NORMATIVE MODELING IN DEMO-ECONOMICS

Frans Willekens
Andrei Rogers

International Institute for Applied Systems Analysis
A-2361 Laxenburg, Austria

PREFACE

Interest in human settlement systems and policies has been a critical part of urban-related work at IIASA since its inception. Recently this interest has given rise to a concentrated research effort focusing on migration dynamics and settlement patterns. Four subtasks form the core of this research effort:

- The study of spatial population *dynamics*;

- The definition and elaboration of a new research area called *demometrics* and its application to migration analysis and spatial population forecasting;

- The analysis and design of migration and settlement *policy*;

- A *comparative study* of national migration and settlement patterns and policies.

This paper, the fourth of a series on policy design and analysis, is an exposition of several fundamental themes in two related but largely independent bodies of literature: the mostly mathematical literature in systems engineering that deals with the control of complex systems describable by sets of differential or difference equations, and the more substantive literature in the formal theory of economic growth and policy. The logical structures of the two paradigms are similar, and, as the paper illustrates, their formalisms can be fruitfully transferred to the field of population policy. To introduce the subject matter to a wider audience, the authors have sought to make the paper relatively self-contained by including some introductory descriptive material on population dynamics and economic growth theory.

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Related papers in the policy analysis series, and other publications of the migration and settlement study, are listed at the back of this report.

Andrei Rogers, Chairman
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SUMMARY

This report is an expository state-of-the-art review of several funda-
mental themes in two related but largely unconnected bodies of literature.
The focus is on recent attempts to design comprehensive dynamic demo-
economic policy models that are formally stated as optimal control prob-
lems. Two groups of models are distinguished: planning-oriented models,
which originated in systems engineering and economics and which are
designed to aid practical problem solving; and theoretically-oriented models,
which have been developed in economic growth theory and which are
intended to provide theoretical insights into the dynamics of economic
systems. The formalisms of both are shown to have natural applications
in demography.

Contents

INTRODUCTION

Social concern with population processes arises when the demographic acts of individuals affect the welfare of others and produce a sharp divergence between the sum of individual (private) preferences and social well-being. In such instances, population processes properly become the subject of public debate and the object of public policy.

Population policy has a special feature that makes it a difficult research topic. Although a central element of any demographic policy is the size and the distribution of population, neither the goals nor the means of such a policy are purely demographic. A population trend is viewed as being good or bad in the light of its presumed social and economic consequences. That is, a population policy is an instrument to achieve non-demographic goals. Davis (1971, p. 7) described a population policy as one that tries to eliminate the demographic causes of the problems to be solved.

The importance of social and economic considerations in formulating population policies was stressed by the World Population Conference in Bucharest in 1974. The Conference strongly endorsed the view that demographic matters considered in isolation from economic and social factors have little significance (Tabah, 1975, p. 380). This is particularly true of migration. It is impossible to determine the goals and means of population distribution policies without considering general economic and social policies. Typically, human settlement programs are part of a regional economic policy, or of a land use policy, or of a physical planning program (for an illustrative review of population distribution policies in several developed countries, see Willekens, 1976a, pp. 31-55).

The interdependence of economics and demography is also reflected in policy-modeling efforts. Models of demo-economic growth and policy are receiving considerable attention in economics and in demography. (For a bibliography see Bilsborrow, 1976.)

In most of modern economic growth theory, population is entered as an exogenous variable affecting economic growth through the labor supply, but itself being unaffected by changes in economic conditions. A few economists have endogenized population in their models by relating it to per capita income or a similar economic index. Although the treatment of population as an endogenous variable in economic growth models is of recent

date, the notion itself has a long history. Classical economists such as Adam Smith, Thomas Malthus, J.S. Mill, and Ricardo all viewed population as being intimately dependent on the state of the economy.

Demographers, too, have only recently attempted to extend purely demographic models of population growth to include economic factors. And, again, the notion is not new. More than fifty years ago, Lotka (1925) already was stressing the importance of an interdisciplinary approach to the study of population. He considered population not in isolation but as part of a larger ecosystem. The demographic growth model he proposed, which has become the basis of modern mathematical demography, was derived as part of a larger ecological study.

This paper reviews several recent policy models in which demographic and economic variables are endogenous. The models investigated are those with explicit policy objectives. To place the models in perspective, however, we first propose a classification of existing demo-economic models. Then, we introduce the basic mathematical apparatus common to all models treated in the remainder of the paper.

Demo-Economic Models: A Classification

Because the number of demo-economic models is large and growing rapidly, a classification may be useful. Realizing that any classification is to some degree arbitrary, and that classifying items is done more for pragmatic than for scientific reasons, we adopt the following three-way classification scheme (for another classification, see McNicoll, 1975):

- Models of demo-economic growth,

- Demo-economic simulation models,

- Models of demo-economic policy.

Models of Demo-Economic Growth

The main purpose of these models is to *describe* or to *explain* demographic and/or economic growth by considering both demographic and economic variables in an interdependent way. Their fundamental feature is the simultaneous endogenous treatment of demographic and economic growth. We shall consider three types of such models: demometric growth models, neoclassical growth models, and dualistic growth models.

The first category contains empirically-oriented models that frequently are developed as part of an inductive investigation of demo-economic growth. They are closely related to econometric growth models. Greenwood's (1975) simultaneous equations model of urban growth and migration is an example.

The second and third categories embrace theoretically-oriented models developed as aids in the deductive analysis of demo-economic growth. Their purpose is not to predict real situations, but rather to gain theoretical insights into real processes.

Neoclassical growth models have received most attention in economic growth theory (see e.g. Burmeister and Dobell, 1970). The term "neoclassical" is used to describe supply-oriented models, in contrast to Keynesian models, in which production and the use of resources are determined by aggregate demand rather than supply (McNicoll, 1975, p. 649).

Models of the dual economy are best known in the economic development literature. Although not generally thought of as demo-economic growth models, they are included in this classification because of their use of migration as an adjustment mechanism assuring labor market equilibrium. Models of the dual economy have been developed by Lewis (1954), Ranis and Fei (1961), and Jorgenson (1961), and extended by Kelley, Williamson and Cheetham (1972), and Todaro (1969), among others.

Demo-Economic Simulation Models

The distinction between simulation models and descriptive and explanatory growth models is fuzzy. Any demo-economic simulation model contains a growth model as its central and vital element. The quality of a simulation model varies directly with the quality of its growth model. The fundamental difference arises from the aims of the two kinds of models. While the second group of models are intended to describe or to explain, simulation models are meant to demonstrate and compare impacts of alternative policies or of alternative trajectories of exogenous variables, and therefore are impact evaluation tools. In this sense, policy simulation models may be descriptive or explanatory models that are adapted to investigate the sensitivity of the system to be studied to changes in predefined instrument and exogenous variables.

Demo-economic simulation models usually are designed to explore the economic implications of alternative population trajectories and trajectories of demographic parameters, and very rarely do they try to describe the evolution of a complete demo-economic system under changing conditions. One of the earliest examples of models of this type is the Coale-Hoover (1958) model for India. It focused on the impact of fertility reduction and the consequent changes in the size and age distribution of the population on economic development, in casu employment (labor), investments (capital) and per capita income. This model has produced a number of progeny (Demeny, 1965; Enke et al., 1968).

A few attempts have been made to simulate in a truly integrated manner the interdependencies in a demo-economic system. The first models of this sort are macro-economic models with population as an endogenous variable. Their principal purpose is not to represent the full complexity of the real situation, but rather to identify important insights about the demo-economic process. This class of macro-simulation models of demo-economic systems is illustrated by the Yap (1976) model for Brazil, designed to simulate the interaction between rural-urban migration and economic development, and by Kelley and Williamson's (1974) model of Meiji, Japan.

Finally, large-scale data-based models have been developed to simulate the evolution of demo-economic systems. The "Bachue" model of the International Labour Office, for example, considers a multisectoral economy and a disaggregated demographic system in a study of alternative employment-generation strategies. Economic development depends on demographic change, and fertility, mortality, and migration patterns are determined by the economy. For a critical review of some of the models, see Arthur and McNicoll (1976).

Models of Demo-Economic Policy

Demo-economic policy models strive to *prescribe* comprehensive demographic and economic policies. Policy objectives and policy instruments are stated explicitly. Objectives may be expressed as a set of targets to be reached, as an overall welfare index to be maximized, or as a combination of both. The models are dynamic in the sense that instruments and objectives belong to different time periods. Formally, the dynamic policy problem is that of choosing time paths for certain variables, called *instrument* or *control variables*, from a given set of feasible time paths, so as to maximize a given objective or to achieve given targets (compare this formal statement with that of Intriligator, 1971, p. 292). When presented in this form, the dynamic policy problem becomes an optimal control or dynamic optimization problem. Therefore, a convenient analytical framework for the study of quantitative dynamic demo-economic policy is the theory of optimal control. In other words, the population policy problem may formally be stated as a problem of optimal control (see Arthur and McNicoll, 1972, p. 2; Willekens, 1976b, p. 86). For the analogy in economic policy, see for example Chow (1973, 1975) and Pindyck (1973a).

Within the formal framework of optimal control, two groups of demo-economic policy models may be distinguished: planning-oriented policy models and theoretically-oriented policy models. The latter are set up to gain theoretical insights into the characteristics of an optimal demo-economic system. Their aggregation level is usually high, and the underlying growth model is generally of the neoclassical type. The first category comprises models designed to aid policy-makers to solve practical problems.

They are usually more disaggregated and imbed an empirically-
oriented or demometric growth model. While most models of the
first kind have been developed by authors more directly inter-
ested in the planning of the growth and the distribution of the
population, most of the second kind originated in economic
growth theory.

     In the remaining sections of the paper, we investigate some
features of demo-economic policy models of the optimal control
type. The first part is devoted to planning-oriented models.
The second part reviews the theoretically-oriented models.
First, however, we need to introduce some of the optimal control
vocabulary.

The Formal Dynamic Policy Problem

     The basic ingredients of an optimal control problem are
(i) a state equation describing the dynamics or "laws of motion"
of the system, (ii) a set of constraints on the state and con-
trol variables, (iii) a set of boundary conditions, and (iv) a
performance index or objective function (see for example Bryson
and Ho, 1969).

State Equation

     Let the vector $\{x(t)\}$ denote the state of the system at
time t. The state vector may refer to the population distribu-
tion by age or region, or to economic stock variables such as
capital. The control vector $\{u(t)\}$ contains the instruments or
policy variables which may be controlled by the policy-maker.
The dynamics of the system are described by a set of differential
or difference equations, the so-called state equations:

$$d\{x(t)\}/dt = f[\{x(t)\},\{u(t)\},t] \tag{1}$$

or

$$\{x(t + 1)\} = h[\{x(t)\},\{u(t)\},t] \quad . \tag{2}$$

In this paper, the state equations usually describe population
growth and capital accumulation. In other words, the state
variables are population and capital.

Constraints

     The dynamics of the state and control variables may be con-
strained for economic, political, or other reasons. Let the

set of admissible state and control variables be defined by the vector-valued function $\{g(.)\}$:

$$\{g(\{x(t)\},\{u(t)\},t)\} \geq \{0\} \quad . \tag{3}$$

Boundary Conditions

The initial state is given:

$$\{x(0)\} = \{x_0\} \quad . \tag{4}$$

Sometimes the values of the state variables must satisfy certain conditions at the planning horizon T. These are described by the vector-valued function

$$\{m(\{x(T)\})\} = \{0\} \quad . \tag{5}$$

Performance Index

The general formulation of the performance index to be optimized is

$$J = \int_0^T U(\{x(t)\},\{u(t)\},t) \, dt \tag{6}$$

for the continuous model, and

$$J = K(\{x(T)\}) + \sum_{i=0}^{T-1} L(\{x(t)\},\{u(t)\},t) \tag{7}$$

for the discrete model.

The dynamic policy or optimal control problem is then formulated as the determination of the control sequence $\{u^*(t)\}$ for $t = 0,\ldots,T-1$, and the corresponding trajectory of the state vector $\{x^*(t)\}$ for $t = 0,\ldots,T$, such that the systems dynamics (1) or (2), the constraints (3), and the boundary conditions (4) and (5) are satisfied and the performance index (6) or (7) is optimized. The sequence $\{u^*(t)\}$ is the optimal control and $\{x^*(t)\}$ is the optimal trajectory. In other words, *the optimal control problem is to steer a dynamic system so as to optimize a performance index, subject to constraints.* This formulation is a very general one and encompasses most dynamic population policy problems.

PLANNING-ORIENTED POLICY MODELS

The models discussed in this section all have a common feature: they may be considered as logical extensions of demographic growth models to the policy domain. To demonstrate this, we will gradually build up policy models of greater degrees of complexity, starting with growth models that have been studied in mathematical demography.

Malthus can probably be credited with formulating the first model of population growth: "Population, when unchecked, increases at a geometrical ratio" (Malthus, 1798, p. 13). Denoting the population size by N and the rate of population growth by n, Malthus' model may be represented by a first-order differential equation

$$\frac{dN(t)}{dt} = nN(t) \quad , \tag{8}$$

with the solution

$$N(t) = N_0 e^{nt} \quad . \tag{9}$$

The discrete form of Malthus' model is the difference equation

$$N(t + 1) = (\bar{n} + 1) \, N(t) = gN(t) \quad , \tag{10}$$

with the solution

$$N(t) = N_0 g^t \quad . \tag{11}$$

More recently, the aggregate model in (10) has been disaggregated to treat population growth by age (Leslie, 1945; Keyfitz, 1968), by region (Rogers, 1968), and by age and region (Rogers, 1975). The disaggregated model takes the form of a set of linear, first-order, homogeneous difference equations with the simple expression:

$$\{\underset{\sim}{K}(t + 1)\} = \underset{\sim}{G}\{\underset{\sim}{K}(t)\} \quad , \tag{12}$$

where the elements of the vector $\{\underset{\sim}{K}(t)\}$ denote the number of

people at time t by age group and/or by region, and $\underset{\sim}{G}$ is the associated population growth matrix.*

The solution of (12) for an unchanging $\underset{\sim}{G}$ is, of course,

$$\{\underset{\sim}{K}(t)\} = \underset{\sim}{G}^t\{\underset{\sim}{K}(0)\} \quad . \tag{13}$$

In terms of the standard optimal control problem presented in the previous section, the growth model (12) constitutes the homogeneous part of the state equation (1). The population distribution vector $\{\underset{\sim}{K}(t)\}$ is the state vector and the growth matrix $\underset{\sim}{G}$ is the transition matrix. The model describes the dynamics of an age- and/or region-specific population system that is undisturbed by exogenous forces (such as external migration) and free of any policy interventions.

The demographic growth model may be converted into a complete policy model in a number of steps. It is the purpose of this section to build up such a policy model and to provide a framework for comparing existing and potential planning-oriented population policy models.

The first step is to transform the growth model (12) into a complete state equation by adding a sequence of vectors describing control actions in time (and space). The simplest model (Rogers, 1966, 1968, 1971) is a purely demographic one; i.e., both the state and the control vector are in terms of demographic variables, such as fertility and migration (see also MacKinnon, 1975). This model and its variants will be reviewed in the first part of this section. The second part extends this policy model to include economic control and state variables, and considers constraints and objective functions explicitly.

A Matrix Model of Population Control

Recall that the growth of a demographic system may be represented by the matrix equation (12). To investigate the effects of a birth or migration control policy on an interregional population system, one may introduce a constant-intervention (control) vector $\{\bar{\underset{\sim}{m}}\}$ that is added to the population in each time period as follows:

$$\{\underset{\sim}{K}(t + 1)\} = \underset{\sim}{G}\{\underset{\sim}{K}(t)\} + \{\bar{\underset{\sim}{m}}\} \quad . \tag{14}$$

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*In this paper, K is used both for population distributed by age and for capital (pp. 26-48). Population by age, however, is always a vector $\{\underset{\sim}{K}(t)\}$. These notations are consistent with conventions in demography and economics.

The vector $\{\bar{m}\}$ may have both positive and negative components. A positive $\bar{m}_i$ indicates the number of people that must be added to a region's population during each unit interval of time; a negative $\bar{m}_i$ denotes the population that has to be periodically withdrawn from region i. In analyses of alternative birth control policies, a negative $\bar{m}_i$ may be interpreted as the number of births that must be prevented from occurring during each unit interval of time.

Beginning with an initial population at some point in time, say, t = 0, we may trace the effects of a policy control measure over time by repeatedly applying (14):

$$\{K(1)\} = G\{K(0)\} + \{\bar{m}\}$$

$$\{K(2)\} = G\{K(1)\} + \{\bar{m}\}$$

$$= G^2\{K(0)\} + G\{\bar{m}\} + \{\bar{m}\}$$

$$\vdots$$

$$\{K(t)\} = G^t\{K(0)\} + \left[\sum_{i=0}^{T-1} G^{T-1-i}\right] \{\bar{m}\} \qquad (15)$$

and

$$\{K(t)\} - G^t\{K(0)\} = \left[\sum_{i=0}^{T-1} G^{T-1-i}\right] \{\bar{m}\} \quad . \qquad (16)$$

Premultiplying both sides of (16) by G, and subtracting the result from (16), gives, for a particular class of G matrices,

$$(I - G)[\{K(t)\} - G^t\{K(0)\}] = (I - G^t) \{\bar{m}\} \quad . \qquad (17)$$

Therefore,

$$\{K(t)\} = G^t\{K(0)\} + (I - G)^{-1}(I - G^t) \{\bar{m}\} \quad . \qquad (18)$$

Note that (18) is the solution to equation (14).

Equation (18) may easily be transformed into a policy model under certain conditions. First, the goals of a population distribution policy must be expressed in terms of population targets

at a planning horizon. Second, the matrix $(I - G^t)$ is non-singular; this is the necessary and sufficient condition for *controllability* of the system described by (14) (Willekens, 1976b, Ch. 2).* Assuming now a vector of target populations at the planning horizon T, $\{\bar{K}(T)\}$, the intervention vector that will assure that the targets are reached is easily computed:

$$\{\bar{m}\} = (I - G^T)^{-1} \left[(I - G) [\{\bar{K}(T)\} - G^T \{K(0)\}] \right] \quad . \tag{19}$$

In applying the population control model (14) to empirical data, it is important to ensure that the interpretation of the intervention vector makes sense. For example, in a "pure" internal migration policy, the total population of the system remains constant. An inmigrant with respect to one region is an out-migrant with respect to another. The sum of inmigrants must equal the outmigrants, i.e.

$$\{1\}'\{\bar{m}\} = 0 \quad . \tag{20}$$

As a consequence, the policy-maker cannot specify a target population for all regions.

The procedure to compute the control vector in the case of a pure migration model is described by Rogers (1971, p. 106) as follows. The migration rates are taken out of the growth matrix and the migration flows are introduced via the control vector. The new growth matrix is S, say. After computing $\{\bar{m}\}$ by (19) with the growth matrix S and a target vector $\{\bar{K}(T)\}$, some elements of $\{\bar{m}\}$ are adjusted such that the constraint (20) holds, and a revised target vector is calculated. The constraints placed on the control variables make the system (14) uncontrollable; i.e., any target population cannot be reached. The problem, therefore, is to find a vector $\{\bar{m}\}$ which, given the equation (14) and the constraint equation (20), brings the population distribution at the horizon T as close to the target population $\{\bar{K}(T)\}$ as possible. This policy problem may be expressed as the following optimal control problem:

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*The system $\{K(t + 1)\} = G\{K(t)\} + B\{u(t)\}$ is said to be controllable if and only if there exists a control $\{u(t)\}$ that transfers any initial state $\{K(t_0)\}$ at time t_0 to any arbitrary final state $\{K(t_1)\}$ at any time $t_1 > t_0$ (Wolovich, 1974, p. 65). In equation (14) $B\{u(t)\} = \{\bar{m}\}$.

$$\min \quad \{\underset{\sim}{K}(T)\} - \{\underset{\sim}{\bar{K}}(T)\} \tag{21}$$

$$\text{s.t.} \quad \{\underset{\sim}{K}(t+1)\} = \underset{\sim}{S}\{\underset{\sim}{K}(t)\} + \{\underset{\sim}{\bar{m}}\}$$

$$\{\underset{\sim}{1}\}'\{\underset{\sim}{\bar{m}}\} = 0 \quad .$$

Policy model (19) and some of its variants have been applied by Drewe (1971) in a feasibility study of population distribution policy in The Netherlands.

An interesting extension of the above policy model follows from relaxation of the assumption of a fixed policy vector. If the degree or level of a population policy may decline over time, then the vector $\{\underset{\sim}{\bar{m}}\}$ is added only at the beginning, and $w\{\underset{\sim}{\bar{m}}\}$ is added during the next time period $(0 < w < 1)$; i.e.,

$$\{\underset{\sim}{m}(t+1)\} = w\{\underset{\sim}{m}(t)\} = w^{t+1}\{\underset{\sim}{\bar{m}}\} \quad , \tag{22}$$

with w being a scalar. The control at a certain time period t is a constant fraction of the control in the previous time period. In other words, the value of the control vector decreases exponentially over time. The impact of an initial policy $\{\underset{\sim}{\bar{m}}\}$ on the population growth path is therefore

$$\{\underset{\sim}{K}(1)\} = \underset{\sim}{G}\{\underset{\sim}{K}(0)\} + \{\underset{\sim}{\bar{m}}\}$$

$$\{\underset{\sim}{K}(2)\} = \underset{\sim}{G}\{\underset{\sim}{K}(1)\} + w\{\underset{\sim}{\bar{m}}\} = \underset{\sim}{G}^2\{\underset{\sim}{K}(0)\} + \underset{\sim}{G}\{\underset{\sim}{\bar{m}}\} + w\{\underset{\sim}{\bar{m}}\}$$
$$\vdots$$
$$\{\underset{\sim}{K}(t)\} = \underset{\sim}{G}^t\{\underset{\sim}{K}(0)\} + \left[\sum_{i=0}^{T-1} w^{T-1-i}\underset{\sim}{G}^i\right]\{\underset{\sim}{\bar{m}}\} \tag{23}$$

and

$$\{\underset{\sim}{K}(t)\} - \underset{\sim}{G}^t\{\underset{\sim}{K}(0)\} = \left[\sum_{i=0}^{T-1} w^{T-1-i}\underset{\sim}{G}^i\right]\{\underset{\sim}{\bar{m}}\} \quad . \tag{24}$$

Premultiplying both sides by $(w\underset{\sim}{I} - \underset{\sim}{G})$ yields

$$(w\underset{\sim}{I} - \underset{\sim}{G})[\{\underset{\sim}{K}(t)\} - \underset{\sim}{G}^t\{\underset{\sim}{K}(0)\}] = (w^t\underset{\sim}{I} - \underset{\sim}{G}^t)\{\underset{\sim}{\bar{m}}\} \quad ,$$

whence

$$\{\underset{\sim}{K}(t)\} = \underset{\sim}{G}^t\{\underset{\sim}{K}(0)\} + (w\underset{\sim}{I} - \underset{\sim}{G})^{-1}(w^t\underset{\sim}{I} - \underset{\sim}{G}^t)\ \{\bar{\underset{\sim}{m}}\}\ \ .\quad (25)$$

Assuming a target vector $\{\bar{\underset{\sim}{K}}(T)\}$, the initial control vector that assures the achievements of the targets under this policy regime is

$$\{\bar{\underset{\sim}{m}}\} = [w^t\underset{\sim}{I} - \underset{\sim}{G}^t]^{-1}\left[(w\underset{\sim}{I} - \underset{\sim}{G})\,[\{\bar{\underset{\sim}{K}}(T)\} - \underset{\sim}{G}^t\{\underset{\sim}{K}(0)\}]\right]\ \ .\quad (26)$$

The policy model (14) may be extended a step further (Willekens, 1976b, pp. 69-71). Instead of assuming a constant relative decline in the value of the control vector, suppose that the control vector at each time period is a linear combination of the control vector at the previous time period, i.e.

$$\{\underset{\sim}{m}(t)\} = \underset{\sim}{W}\{\underset{\sim}{m}(t-1)\}\ \ ,\quad (27)$$

and

$$\{\underset{\sim}{m}(t)\} = \underset{\sim}{W}^t\{\underset{\sim}{m}(0)\} = \underset{\sim}{W}^t\{\bar{\underset{\sim}{m}}\}\ \ ,\quad (28)$$

where $\underset{\sim}{W}$ is assumed to be nonsingular. Introducing (28) into (14) and solving yields

$$\{\underset{\sim}{K}(t)\} = \underset{\sim}{G}^t\{\underset{\sim}{K}(0)\} + \left[\sum_{i=0}^{T-1}\underset{\sim}{G}^i\underset{\sim}{W}^{T-1-i}\right]\{\bar{\underset{\sim}{m}}\}\ \ .\quad (29)$$

The control vector at the initial time period yielding a target population distribution $\{\bar{\underset{\sim}{K}}(T)\}$ at horizon T may be computed easily.

Note that equations (18) and (25) are particular cases of (29). In the case of a constant control vector, the matrix $\underset{\sim}{W}$ is an identity matrix. In the case of an exponentially declining control vector, on the other hand, $\underset{\sim}{W}$ is a scalar matrix, i.e. $\underset{\sim}{W} = w\underset{\sim}{I}$.

The policy models discussed in this section have an interesting common feature. Since the matrices $\underset{\sim}{G}$ and $\underset{\sim}{W}$ are assumed to be time-invariant, the matrix sum in (29) depends only on the planning horizon T. Let

$$\sum_{i=0}^{T-1}\underset{\sim}{G}^{T-1-i}\underset{\sim}{W}^i = \underset{\sim}{A}(T)\ \ ;$$

then equation (29) may be written for t = T as

$$\{\underset{\sim}{K}(T)\} = \underset{\sim}{G}^T\{\underset{\sim}{K}(0)\} + \underset{\sim}{A}(T)\ \{\underset{\sim}{\bar{m}}\}\ ,\qquad\qquad (30)$$

where

$\quad\{\underset{\sim}{K}(T)\}$ is the population distribution at time T,
$\quad\{\underset{\sim}{\tilde{K}}(0)\}$ is the initial population distribution,
$\qquad\{\underset{\sim}{\bar{m}}\}$ is the control vector in the initial time period.

Hence, although a population policy is implemented in each time period, the population trajectory is completely determined once the control vector in the base year is fixed. The dynamic, multiperiod population policy problem reduces therefore to a single-period problem. Policy models, where the control at t is a fixed linear combination of the control vector in the initial period, have been called *initial period control models* (Willekens, 1976b, p. 69). In the next section, we will drop the constraint on the control vector and introduce the possibility of intervening in population redistribution by applying economic policy instruments.

Elaboration of the Matrix Model of Population Control

The expansion of the above matrix model and its variants to a complete dynamic policy problem would involve (Willekens, 1976b, Ch. 3):

- Introducing economic control variables and the specification of their impact on the population distribution.

- Dropping the stringent constraints on the control vector, i.e., the extension of the initial period control problem to a truly dynamic control problem.

- Allowing for other constraints on both the state and the control variables, and for formulations of the policy objectives other than in terms of targets.

Introduction of Non-Demographic Control Variables

It was stressed in the introduction that a fundamental feature of population policy is that it does not occur in a vacuum. The ultimate goals of demographic intervention are non-demographic in nature, and the instruments are socio-economic. Policy models must reflect this connection. The first link between population policy and socio-economic policy lies in the instruments. Policy-makers usually do not directly alter the volume of migration in order to mold a population distribution

into a desired pattern. Rather, the intervention is indirect, through economic variables such as regional income, employment, housing, government expenditures, and so on. Therefore, $\{\bar{u}\}$ is a vector of *socio-economic* control variables, and the impacts of the instruments on the population distribution in the next time period is given by vector function

$$\{\bar{m}\} = \{h(\{\bar{u}\})\} \quad . \tag{31}$$

For the sake of simplicity, we will assume a linear relationship,

$$\{\bar{m}\} = B\{\bar{u}\} \quad , \tag{32}$$

say, where B is a time-invariant coefficient matrix of dimension $N \times K$, where N is the dimension of $\{\bar{m}\}$ and hence the dimension of the state vector (e.g. number of regions), and K is the dimension of $\{\bar{u}\}$ or the number of instruments. An element b_{ij} denotes the impact of the j-th control variable on the i-th element of $\{\bar{m}\}$. The ratio $-b_{ij}/b_{ik}$ is the amount by which the j-th instrument may be cut down without changing the level of the i-th element of $\{\bar{m}\}$, if the value of the k-th instrument is increased with one unit. It is, therefore, the marginal rate of substitution between the two instruments (Fromm and Taubman, 1968, p. 109). Introducing (32) into (14) gives

$$\{K(t + 1)\} = G\{K(t)\} + B\{\bar{u}\} \quad . \tag{33}$$

Equation (33) relates the population distribution in a certain time period to the population distribution in the previous time period and to socio-economic policies. Since $\{\bar{u}\}$ may contain lagged policy variables, the direct effects of earlier policies may be included. If $\{\bar{u}\}$ has no lagged instruments, B coincides with what is known in economics as the matrix of impact multipliers or the *matrix multiplier*. The matrix multiplier plays a pivotal role in the study of the controllability of dynamic systems (Willekens, 1976b, Ch. 2; Aoki, 1976).

At this point, two remarks are in order:

- If the population policy is purely demographic, then (33) reduces to the basic matrix model of population control or its variants. In the basic model $\{\bar{m}\} = \{\bar{u}\}$ and the matrix multiplier is a diagonal matrix. In the intervention model with exponentially declining policies, B reduces to a scalar matrix. However, this matrix is no longer time-invariant.

- The policy problem represented by (33) is still an initial period control problem. It is closely related to the static policy model developed by Tinbergen (1963), in which $\{K(t + 1)\} = \{y\}$ is the vector of target variables and $\{K(t)\} = \{z\}$ is the vector of uncontrollable exogenous variables:

$$\{y\} = G\{z\} + B\{\bar{u}\} \quad . \tag{34}$$

The Tinbergen policy model is therefore a special case of (33) in which there is only one time period. A solution to (34) exists if the rank of B is equal to the number of targets. The solution is unique if B is nonsingular, or, in the words of Tinbergen, if the number of instruments is equal to the number of targets. Then,

$$\{\bar{u}\} = B^{-1}[\{y\} - G\{z\}] \quad . \tag{35}$$

The Multiperiod Control Problem

The policy models considered thus far are not really dynamic. Although there is a control vector for each time period, the trajectory of the controls is fixed such that the only freedom the policy-maker has is in choosing the instruments of the initial time period. Once the initial controls are chosen, future values of the controls, and hence of the state variables, follow automatically. In this section, the assumption of dependency of controls is dropped. The state equation (33) becomes

$$\{K(t + 1)\} = G\{K(t)\} + B\{u(t)\} \quad . \tag{36}$$

The solution to this truly dynamic policy model is

$$\{K(t)\} = G^t\{K(0)\} + \sum_{i=0}^{t-1} G^{(t-1-i)} B\{u(i)\} \quad . \tag{37}$$

From the model (36) and its solution (37), two multiperiod policy problems may be derived:

- The *horizon-oriented policy problem* may be formulated as follows: given the initial condition $\{K(0)\}$ and the assumption of time-invariance of the coefficient matrices, which sequence of control vectors $\{u(i)\}$ ensures that a target vector at a predefined horizon T will be reached?

- In a *trajectory-oriented policy*, the question is whether there exists a sequence of control vectors {u(i)} such that, for a given initial condition and for time-invariant coefficient matrices, any sequence of target vectors {K̃(t)} can be realized.

In mathematical systems theory, the first policy problem is known as state controllability (Wolovich, 1974). The second problem will be denoted as complete state controllability.

Both policy problems will now be treated in formal terms.

Horizon-Oriented Policy: Equation (37) may be written for t = T as

$$\{K(T)\} - G^T\{K(0)\} = [B \vdots GB \vdots \cdots \vdots G^{T-1}B] \begin{bmatrix} \{u(T-1)\} \\ \vdots \\ \{u(1)\} \\ \{u(0)\} \end{bmatrix} \quad (38)$$

$$= D\{\hat{u}\} \quad , \quad \text{say} \quad . \quad (39)$$

The system (39) is state controllable if the N × KT matrix D is of rank N, where N is the dimension of the target vector {K̄(T)} (Wolovich, 1974, p. 65). If D is nonsingular, there is a unique control sequence, which is given by

$$\{\hat{u}\} = D^{-1}[\{\bar{K}(T)\} - G^T\{K(0)\}] \quad . \quad (40)$$

In the dynamic policy model (36) and (39), it is the combined magnitude of the number of instruments and the planning horizon that determines state controllability. Any target vector may be reached by only one instrument (K = 1) provided the planning horizon is not less than N and certain other conditions hold (Preston, 1974, p. 70). Also, any set of targets may be reached in only one time period if the policy-maker can manipulate at least N instruments.*

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*This is exactly the controllability condition derived by Tinbergen (1963) for a static policy model. For t = 1, D coincides with B.

Trajectory-Oriented Policy: The policy problem discussed
in the previous section dealt with the existence of a sequence
of control vectors, necessary for the achievement of the desired
target vector at a predefined planning horizon. In practice,
policy-makers would be interested not only in achieving desired
target values, but also in keeping them on some desired trajec-
tory once achieved, or in achieving the targets along a desired
path. It is not uncommon in politics that short-term objectives
conflict with long-term goals: long-term goals may become un-
attainable because of short-term policies. Consequently, not
only is the state at the planning horizon of interest, but also
the trajectory. It is, therefore, relevant to consider the
policy problem in which targets are formulated at each time
period.

Writing (38) for each time period gives

$$\begin{bmatrix} \{K(T)\} - G^T\{K(0)\} \\ \{K(T-1)\} - G^{T-1}\{K(0)\} \\ \vdots \\ \{K(1)\} - G\{K(0)\} \end{bmatrix} = \begin{bmatrix} B & GB & G^2B & \cdots & G^{T-1}B \\ 0 & B & GB & \cdots & G^{T-2}B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B \end{bmatrix} \begin{bmatrix} \{u(T-1)\} \\ \{u(T-2)\} \\ \vdots \\ \{u(0)\} \end{bmatrix}$$

(41)

$$\{\hat{d}\} = F\{\hat{u}\} \quad , \quad \text{say} \quad . \tag{42}$$

The system (41) is controllable if the NT × KT matrix F is
of rank NT (for T < N). If F is nonsingular, then there exists
a unique control sequence:

$$\{\hat{u}\} = F^{-1}\{\hat{d}\} \quad . \tag{43}$$

Note that horizon-oriented policy problems form a special case
of the trajectory-oriented policy problem. If, in (41), $\{K(t)\}$
is not predefined for $t = 1,\ldots,T-1$, then it reduces to (38),
i.e. the horizon-oriented policy problem. Therefore, complete
state controllability implies state controllability.

The computation of the unique policy sequence is straight-
forward once the *existence* of such a policy is demonstrated.
But what if F (or D or B) is singular? In this case, there may
be an infinite number of control sequences that give the desired

values of the state vectors; or there may be no control sequence
at all that reaches the targets. Two cases are considered in the
following.

(i) Case 1: NT < KT

If $\underset{\sim}{F}$ is rectangular and of rank NT, the number of controls
exercised, KT, exceeds the number of targets specified, NT.
Consequently, there are an infinite number of solutions to (42)
and, therefore, an infinite number of control sequences that
lead to the predefined targets. All the solutions to (42) may
be expressed using the notion of a generalized inverse. If $\underset{\sim}{F}^{(1)}$
is a generalized inverse of $\underset{\sim}{F}$, satisfying $\underset{\sim}{F}\,\underset{\sim}{F}^{(1)}\underset{\sim}{F} = \underset{\sim}{F}$, and if
$\{\underset{\sim}{c}\}$ is an arbitrary vector of dimension KT, then the general
solution to (42) is (Rogers, 1971, p. 258)

$$\{\hat{\underset{\sim}{u}}\} = \underset{\sim}{F}^{(1)}\{\hat{\underset{\sim}{d}}\} + [\underset{\sim}{I} - \underset{\sim}{F}^{(1)}\underset{\sim}{F}]\ \{\underset{\sim}{c}\}\ . \tag{44}$$

Out of the infinite number of feasible control sequences,
the policy-maker must choose a single one. In order to get a
unique solution to (42), he may force the number of instruments
exercised, KT, to be equal to the number of targets specified,
NT, by deleting some instrument variables at certain time peri-
ods; or he may constrain the values that the instrument vari-
ables can take on. A wide variety of possible constraints exists,
but we consider only two categories: linear dependency among
several instruments, and definition of acceptable values of the
instruments.

By making some instruments linearly dependent, the freedom
of policy action is reduced such that only one control sequence
is available to achieve the targets. An illustration of this
type of constraint has been given in the previous section.

In many cases, the policy-maker has a good idea of what
levels of control variables are acceptable politically and eco-
nomically. Minimizing some measure of deviation between the
realized and the most acceptable values assures a unique sequence
of instrument vectors. For example, the solution of the following
tracking problem may yield a unique control sequence:

$$\min\quad W = \sum_{t=0}^{T-1} [\{\underset{\sim}{u}(t)\} - \{\bar{\bar{\underset{\sim}{u}}}(t)\}]'\ \underset{\sim}{R}[\{\underset{\sim}{u}(t)\} - \{\bar{\bar{\underset{\sim}{u}}}(t)\}] \tag{45}$$

$$\text{s.t.}\quad \{\underset{\sim}{K}(t+1)\} = \underset{\sim}{G}\{\underset{\sim}{K}(t)\} + \underset{\sim}{B}\{\underset{\sim}{u}(t)\}\ .$$

In equation (45), the objective is to minimize the sum of the squared deviations between $\{u(t)\}$ and an acceptable or desired control vector at time t, $\{\bar{u}(t)\}$. It is a simple linear quadratic control problem.

The use of a quadratic objective function with linear constraints is common in economic policy analysis.* It is based on two assumptions: that the policy-maker's preferences are quadratic, and that each of the targets depends linearly on all of the instruments, the coefficients of these linear relations being fixed and known.

(ii) Case 2: NT > KT

If the number of targets specified exceeds the number of instruments exercised, the system (42) is inconsistent, and not all the target values can be reached. This poses an additional decision problem for the policy-maker. Does he give up some targets in order to reach others, or does he want to approximate all of the targets as closely as possible? In the latter case, we again have a tracking problem, but now in the state variables instead of in the controls. A policy model analogous to (45) may be formulated as follows:

$$\min \quad W = \sum_{t=1}^{T} [\{K(t)\} - \{\bar{K}(t)\}]' \; Q[\{K(t)\} - \{\bar{K}(t)\}] \qquad (46)$$

$$\text{s.t.} \quad \{K(t + 1)\} = G\{K(t)\} + B\{u(t)\} \quad .$$

A combination of tracking problems (45) and (46) leads to the dual tracking problem. Desirable values are given for the trajectories of both the state and the control variables. Some extensions of the dual tracking problem are given by Willekens (1976b, pp. 98-101).

The Generalized Dynamic Policy Problem

In the policy problems considered thus far, it was assumed that the policy-maker's preference system could be expressed completely in terms of target values for the state variables,

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*Theil's quadratic programming model for static and dynamic policy analysis (Theil, 1964, pp. 34-35 and Ch. 4; Friedman, 1975, pp. 158-160) is frequently used, as is the linear-quadratic control model (Sengupta, 1970; Pindyck, 1973a, 1973b; Vishwakarma, 1974; Garbade, 1975; Chow, 1972, 1975, Ch. 9). The linear quadratic control model is particularly successful in applied problems of quantitative economic stabilization policy.

and that the achievement of these targets was constrained only
by the "law of motion" or state equation describing the dynamic
behavior of the system. The advantage of this formulation of the
policy problem is that its solution can be investigated analyti-
cally. It has been shown that the existence of a control vector,
or of a sequence of control vectors, ensuring achievement of the
targets is determined by the rank of the matrix multipliers B,
D or F. In other words, the ranks of the matrix multipliers˜
determine the controllability of the dynamic demo-economic system.
Once the existence of a feasible policy has been demonstrated,
the computation of the control vector, or sequence of control
vectors, is straightforward. The design of an optimal policy
is particularly simple if the matrix multiplier is nonsingular.
In this case, only one feasible combination of controls exists:
the optimal combination.

In the previous sections, no direct constraints were imposed
on the state variables. The control variables were constrained
in a very simple way; namely, through the introduction of linear
dependency. In this section, we expand the possible constraints,
thus reducing the set of feasible control vectors. In addition,
more realistic policy objectives are discussed.

In practical policy-making, the values that the state and
control vectors can take on are restricted by political, economic,
and social considerations. For example, it is politically un-
acceptable for the values of policy instruments to fluctuate
heavily from one period to another. To remedy possible problems
of instrument instability, Holbrook (1972, p. 57) proposes to
include the instruments in the policy-maker's preference function.
Each element of the control vector also may be required to lie
within a lower and an upper boundary:

$$\underline{u}_i(t) \le u_i(t) \le \bar{u}_i(t) \quad . \tag{47}$$

Population policy is not cost-free; imposing controls implies
the incurring of costs. It is, therefore, natural to assume a
budget constraint that limits the action space of the policy-
maker. We distinguish between a budget constraint for each
period:

$$\{\underset{\sim}{c}(t)\}' \ \{\underset{\sim}{u}(t)\} \le C(t) \quad , \tag{48}$$

and a global budget constraint:

$$\sum_{t=0}^{T-1} \{\underset{\sim}{c}(t)\}' \ \{\underset{\sim}{u}(t)\} \le C \quad . \tag{49}$$

The cost vector $\{c(t)\}'$ contains the unit costs of each instrument.

Constraints (47) to (49) are related to the control vector. Frequently, the state vector itself, i.e. the population distribution, is constrained in addition to the control vector. For example, the policy-maker may want to put upper and lower limits on the population in each region in order to avoid the social costs of excessive density or of depopulation. Other constraints on the state vector may be formulated. The general formula expressing constraints on state and/or control vectors is given by equation (3). Usually, however, such constraints take the form of a set of linear inequalities. Together with the boundary conditions and the state equations these delineate the feasible set of controls out of which an optimal control vector or control sequence may be chosen according to an objective function.

In the previous section, quadratic objective functions have been considered. Other illustrations of formal population policy problems with quadratic objectives and linear constraints are given by Evtushenko and MacKinnon (1975) and by Mehra (1975). If constraints and objectives are both linear, the policy model takes the format of a dynamic linear programming problem (Propoi and Willekens, 1977).

The most general formulation of a dynamic policy problem is presented by equations (1) to (7) of the first section of this paper. Neither constraints nor objectives need take simple linear or quadratic forms. In general, however, simplifications are adopted to facilitate the computational task of finding the optimum. Solution algorithms for dynamic mathematical programming or optimal control problems are beyond the scope of this paper. Descriptions and numerical illustrations may be found in textbooks, such as those of Bryson and Ho (1969), McReynolds (1970), and Noton (1972).

THEORETICALLY-ORIENTED POLICY MODELS

Theoretically-oriented policy models have been developed to gain insights into the characteristics of an optimal demo-economic system. Most originated in economics, particularly in the field of economic growth theory. Their main concern is the study of the existence, uniqueness, stability, and efficiency properties of equilibrium growth paths (McNicoll, 1975, p. 651).

The basic format of these models is that of an optimal control problem, as described in equations (1) to (7). As in the previous section, we will begin our exposition with the simple economic growth model that underlies most theoretically-oriented policy models, the neoclassical growth model, and then gradually build up policy models of greater complexity.

In contrast to demographic growth, there is no unique indicator that measures economic growth. Gross product or output, value added, consumption, and other such variables all have been used. Consider, for example, the activity of production. The output of a production process is described by the production function. Usually, only two production factors are considered, capital and labor. The production function, therefore, may be expressed as

$$Q(t) = F[K(t), L(t)] \quad , \tag{50}$$

where $K(t)$ denotes the capital stock at time t, and $L(t)$ is the corresponding stock of labor.

In a well-defined production function, growth of total output is uniquely determined by the growth of the factors of production. Solow (1956) suggested simple hypotheses about the development of factor endowments that close the system and enable a study of the growth path generated by the model economy. Meade (1961) and Swan (1956) independently developed similar models leading to the same conclusions. Their model is known as the neoclassical or Solow-Swan growth model.

Assume a neoclassical production function, and a growth path of capital labor obeying the following assumptions.

First, the labor force $L(t)$ grows at a constant relative rate n, which is equal to the growth rate of the population and is given exogenously:

$$\frac{dL(t)}{dt} = nL(t) \quad , \qquad L(0) = L_0 > 0 \quad . \tag{51}$$

The labor supply function is the solution of the differential equation in (51):

$$L(t) = L_0 e^{nt} \quad . \tag{52}$$

Second, a constant fraction s of the total output flow $Q(t)$ is saved and all the savings are invested in the capital stock $K(t)$. Assuming moreover that capital does not depreciate, the growth of the capital stock is given by the investment function:

$$\frac{dK(t)}{dt} = sQ(t) \quad , \qquad K(0) = K_0 > 0 \quad . \tag{53}$$

The growth of the output is

$$\frac{dQ}{dt} = \frac{dF(.)}{dK(t)} \frac{dK(t)}{dt} + \frac{dF(.)}{dL(t)} \frac{dL(t)}{dt}$$

where $dK(t)/dt$ and $dL(t)/dt$ are given by (51) and (53) respectively. Note that this growth model applies only to an economy in full employment. The labor supply is completely inelastic. The labor supply curve is a vertical line that shifts to the right over time as the quantity of labor increases. Wages have no impact on labor supply or on labor demand. The real wage rate, or equivalently the marginal productivity of labor, adjusts at each time period so that all available labor is employed.

If there is unemployment, the wage rate should fall. Labor becomes cheaper and induces a substitution of labor for capital. This lowers the capital-labor ratio until full employment is restored.

Inserting (50) and (52) in (53) gives

$$\frac{dK(t)}{dt} = sF[K(t),L_0 e^{nt}] \quad . \tag{54}$$

This is the basic equation determining the time path of capital accumulation that must be followed for full employment to be maintained. For each time t, the supply of labor and capital is inelastic. Labor is given by (52) and the capital stock is a result of previous accumulation. All labor and capital that exist at t will be fully employed. This is brought about by an adjustment of the marginal productivities. For each t, the output may be computed by the production function. How much of the output will be saved and reinvested is given by (54). This investment adds to the capital stock of the next period.

In the Solow-Swan model of economic growth, the possibility of factor substitution assures that full employment is maintained. The burden of adjustment falls on the marginal productivities of capital and labor, or equivalently, on the marginal capital-output ratio. To study the relationship between the time path of capital accumulation and population growth, we express the capital stock at time t as

$$K(t) = kL(t) \tag{55}$$

where k is the capital-labor ratio, defined as $k = K(t)/L(t)$. Differentiating both sides of (55) gives

$$\frac{dK(t)}{dt} = \frac{dk}{dt} L(t) + k \frac{dL(t)}{dt} \quad . \tag{56}$$

Substitution of (52) in (56) gives

$$\frac{dK(t)}{dt} = \frac{dk}{dt} L_0 e^{nt} + knL_0 e^{nt} \quad . \tag{57}$$

Equating (57) to (54) yields

$$\left[\frac{dk}{dt} + kn\right] L_0 e^{nt} = sF[K(t), L_0 e^{nt}] \quad . \tag{58}$$

The production function is neoclassical with constant returns to scale. If capital and labor are multiplied by some constant, the output is multiplied by the same constant. Therefore, we may divide both sides of (58) by $L_0 e^{nt}$:*,**

$$\frac{dk}{dt} + kn = sF[k,1] \quad .$$

The time path of the capital-labor ratio is given by the differential equation:

$$\frac{dk}{dt} = sF(k,1) - nk \quad . \tag{59}$$

The function $F(k,1)$ is the *per capita production function*. It is the total production that arises as varying amounts k of capital are employed with one unit of labor. In other words, it gives the output per worker as a function of capital per worker. It depends only on the capital-labor ratio or capital-output ratio because of constant returns to scale. Equation (59) shows that the growth rate of the capital-labor ratio (dk/dt)/k is equal to the growth rate of capital, or the rate of capital accumulation $sF(k,1)/k$, minus the growth rate of labor, n. Note that $k/F(k,1)$

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*If there are no constant returns to scale, we must consider (50) and (52) directly without this simplification.
**In the economic literature, the per capita production function is usually denoted by f(k). The notation F[k,1] is to remind the reader of the existence of the production factor labor.

is the capital-output ratio, say C. Hence, we may write

$$\frac{1}{k}\frac{dk}{dt} = s/C - n \quad . \tag{60}$$

The quantity s/C is the *warranted rate of growth* (Harrod, 1970, p. 47). In Harrod's version of the Harrod-Domar model, it is the rate of growth of output for which the actual level of production coincides with the expected demand. The producer produces neither more nor less than the right amount.

If dk/dt = 0, the capital-labor ratio is constant, and the capital stock must be expanding at the same rate as the labor force, namely n, to maintain full employment. But this is exactly the formulation of the Harrod-Domar consistency condition that an economic system must satisfy in order for steady-state growth to be possible under a fixed capital-output ratio, a constant savings rate, and full employment. In Harrod's formulation, the condition is that the warranted rate of growth equals the *natural growth rate* n (the growth rate of the population):

$$\frac{s}{C} = n \quad . \tag{61}$$

This equilibrium situation is labeled by Robinson (1970, p. 133) the *golden age*, to indicate a steady, smooth growth with full employment. In Solow's neoclassical model, the capital output ratio is not fixed but changes automatically in response to changes in factor supplies (measured by changes in s or n).

It can be shown that, for any positive s and n satisfying

$$0 < \frac{n}{s} < F'(0) \quad , \tag{62}$$

there exists a unique positive capital-labor ratio k* such that dk*/dt = 0 (Burmeister and Dobell, 1970, p. 25), such that equilibrium or steady state is feasible. How the capital-labor ratio changes as the system converges to its equilibrium position is portrayed in Figure 1. It is the phase diagram for the differential equation in (59).

For any point on the dk/dt curve, there is full employment and hence short-run equilibrium. The position of the economy is described by k, and its growth by dk/dt. In the long-run equilibrium, dk/dt = 0, the capital-labor ratio is constant, and capital grows at the same rate as labor.
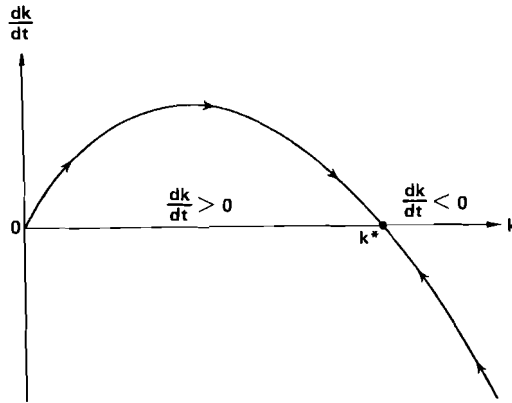
Figure 1.   Phase diagram for Solow's fundamental differential

equation ($\frac{dk}{dt} = sF(k,1) - nk$).

Source: Burmeister and Dobell, 1970, p. 26

Equations (59) and (60) describe the growth rate of the economy in terms of the growth rates of factor supplies. It presents the "law of motion" of the economic system.

In these models, and in various extensions, labor grows at a constant rate. To convert the neoclassical growth model (59) into a complete demo-economic policy model, we may consider a number of additions. The first is to drop the assumption of exogenously defined labor increase by endogenizing the growth rate of labor; the second is the introduction of policy-objectives and of other constraints.

Economic Growth with Population Endogenous

Classical economists such as Adam Smith viewed population size as being positively related to wage level. High wages would affect birth and death rates; they would encourage early marriage, and hence higher birth rates. In addition, children would become more valuable as future workers and as a form of retirement in-surance. This would induce parents to take greater care of their children, and would thereby diminish the infant death rate.

Ricardo considered a third factor of production, land, whose total supply is fixed. Constant returns to scale are assumed for the three factors: land, capital, and labor. Therefore, a pro-duction function containing only capital and labor exhibits

decreasing returns. As did Smith, Ricardo linked population
growth to wage level. He assumed that there was a subsistence
wage; if the actual wage fell below the subsistence wage, women
would adopt a net reproduction rate of less than unity. The con-
clusion of Ricardo's analysis was that population and the economy
would approach a stationary state (n = 0), with wages at a sub-
sistence level.*

Although Smith and Ricardo both devoted some attention to
the economics of population growth and indicated that population
is endogenous to economic growth, Malthus was the first to succeed
in systemizing a general theory of population. According to
Malthus, birth rates are biologically determined, but death rates
are affected by economic conditions.

The formal treatment of an endogenous population in economic
growth models is of a more recent date. This section reviews
some attempts to endogenize the demographic component. In addi-
tion, it investigates the impact of an endogenously changing
labor force participation rate and of the explicit consideration
of consumption.

The Neoclassical Model with Population Endogenous

To illustrate how population growth may affect economic
growth, consider the fundamental equation of the Solow-Swan model
where the population growth rate is a function of the wage rate
w(k), or of per capita income or consumption (Solow, 1970,
p. 189):

$$\frac{dk}{dt} = sF(k,1) - kn[w(k)] \quad .$$
(63)

Assume that, when the capital-labor ratio k, and hence the wage
rate w(k), is low, the population is unable to maintain itself,
and the growth rate of labor is negative. As wages rise, the
population growth rate increases until the wage rate reaches
such a level that the wealthy population decides to cut down its
growth rate. Such a case might be represented by a growth rate
equation for n[w(k)] or n(k), or by a phase diagram such as
Figure 2.

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*A production function with decreasing returns to scale is,
somewhat surprisingly, not a sufficient condition to ensure that
a stationary population will be approached (Niehans, 1963; Enke,
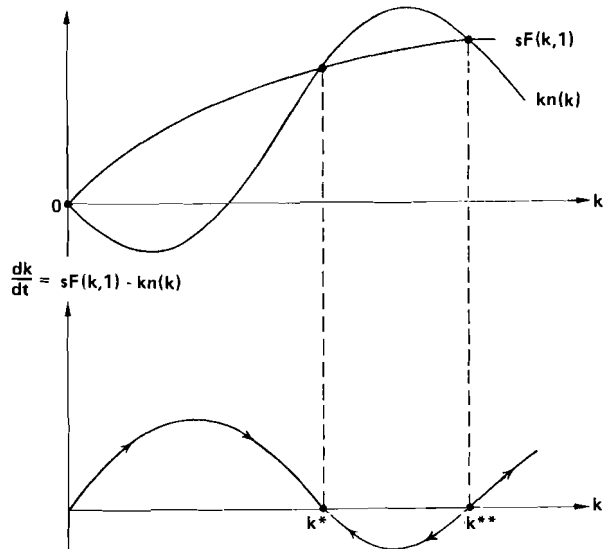1963; Pitchford, 1974, p. 56-70).

Figure 2. Phase diagram for the Solow-Swan model with population growth endogenous (equation $\frac{dk}{dt} = sF(k,1) - kn(k)$).

As Figure 2 shows, there is the possibility of multiple equilibria and hence also of unstable ones. Below the equilibrium point k** small perturbations of k force the capital-labor ratio to an equilibrium value k*. When the capital-labor ratio increases beyond the unstable equilibrium point k**, the economy is on a path with a perpetually rising capital-labor ratio and hence per capita income.

Instead of focusing on the wage rate, one may make population growth depend on per capita income or consumption. In his original article, Solow (1970, p. 188) treats these equivalent cases. In general, the population growth equation becomes

$$\frac{dL}{dt} = L(t) \; n(\xi) \tag{64}$$

where ξ denotes wages, per capita income, or per capita consumption.

In the Sato and Davis (1971, p. 881) model, ξ denotes per capita income y or $F(k,1)$. The economic dynamics is therefore given by

$$\frac{dk}{dt} = sF(k,1) - kn[F(k,1)] \quad . \tag{65}$$

The function n(.) is monotonic (n' $\geq$ 0). The logistic growth curve in Figure 3 reflects the fact that the death rate decreases with income and that the birth rate increases up to a certain level of y and declines thereafter.
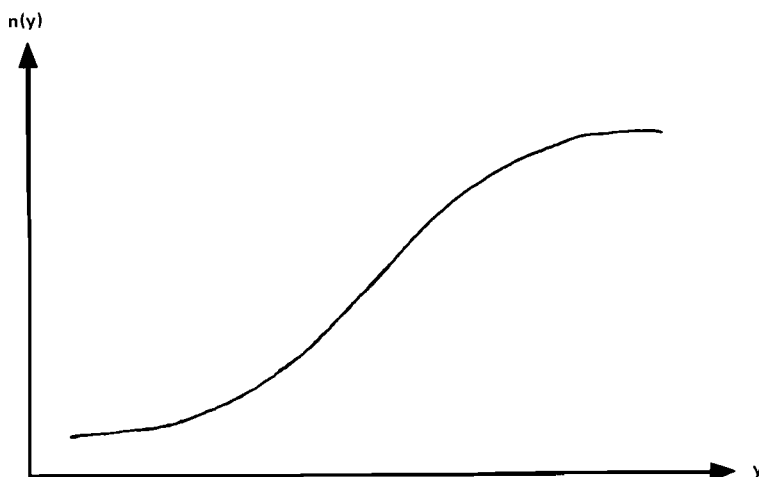
n(y)

Figure 3. Relation between population growth and per capita income.

The assumption that population growth depends on per capita income and that the labor force participation rate is constant has also been made by Lane (1975, p. 58) and Pitchford (1974, p. 167).

In recent years there has been revived interest in population as an endogenous variable in economic growth models, particularly within the perspective of policy formulation. The rationale for making population an endogenous variable of dynamic economic growth models has been given by Dasgupta. His approach is to treat capital accumulation and population growth as interdependent: "The economic welfare of a community is affected by policies that determine (1) the rate of capital accumulation; and (2) the rate of growth of population. At any moment of time the optimum size of population will depend on the size of the existing capital stock and the optimum rate of savings will depend on the existing number of people. In this sense a population policy cannot be formulated without a concurrent savings policy. The two must be considered together." (Dasgupta, 1969, p. 295.)

Population Growth and Labor Force Participation

An assumption frequently made by authors attempting to endogenize population growth in economic models is that labor force and population are interchangeable variables. The rationale for this is given by Pitchford (1974, p. 55): "If the proportional rate of growth of population is constant and has been for a long time, it is not unreasonable to assume that a fixed ratio between the work force and population has been established." In terms of mathematical demography, it means that the population is assumed to be stable. In the stable population, the age composition is constant, and hence the population in the active age groups is a fixed proportion of the total population:

$$L(t) = pN(t) \tag{66}$$

where

$L(t)$ is the labor force,
$N(t)$ is the population at time t,
    p is the labor force participation rate.

If population grows at a constant rate n, and p is constant, the labor grows at the same rate n.

In reality, the labor force participation rate is not constant, but depends on the age composition of the population and on economic conditions. We may therefore ask how economic growth would be affected if the labor force participation rate varied with changes in the economic situation.

Consider the neoclassical growth model in (59):

$$\frac{dk}{dt} = sF(k,1) - nk \quad . \tag{59}$$

Assume that the labor force participation rate depends on the prevailing wage $w(k)$, which depends on the factor shares. The labor force at time t then is

$$L(t) = p[w(k)] \, N(t) = p[w(k)] \, N_0 e^{nt} \quad . \tag{67}$$

Therefore,

$$\frac{dL(t)}{dt} = \frac{dp}{dw} \cdot \frac{dw}{dk} \cdot \frac{dk}{dt} N_0 e^{nt} + pN_0 n e^{nt}$$

or

$$\frac{dL(t)}{dt} \cdot \frac{1}{L(t)} = \frac{dp}{dw} \cdot \frac{dw}{dk} \cdot \frac{dk}{dt} \cdot \frac{1}{p} + n \quad , \qquad (68)$$

$$\frac{dk}{dt} = \frac{dK}{dt} \frac{1}{L} - \frac{dL}{dt} \frac{1}{L} k \quad . \qquad (69)$$

Substituting (59) and (68) in (69) gives

$$\frac{dk}{dt} = sF(k,1) - \left(\frac{dp}{dw} \frac{dw}{dk} \frac{dk}{dt} \frac{1}{p} + n\right) k$$

$$= sF(k,1) - \left[\left(\frac{dp}{dw} \cdot \frac{w}{p}\right) \left(\frac{dw}{dk} \cdot \frac{k}{w}\right) \left(\frac{dk}{dt} \cdot \frac{1}{k}\right) + n\right] k$$

$$= sF(k,1) - z_p z_w \frac{dk}{dt} - nk \qquad (70)$$

where

z_p represents the elasticity of the participation rate with respect to wages,

z_w represents the elasticity of wages with respect to the capital-labor ratio.

The growth path of the capital-labor ratio reduces to

$$\frac{dk}{dt} = \frac{sF(k,1) - nk}{1 + z_p z_w} \quad . \qquad (71)$$

Balanced growth equilibrium, i.e., where $dk/dt = 0$, occurs as in the Solow-Swan model. Convergence to the equilibrium point, however, takes more time. Since dp/dw may be expected to be positive, and since dw/dk is positive, z_p and z_w are positive, making the denominator of (71) greater than one and thus slowing the speed of convergence. This reflects the fact that rising wages, associated with a rising capital-labor ratio, induce the entry of a larger proportion of the population into the labor force.

The Neoclassical Model with Consumption

In the Solow-Swan model, the savings rate s was held constant. Different values of s would lead to different allocations of capital in the golden-age growth path, and therefore to different equilibrium capital-labor ratios, wage rates, personal income streams, and consumption rates. A logical extension of the neoclassical model, therefore, is to introduce consumption explicitly. One may begin by expressing total savings as

$$S = sF(K,L) = F(K,L) - cN \quad , \tag{72}$$

where c is consumption per capita and N is the total population. Assuming a labor force participation rate of unity, i.e. $L = N$, and constant returns to scale, (72) may be written as

$$S/L = sF(k,1) = F(k,1) - c \quad . \tag{73}$$

Substituting (73) in the Solow-Swan growth model in (59) gives*

$$\frac{dk}{dt} = F(k,1) - c - nk \quad . \tag{74}$$

There is a different golden-age growth path for every k. For a given consumption rate c, the steady-state capital-output ratio is

$$C = \frac{F(k,1)}{k} = \frac{c}{k} - n \quad , \tag{75}$$

where c/k is consumption as a fraction of total income. On the other hand, for a given capital-output ratio, or equivalently, capital-labor ratio, the steady-state consumption rate is

$$c = F(k,1) - nk \quad . \tag{76}$$

Equation (76) provides a direct link between growth theory and growth policy. Although the rate of population growth, n, is held constant, one can derive rules for capital accumulation

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*Recall that this implies full employment and that all savings are invested.

that may be compared with those obtained with different rates of population growth. In addition, (76) or (74) show up as constraints in a number of demo-economic optimization models.

Optimization and policy considerations come in when both per capita consumption and the capital-labor ratio are allowed to vary. The relevant policy question is: what steady-state capital-labor ratio is able to sustain a maximum per capita consumption? The first-order condition for maximizing per capita consumption is

$$\frac{\delta c}{\delta k} = 0 = \frac{\delta F(k,1)}{\delta k} - n$$

or

$$F'(k,1) = n \quad , \tag{77}$$

where

$$\frac{\delta F(k,1)}{\delta k} = F'(k,1)$$

is the marginal product of capital. This means that the interest rate equals the rate of labor force growth. The capital output-ratio maximizing c will be denoted k*.

Equation (77) is the *golden rule of capital accumulation.* It has been discovered independently by Swan (1956) and by Phelps (1961; see also 1970, p. 198), and was already implicit in Ramsey's (1928) work. Among all the possible golden-age paths of natural growth, that golden age is "best" which practices the golden rule: the investment made by each generation is such that the next generation has the highest possible per capita consumption. Under the golden rule, the relative share of output going to capital is the optimal savings ratio

$$s = \frac{k*F'(k*,1)}{F(k*,1)} = \frac{k*n}{F(k*,1)} \quad . \tag{78}$$

(See also Burmeister and Dobell, 1970, pp. 49-53.) Therefore, the savings rate that maximizes per capita consumption in the long run is equal to the share of profit in national income.

The golden-rule consumption per head is given by

$$c^* = F(k^*,1) - nk^* \quad . \tag{79}$$

How does an increase of the population growth rate n affect c* (Phelps, 1966, pp. 178)? If k* > 0, an increase of n leads to an increase in F'(k*,1), by (77). But F'(k*,1) is decreasing in k, so k* will decrease, which implies a decline in per capita output F(k*,1). The net effect upon c* of increasing n may be found by differentiating (79) with respect to n. This yields:

$$\frac{dc^*}{dn} = F'(k^*,1) \frac{dk^*}{dn} - \left(k^* + n \frac{dk^*}{dn}\right)$$

$$= [F'(k^*,1) - n] \frac{dk^*}{dn} - k^* \quad ,$$

which, after applying (77), gives:

$$\frac{dc^*}{dn} = - k^* \quad . \tag{80}$$

Therefore, the golden-rule per capita consumption declines as the population growth rate rises, and the lower the population growth rate, the better.

In Phelps' model population growth is given exogenously. Davis (1969) has extended the model to allow the population growth rate to vary with per capita income, i.e. n = n[F(k,1)]. When population grows endogenously, the golden-rule savings rate is no longer equal to (78) but is modified to

$$s = \frac{k^* F'(k^*,1)}{F(k^*,1)} \left[1 - k^* n'[F(k^*,1)]\right] \quad . \tag{81}$$

The ratio

$$\frac{k^* F'(k^*,1)}{F(k^*,1)}$$

is the relative share of output going to capital. Whether the modified golden-rule savings rate is greater or less than the relative share of capital is determined by the sign of n'(.). Correspondingly, the growth rate of the economy under endogenous labor supply is greater than or less than the marginal productivity

of capital, depending upon whether n'(.) is negative or positive.*
A positive n'(.) implies a monotonic function describing a posi-
tive relationship between the population growth rate and per
capita income.

Age Specificity in Demo-Economic Growth

Population growth and capital accumulation are represented
by simple aggregate models. In the models of economic growth
discussed so far, capital and labor are assumed to be homogeneous.

A few growth models have focused on a disaggregation of
population by age. Samuelson (1958, 1975) and Arthur and McNicoll
(1977, 1978) have shown how the introduction of age groups affects
the optimum rate of population growth. In the aggregated version
of the growth model, given by (74), a small population growth
rate is preferable, since an increase in population growth calls
for a greater investment to maintain the capital-labor ratio, or
capital per head. This capital widening diverts resources from
consumption and from capital deepening (increase in capital per
head). Samuelson (1958) has shown that the introduction of aging
with the possibility of transfers between age groups may alter
the conclusion. We will return to age-specific demo-economic
policy models in the next section.

Optimum Demo-Economic Growth

Heretofore, we have investigated several models of demo-
economic growth and their steady-state properties. Policy objec-
tives were introduced to select a unique steady-state or golden-
age growth path. The focus was not on the objectives themselves,
but on the characterization of a unique steady state.

The transformation of these models into truly dynamic demo-
economic policy models requires:

- The introduction of an explicit population control
  variable;

- The introduction of explicit *intertemporal* policy objec-
  tives. This involves problems of definition of the
  welfare criterion and of the social rate of discount;

- An allowance for other constraints on both the state
  and control variables (this will not be studied here);

- A more realistic description of the population system
  by introducing age specificity.

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*It is assumed that $[1 - k*n'(.)] > 0$.

These requirements will be dealt with below.

The Population Control Variable

In the models considered thus far, population was treated
exogenously or endogenously, but was not considered as a direct
policy variable. The problem was to determine the optimal
capital-labor ratio or the savings rate for a given population
growth rate. Recall, for example, the golden rule of capital
accumulation. Associated with each population growth rate n is
a golden-rule state. The golden-rule consumption per head is
given by (79) and depends on n, i.e. $c^* = c(n)$. For each n,
the implied optimal savings rate is easily derived, since
$c = (1 - s) F(k,1)$. This savings rate is optimal in the sense
that it maximizes per capita consumption under the given regime
of population growth.

Phelps (1966, pp. 179-182) went a step further. He addressed
the policy problem of finding the growth rate n yielding a golden-
rule state that is socially preferred. This step completely inte-
grates economic (savings) policy and population policy. Which
golden-rule state is preferred depends of course on the objectives.
According to Phelps, society not only wants to consume as much as
possible, but also wants to grow, i.e. wants to have children.
Social welfare is, therefore, a function of both the consumption
per head and the population growth rate. Hence, the function to
be maximized is $u(n,c^*)$. Assuming a constant mortality rate,
the welfare function may be written in terms of the birth rate b.
Writing $c^* = h(b)$, the policy problem becomes

$$\max_{b} u[b,h(b)] \quad . \tag{82}$$

The problem now is to find the optimal combination of fertility
and consumption in a situation of balanced growth (i.e. when out-
put, capital, and consumption all grow at the same rate as labor).
It is a simple but complete demo-economic policy problem. The
optimality condition is

$$\frac{\delta u}{\delta b} = 0 = u_b + u_c \, h'(b^*) \quad ,$$

or

$$- \frac{\delta c^*}{\delta b} = -h'(b^*) = u_b/u_c \quad . \tag{83}$$

Equation (83) states that for the social welfare function in (82) to be at a maximum, the birth rate must be such that the marginal cost of the birth rate (per capita births), in terms of golden-rule consumption per head, is equal to the marginal rate of substitution between the birth rate and the golden-rule consumption per head. The birth rate b* that yields the golden-rule state in which (82) is maximum is the *golden rule of procreation*.

Phelps' golden rule of procreation gives the optimum population growth rate under the assumption that population policy is costless. Changing the population growth rate, however, requires resources that could have been directed to productive investments. The portion of the income allocated to population control is denoted by McNicoll (1975, p. 671) as *demographic investment*.* The function describing the relationship between demographic investment per capita at time t, j(t) say, and the population growth rate n is n[j(t)]. In other words, j(t) denotes the per capita expenditures required to reach a population growth rate n. If j(t) = 0, n is equal to the natural rate of population growth. Total savings are now divided among investment in the capital stock and demographic investment. The basic technological relation (59) is then

$$\frac{dk}{dt} = sF(k,1) - kn[j(t)] - j(t) \quad . \tag{84}$$

In the simple case, j(t) is a constant fraction of per capita income, i.e. j(t) = gF(k,1); hence the demographic response function is n[F(k,1)] and per capita consumption is

$$c = (1 - s - g) F(k,1) \quad . \tag{85}$$

To find the steady-state or equilibrium level of demographic investment we recall Phelps' golden rule of procreation. The population growth rate or the birth rate that maximizes the social welfare derived from both per capita consumption and growth rate is such that

$$\frac{\delta c^*}{\delta b} = h'(b^*) = - u_b/u_c \quad . \tag{86}$$

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*McNicoll's definition of demographic investment differs from that of Sauvy (1976, p. 64), who considers it to be that part of total investment required to maintain the standard of living or the capital-labor ratio. The latter perspective is identical to capital widening.

The demographic investment associated with a birth rate b* is
j*. Hence

$$\frac{\delta c^*}{\delta b} = h'[b(j^*)] = - u_b/u_c \quad , \qquad (87)$$

and

$$h'[b(j^*)] \ u_c = - u_b \quad . \qquad (88)$$

The optimal demographic investment is such that the loss in
utility from reducing the population by one unit is exactly equal
to the utility derived from the higher per capita consumption.
Therefore, at the optimum, a given increment of investment has
the same impact whether allocated to production (and consumption)
or to population control.

Another approach to finding the optimum level of demographic
investment has been taken by Sato and Davis (1971, p. 890). The
authors assume that population grows endogenously but can be in-
fluenced by direct policy intervention. The demographic response
function is therefore n[f(k,1),g], where g is the fraction of per
capita income allocated to population control. This is an exten-
sion of the problem, discussed in the previous section, of deter-
mining an optimum savings rate when population grows endogenously.
Maximization of per capita consumption (85) yields the "modified"
golden rule of capital accumulation:

$$s^* = \frac{k^*F'(k^*,1)}{F(k^*,1)} \ (1 - g - k^*n') \qquad (89)$$

and

$$\frac{\delta n}{\delta g} = \frac{F(k,1)}{k} = - \frac{1}{C} \qquad (90)$$

where

$$n' = \frac{\delta n[F(k,1),g]}{\delta F(k,1)} \quad .$$

The introduction of direct population control reduces the
optimum savings rate even further than before (compare (89) with
(78)). Population policy should be implemented until the marginal

impact of public expenditures is equal to the average productivity of capital (output-capital ratio 1/C). This implies that at the optimum, per capita income or output is equal to the product of the capital-labor ratio and the marginal impact of population control expenditures. Consequently, since in the steady state n = s/C, the equilibrium growth rate of population must be

$$n^* = - s \frac{\delta n}{\delta g} \quad .$$ (91)

Since s/C is Harrod's warranted rate of growth, the quantity $-s(\delta n/\delta g)$ may be called the modified warranted rate of growth for endogenous population increase and direct policy intervention.

The population policy variable or decision rule considered by Phelps is the growth rate (or birth rate if mortality is constant). Other authors have addressed the question of the optimum size or density of the population. For example, Dasgupta (1969) treats the problem first formulated by Wicksell; namely, what size (density) of population under given circumstances is the most advantageous? Posing the question of optimum population size implies the assumption that a zero growth rate is best, since only if n = 0 can an optimum population be maintained (other conditions being equal). The concept of optimum size is more suitable for "classical" economic regimes in which the reality of finite resources or of some fixed production factors such as land eventually leads to diminishing returns. The concept of optimum growth rate, on the other hand, fits the neoclassical regime, with no resource constraints but with constant returns to the production factors.*

Policy Objectives

What is the optimum population size or the optimum growth rate of the population? According to Phelps, an optimum growth rate is one that maximizes (82). In the demo-economic policy literature, the policy objective usually involves a measure of per capita consumption. Two types of welfare indices are used frequently: social welfare as a direct function of per capita utility; and social welfare as a weighted function of per capita utility, the weight being the population size. We consider both indices in a static and a dynamic framework.

Welfare is usually expressed in terms of consumption. The total consumption stream available to the population is equal to the total production F[K(t),L(t)] less the investment I(t) = dK(t)/dt. Both the per capita and the total utility criteria may be considered in a static and a dynamic framework.

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*These definitions of classical and neoclassical economic regimes follow Arthur and McNicoll (1977, p. 114).

In static analyses, the per capita welfare criterion is simply

$$u(c) = [F(K,L) - I]/L \quad .$$

The optimum population size is obtained when per capita consumption is at a maximum, i.e.

$$\frac{\delta c}{\delta L} = \frac{\delta [F(K,L) - I]/L}{\delta L} = \frac{1}{L} \left[\frac{\delta F(K,L)}{\delta L} - c \right] = 0 \qquad (92)$$

or

$$c = \delta F(K,L)/\delta L \quad .$$

Therefore, the optimum population size is reached when the contribution to production of a marginal person is equal to his consumption (which is the average consumption).

The total-utility criterion supposes that social welfare is equal to the average individual utility weighted by population size. The use of such an approach has been strongly endorsed by Meade (1955). Meade performed a static analysis. The objective function $U(C)$ is simply the product $u(c) \times L$, where L denotes the population size. This is known as the Bentham criterion (McNicoll, 1975, p. 666).* The optimum population size is given by the condition

$$\frac{\delta [u(c)L]}{\delta L} = 0 = u(c) + Lu'(c) \frac{\delta c}{\delta L} \quad , \qquad (93)$$

where $\delta c/\delta L$ is given by (92). Hence optimality requires that

$$u(c) = u'(c) \left[c - \frac{\delta F(K,L)}{\delta L} \right] \quad , \qquad (94)$$

which means that the utility of a marginal individual entering the population (and consuming at the average level) must equal

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*According to Meade (1955, p. 88), there exists a consumption level C_0 at which life is just enjoyable, i.e. $U(C_0) = 0$. The quantity C_0 is referred to as the welfare subsistence level.

the disutility he causes to the other members of the population.

In dynamic analyses, the utility from consumption at different time periods must be considered. This implies the formulation of a measure of time preference. If ρ denotes the time preference or the rate of discount, then the per capita utility criterion is

$$\int_0^T e^{-\rho t} u[c(t)] \, dt \quad , \tag{95}$$

where

      $c(t)$ is the per capita consumption at time t, i.e.
          $c(t) = [1 - s(t)] \, F(k,1)$,
      $u(.)$ is the utility function, and
        T is the planning horizon.

The total utility criterion, on the other hand, is

$$\int_0^T e^{-\rho t} u[c(t)] \, L(t) \, dt \quad , \tag{96}$$

where $L(t)$ is the population at time t.

Both of these criteria are frequently used in dynamic policy models. Dasgupta (1969, p. 297) compares alternative policies by assessing their impact on total welfare (96). He argues that it is a better measure to compare the ultimate value of having one more person in the world with the ultimate value of present people having a bit more to consume. Both population size and utility from per capita consumption enter the objective function directly. Sato and Davis (1971) compare the theoretical implications of both welfare indices on the optimum policy, under the assumption that population grows endogenously, i.e. that the economic dynamics are those given by (65). Maximization of (95) subject to (65) yields an optimum steady-state savings rate equal to

$$s* = \frac{k*[F'(k*,1)(1 - k*n') - \rho]}{F(k*,1)} \quad . \tag{97}$$

The optimum per capita consumption is of course $c* = F(k*,1) - k*n$. Note that for $\rho = 0$, (97) reduces to the modified golden rule of capital accumulation (81).

Maximization of (96) subject to (65), on the other hand, yields quite different results. The steady state is given by the following relationship:

$$\frac{u}{u'} \ n' = \frac{(\rho - n)}{F'(k^*,1)} \ [\rho - F(k^*,1)(1 - k^*n')] \quad , \qquad (98)$$

where u is the utility function. Note that when population grows exogenously, n' = 0 and (98) reduces to

$$F'(k^*,1) = \rho \quad ;$$

i.e., the marginal product of capital must be equal to the rate of discount or time preference.* The steady-state capital-labor ratio is determined entirely by the discount rate. The form of the utility function has no effect at all.** But if population grows endogenously, the utility function does play a role.

The choice of the rate of discount has been an element of debate in the growth literature. Ramsey (1928) found discounting the future at a positive rate "...a practice which is ethically indefensible". Other economists, such as Harrod, have taken a similar stand. Whether a positive rate of time preference is unethical and what the discount rate should be are questions beyond the scope of this paper. The fact is that most economists today introduce some discounting in optimal policy models. Some authors investigate and compare the theoretical features of both cases $\rho > 0$ and $\rho = 0$ (see e.g. Dasgupta, 1969; Sato and Davis, 1971).

Age Composition and Demo-Economic Policy

The optimal decision rules of demo-economic policy, studied in the previous sections, are based on the assumption of homogeneous capital and labor. The validity of the optimal policies depends on the value of this underlying assumption.

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*This is the Ramsey rule. The rate of time preference also may be written as

$$- \ \frac{1}{p(t)} \ \frac{dp(t)}{dt} \quad ,$$

with $p(t) = e^{-\rho t} u'[c(t)]$; i.e., the rate of time preference is the rate of decline in the discounted marginal utility.
**The utility function is, of course, of central importance for the optimum trajectory to equilibrium.

Recently, some effort has been devoted to the analysis of more disaggregated policy models, in particular, age-specific models. The following discussion is based on work carried out by Samuelson and Arthur and McNicoll.

Samuelson considers only two age groups.* The young age group consists of the working population, while the old age group contains only retired people. In this simple model, the working population supports the retired population through "consumption" loans. Repayment can be expected when this working population retires. Therefore, each generation is supported by the following generation. The support or consumption transfers received by the retired population increases if the proportion of the young population expands, which is the case if the population growth rate rises. In Samuelson's two-age model with inter-generational transfer, therefore, the greater the population growth rate, the better. This conclusion is the opposite of the optimum defined in Solow's neoclassical model and its extensions (see for example equation (80)).

A combination of the Solow model in (59) with the Samuelson model of overlapping generations yields an intermediate result. Recall equation (72). There total output is equal to consumption and investment (savings):

$$F(K_t, L_t) = C_t + K_{t+1} - K_t \quad . \tag{99}$$

Consumption at time t is the sum of the consumption of the young and the old populations. In the absence of mortality, and for time intervals equal to age intervals, the number of old people at time t is equal to the number of young people at time t - 1. Let L_{t-1} denote the young population at time t - 1, and c_t^1 and c_t^2 the per capita consumption of the young and old populations respectively; then (99) becomes

$$F(K_t, L_t) = L_t c_t^1 + L_{t-1} c_t^2 + K_{t+1} - K_t \quad . \tag{100}$$

Dividing by L_t yields

$$F(k_t, 1) = c_t^1 + \frac{1}{1 + n} c_t^2 + \frac{K_{t+1} - K_t}{K_t} k_t \quad ,$$

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*At the micro-level, the introduction of aging is identical to the consideration of an explicit life cycle (e.g. childhood, work, childbearing, retirement).

where n is the growth rate of labor. In the steady state, capital
and labor grow at the same rate; hence

$$F(k,1) = c^1 + \frac{1}{1+n} c^2 + nk \quad .$$
(101)

Compare (101) with (76). Note that each combination of n,
c^1 and c^2 defines a golden state. Each golden state is charac-
terized by a constant capital-labor ratio.

Following Phelps, Samuelson asks which golden state yields
maximum utility from consumption. The utility function must of
course reflect lifetime consumption, i.e. $u = u[c^1,c^2]$. There
is a unique relation between c^1 and c^2, given by (101):

$$c^1 = F(k,1) - \frac{1}{1+n} c^2 - nk \quad .$$
(102)

The utility function, therefore, is

$$u[c^1,c^2] = u\left[F(k,1) - \frac{1}{1+n} c^2 - nk, c^2\right] \quad ,$$
(103)

and maximization with respect to k, c^1 and c^2 yields the follow-
ing optimality conditions:

$$\frac{\delta u}{\delta k} = 0 = F'(k*,1) - n$$
(104)

$$\frac{\delta u[*c^1,*c^2]}{\delta c^1} = (1+n) \frac{\delta u[*c^1,*c^2]}{\delta c^2} \quad .$$
(105)

Condition (104) is Phelps' golden rule of capital accumula-
tion. Equation (105) states that for utility to be a maximum,
the discounted marginal utility of consumption must be the same
for all ages. This relation is the *biological interest rate
relation* of Samuelson. The two conditions together constitute
the golden rule.

There is a golden-rule state associated with every population
growth rate n. One may, therefore, be interested in selecting a
rate n that is socially desirable. This problem has been addressed
by Phelps and has led him to the derivation of the golden rule
of procreation. Phelps included both per capita consumption and

the population growth rate in the welfare function. Samuelson (1975, p. 534), however, kept to the function (103). Maximization of (103) with respect to n yields:

$$\frac{\delta u}{\delta n} = 0 = \frac{\delta u}{\delta c^1} \cdot \frac{\delta c^1}{\delta n}$$

or

$$0 = -k* + \frac{*c^2}{(1 + n**)^2} \quad ,$$

and

$$1 + n** = \left[\frac{*c^2}{k*}\right]^{1/2} \quad . \tag{106}$$

The growth rate n** that maximizes lifetime welfare is denoted by Samuelson as the *goldenest golden-rule state*. Moreover, at a growth rate n**, private lifetime savings will be just sufficient to support the goldenest golden-rule state.

Since k and c^2 are themselves functions of n in (104) and (105), (106) is an implicit function of n**. To find the true maximum, second-order conditions must be supplemented. Note that rapid or slow growth no longer is "better", but that the value of n** is determined by the utility function and the production function. (Compare this result with Phelps' golden rule of procreation.)

Arthur and McNicoll (1978) have generalized the two-age life cycle model to one with a continuous-age life cycle. This generalization allows for an inclusion of child-dependency costs. The intergenerational transfer is not only from working population to old, but also from working population to children. Therefore, the net intergenerational transfer effect of growth is no longer necessarily positive; hence the inclusion of transfers to younger people therefore tends to result in lower optimal growth.

The authors consider continuous intervals for both time and age. Equation (99) becomes

$$F(K_t, L_t) = C_t + \frac{dK}{dt} \quad . \tag{107}$$

The population is assumed to be stable, i.e. with constant age-specific rates of fertility and mortality, constant age distribution, and growing at a constant rate n (see example, Keyfitz,

1968; Rogers, 1975). At stability, the population at exact age x at time t is

$$\ell(x,t) = e^{-nx}p(x)\ell(0,t) \quad , \tag{108}$$

where

> $\ell(0,t)$ is the number of births at time t,
> $p(x)$ is the probability of survival from birth to age x.

Let $c(x,t)$ denote age-specific per capita consumption at time t. Equation (107) may be expressed as follows:

$$F(K_t,L_t) = \int_0^\omega \ell(x,t)c(x,t)dx + \frac{dK_t}{dt} \quad , \tag{109}$$

where ω is the last age group.

Dividing by the amount of labor L_t, where

$$L_t = \left[\int_0^\omega e^{-mx}p(x)\lambda(x)dx\right]\ell(0,t) \tag{110}$$

and $\lambda(x)$ is the age-specific labor force participation rate, we have that

$$F(k_t,1) = \bar{c}(t) + \frac{1}{K_t}\frac{dk_t}{dt}k_t \quad , \tag{111}$$

where $\bar{c}(t)$ is the average consumption per *worker* at time t, i.e.,

$$\bar{c}(t) = \frac{\displaystyle\int_0^\omega e^{-nx}p(x)c(x,t)dx}{\displaystyle\int_0^\omega e^{-nx}p(x)\lambda(x)dx} \quad .$$

In steady state, capital and labor grow at the same rate n, and $c(x,t) = c(x)$; hence*

$$F(k,1) - \bar{c} = nk \quad .$$
(112)

Each combination of $\bar{c}$ and n defines a golden state. Note that (112) is the familiar Solow condition, $sF(k,1) = nk$ (see also Arthur and McNicoll, 1977, p. 116).

Which golden state yields a maximum lifetime utility from consumption? If $u[c(x)]$ is the utility from consumption at age x, then a baby just born has a probability $p(x)$ of enjoying this consumption. Maximization of

$$U[c(x)] = \int_0^\omega p(x) \; u[c(x)] \; dx$$
(113)

subject to (111) yields the following optimality conditions for all x:

$$\frac{\delta u[c^*(x)]}{\delta c(0)} = e^{nx} \; \frac{\delta u[c^*(x)]}{\delta c(x)} \quad ;$$
(114)

i.e., the discounted marginal utility of consumption must be the same for all ages. In other words, for the lifetime welfare to be a maximum, the disutility of there being one unit of consumption less at age 0 (loan) must be offset by the utility of the consumption of this unit at age x, multiplied by the interest (repayment). This relation is the "biological interest rate" condition, similar to (105).

As in the case of two ages, we search now for the growth rate n that yields the most golden golden-rule state. Maximizing (113) with respect to n gives

$$\frac{\delta U[c^*(x)]}{\delta n} = \int_0^\omega p(x) \; \frac{[\delta u(c^*(x))]}{\delta c(x)} \; \frac{\delta c(x)}{\delta n} \; dx$$

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*Equation (112) is equivalent to equation (8) of Arthur and McNicoll (1977, p. 116). Whereas the original article used the average consumption per person, we use the average consumption per worker. This simplifies the result.

$$= \frac{\delta U[c^*(x)]}{\delta c(0)} \left[\int_0^\omega e^{-nx} p(x) \frac{\delta c(x)}{\delta n} dx \right] \quad .$$

Transforming this expression gives

$$\frac{\delta U[c^*(x)]}{\delta n} = \frac{\delta U[c^*(x)]}{\delta c(0)} \bar{\lambda} [\bar{c}^* (A_c - A_L) - k] \quad ;$$

here $\bar{c}^*$ is the optimal average consumption per worker, $\bar{\lambda}$ is the average labor force participation rate, and A_c and A_L are, respectively, the mean age of consuming and of the labor force. Therefore, the lifetime welfare effect of changing the population growth rate is equal to an intergenerational transfer effect (the difference in A_c and A_L)* and a capital widening effect. The latter effect is always negative, while the intergenerational transfer effect can be either positive or negative.

CONCLUSION

The purpose of this paper was to review the existing links between formal representations of population policies and economic policies. This bringing together of the work of demographers and economists aims to contribute to better policy-making.

The common feature of the models reviewed is the underlying mathematical paradigm. Any dynamic policy problem may formally be stated as an optimal control problem, and the theory of optimal control provides the apparatus necessary to solve for the optimal values of the policy variables.

Two groups of demo-economic policy models have been examined, each in increasing order of complexity: planning-oriented models and theoretically-oriented models. Planning-oriented models may be viewed as logical extensions of mathematical demographic growth models to the policy domain. The demographic growth model itself is imbedded in the policy models as the homogeneous part of the state equation. The discussion focused on two major issues in dynamic policy modeling: existence and design. Systems theory provides the necessary mathematical tools.

Theoretically-oriented models originated in the theory of economic growth and have a much higher level of abstraction than planning-oriented models. Studies of their underlying theoretical concepts, of the structure of their policy problem, and of

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*The average age of consuming is usually three to four years below the average working age (Arthur and McNicoll, 1978).

the existence and stability of their optimal policies have re-
ceived much attention.

This paper presents some approaches to linking demo-economic
growth and policy in formal models. Fundamental differences in
the two approaches do not permit a complete synthesis at this
time. Synthesis may not even be desirable since the two ap-
proaches serve a different purpose.

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