\chi^t, \chi^t, \chi^t, \chi^t), \tag{7.7}$$

where χ is an eigenvector of the matrix A with eigenvalue 1 and $\Sigma(\chi) = 1$. This limiting matrix A^∞ again has the material conservation property.

Finally, it can be shown that without any outside disturbances *any* state of the system is transformed into an eigen or equilibrium state and that the speed with which this transformation takes place is of the order of the reciprocal of the smallest exchange coefficient of the system.[14] This means that if we disturb the CO_2 cycle of the earth by introducing CO_2 into the atmosphere (as a result of the burning of fossil fuels) and then stop doing it, after some time an equilibrium will again be reached – naturally with an increased overall CO_2 inventory.

So far we have considered only equilibrium states. Let us assume now that at time t_0 the system is in an equilibrium state I_0, and that from t_0 in the interval (t_i, t_{i+1}) the amount $n_{i,i+1}$ of CO_2 is released into the atmosphere for $i = 1, 2, \ldots$. Then one can describe the state of the system at time t_l, $l > 0$, by the following formula:

$$I_l^t = I_0^t + \sum_{i=1}^{l} A^{l-i} * N_{i-1,i}^t, \qquad l = 1, 2, \ldots.$$

where $N_{i,i+1} = (n_{i,i+1}, 0, 0, 0)$. This can be verified immediately by complete induction: For $l = 1$ we get,

$$I_1^t = I_0^t + N_{0,1}^t = A * I_0^t + N_{0,1}^t,$$

which is consistent with Eq. (7.2), as we assumed I_0^t to be an equilibrium state. Furthermore,

$$\begin{aligned} I_{l+1}^t &= A * I_l^t + N_{l,l+1}^t \\ &= A(I_0^t + \sum_{i=1}^{l} A^{l-i} * N_{i-1,i}^t) + N_{l,l+1}^t \\ &= I_0^t + \sum_{i=1}^{l+1} A^{l+1-i} * N_{i-1,i}^t. \end{aligned}$$

At this point it should be noted that this way of analyzing mass balances in box models has a wide range of applications in various fields at small and large scales – in fact, we have already discussed such applications in Chapter 6 (see Figure 6.3).

As an illustration of this wide variety of applications, we describe another completely different application for the analysis of mass flows in closed systems with the help of a box model, namely the oxygen exchange in a closed life-support system.[16] This system is modeled by 6 boxes (O_2 collector, man, O_2 regenerator, end product transformer, plant mass, and separation unit); the system of equations describing the time dependence of the O_2 content $x_i(t)$ of the ith box is analogous to the system (7.1) given by

$$\frac{dx^t(t)}{dt} = F * x^t + C^t(t),$$

where $x(t) = (x_1(t), \ldots, x_6(t))$, where F is a 6×6 matrix with time-independent elements, and where the vector $C(t) = (k_1(t), \ldots, k_6(t))$ represents various control functions (e.g., for the oxygen demand of the man).

In reference 16, additional mass flows (for CO_2, N, and other elements) are considered. The main purpose of the analysis is to determine conditions for stability; because of the existence of control functions, this is much more complicated than in our case; therefore, we will not go into detail but refer the reader to the original paper.

7.2.2 UNCERTAINITY CONSIDERATIONS

We have not yet discussed the validity of the assumptions underlying our model (only the question of the appropriate time steps has been mentioned). There are many sources of error that might influence the results; these errors fall into two categories:

Errors in the model (adequacy of linear relationships and of homogeneous boxes; with respect to this question see, for example, Oeschger *et al.*,[13] where eddy diffusion has been taken into account explicitly)

Errors in measurements and estimates of transition coefficients, inventories, and fossil-fuel-caused CO_2 inputs into the atmosphere

In line with our reasoning in Chapter 2, we will only discuss the second category.

First, we draw some general conclusions from the results of the preceding section. If we know that the system is in the equilibrium state, then, according to Eq. (7.6), we know the *relative* inventories if the exchange coefficients are known sufficiently well. The same holds if the equilibrium state is disturbed by an input of finite size from outside – again, we know some time after the end of the disturbance the relative inventories (which in fact are the same as before the disturbance).

Second, in order to make some statements about the *absolute* inventories, we proceed along the lines laid out in Chapter 2. We assume that the exchange coefficients are known precisely relative to the inventories and that inputs from outside, if any, are also known precisely; thus, equilibrium as well as disturbed states are described. We assume that at time t_0 the real inventories $I_0 = (I_0^a, I_0^b, I_0^m, I_0^d)$ are estimated independently. With the help of these estimates, we form estimates for the book inventories $B_1 = (B_1^a, B_1^b, B_1^m, B_1^d)$ at time t_1 in the following way, according to (7.7):

$$B_1 = A^* I_0 + N_1, \tag{7.8}$$

or, explicitly,

$$\begin{aligned} B_1^a &= I_0^a - k^{ab} \cdot I_0^a - k^{am} \cdot I_0^a + k^{ba} \cdot I_0^b + k^{ma} \cdot I_0^m + n_{0,1} \\ B_1^b &= I_0^b + k^{ab} \cdot I_0^a - k^{ba} \cdot I_0^b \\ B_1^m &= I_0^m + k^{am} \cdot I_0^a + k^{dm} \cdot I_0^d - k^{ma} \cdot I_0^m - k^{md} \cdot I_0^m \\ B_1^d &= I_0^d + k^{md} \cdot I_0^m - k^{dm} \cdot I_0^d . \end{aligned} \tag{7.9}$$

These book inventories are compared with the real inventories I_1^a, I_1^b, I_1^m and I_1^d that are measured at t_1: according to Chapter 2 the differences $MUF_1 = (MUF_1^a, MUF_1^b, MUF_1^m, MUF_1^d)$ are defined by

$$MUF_1 = B_1 - I_1, \tag{7.10}$$

and tests of significance are performed for the null hypothesis $E(MUF_1) = 0$. If there are no significant differences between I_1 and B_1 (if there are, the whole scheme has to be reviewed), then the starting inventory S_1 for the second inventory period (t_1, t_2) is estimated in the form of a "minimum variance unbiased estimate," in the sense of section 2.3:

$$S_1^x = a_1^x \cdot I_1^x + (1 - a_1^x) \cdot B_1^x, \qquad 0 \leqslant a_1^x \leqslant 1,\ x = a, b, m, d, \tag{7.11}$$

such that the variance of S_1^x is minimized. Generally, the book inventory vector B_{i+1}^t at t_{i+1} is given by

$$B_{i+1}^t = S_i^t + D_{i,i+1}^t, \tag{7.12}$$

where the "throughput vector" $D_{i,i+1}^t$ is given by

$$D_{i,i+1}^t = (A - E) * I_i^t,$$

where E is the unity matrix and where the matrix A is given by (7.3). The xth component S_i^x is determined by the recursive relation

$$S_i^x = a_i^x \cdot I_i^x + (1 - a_i^x) \cdot (S_{i-1}^x + D_{i-1,i}^x), x = a, b, m, d.$$

The variance of S_i^x is given by

$$\begin{aligned} \text{var}(S_i^x) &= a_i^{x2} \cdot \text{var}(I_i^x) + (1 - a_i^x)^2 \cdot (\text{var}(S_{i-1}^x) + \text{var}(D_{i,i-1}^x) \\ &\quad + 2 \cdot a_{i-1}^x \cdot c^x \cdot \text{var}(I_{i-1}^x)) \end{aligned} \tag{7.13}$$

where

$$c^x = \begin{cases} -k^{ab} - k^{am} & x = a \\ -k^{ba} & x = b \\ -k^{ma} + k^{md} & \text{for} \quad x = m \\ -k^{dm} & x = d \end{cases}$$

Therefore, the minimization of $\text{var}(S_i^x)$ leads to the following recursive relations of S_i^x and a_i^x (we omit the indices $x = a, b, m, d$):

$$a_i = \frac{\text{var}(S_{i-1}) + \text{var}(D_{i-1,i}) + 2a_{i-1} \cdot c \cdot \text{var}(I_{i-1})}{\text{var}(I_i) + \text{var}(S_{i-1}) + \text{var}(D_{i-1,i}) + 2a_{i-1} \cdot c \cdot \text{var}(I_{i-1})}, \tag{7.14}$$

$$\text{var}(S_i) = a_i^2 \cdot \text{var}(I_i) + (1 - a_i)^2 \cdot \text{var}(S_{i-1}) + \text{var}(D_{i,i-1}) + a_{i-1} \cdot c \cdot \text{var}(I_{i-1}).$$

For $\text{var}(I_i) = \text{var}(I)$, $\text{var}(D_{i,i-1}) = \text{var}(D)$, i.e., independent of i, the asymptotic values of $\text{var}(S)$ and a are given by the following relations:

$$a = \frac{\text{var}(S) + \text{var}(D) + 2a \cdot c \cdot \text{var}(I)}{\text{var}(I) + \text{var}(S) + \text{var}(D) + 2a \cdot c \cdot \text{var}(I)},$$

$$\text{var}(S) = a^2 \cdot \text{var}(I) + (1 - a)^2 \cdot \text{var}(S) + \text{var}(D) + a \cdot c \cdot \text{var}(I).$$

The solution of these equations is given by

$$\begin{aligned} \text{var}(S) &= a \cdot \text{var}(I), \\ a^2 \cdot \text{var}(I) + (a - 1) \cdot [\text{var}(D) + 2a \cdot c \cdot \text{var}(I)] &= 0. \end{aligned} \tag{7.15}$$

As $0 \leqslant a \leqslant 1$, we get in fact a reduction of the uncertainty of the inventories.

TABLE 7.1 Relative Inventories and Transition Coefficients for the Four-Box Model, According to Sawyer,[6] and Consistent with Eq. (7.6)

Source	I^a	I^b	I^m	I^d	k^{ab}	k^{ba}	k^{am}	k^{ma}	k^{md}	k^{dm}
Sawyer[6]	1	1.2	1.2	58	$\frac{1}{33}$	$\frac{1}{40}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{300}$
Consistent with Eq. (7.6)	1	1.21	1.2	58.06	$\frac{1}{33}$	$\frac{1}{40}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{6.2}$	$\frac{1}{300}$

It should be mentioned that alternative statistical models can be developed. Instead of the book inventory B^t defined by Eq. (7.8) we may write

$$B_{i+1}^t = A * S_i^t,$$

where the matrix A is given by (7.3) and where S_i^t is defined by Eq. (7.9). Which of the two procedures leads to a smaller variance of S_i^t depends on the numerical values of the parameters.

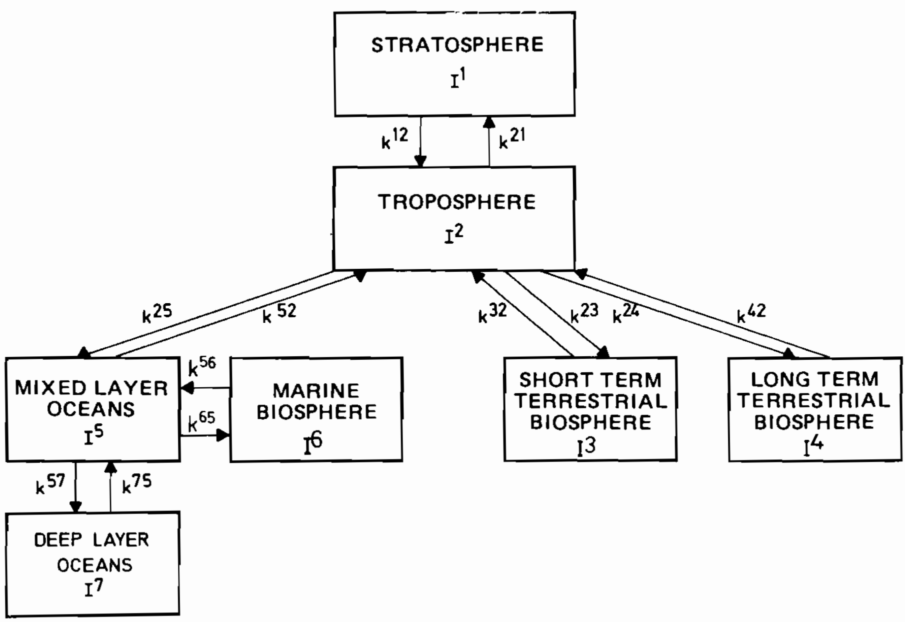

FIGURE 7.3 Natural reservoirs of carbon dioxide. I^x is the content of reservoir; x, k^{xy} describes the transition from reservoir x to reservoir y. (Adapted from Machta.[17])

7.2.3 NUMERICAL CALCULATIONS AND APPLICATIONS

As a first application of the theoretical results offered above, we will consider some carbon dioxide data reported in the literature and check their consistency by seeing whether they fulfill the condition (7.6).

In Table 7.1 the exchange coefficients and the inventories as given by Sawyer[6] are listed. It can be seen easily that these data are only partly consistent in the sense of formula (7.6). Therefore, a consistent set of exchange coefficients is also given in Table 7.1. However, it should be noted that this set cannot be determined uniquely. We have changed the coefficients such that as few data as possible had to be changed and, in addition, such that the inventory of the deep sea, the value of which is consistent with data reported by Zimen and Altenhein[8] and Machta,[17] remain unchanged.

Figure 7.3 illustrates the seven-box model developed by Machta[17]; this model takes into account the following reservoirs:

Stratosphere (1)	Mixed-layer oceans (5)
Troposphere (2)	Marine biosphere (6)
Long-term terrestrial biosphere (3)	Deep-layer oceans (7)
Short-term terrestrial biosphere (4)	

Without writing down the system of equations that corresponds to the system (7.1) and that can be derived immediately from Figure 7.3, we give here only the equivalent of formula (7.6), i.e., the relative size of the inventories in the equilibrium state:

$$I = (1, k^{12}/k^{21}, k^{12}/k^{21} \cdot k^{23}/k^{32}, k^{12}/k^{21} \cdot k^{24}/k^{42}, k^{12}/k^{21} \cdot k^{25}/k^{52},$$
$$k^{12}/k^{21} \cdot k^{25}/k^{52} \cdot k^{56}/k^{65}, k^{12}/k^{21} \cdot k^{25}/k^{52} \cdot k^{57}/k^{75}). \qquad (7.6')$$

Table 7.2 shows the data given by Machta[17] together with those data that would be consistent with formula (7.6′). The large differences with respect to the exchange coefficient k^{dm} (Machta's k^{75}) between Sawyer's and Machta's data will be important later.

As a second application, we ask how fast the system will return to the equilibrium state after a disturbance from outside. As already mentioned, the speed of approaching the equilibrium is determined by the reciprocal of the smallest exchange coefficient. A numerical illustration is given in Tables 7.3 and 7.4, where the convergence of the matrix A^n toward the matrix A^∞, defined by Eq. (7.7), is demonstrated both for Sawyer's and for Machta's data. According to Sawyer's data, nearly 300 years is needed for reaching a new equilibrium, whereas according to Machta's data nearly 1,600 years is needed. As the consistency relations (7.6) or (7.6′) are not sufficient to determine k^{dm} uniquely unless all other inventories and exchange coefficients are known precisely, it would be extremely interesting in view of the problems connected with the CO_2 cycle to have more and better data.

TABLE 7.2 Relative Inventories and Exchange Coefficients for the Seven-Box Model, According to Machta,[17] and Consistent with Eq. (7.6′)

Source	I^1	I^2	I^3	I^4	I^5	I^6
Machta[17]	1	5.7	11.1	0.7	30	0.2
Consistent with Eq. (7.6′)	1	5.75	11.2	0.6	30.4	0.22

Source	I^7	k^{12}	k^{21}	k^{23}	k^{32}	k^{24}
Machta[17]	366.7	$\frac{1}{2}$	$\frac{1}{11.5}$	$\frac{1}{24.4}$	$\frac{1}{40}$	$\frac{1}{19}$
Consistent with Eq. (7.6′)	366.7	$\frac{1}{2}$	$\frac{1}{11.5}$	$\frac{1}{24}$	$\frac{1}{47}$	$\frac{1}{19}$

Source	k^{42}	k^{25}	k^{52}	k^{56}	k^{65}	k^{57}	k^{75}
Machta[17]	$\frac{1}{2}$	$\frac{1}{1.1}$	$\frac{1}{5.9}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{50}$	$\frac{1}{1600}$
Consistent with Eq. (7.6′)	$\frac{1}{2}$	$\frac{1}{1.25}$	$\frac{1}{6.6}$	$\frac{1}{273.8}$	$\frac{1}{2}$	$\frac{1}{132.8}$	$\frac{1}{1600}$

TABLE 7.3 Convergence of the Matrix A^n toward A^∞ (Defined by Eq. 7.7) for the Four-Box Model, Based on Sawyer's Data[6]

A				A^{100}			
.7697	.0250	.1667	.0000	.0241	.0466	.0200	.0153
.0303	.0975	.0000	.0000	.0564	.1598	.0374	.0156
.2000	.0000	.6720	.0033	.0240	.0370	.0216	.0188
.0000	.0000	.1613	.9967	.8954	.7567	.9210	.9502
A^{200}				A^∞			
.0174	.0209	.0167	.0160	.0163	.0163	.0163	.0163
.0253	.0414	.0223	.0189	.0197	.0197	.0197	.0197
.0201	.0221	.0197	.0193	.0196	.0196	.0196	.0196
.9373	.9157	.9413	.9458	.9444	.9444	.9444	.9444

TABLE 7.4 Convergence of the Matrix A^n toward A^∞ (Defined by Eq. 7.7) for the Four-Box Model, Based on Machta's Data[17]

A				A^{100}			
.7697	.0169	.0455	.0000	.0881	.0991	.0844	.0065
.0303	.9831	.0000	.0000	.1777	.3226	.1540	.0065
.2000	.0000	.9469	.0006	.3709	.3776	.3606	.0317
.0000	.0000	.0076	.9994	.3634	.2007	.4010	.9554
A^{1000}				A^∞			
.0169	.0173	.0168	.0158	.0159	.0159	.0159	.0159
.0310	.0320	.0308	.0282	.0285	.0285	.0285	.0285
.0740	.0755	.0736	.0694	.0699	.0699	.0699	.0699
.8781	.8752	.8787	.8867	.8857	.8857	.8857	.8857

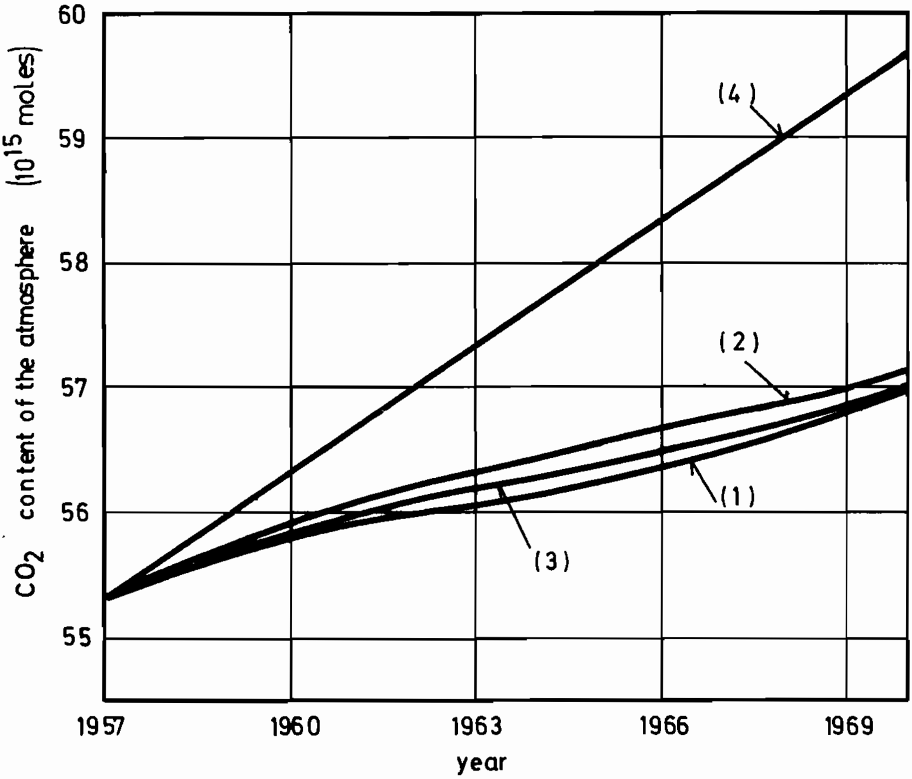

FIGURE 7.4 Comparison of measured and theoretical data for the CO_2 content of the atmosphere. (1) Experimental data after Keeling.[19] (2) Theoretical data using formula (7.1). (3) Theoretical data using the Runge–Kutta procedure for solving the system of differential equations (7.1′). (4) CO_2 from burned fossil fuels kept in the atmosphere.

As a third application, we compare the data on the CO_2 content of the atmosphere from 1958 to 1970 as measured by Keeling[19] at Mauna Loa with the theoretical values obtained from formula (7.8) on the basis of Sawyer's and Machta's data. To be able to do this, we take, after Fairhall,[7] for the CO_2 content of the atmosphere

$$I^a_{1958} = 312 \text{ ppm.}^\dagger$$

We also take, after Keeling,[19] for the annual input n^a of CO_2 into the atmosphere as a result of the burning of fossil fuels

$$n^a_{1958} = 0.248 \cdot 10^{15} \text{ moles/yr}$$

$$n^a_{1970} = 0.425 \cdot 10^{15} \text{ moles/yr.}$$

† An increase of 1 ppm/year corresponds to an increase of 5.64×10^{15} moles/year.

It is clear that if we start at t_0 with the equilibrium inventories and add in (t_0, t_1) the amount $n_{0,1}$ of CO_2 to the atmosphere, then according to Eq. (7.1) we have at t_1 an atmospheric inventory $I_1^a = I_0^a + n_{0,1}$. Only after t_1 is transport of the additional CO_2 into the other boxes started, because now the relative biospheric and mixed sea inventories are smaller than before and therefore, according to Eq. (7.1), the backflow into the atmosphere is smaller.

The results of these calculations are represented in Figure 7.4, along with the CO_2 content of the atmosphere that would result if all CO_2 from burned fossil fuels remained in the atmosphere (curve 4). One sees that the material balance model (curve 2) describes the measured data (curve 1) much better. It should be noted that curve 2 is obtained (within drawing accuracy) for both Sawyer's and Machta's data. One also sees that curve 3, based on the continuous version [Eq. (7.1′)] of Eq. (7.1), has more or less the same form as that based on the discrete version.

With the help of these calculations, we may discuss the question of uncertainty. According to Eq. (7.10), we have

$$MUF_i^a = I_{i-1}^a - I_i^a - (k^{ab} + k^{am}) \cdot I_{i-1}^a + k^{ba} \cdot I_{i-1}^b + k^{ma} \cdot I_{i-1}^m + n_{i,i+1}.$$

Now, using the data given above, we get for the years 1958 to 1970

$$I_{i-1}^a - I_i^a = -0.002 \cdot I_{i-1}^a \qquad \text{and} \qquad n_{i,i+1} = 0.007 \cdot I_{i-1}^a$$

Therefore, in the ideal case we should get, with $MUF_i = 0$;

$$-(k^{ab} + k^{am}) \cdot I_{i-1}^a + k^{ba} \cdot I_{i-1}^b + k^{ma} \cdot I_{i-1}^m = -0.005 \cdot I_{i-1}^a.$$

Inversely, if we have annual inventory estimates, we can check whether the null hypotheses $E(MUF_i) = 0$ have to be rejected (i.e., whether the model and the data are correct).

So far, little information has been given in the literature about the uncertainty of transition coefficients and inventories.[20,21] One might argue that because of this very lack of data statistical analyses are mandatory; in fact, if we assume

$$\text{var}(I^a) = \text{var}(I^b) = \text{var}(I^m),$$

then we get for the asymptotic state, according to Eq. (7.15) and with Sawyer's data[6]

$$k^{ab} = \frac{1}{33}, \qquad k^{am} = \frac{1}{5},$$

the result

$$\text{var}(S^a) = 0.383 \cdot \text{var}(I^a).$$

That is, the standard deviation of the atmospheric CO_2 inventory is reduced by a factor of three.

As a last application, we ask what – according to our model – the asymptotic value of the carbon dioxide content of the atmosphere would be if all of the fossil fuel now known were to be burned. According to Zimen,[8] this would correspond to a final cumulated input of $N = 600 \cdot 10^{15}$ moles. If we start with $I_0^a = 51.4 \cdot 10^{15}$

moles in preindustrial time, i.e., until 1860,[18] then we obtain with Sawyer's data

$$I_0^b = 62.2 \cdot 10^{15}, I_0^m = 61.7 \cdot 10^{15}, I_0^d = 2{,}985.4 \cdot 10^{15} \text{ moles.}$$

This gives a total inventory I_0 of

$$I_0 = 316.07 \cdot 10^{15} \text{ moles.}$$

To answer our question, we have to add to this inventory the CO_2 from the burned fossil fuels and distribute this total inventory according to Eq. (7.6). The result is

$$I_\infty^a = 61.2 \cdot 10^{15}, \; I_\infty^b = 74 \cdot 10^{15}, \; I_\infty^m = 73.4 \cdot 10^{15}, \; I_\infty^d = 3{,}552 \cdot 10^{15} \text{ moles.}$$

This means that in the asymptotic state $567 \cdot 10^{15}$ moles from the $600 \cdot 10^{15}$ moles go into the deep sea and furthermore that the atmospheric content rises from the 1958 value of 312 ppm to 345 ppm in the asymptotic state.

These results, together with those for the speed of convergence, are especially interesting in view of proposals that have been made recently, namely the direct introduction of the CO_2 from the burned fossil fuels into the deep sea.[22] If such a scheme were feasible, then the figures given above indicate what fraction of the buried CO_2 will re-enter the atmosphere and at what speed. Inversely, if all the CO_2 from the burned fossil fuels were released to the atmosphere, one gets an idea of how long the CO_2 would stay in the atmosphere until it went into the deep sea; in other words, one gets an idea of how long mankind has to live with an atmospheric CO_2 content higher than the equilibrium value.

In this connection, one final remark should be made. Up to now practically all atmospheric CO_2 research has concentrated on the dynamics of the undisturbed state as well as on the future development of the disturbed state. Therefore, the question of whether a final equilibrium concentration of 345 ppm of the atmosphere would be tolerable has not yet been treated comprehensively from a decision theoretical point of view, even though some work along this line has been initiated recently.[23,24] It is clearly necessary that in these analyses that are ultimately meant to help in preparing policy decisions concerning such sensitive issues as fossil fuel consumption one must explicitly take into account uncertainties of the knowledge, as well as random variations in flows and inventories.

7.3 REGIONAL AND LOCAL EXAMPLES

We have considered *global* natural material balances that are more and more influenced by human activities. We started from an equilibrium state and treated human influence as a disturbance of this (preindustrial) equilibrium state. In this section we will consider the accountability of pollutants on *regional* and *local* levels. This means that we do not have a natural equilibrium that gives us a measure for the orders of magnitude in question; rather, the analyses as well as the measures to be taken have to be based completely on what is observed and on the consequences of relevant throughputs and inventories. Again we will illustrate the ideas by means of a specific example with a rich data base.

7.3.1 REGIONAL SULFUR DIOXIDE ACCOUNTABILITY

As a consequence of the burning of fossil fuels, sulfur is released into the air in the form of sulfur dioxide, which represents – unlike CO_2 – an immediate hazard to human beings.[25] It has, therefore, been the subject of major research.

In order to perform a SO_2 accountability for a region, we consider the air volume over that region as a material balance area in the sense of Chapter 2. Establishing a material balance for the atmospheric SO_2 in this material balance area means determining

The inputs over a given period of time (i.e., all the inputs into that volume)

The outputs over a given period of time (i.e., the removal by washing out by rainfall, decay, and so on)

The physical inventory at the beginning and at the end of an inventory period

The inputs can be measured at least in cases where there are point sources, and, they can be estimated fairly accurately if they come from spread sources (e.g., from households or from transportation). The inventory can be measured by monitoring

TABLE 7.5 SO_2 Data[a]

General		
Natural SO_2 concentration ($\mu g/m^3$)		10
Limit for long-term SO_2 concentration		
VDI ($\mu g/m^3$)		230
U.S. EPA ($\mu g/m^3$)		80
SO_2 residence time in the atmosphere (days)		1–6
Ruhr area		
EMISSION (10^6 tons/yr)		
Power plants		80·0.44
Residential		80·0.13
Industry		80·0.64
Transportation		80·0.01
TOTAL		80·1.22
INVENTORY ($\mu g/m^3$)		
Power plants	80·1.79	80·13.7
Residential	80·0.53	80·4.0
Industry	80·2.60[b]	80·20.0[c]
Transportation	80·0.64	80·0.3
TOTAL	80·4.96	80·38

[a] Data from Büker *et al.*[26] [b] At a wind speed of 4.5 m/sec.
[c] On the seventh day of a temperature inversion.

systems like those established in certain regions (Los Angeles and the Ruhr area, for example). The outputs cannot be measured directly. One may argue that the outputs can be evaluated over a fixed period by determining the difference between beginning inventory plus inputs and ending inventory; however, such a determination does not meet the requirements of the material balance, as full knowledge of the processes involved is impossible.

To solve this output problem in the case of environmental accountability, one needs models that describe the processes of pollutant formation and decay in the atmosphere, as well as the washing out of pollutants from the atmosphere. Once such models have been fully established, they can take the place of the missing "output" measurements. However, such models must be tested, and in principle they can be tested only if a complete measurement system is available. This problem can be solved only by an iterative process.

Work on models of the type postulated has already begun. In Table 7.5, which summarizes calculations performed by Büker *et al.*,[26] results from a crude model for the Ruhr area are shown. They correspond satisfactorily with the results of measurements. A similar but more detailed study of the SO_2 contained in the air volume above Great Britain has been published only recently.[27] More complex models are being established, for the Tennessee Valley region,[28,29] for example, where the emphasis is on radioactive discharge. An interesting attempt to balance the lead contained in the air over a given region (Southern California Bight) was published in 1974.[30]

Once the SO_2 balance for a region is established, questions similar to those mentioned in the foregoing section again arise. What are tolerable levels of the atmospheric SO_2 concentration, and how can these levels be achieved in an optimal way?

It is clear that the best solution of the atmospheric SO_2 problem would be to stop the emission, i.e., to desulfurize oil and coal directly. However, in the case of coal this is not possible, and in the case of oil it is very expensive: it has been estimated that the reduction of sulfur emissions by 1 ton of SO_2 would cost about $1000 (U.S.).[31] Additionally, there is the question of what could be done with the enormous amounts of sulfur that would become available in case of complete desulfurization of raw oil in the refineries: 50 million tons of sulfur would be obtained annually this way, and the world market amounts to only about 30 million tons.[32]

Thus, on the basis of the physical properties of atmospheric SO_2 throughputs and inventories (analyzed with the help of material accountability considerations) the trade-off between technological possibilities, economic necessities, and human health hazards has to be faced once again.

7.3.2 SULFUR ACCOUNTABILITY IN POWER PLANTS AND REFINERIES

The control of the SO_2 input into the atmosphere of a region means primarily the control of the SO_2 emissions from refineries and fossil-fueled power stations. This means, however, that we now go back to the plant level: one plant is considered as a material balance area in the sense of Chapter 2, and sulfur is the material to be safeguarded.

Consider the case of an oil-fired power plant. Sulfur comes in with the oil and leaves the plant through the stack in the form of SO_2 together with the offgases if there are no filters to trap it. If there are filters, then a certain percentage of the sulfur is removed from the offgas; however, it is kept in the filters, and one has to ask where the sulfur goes from here. If oil is desulfurized – which transforms a power plant into a chemical facility[32] – then the sulfur is kept in the form of chemical compounds before the oil goes into the process, and one has to ask what happens to these chemical compounds. In any case, it is important to observe the flow of the sulfur, including the final deposition of the sulfur compounds removed from the offgas or from the oil. Otherwise, one would have kept the sulfur out of the air but sent it eventually in the form of chemical discards into the groundwater; in other words, one would not change the final effect.

It is obvious that the establishment of the material balance for sulfur in an oil-fired power plant is very similar to the case, for example, of the plutonium material balance in a reprocessing plant. Thus, in this context, too, integral experiments have already been proposed; according to Kneese *et al.*[33]: "Our knowledge of the flow of materials through the economic system and their loss or purposeful discharge to the environment is extremely limited, especially with respect to the industry." The final element in verification of the material balance through inspections – so important in safeguards – may at present not be relevant to the discussion of pollutant accountability. Only if strict standards for the emission of pollutants are set, standards that impose an economic burden on the plant operators, the question of deliberate illegal release of pollutants and, therefore, of the verification of data may become important.

7.3.3 NOTE ON SPECIAL METAL BALANCES

In recent years one particular area of environmental accountability has become of major interest – the establishment of balances for special metals used in all advanced economic systems. Weise[34] has pointed out the need for such a balance for mercury, and several countries have begun to implement such balances for various metals that have an impact on the environment and therefore on man or that pose a direct hazard to human health if not controlled properly. A recent publication[35] reports a material balance scheme for cadmium, but other elements – particularly mercury – have also been the subject of study.

The idea of this special approach is to analyze carefully the flow of the specific element under consideration in a national economy: physical inventory at a given point in time (clearly, only the material in the human environment and not the minerals in the ground), imports, exports, releases to air and water, and finally physical inventory at the end of the inventory period. With the help of these balances one can investigate whether one really knows where the material flows, where it comes to rest, whether these flows, especially the releases to the environment, are in accordance with the relevant regulations, and finally, whether the burden on man and environment imposed by these materials is tolerable.

It is clear that here the long-term aspects of the use of specific materials are considered, and not the timely detection of any illegal use of processing. Therefore, it seems natural to take 1 year as an inventory period. It is also clear that a national economy represents the appropriate material balance area because imports and exports can be measured easily. A major problem is the measurement of the many small mass flows related to small processing units, like laboratories, and especially the releases to the environment.

In the papers mentioned earlier there are illustrations of the difficulties of determining the uncertainty of the mass balances to be established. Therefore, only very large amounts missing could be detected with a reasonable probability under present conditions. On the other hand, at the moment there exists no quantitative theory about the significance of missing amounts of material; in other words, one still has to develop ideas about what amounts of mercury or cadmium, if missing,

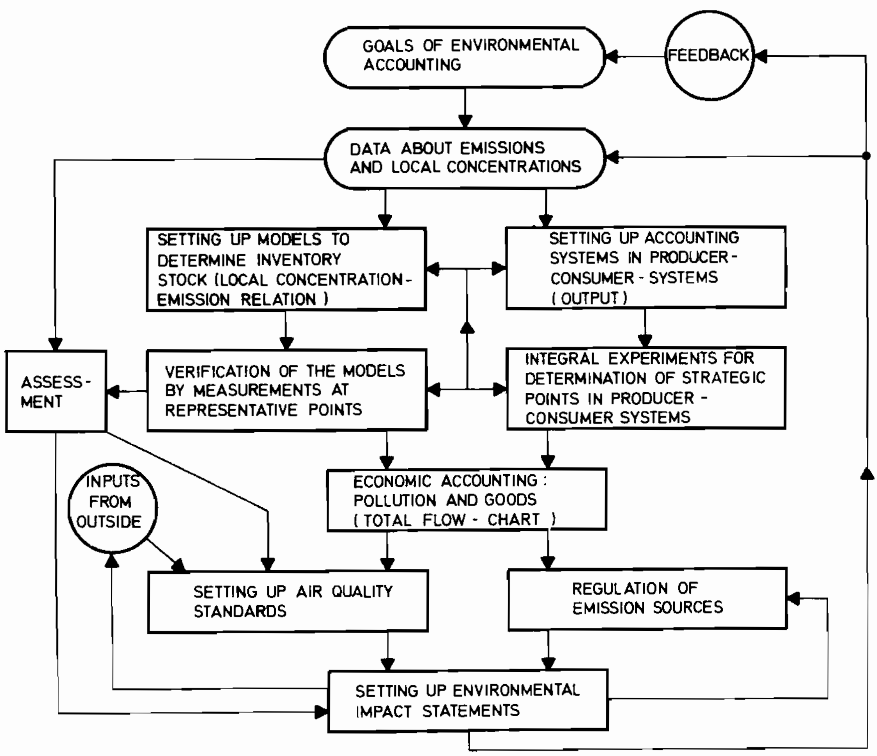

FIGURE 7.5 Environmental accountability. (From Gupta *et al.*[36])

cannot be tolerated by a national economy. It is clear that in this important area work has only just begun, and one must hope that the success of mass balance considerations elsewhere will stimulate progress here as well.

7.4 CONCLUSION

We have surveyed natural material balances, and we have seen how human activities increasingly disturb these balances on a global scale. Furthermore, we have looked into environmental problems of regional and local scale, and we have demonstrated that the idea of material accountability, which has been propounded by several groups and which has proven successful in the case of nuclear material safeguards, could be valuable in diverse situations in the natural environment and in the economy.

Figure 7.5 represents in the form of a flowchart the complex interrelations among the main aspects of the problem of environmental balances:

- Material balance models
- Verification of the models by measurements
- Technology and economics

In the preceding chapters examples of these individual aspects have been given. The diagram illustrates the importance of achieving an appropriate balance of theoretical models, of monitoring systems, and of environmental standards of any kind.

It seems too early for an international accountability system, for example, for pollutants, although this has already been envisioned by some groups.[2,33] Even if the data base exists, there may be difficulties of access to it and problems in organizing a worldwide reporting system. In this connection it is important to realize that in a specific case such an international accountability system does exist, that it has proven workable for the plant operators and for the safeguards authority, and that it can be implemented with a reasonable budget (in 1974 the staff numbered 100 and the budget amounted to $3 million, including R&D).

It must be stressed that international political considerations urgently called for setting up the IAEA safeguards system. At present the issue of environmental pollution control may not be seen as so urgent, so it is not surprising that the international demand that is a prerequisite for the establishment of an international safeguards system is lacking. Even with radioactive waste discharges, whose danger is universally acknowledged, the problems of establishing a register were great. Despite political unwillingness to recognize the patent realities of the situation, it is vital that steps be taken immediately to ameliorate this problem, whose magnitude increases daily.

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8 Arms Control

Arms control measures as a means of reducing political tensions entered the international political scene in the years after the First World War. At that time, most of the effort was devoted to finding formulae capable of expressing equivalencies that were negotiable. Thus, at the Washington Conference on the Limitation of Naval Ships (1921–1922), for instance, a formula for capital ships was agreed upon, but little attention was given to the need for information about compliance or the preparation of responses in the event of noncompliance.

After the Second World War, when arms control discussions started again, the question of possible verification measures was raised immediately – partly perhaps because disarmament had failed to prevent the Second World War and because clandestine preparations for a war had been carried out so effectively by some nations. There are many more political and technical reasons, but they will not be discussed here; it is sufficient for our purposes to note that today security in disarmament is identified with adequate inspection. At this point those aspects discussed in Chapter 5 become relevant – for instance, what is the minimum level of inspection consistent with security requirements on one hand and with acceptability on the other?

Even though no large-scale implementation of the control of conventional arms has been achieved so far, it is widely held that some kind of accountability measures will be used if arms control or disarmament verification agreements become effective. Therefore, and simply because of the vital importance of these problems, we will describe some ideas developed so far.

In this chapter, we will offer some remarks about the problem as a whole, and then discuss some earlier ideas about accountability and inspection measures. Finally, we will report on a model in which all the aspects of the verification of arms reduction measures in Central Europe, at least, have been tackled.

It should be noted here that in this chapter we use the term "verification" in a much more general sense than in the foregoing chapters, where we limited its use to the control of data generated by the inspected party. Here, we will use this word for all measures taken in order to get any information about compliance with the provisions of arms control and disarmament treaties and agreements. This is the usage employed by the U.S. Arms Control and Disarmament Agency,[1] who have given an excellent outline of the political problems connected with the ideas to be discussed in this chapter.

For those interested in the current state of arms control and disarmament the journal *Arms Control Today*[2] is recommended as a source for general references. Detailed analyses of the various aspects of the whole problem are given in a special issue of *Daedalus,*[3] and information about technical means may be found in Greenwood.[4] Saaty[5] and especially Mathematica[6] offer a unique collection of mathematical models for a series of specific problems of this field.

8.1 SURVEY ON ARMS CONTROL ANALYSES

We have already demonstrated in Chapter 4 that one cannot analyze verification measures effectively if one does not analyze – at least in a cursory way – the whole problem, i.e., those arms control and disarmament measures that are to be agreed upon and that have to be verified. Therefore, we first discuss some of the current ideas in that field and then some verification models.

8.1.1 MODELS DESCRIBING ARMS CONTROL AS A CONFLICT SITUATION

In order to determine the frame or, in the language of Chapter 4, the "global parameters" of a verification system, it is necessary to have an idea of the possible gains and losses of the parties who are considering entering into an arms control agreement, even if these gains and losses can be described only very vaguely.

In 1963, Maschler developed a very detailed model to analyze some aspects of a treaty between the United States and the Soviet Union concerning a ban on testing nuclear weapons.[7] The model treated the subjective beliefs and predictions of the decision-making authorities of the United States on the basis of a two-person non-zero-sum game. For this purpose an attempt was made to order the subjective utility payoffs for the various policies that could be adopted by the two parties to the game, and some conclusions were reached about whether and under what conditions it would be advisable for the United States to sign such a treaty.

Whereas Maschler based his analyses on a mathematical model, Saaty[8] discussed in general terms the possible advantages and disadvantages of an arms control agreement for a state. He began with an evaluation of the expense of weapons systems compared to their destructive power, continued with considerations on the value of human life, and ended with such qualitative criteria as reduction of international tensions and improvement of the capacity to develop in peaceful directions by establishing machinery for handling conflicts by means other than war.

It seems to be a hopeless enterprise to translate the idealistic goals of arms control and disarmament as expressed above into a concrete bilateral or multilateral agreement that includes quantitative details about verification measures. In fact, no quantitative models are known that have made an attempt of this kind and thus can offer algorithms that are derived from first principles and that are able, among other things, to determine necessary effort. In this connection, it seems worthwhile to mention the ideas on *mathematical style* and *political style* expressed in ACDA/ST −37:[9]

> The mathematical style modelizes the questions of inspection effort, cheating strategies, effectiveness within a given, closed, fully delineated set of conditions. These questions are studied, then, as a quantifiable relationship between the number of opportunities to violate and the number of inspections allowed. The political style examines the psychological and political milieu within which the decisions to cheat or comply are made. Therefore, such complicating variables as incentive to cheat, incentive to detect, longer-term objectives, deterioration of existing weapons systems, political repercussions, and the domestic political system must be taken into account when assessing how much inspection is needed.

We realize that it was exactly in the sense of the mathematical style as defined here that we assumed in our analysis of the nuclear material safeguards system that the values of some global parameters are fixed by political decisions and that thereafter the systems analyst optimizes the available resources according to some derived criterion. We will come back to this point once more later on.

8.1.2 MATERIAL ACCOUNTABILITY AS AN ARMS CONTROL VERIFICATION TOOL

The major part of the published literature dealing with arms control verification measures considers nuclear weapons test ban verification measures. This may result from the fact that such agreements indeed were reached and therefore called for detailed analyses. Furthermore, the verification measures in question here (for instance, installation of seismic equipment) lent themselves in a comparatively natural way to mathematical treatment: the determination of the necessary number of stations and their optimal distribution such that any clandestine weapons test is detected with a given probability is a well-defined mathematical problem.

In addition to these test ban analyses, there are several approaches to other, more general arms control and reduction schemes, and here material accountability considerations enter the scene at a very early stage. There is one example[10] in which a factory is considered that produces some materials that are of strategic importance; the factory is subject to control in the form of a monitoring system that guarantees an account of all the material entering and leaving the factory by applying the accountability principles already well known to us.

There is another example where a pure record-report system forms the basis of a verification system[11] and where the consistency of the records that are generated by the inspected party and that are reported to the control authority is taken as a criterion of compliance with an arms control agreement. Clearly, this means that some material or troop unit that has been recorded as having left one place must be recorded as having arrived at another place – again the old principle in a somewhat different form. The same reasoning as used in the case of the test ban agreements may explain why there has not been more effort devoted to these important questions. Up to now there are not agreements on the control or reduction of conventional military forces that are subject to verification procedures.

8.1.3 DIRECT INSPECTIONS

Next to test ban verification measures, direct inspections have been a major object of mathematical studies.[6] Here, some very detailed game theoretical models on specific aspects have been developed that have gained the attention of a wide scientific audience. In this connection it should be noted that formulae of the type of the formulae (3.40) have been developed independently in the course of such studies.[12]

8.1.4 CONCLUSIONS

Let us summarize this section by saying that there exists a large literature on arms control and disarmament verification measures both in the mathematical and in the political style. Probably much more work has been done that has not yet been published in the open literature. There is still some lack of realistic applications; this is a result of the fact that international verification systems have not yet been installed. However, this would change immediately as soon as agreements of this kind were negotiated.

8.2 VERIFICATION OF MUTUAL BALANCED FORCE REDUCTION IN CENTRAL EUROPE

In June 1968, the North Atlantic Treaty Organization (NATO) and the Warsaw Pact began discussions about Mutual Balanced Force Reduction (MBFR) in Central Europe. Since that time there have been ongoing discussions about this issue, and formal talks opened in Vienna in 1973. Even though there is still no final agreement in sight in Vienna, and the question of verification has not yet been discussed formally, some unilateral efforts have been made to analyze the effect of specific verification measures that one might imagine to be negotiable. As these analyses contain all the elements of material accountability and its verification that have been described in the earlier chapters of this monograph, we will merely sketch them here. In so doing, we strictly adopt the mathematical style, i.e., we do not discuss the whole problem with all its political implications, but concentrate on one aspect, namely, the technical problems of verification.

In the following we will report briefly on some MBFR models that have been developed so far. We will then discuss the problem of verification of MBFR agreements in some detail. It will be shown that "material" accountability procedures could indeed play an important role in this area.

8.2.1 MBFR MODELS

Analyses on the subject of MBFR in the political style may be found in *Daedalus*[3] and especially in Volume 6 of *Arms Control Today*;[2] the latter contains an updated list of references from which one may get an excellent impression of the current state of MBFR. We will not discuss these analyses here but instead try to give an idea about models in the mathematical style that have been reported by Bellany;[13,14] these models seem to give a kind of quantitative framework or reference for what will be said about verification procedures.

A first approach is to assume that there exists a balance that can be represented by an equation of the form

$$g(M_e, T_e, P_e, \ldots) = f(M_w, T_w, P_w, \ldots),$$

where g and f are the warmaking capabilities of the Eastern (e) and Western (w) forces, respectively, and are functions of troops (M), tanks (T), aircraft (P), and the like, deployed by each side. (We will not discuss here what is meant by warmaking capability, but we will come back to this point in the next section.) It follows from this equation that after force reductions ΔM, ΔT, and so on, there will be a new balance if

$$\Delta M_e / \Delta M_w = \partial M_e / \partial M_w$$

and so on, where ΔM_e and ΔM_w represent small reductions in the number of troops of East and West, respectively.

The value of $\partial M_e / \partial M_w$ can be determined only if the functional relationship between troop numbers on each side is known, but because of insufficient published data this problem cannot be tackled directly. However, if one made the additional assumption that the existing level of forces came about via an arms race process obeying either the Richardson model[15] or the less crude and more recent variant of Caspary[5] and if one also assumed that the levels of forces had reached a stage of zero growth rate, then one would get

$$M_e = k \cdot M_w \text{ (approximately)},$$

and one would get the following relation for small balanced reductions in troops:

$$\Delta M_e / M_e = \Delta M_w / M_w$$

which means that balanced reductions should take place in the form of equal percentage reductions.

A second approach is to assume that there exists a stable balance between the two opposing forces in the particular sense that an actual outbreak of hostilities would lead to no advantage, initially at least, to either side. In other words, one tries to estimate the initial losses in a battle and applies the results to peacetime reductions. Because of unavailability of published data, one has to attempt to describe mathematically the course of a very simple battle.

One way of doing this is to regard the battle as a tank battle and to begin by writing the rate of tank losses as

$$dT_e/dt = -p \cdot T_e \cdot T_w; \qquad dT_w/dt = -q \cdot T_e \cdot T_w,$$

where $T_e \cdot T_w$ is a measure of the encounter probability of one tank with an enemy tank; and p and q are measures of the respective rates at which West's tanks destroy East's tanks and vice versa. Then losses ΔT_e and ΔT_w in the first moments of the battle emerge as

$$\frac{\Delta T_e}{T_e} = \frac{\Delta T_w}{T_w} \cdot \frac{p}{q} \cdot \frac{T_w}{T_e}.$$

At this stage, following Bellany,[13] we need to make a further assumption about our tank battle, namely that the numerically larger tank force, East's, is the attacker. It then becomes possible to assign a value to the ratio p/q, provided we accept the fairly widely held belief that tanks can more effectively destroy other tanks from a defensive position than they can themselves be destroyed by attacking tanks. If current NATO tank strength in the Central Region of Central Europe is assumed to accurately reflect this superiority of the defense in the sense that $p/q = T_e/T_w$, then the losses (and therefore the appropriate peacetime reductions) become

$$\Delta T_e/T_e = \Delta T_w/T_w.$$

If on the other hand, p/q is taken to be a fixed number on average, derivable from the study and observation of historical conflicts and weapons trials, and it seems that students of such matters put its value somewhere between 3 and 4 in this context, then we get

$$\frac{\Delta T_e}{T_e} = \frac{\Delta T_w}{T_w} \cdot r \cdot \frac{T_w}{T_e}$$

where $3 < r < 4$. Inserting now the actual ratio of the NATO to Warsaw Pact strengths in the Northern and Central regions of Central Europe together, one obtains

$$\Delta T_e/T_e = s \cdot \Delta T_w/T_w,$$

where $1.1 < s < 1.4$.

There are more models of this kind, but we will stop the description here because the intention was only to give an idea of the ways of thinking that could lead to reduction figures – a problem similar in magnitude to that of fixing absolute figures for the inspection effort. We also will not discuss here the validity of assumptions and the many caveats that have to be observed at any stage of such an analysis; instead we will concentrate on the verification aspects of any arms reduction agreement.

8.2.2 FORMULATION OF THE VERIFICATION PROBLEM

Let us now assume that an arms reduction is agreed upon in the following form. A fraction of the *stationed military forces* (troops and heavy equipment) is withdrawn from a certain area in Central Europe, which we call the *reduction area*. Furthermore, an upper level for the military forces is agreed upon that cannot be passed in the future; any attempt at increasing the forces beyond this level would be considered an act of noncompliance with the agreement.

It is clear that the reduction itself, i.e., the withdrawal of forces, need not be subject to sophisticated verification measures, as such an action can always be demonstrated to representatives of neutral states, or to the press, or to any observers. The problem for the adversary states consists in continuously maintaining the conviction that the agreed force level is not passed. Therefore, it is necessary to undertake verification measures whose goal is early detection of any significant increase of the military forces beyond the agreed level.

Before we can talk about technical details of verification measures and their compliance with the goal spelled out above we have to define what we mean by "military forces." Obviously, the number of troops alone or of tanks and guns does not fully describe the strength of military forces. On the other hand, it makes no sense to define the strength of military forces in a comprehensive and therefore sophisticated way if there exists no possibility of verifying the so-defined military forces with the means available. The solution analyzed so far is indeed to count only troops and tanks, keeping in mind that this can be only a first-order solution to the problem.

It should be noted that quite similar difficulties exist if one wants to verify some environmental standards such as air or water standards. Air or water pollution can be defined only in accordance with appropriate measurement methods (e.g., for SO_2 or dust particle content in the case of air), which means that one has to limit the definition in such a way that the most important features of the problem are caught.

What are the possible verification measures? In *Daedalus*[3] a long list of possible technical means is given, including satellites and ground sensors. However, in the MBFR case there are many reasons that make these means unacceptable either to one or to all parties; therefore, a system whose basic elements are a record system and physical inspections seems to be the best chance of acceptance, at least at the moment. Still, a rich portfolio of possible combinations of these elements exists, as we shall see, and the question is which combination best meets the verification goal under the given boundary conditions.

8.2.3 BASIC MEANS AND PROCEDURES

In the years 1973 to 1976 there were major studies that analyzed the basic means and procedures, the boundary conditions, and the effectiveness of an MBFR-verification system for Central Europe. We will report here about some elements of these unpublished studies because they represent a remarkable effort toward possible solution of these problems of vital interest and because they are an illuminating example of large-scale use of the material accountability principle.

The studies start with the assumption that for a given reduction area a force level has been agreed upon and, furthermore, that after withdrawal of the surplus of troops and tanks, at time t_0 a certain force level L is guaranteed in this area. In the following we restrict our discussion, for the sake of simplicity, to troops. In the time interval (t_0, t_1) an exchange of stationed troops takes place across the boundaries of the reduction area, and the problem arises of how to guarantee that the net input of foreign troops into the reduction area in this interval of time is not significantly greater than zero.

The solution proposed at the moment is to limit the exchange of troops to so-called declared *exit–entry points* where there are inspectors to observe the flow of troops. In addition, some measures are proposed that will guarantee that the possibility of crossing the boundary at undeclared points is excluded. We will call these two basic measures *transfer measurement* and *transfer safeguards.*

It should be noted that the exchange of troops can take place via road, railroad, sea, and air. Whereas the situation is relatively clear in the first three cases, it is more complicated in the fourth. We can maintain the scheme only if we define the boundaries of the airports as boundaries of the reduction area. It is clear that in this case the transfer safeguards pose a difficult problem.

Another point worth mentioning is that the major problem here is to guarantee that troops cannot enter the reduction area through undeclared entry points without being detected or, in other words, that no undeclared troops can enter the reduction area. Contrary to this situation, in the nuclear material safeguards system only the declared material, i.e., the material that has been recorded and reported before entering the fuel cycle, is subject to safeguards. The fact that in the arms control case such a limitation cannot be accepted may be seen as reflecting the difference in importance between these two control problems.

In principle, the two basic means, transfer measurement and safeguards, would be sufficient if there were no counting or measurement errors. However, as these errors cannot be avoided, after some time they accumulate in such a way that the uncertainty about whether the actual force level is enhanced significantly is no longer tolerable. Therefore, from time to time one has to perform a physical inventory of the stationed troops in order to "recalibrate" the information. Obviously, the same problem as before arises: it is not sufficient to perform a physical inventory with respect to the declared troops; one also has to be sure that there are no undeclared inventories. Thus, some inventory safeguards measures have to be taken. To find the appropriate tools seems to be the most difficult problem posed by this concept of arms control verification. In all material accountability problems considered so far the physical inventory was a constitutive element of the material balance establishment. Here, however, we have a case in which it plays only an auxiliary role, as all material flows are measured against a fixed nominal inventory, namely the agreed force level.

TABLE 8.1 Means and Procedures for Verification of an Arms Control Agreement

I	*Transfer Measurement* Official exit–entry points Accompanying documents to facilitate verification Permanent stationary inspections
II	*Transfer Safeguards* Mobile ground teams Aerial (or satellite) reconnaissance Observation of the air space
III	*Inventory Measurement* Declaration of official installations Overall inspection *or* notifications combined with spot checks *Ad hoc* inspections *Ad hoc* aerial (or satellite) inspections
IV	*Inventory Safeguards* *Ad hoc* aerial (or satellite) reconnaissance Mobile ground teams (*ad hoc*)

Before going into some mathematical details, we will say a few words about *procedures*. There are several possibilities, and since there is no solid information about what can and what will be negotiated, we will give two extremes. One possibility is that the inspection authority generates all the information necessary by its own means: inspectors count the troops crossing the exit–entry points; they take the physical inventories; and, in addition, some measures are taken in order to detect any undeclared inventories. Naturally, such a procedure would be extremely *intrusive* if it were completely effective; we will come back to this point. Another possibility is that the party subject to the control generates all the necessary information and that the inspection authority mainly performs consistency checks using the reported information, and performs only marginal measurements on its own. The question arises of whether the inspected party is willing to give detailed enough information that these consistency checks result in satisfactory effectiveness. Here, we are led to fundamental problems of the system, which we will discuss in the last section.

Table 8.1 summarizes the four basic components of a complete system for the verification of an arms control agreement in a well-defined reduction area; some procedural aspects are also included in the table. It should be noted here that in establishing this system the emphasis was on completeness; the aspect of acceptability has not been considered so far.

8.2.4 MATHEMATICAL CONSIDERATIONS

Because no MBFR verification system has been been established, we cannot yet perform a mathematical analysis as we did for the nuclear material safeguards system.

Therefore, we will limit our considerations to some aspects of the whole problem that we can handle with the tools developed in the first chapters.

Transfer Measurements

Let us assume that we have R exit–entry points and that in a given reference time, at the ith point, $i = 1, \ldots, R$, N_i transfers take place. We consider for illustrative purposes troop transfers, but we also could imagine others. The reference time (e.g., 1 year) is defined here as a period during which a cycle of transfers is completed and after which a similar cycle begins. Furthermore, let us assume that at the ith exit–entry point the transfers are always of the same order of magnitude; that the number of troops in a transfer counted by the inspector is T_{ij}, $j = 1, \ldots, N_i$, $i = 1, \ldots, R$, where T_{ij} has a sign according to the direction of the transfer; and that the variance of the random counting error for one transfer is σ_i^2, $i = 1, \ldots, R$.

It should be noted that the estimation of these variances requires experiments. In fact, a series of so-called *field tests* has given some indication of the possibilities and limitations of physical inspections. One must admit, however, that we are still far from being able to give satisfying quantitative estimates of the accuracy of inspection measures. It is for this reason that we do not take into account systematic counting errors here, even though they could easily be included with the help of the formalism given in Chapter 2.

According to the discussion in the foregoing section one can imagine the following verification schemes:

1. As a first possible procedure we assume that no data are reported by the inspected party to the inspection authority, which means that the inspection authority must generate all necessary data itself. However, the inspection authority has the capability of acquiring all data needed. As measurement accuracy is limited, a significance test has to be performed to determine whether the transfers during the reference time lead to a strength of forces beyond the agreed level.

Let I_0 be the initial inventory, and σ_{I0}^2 the variance of the error of the determination of the initial inventory. We will not discuss here the way in which the initial inventory is determined, whether by counting or by other means, and we will therefore not discuss the reasons for these errors, which are characterized by the variance σ_{I0}^2.

The book inventory B at the end of the reference time is given by

$$B = I_0 + \sum_{i=1}^{R} \sum_{j=1}^{N_i} T_{ij}, \tag{8.1}$$

and the variance σ_B^2 of the book inventory is given by

$$\operatorname{var}(B) = \sigma_{Ie}^2 + \sum_{i=1}^{R} N_i \cdot \sigma_i^2 = :\sigma_B^2. \tag{8.2}$$

The problem consists in testing whether this book inventory is significantly greater than the agreed force level L. This means that a test has to be performed with respect to the null hypothesis H_0 (that the expected value of the difference between the book inventory and the agreed level is zero) and the alternative hypothesis H_1 (that the expected value of this difference is a value greater than zero):

$$H_0 : E(B-L) = 0; \quad H_1 : E(B-L) = M > 0. \tag{8.3}$$

In other words, M is the difference between the agreed and the actual force level. If the counting errors are normally distributed, we get, by analogy to formula (2.15), the following expression for the probability of detection

$$1-\beta = \Phi\left(\frac{M}{\sigma_B} - U_{1-\alpha}\right), \tag{8.4}$$

where α is the false alarm probability. It should be kept in mind that this probability of detection refers only to the strategy of passing the agreed force level by introducing troops into the reduction area via agreed exit–entry points. We will come back to this point.

So far, we have tacitly assumed that the inspection authority wants to know whether the agreed force level is surpassed only at the end of the reference time. However, if the reference time is one year, the "critical time" might be one week, perhaps even one day. Thus, we have the same problem that we treated in Chapter 2: if we fix the overall false alarm probability for the reference time, we have to choose the false alarm probabilities for the single "inventory periods" appropriately.

Without repeating the formulae here, we mention only that for a fixed value of M and increasing reference time the probability of detection $1-\beta$ decreases such that it will eventually no longer be satisfying. When this happens, it is time to perform a physical inventory in order to recalibrate the information, or, in other words, to raise the value of the probability of detection to a satisfying level.

2. As a second possible procedure we assume that sometime before each transfer the corresponding transfer data of the inspected party (expected nominal value E_{ij} and variance σ_i^2 of the jth transfer at the ith exit–entry point) are reported to the inspection authority. The inspection authority then decides on the basis of a random sampling plan whether it will verify the data of this transfer by checking the reported data with the help of independent measures.

It should be noted that the variance σ_i^2 need not necessarily be understood as the variance of a counting error; it may also be the variance of an error caused by imperfect knowledge about the transfer at the time the data have to be reported.

Let us assume for the moment that there exists only one exit–entry point; therefore, let us omit for the moment the index i. Furthermore, let us assume at the moment that one inspection takes ϵ hours and that C hours are permitted for the inspection of the transfers at this exit–entry point. This means that n transfer data can be verified where

$$C = \epsilon \cdot n. \tag{8.5}$$

Let the variance of the inspector's estimate of one transfer datum be σ_2^2 and that of the reported datum σ_1^2. Then the variance of the difference D between one reported and one estimated datum is

$$\sigma^2 = \sigma_1^2 + \sigma_2^2. \tag{8.6}$$

If this variance is known, the inspection authority (the inspector at the exit–entry point) can perform a difference test with respect to the null hypothesis H_0 (that the expectation value of the difference between inspector's estimate and reported datum is zero) and with respect to the alternative hypothesis H_1 (that this expectation value is either larger or smaller than zero), depending on whether the transfer entered or left the reduction area:

$$H_0 : E(D) = 0; \qquad H_1 : E(D) = \mu. \tag{8.7}$$

The analysis of this test leads, under appropriate normality assumptions, to the already well-known formula for the probability of error of the second kind:

$$\beta' = \Phi\left(U_{1-\alpha'} - \frac{M}{\sigma}\right), \tag{8.8}$$

where α' is the probability of error of the first kind.

According to our assumptions, the whole verification procedure consists of n single difference tests. If we assume that one transfer datum can be falsified by the amount μ, then r falsifications are necessary in order to reach the amount M, where

$$M = \mu \cdot r. \tag{8.9}$$

Here, the falsifications of data of transfers going into or leaving the reduction area have to be taken with the appropriate sign.

If the inspected party has decided at the beginning of the reference time to pass the agreed level by the amount M, the probability of detection $1-\beta$ is given by formula (3.18):

$$\beta = \sum_i \Phi\left(U_{\sqrt[n]{1-\alpha}} - \frac{\mu}{\sigma}\right)^l \cdot (1-\alpha)^{1-\frac{l}{n}} \frac{\binom{r}{l}\cdot\binom{N-r}{n-l}}{\binom{N}{n}} \tag{8.10}$$

where

$$\alpha = 1-(1-\alpha')^n \tag{8.11}$$

is the resulting false alarm probability.

For R different exit–entry points the analysis becomes very difficult, as already detailed in Chapter 3; the boundary conditions that correspond to Eqs. (8.5) and (8.9) are given here by

$$C = \sum_i \epsilon_i \cdot n_i, \qquad M = \sum_i \mu_i \cdot r_i. \tag{8.12}$$

Therefore, it seems that in a real situation the only way to determine the optimum distribution of a given inspection effort on R exit–entry points and the probability of detecting noncompliance with the agreed force level of size M is to perform a simulation study.

3. A third possibility is that no detailed data are reported to the inspection authority and, in addition, the available inspection effort is too small for full coverage of all transfers. We assume furthermore that either the inspected party reports some general data on transfers (expected values and variances) or the inspection authority has built up some experience about such general data. This means that, as in case 1, the inspection authority has to form an estimate of the net transfer during the reference time with the help of its own findings and the general data; at the end of the reference time, the inspection authority must perform a significance test similar to that in case 1. With the appropriate assumptions this problem can be treated in exactly the same way as the one treated in section 4.4, so we will not present the mathematical apparatus again.

Inventory Measurements

What has been said about transfer measurements applies to inventory measurements, except that here we do not have the analytical problems caused by the sequential nature of transfer measurements. There is, however, a difference between these two types of measurements that might require completely different technical solutions: while one can in the transfer measurement case imagine that an inspection or data verification of the type discussed above could be accepted by the inspected party, there seem to be major difficulties in the inventory measurement case. Probably at least some parts of barracks, installations, or ground training areas would not be accessible to inspectors, and it is not clear at the moment if such a restriction could simply be treated as a kind of systematic error or whether it would necessitate a completely new approach.

Transfer and Inventory Safeguards

According to the foregoing section safeguards measures are necessary in order to guarantee that no military forces enter the reduction area through undeclared entry points and to ensure that there are no undeclared forces in the reduction area during the time of inventory taking. We have mentioned the difficulties of finding appropriate (acceptable) measures; therefore, we will not go into mathematical discussion before there is clarification about the subject. Still, two points are worth mentioning. First, unlike the transfer and inventory measures, no quantitative statements are necessary here; it is sufficient that either the statement "nothing happened" or the statement "something happened" can be made with some accuracy, which means a simplification of the problem. Second, again in contrast to the transfer and inventory measurements, here the problem of circumvention of safeguards becomes relevant: whereas in the case of measurements the situation seems to be quite clear, one never knows in the case of safeguards if there is any chance to detect something or if, on the contrary, an illegal action is performed in such a way that the safeguards are meaningless. In other words, it is not clear to what degree there is a statistical independence between illegal actions and verification measures; if such independence exists, the probability of detection is no longer an appropriate criterion of optimization.

8.2.5 SYSTEM PROBLEMS

On the foregoing pages we have touched several times upon aspects of the system as a whole that were treated in Chapter 4. We will discuss some of them once more in this connection even though we will not be able to offer satisfying solutions.

We have defined the goal of the MBFR verification measures as the timely detection of any significant increase of forces beyond the agreed level. This goal must be put into the context of the goals of MBFR itself, which cannot be specified so clearly and so easily; they range from such concrete goals as saving of national financial resources through the reduction of arms to such qualitative ones as reduction of tensions and building up of mutual confidence.[7] At this point the first "system problem" arises: How can we achieve accord between MBFR goals and MBFR verification goals? We can fulfill the verification goal by investing heavily in the inspection effort, but will such an *intrusive* verification system have any chance of *acceptance* by the inspected parties? Apparently, in addition to the verification goal some boundary conditions must be formulated to guarantee a reasonable balance between high verification efficiency on one hand and high chance of acceptability on the other.

Let us assume that there is agreement about the basic measures and procedures of an MBFR verification system – for example, that some kinds of physical inspection and counting measures are accepted but that there is to be no satellite surveillance because the evaluation of the data generated by these means is not accessible to all parties to the game. The problem then arises of how to establish the system as a whole such that the above-mentioned balance is guaranteed. One approach to this problem (discussed in Chapter 4) is to look at the global parameters of the verification system. In the case of an MBFR verification system we have the following global parameters:

- Length of the inventory period (which has here a somewhat different meaning from its meaning in Chapter 2)
- Probability of detection for the whole system
- False alarm probability for the whole system
- Increase of forces beyond the agreed level
- Inspection effort (expressed in inspection man-hours and/or monetary terms)

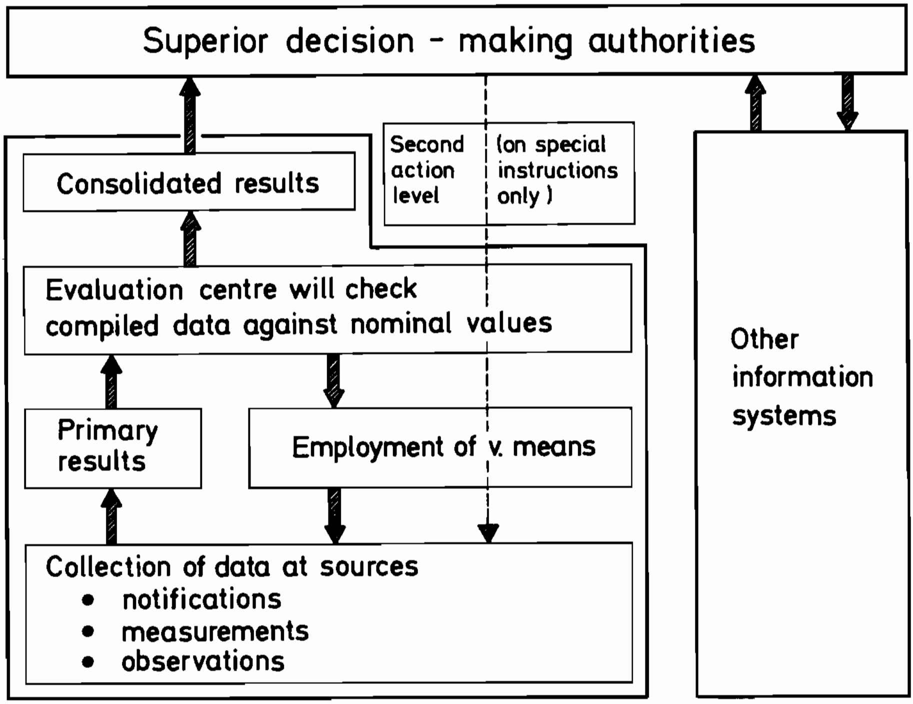

FIGURE 8.1 Suggested MBFR verification system.

As we have only one relation between these global parameters, once we have put together all the subsystems we have to choose the value of four of the five parameters independently, and it is at this point that we have to achieve the balance between the different goals and boundary conditions. Naturally, a high value of the overall false alarm probability leads to a high probability of detection; however, mutual confidence will not benefit very much from a large number of false alarms even if the alarms are later recognized as false. Similar arguments hold for the choice of the values of length of inventory period and inspection effort. Thus, one could imagine that in designing a verification system one proceeds as follows. One first asks what increase beyond the agreed force level in which period of time would be considered critical. Next, the values of overall false alarm probability and inspection effort are fixed, and the overall probability of detection is calculated with the help of an optimization procedure of the kind described earlier. If one thinks that the value of the probability of detection is high enough, then the system could be implemented; otherwise, the parameter values have to be reviewed and the procedure has to be continued until a satisfying and consistent set of parameter values is reached.

In conclusion, some remarks about action levels are in order, as this aspect is of central importance to the whole system. How many levels, and of what organizational form, are necessary to convince a government that the alarm raised by the system is a real alarm, or only a false alarm? So far, there is only vague understanding of this area; one suggested system, based on studies already mentioned, is represented in Figure 8.1. It seems too early to go into more detail here, especially if one remembers that even the well-defined and widely accepted nuclear materials safeguards system is still rather vague in light of the degree of clarification necessary, according to the discussion in Chapter 4, to full understanding of the structure of the system.

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List of Symbols

GENERAL LOGICAL SYMBOLS

$A: = B$	Defining colon: A is defined by B
$a = b$	Equality: a is equal to b
$a < b$	Inequality: a is smaller than b
$a \leqslant b$	Inequality or equality: a is smaller than or equal to b
$a \ll b$	Extreme inequality: a is much smaller than b
$A: = \{a_1, a_2 \ldots\}$	Set of elements: the set A consists of the elements $a_1, a_2, \ldots$
$B: = \{a: 0 \leqslant a \leqslant 1\}$	Set of all real numbers between zero and one
$a \in A$	a is an element of the set A
$A \cup B$	Union of the sets A and B
$A \wedge B$	Intersection of the sets A and B

MATHEMATICAL SYMBOLS

$\sum_{i=1}^{n} a_i :=a_1 + a_2 + \ldots a_n$	Sum of $a_1, a_2, \ldots, a_n$
$\prod_{i=1}^{n} a_i :=a_1 \cdot a_2 \cdots a_n$	Product of $a_1, a_2, \ldots, a_n$
$y = f(x)$	Function: (dependent) variable y is a function of (independent) variable x
$f'(x) := \frac{df(x)}{dx}$	Derivation (derivative) of function $f(x)$
$\frac{\partial f(x,y)}{\partial x}$	Partial derivation of function $f(x, y)$
$\int_a^b f(x)dx$	Integral of the function $f(x)$
$\exp(x) := \sum_i \frac{x^i}{i!}$	Exponential function
$i! := 1 \cdot 2 \cdot 3 \cdots i$	Factorial
$\binom{i}{j} := \frac{i!}{j!(i-j)!}$	Binomial coefficient

STATISTICAL SYMBOLS

prob $\{A\}$	Probability of event A
prob $\{A/B\}$	(Conditioned) probability of event A under the condition that B holds
$F_X(x) :=$ prob $\{X \leqslant x\}$	Distribution function of the random variable X
$f_X(x) := F'_X(x)$	Density of the distribution function $F_X(x)$ (if existing)
$E(X) := \int_{-\infty}^{\infty} x \cdot f_X(x)dx$	Expected value of the random variable X, frequently called μ

$E(X-\mu)^2 := \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f_X(x)dx = E(X^2) - E^2(X)$	Variance of the random variable X, frequently denoted as σ^2
$\sqrt{E(X-\mu)^2}$	Standard deviation of the random variable X
$\mathrm{cov}(X, Y) = \int_{-\infty}^{\infty} (x-\mu_X)\cdot(y-\mu_Y)\cdot f(x,y)dxdy$	Covariance of the random variables X and Y
$\rho := \frac{\mathrm{cov}(X, Y)}{\sigma_x \cdot \sigma_y}$	Correlation of the random variables X and Y
$X \sim N(\mu, \sigma^2)$	Gaussian or normal distributed random variable (i.e., $F_X(x) = \frac{1}{\sqrt{2\Pi} \cdot \sigma} \int_{-\infty}^{x} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) dt$)
$\Phi(x)$	Distribution function of the Gaussian distributed random variable with $\mu = 0, \sigma^2 = 1$
U_x	Inverse of the Gaussian distribution
H_0	Null hypothesis
H_1	Alternative hypothesis
α	Probability of error of the first kind
β	Probability of error of the second kind

Index

Abramowitz, M., 88
Accountability tank, for nuclear material flow measurement, 96
Accountancy, bad, 4
Ackley, W.R., 118, 136
Action level, second, and false alarms, 34
Aganbegyan, A. 132, 137
Altenhein, F.K., 1, 7, 140, 159
Alternative hypothesis, definition of, 13
Anderson, T.W., 19, 32
d'Arge, R.G., 7, 159, 160
Arms control, 161–176. *See also* Non-proliferation treaty; Nuclear material safeguards
  direct inspections and, 164
  history of, 161
  material accountability as a verification tool for, 163–164
  survey of analyses of, 162–164
Arms Control Today, 162, 165, 176
Avenhaus, R., 22, 32, 37, 57, 65, 76, 88, 97, 113, 126, 136, 142, 159
Ayres, R.N., 5, 7, 114, 129, 131, 136, 159, 160

Bain, E.E., 132
Balances, intersectoral, in socialist economies, 132
  monetary, in socialist economies, 132–133
    interaction between physical balance and, in socialist economies, 134
  physical, in socialist economies, 133–134
    interaction between monetary balance and, in socialist economies, 134
Balzer, D., 160
Bandeen, W.R., 159
Baranov, E.F., 132, 137
Battle, tank, and MBFR modeling, 166
Bayer, G., 160
Beightler, G.S., 65
Bellany, I., 165, 176
Benedict, U., 122, 136
Bennett, C.A., 7, 88, 112
Beyrich, W., 32, 96, 113
Bierlein, D., 68, 88
Blankenburg, H., 135, 137
Book inventory, definition of, 10
  correlation with physical inventory, 18–19
Bolin, B., 159
Bottle industry, Process Chain Evaluation Model in, 130–132
Bouchey, G.D., 54, 65
Branson, J.R., 160
Broecker, W.S., 159
Brownlee, K.A., 7

Buckley, P.S., 115, 116, 117, 136
Büker, H., 154, 155, 160
Burchard, H.-J., 160

Cadmium, need for accountability of, 156
Carbon dioxide, natural reservoirs of, seven-box model of, *148*,* 149–150
in atmosphere, effect of, on global climate, 1
fossil fuels and, 1, 152–153
increase in, 141
measured and theoretical data on, 151–152, *151*
Carbon dioxide balance of earth, models of, 140–141
Carbon dioxide cycle of earth, 142–153
and four boxes model, 143–145, 150
uncertainty considerations in, 145–148
Carbon dioxide system, proposed measurements for, 142
Chemical industry, material accountability in, 115–123
Chien, H.H.Y., 117, 136
Climate, global, carbon dioxide in atmosphere and, 1
Cochran, W.G., 64, 65
Computer-aided material accountability procedures, 134–136
Containment, nuclear material safeguards and, 91, 93
Cutten, C.W., 135, 136

Daedalus, 162, 165, 167, 176
Data, operator's and inspector's, comparison of, 36–44
with *D*-statistic, 36–41
with *P*-statistic, 41–44
Data verification. *See* Verification
Davis, M.D., 176
De Armas, C.M., 117, 136
Debreu, G., 159
Deming, W.E., 64, 65
Detection of diversion, by means of accountability, 19–24
overall probability of, in nuclear material reprocessing plant, 109–111
probability of, definition of, 14, *15*
Detection time, expected, and frequency of inventory, 24–29, *28*
Digital accountability, in mints, 125
Distillation, material accountability in, 115–119
Diversion of material, 4
detection of, by material balance establishment, 35
frequency of inventory and, 24
material accountability and, 19–24
from mints, 125
measurement errors, random losses, and, 11–17
overall probability of detection of, in nuclear material reprocessing plant, 109–111
strategy for, 21
Doub, W.O., 113
Dresher, M., 60, 65
Drosselmeyer, E., 96, 113
Dukert, J.M., 113

Economics, and technology, material accountability in, 114–137
Effluent charges, 131
Einstein, A., 1
Environment, global changes in, 138
Environmental material accountability, 138–160
global, 142–153
international system for, 158
regional and local, 153–158
Environmental balances, main aspects of, *157*, 158
Error, of the first kind, probability of, definition of, 12
measurement, random losses, diversion and, 11–17
random and systematic, 16
of the second kind, probability of, definition of, 13
systematic measurement, importance of, 16–17
EURATOM, and nuclear material safeguards, 93–94

*Numbers in italic indicate pages on which figures appear.

Fairhall, A.W., 140, 151, 159
False alarm, and second action level, 34
False alarm probability, definition of, 14
Falsification of data by operator, procedure for, 46
Faude, D., 160
Fenske, M.R., 136
Flow measurement
  verification of, 58–64
    dependent sequential, 62–64
    independent sequential, 59–62
    independent sequential, with one class of material, 60–61
    independent sequential, with several classes of material, 61–62
Forder, G.J., 137
Franklin, N.L., 7
Frederiksen, P., 112
Frequency of inventory, optimization criterion for, 24–27
Frick, H., 22, 32, 88, 113

Garland, J.A., 160
Geitz, H.G., 136
Göran, K.G., 6, 7
Gozani, T., 137
Granberg, A.G., 132, 137
Granquist, D., 32
Greenhouse effect, 141–142, *141*
Greenwood, T., 162, 176
Gupta, D., 113, 157, 160
Gugelmann, A., 159

Häfele, W., 92, 112, 113, 139, 141, 159, 160
Halbritter, G., 160
Harris, L., 135, 137
Hartmann, G., 113, 126, 136, 142, 159
Herfindahl, O.G., 7
Herstein, I.N., 159
Hinckfuss, D.A., 136
Höpfinger, E., 60, 65
Hough, C.G., 79, 88, 112

Input, in nuclear material reprocessing plant, 101–102
International Atomic Energy Agency, nuclear material safeguards and, 1, 91. *See also* Nuclear material safeguards; Nuclear nonproliferation treaty
Inspection, and arms control, 161
Inspection effort, maximum routine, in UN model agreement, 93
Inspection team, strategy of, for detection, 23
Inventory, book. *See* Book inventory
  frequency of, 24–29
  hidden, in mint, 127
  physical. *See* Physical inventory
Inventory measurements, for MBFR verification, 173
Inventory periods, sequence of, 17–19
Inventory verification, with one class of material, 33–44
  with several classes of material, 44–58
    approximate solution for the general case of, 50–58
    exact solution for a special case of, 45–50
Isotope separation, material accountability in, 119–123

Jaech, J.L., 88, 96, 113
Jakoby, U., 135, 137
Janisch, D.B.B., 112
Jansen, P., 160
Jennekens, J.H., 112
Jewell, W.S., 80, 89
Johnson, A.I., 137
Jones, J., 96, 113

Kabel, K., 135, 137
Kahn, P.M., 88
Kaiser, D., 136
Karlin, S., 47, 65
Katz, F., 113
Keeling, Ch.D., 151, 159
Kellner, S., 136
Kellog, W.W., 7, 159
Kimball, A.W., 56, 65
Kneese, A.V., 3, 7, 138, 156, 159, 160
Koen, B.V., 65
Kossov, V.V., 132, 137
Kraemer, R., 32, 96, 113
Krieger, D., 111, 113
Kull, L.G., 137

Lavoisier, A.L., 1
Lead. *See* Metal

Lendvai, O., 112
Light water reactor, and nuclear fuel cycle, *94*, 95
Losses, in mint, 127
  random, measurement errors, diversion and, 11–17

Machta, L., 148, 149, 150, 159
Man, impact of, on natural material balances, 139–142
Marsden, I.R., 6, 7
Märzendorfer, H., 2, 7, 160
Maschler, M., 162, 176
Material accountability, for detecting diversion of material, 19–24
  environmental, 138–160
  history of use of, 2–3
  and nuclear materials safeguards, 91–93, 111–112
  and nuclear weapons non-proliferation treaty, 3–4
  purpose of, 20
  and rare metal resources, 2
  reasons for, 2
  and regulations, 3
  in technology and economics, 114–137
Material accountability, principles, applications of, 5
  formulation of, 4
  large-scale use of, in MBFR verification, 168
Material accountability procedures, computer-aided, 134–136
  at nuclear material reprocessing plant, 100–104
Materials–Process–Product Model, 114, 129–132, *129*
Material balance, and detection of diversion, 35
  and economics, 3
  man's impact on natural, 139–142
  in socialist economies, 132–134
Material balance area, definition of, 9–10
  for special metal balances, 157
  subdivision of, 29–32, *30*
    for localization of loss or diversion, 31
  for sulfur dioxide, 154
Material balance concept, 8–32, *11*
  usefulness of, 8–9
Material balance establishment, correlation between data verification and, in nuclear material reprocessing plant, 106–109
Material balance principle, for one inventory period, 9–11
Material unaccounted for, definition of, 10
  in socialist economies, 134
Mathematica, Inc., 162, 176
Matthews, W.H., 1, 7, 159
MBFR. *See* Mutual Balanced Force Reduction
McCracken, D.I., 135, 137
Measurement errors, random losses, diversion and, 11–17
  systematic, importance of, 16–17
Mercury, need for accountability of, 156
Metal balances, and nuclear material balances, 124
  special, 156–158
Metal industry, losses in, 124
  material accountability in, 123–129
Metals, rare, material accountability system for, 2
Military forces, definition of, for MBFR verification, 167
Miller, R.G., 75, 88
Mints, digital accountability in, 125
  external material balance in, 126
  importance of material balance in, 126
  internal material balance in, 126
    accuracy of, 128
    measurement of, 126, *127*
  products of, 125
  verification procedure in, 127
Mint material accountability, 125–129
  and nuclear material accountability, 125
Models, of arms control as conflict situation, 162–163
  box, applications of, 145
  of carbon dioxide balance of earth, 140–141
  four-boxes, and carbon dioxide cycle of earth, 143–145, 150
  "mathematical," of arms control, 163
  of MBFR, 165–167

"political," of arms control, 163
Process Chain Evaluation, 130–132
radiative equilibrium, and greenhouse effect, 141
seven-box, of carbon dioxide cycle of earth, 149–150, *148*
Montias, J.M., 132, 137
Morgan, F., 32, 112
Morgan, J.I., 137
Mutual Balanced Force Reduction, models of, 165–167
and nuclear materials safeguards systems, 168
verification of, 164–176
verification system, action levels of, 175–176
components of, 169
design of, 175
global parameters of, 174–175
goal of, 174
goal of, and MBFR goals, 174
and illegal actions, 174
inventory measurements for, 173
mathematical considerations for, 169–174
problems of, 174–176
transfer and inventory safeguards for, 173–174
transfer measurements for, 170–172

Nakicenovic, N., 76, 88, 97, 113
Nentwich, D., 113
Niehaus, W., 159
Nordhaus, W., 159
Nuclear fuel cycle, light water reactor and, *94*, 95
measurement methods for nuclear material in, 95–97
problems of, 96
measurement of nuclear material flow in chemical reprocessing plant, 96
parts of, 95
physical inventory during, 96
plutonium in, 95
Nuclear material accountability, and mint material accountability, 125
numerical example of, 97–111
Nuclear material balances, and metal balances, 124
Nuclear material flow measurement, in accountability tank, 96
Nuclear material measurement, nuclear fuel cycle and, 95–97
Nuclear material reprocessing, material accountability procedures in, 100–104
at Nuclear Fuel Services plant, 97–111, *98*
Nuclear material reprocessing plant, correlation between data verification and material balance establishment in, 106–109
data verification procedures in, 104–106
input in, 101–102
material unaccounted for in, 103–104
overall probability of detection in, 109–111, *110*, *111*
physical inventory in, 101
product of, 102–103
waste in, 102

Nuclear material safeguards, containment and, 91, 93
development of, 91
goal of, 1
international, 90–113
development of, since 1971, 93–94
efficiency of, 94
and nonproliferation treaty, 90–95
importance of, 90
material accountability and, 91–93, 111–112
and MBFR, 168
national, 111
procedures for, in UN model agreement, 93
structure of, 91, *92*
surveillance and, 91, 93
Nuclear material safeguards agreement, model, 91–94
Nuclear weapons nonproliferation treaty, 1, 3–4
and IAEA nuclear materials safeguards system, 90–95
Null hypothesis, definition of, 12

Oeschger, H., 146, 159
Ollivier, R.T., 137
Operator, strategy of, for diversion, 23
with several classes of material, 45
Otto, H.E., 113
Oxygen in atmosphere, measurements of, 142·
Oxygen, and carbon dioxide cycle, 139–142, *140*

Patterson, G., 160
PCEM. *See* Process Chain Evaluation Model
Pearman, G.I., 159
Peng, T.-M., 159
Physical inventory, correlation with book inventory, 18–19
definition of, 10
during nuclear fuel cycle, 96
in nuclear material reprocessing plant, 101
of troops in MBFR, 168
Pigford, T.H., 122, 136
Pingry, D.E., 7
Plotnikov, K.N., 137
Plutonium, measurement system for, in reprocessing plant, 100
in nuclear fuel cycle, 95
and nuclear material reprocessing, 97
Plutonium inventory of reprocessing plant, estimate of, 98–100
Pollutants, accountability of, 153–158
Pommernacki, G.L., 137
Ponssard, J.-P., 59
Powley, G., 135, 137
Preuschen, G., 159
Process Chain Evaluation Model, alternative forms of taxation and, 131
in bottle industry, 130–132
effluent charges and, 131
technological substitution and, 131–132
Pushkov, A., 112

Radioactive material. *See* Nuclear material
Radioactive waste, disposal of, 95
Radiation balance of earth, satellite observations and, 142
Random losses, measurement error, diversion and, 11–17
Random sampling, with one class of material, 34–35, *35*
Raschke, E., 159
Rauhut, A., 160
Repenning, K.-D., 160
Resources For the Future (RFF), 3
Richardson, L.F., 176
Rippin, D.W.T., 137
Roberts, A.N., 136
Robinson, G.D., 7, 159
Rudner, K.N., 136, 137

Saaty, T.L., 162, 176
Safeguards, international nuclear material, 90–113. *See also* Nuclear material safeguards
Safeguards authority, choice of sample size by, with several classes of material, 45
Safeguards procedures, for rare or dangerous industrial substances, 34
Sassin, W., 160
Satellites, use of in understanding earth–atmosphere system, 142
Sawyer, J.S., 140, 148, 149, 150, 152, 159
Saxton, J., 7, 136
Scheffée, H., 32
Scheuerpflug, W., 113
Schikarski, W., 160
Schotterer, U., 159
Schneider, R., 32, 88
Settle, D., 160
Shande, S., 160
Shea, T.E., 136
Siegenthaler, U., 159
Silver. *See* Mint
Simmons, R.T., 137
Skinskey, T.G., 115, 136
Singh, M., 32
Socialist economies, material balance in, 132–134
Spannagel, G., 160
Sparrow, R.E., 135, 137
Starting inventory, problem of, 17–19
Stegun, I.A., 88
Stehfest, H., 160
Stern, M., 7, 136

Stewart, K.B., 10, 17, 32, 36, 54, 65, 88
Stump, N.W., 136
Sulfur accountability, in power plants and refineries, 155–156
Sulfur dioxide in atmosphere, material accountability system for, 2
  regional accountability of, 154–156
Surveillance, and nuclear material safeguards, 91, 93

Taxation, alternative forms of, 131
Taylor, T.B., 111, 113
Technological substitution, PCEM and, 131–132
Technology and economics, material accountability in, 114–137
Tereshchenko, A.P., 159
Throughput, definition of, 10
Transfer and inventory safeguards, for MBFR verification, 173–174
Transfer measurements, in MBFR verification, 170–172
Treaty on the Non-Proliferation of Nuclear Weapons. *See* Nuclear weapons non-proliferation; Nuclear material safeguards
Treyball, R.E., 115, 136
Tyndall, 141

Ungerer, W., 112
Uranium, extraction of, 95
Uranium isotope separation, 119–123

Vaclavek, V., 136
Valentin, K., 10
Verification, in context of arms control, 161–162
  of flow measurement, 58–64
  independent sequential, 59–62
  of inventory. *See* Inventory verification
  material accountability as a tool of, for arms control, 163–164
  of MBFR in Central Europe, 164–176
    basic means and procedures for, 167–169
    formulation of problem of, 167
  of source data, 5
Verification procedures, in mint, 127
  in nuclear material reprocessing plant, 104–106
Vosolsobe, I., 136
Volynets, V.M., 159

Walsham, G., 135, 137
Walter, H., 136
Waste, in nuclear material reprocessing plant, 102
Weise, E., 156, 160
Whinston, A.B., 7
Willrich, M., 111, 113
Winkler, D., 136
Winler, H., 113
Wintzer, D., 160
Wurth, G., 136

Zajonc, H., 160
Zimen, K.E., 1, 7, 140, 152, 159
Zinc. *See* Metal

(contents continued)

5.3 Numerical example
5.4 Conclusion

6 Material Accountability in Technology and Economics
6.1 The chemical industry
6.2 The metal industry
6.3 The Materials-Process-Product Model
6.4 Use of material balances in socialist economies
6.5 Computer-aided material accountability procedures

7 Environmental Accountability
7.1 Man's impact on natural material balances
7.2 A global example: the carbon dioxide cycle of the earth
7.3 Regional and local examples
7.4 Conclusion

8 Arms Control
8.1 Survey on arms control analyses
8.2 Verification of Mutual Balanced Force Reduction in Central Europe

List of Symbols

Index

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