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On the Exact Stabilization of an Uncertain Dynamics

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Abstract

The study is motivated by the problem of stabilizing the concentration of atmospheric carbon, which is widely discussed in the context of global warming nowadays. A key difficulty in the design of stabilization strategies is the uncertainty of the underlying physical model. In the present paper, a general problem setting is suggested and a relevant alanytic framework elaborated. Analysis employs specific qualitative features of an uncertain dynamics, including automatic stabilization of the trajectories in the absence of input disturbances. An asymptotic version of Krasovskii's extremal shift control principle is developed and model-robust strategies stabilizing a state coordinate at a prescribed level are constructed.

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On the Exact Stabilization of an Uncertain Dynamics*

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1 Motivation

In the context of global warming, a considerable interest has been drawn to the problem of stabilization of the concentration of greenhouse gases in the atmosphere. A vast literature is devoted to this issue. Here, we refer to Svirezhev, et. al., 1999 (providing a list of relevant publications), in which the problem is analyzed using an ODE model of the global dynamics of carbon, the major greenhouse gas in the biosphere.

The model's state variables are the amounts of carbon in the atmosphere, $x(t)$, and in the ocean, $y(t)$, and the average surface/air temperature, $z(t)$; here t is the time variable. The state variables are scaled so that their zero values correspond, respectively, to the absolute value of carbon in the atmosphere, the absolute value of carbon in the ocean and the average surface/air temperature in the pre-industrial period. Annual antropogenic emissions of CO_2, $\varphi(t)$, act as controls regulating the dynamics of carbon in the atmosphere. The model has the form

$$\begin{aligned} \dot{x}(t) &= \varphi(t) - \alpha_1 x(t) + \alpha_2 y(t), \\ \dot{y}(t) &= \alpha_1 x(t) - \alpha_2 y(t), \\ \dot{z}(t) &= \alpha_3 x(t) - \alpha_4 z(t) \end{aligned} \tag{1.1}$$

where α_1, α_2, α_3 and α_4 are positive parameters. The initial state of the model represents the amounts of carbon in the atmosphere and in the ocean and the average temperature at time 0 (corresponding to the year 2000):

$$x(0) = x^0, \quad y(0) = y^0, \quad z(0) = z^0.$$

Svirezhev, et. al., 1999, analyze emission control scenarios $\varphi(t)$ that keep the temperature, $z(t)$, within a prescribed interval $[z^-, z^+]$, the so-called *tolerable window*, which prevents the occurrance of harmful impacts of global warming. (Generally, the tolerable window approach imposes also constratins on the rate of change of the temperature, $\dot{z}(t)$; see WBGU, 1995; Bruckner, et. al, 1999). A reasonable scenario consists in stabilizing the amount of carbon in the atmosphere, $x(t)$, at a prescribed limit value $\hat{x}$ as time goes to infinity:

$$\lim_{t\to\infty} x(t) = \hat{x}. \tag{1.2}$$

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Then, as (1.1) shows, the temperature, $z(t)$, is stabilized at $\hat{z} = \alpha_3 \hat{x}/\alpha_4$. Moreover, if one guarantees $\xi^-(t) \leq x(t) - \hat{x} \leq \xi^+(t)$ with explicit bounds $\xi^-(t), \xi^+(t)$ converging to 0 as $t \to \infty$, then similar exlicit bounds $\zeta^-(t), \zeta^+(t)$ (converging to 0 as $t \to \infty$) can easily be derived for $z(t)$: $\zeta^-(t) \leq z(t) - \hat{z} \leq \zeta^+(t)$; and the tolerable window constraints are met provided $[\hat{z} + \mu^-(t), \hat{z} + \mu^+(t)] \subset [z^-, z^+]$ for all $t \geq 0$.

Our study relates to "post-planning" decisionmaking. Assuming that an emission scenario $\varphi(t)$ that ensures (1.2) is found, we address the question of the practical realization of (1.2). The question immediately becomes nontrivial if we take into account that the model is inaccurate and does not present us the real dynamics. It is clear that (1.2) is violated if we implement $\varphi(t)$ for even a sligtly perturbed model. In practice, the uncertainties in the model (reflecting highly complex processes in the environment) should be viewed as large enough. An adequate assumtion is that the "real system" is not known to us; instead, we are given a (relatively broad) class of "admissible" systems, which includes the real one. This assumption implies that a desired emission control policy should guarantee (1.2) for every admissible system chosen beforehand.

Since in (1.1) the evolution of the stabilized variable $x(t)$ does not depend on $z(t)$, the dynamics of $x(t)$ and $y(t)$ is of practical interest. One can assume that admissible systems describing a variaty of admissible dynamics for $x(t)$ and $y(t)$ can include nonlinear models much more complex than the linear one given in (1.1). For example, the admissible systems may have the form

$$\begin{aligned} \dot{x}(t) &= \varphi(t) + cu(t) + g(x(t), y(t)), \\ \dot{y}(t) &= -g(x(t), y(t)) \end{aligned} \tag{1.3}$$

where c is a positive parameter and $g(x, y)$ is an (uncertain) function decreasing in x and increasing in y. The parameter $u(t)$ acts as a "scenario correction" control intended to compensate the uncertainty of the model. Using currently available data on the trajectory of the "real" system, the controller forms $u(t)$ and modifies the original emission scenario $\varphi(t)$ with the intension to ensure (1.2). The initial state

$$x(0) = x^0, \quad y(0) = y^0 \tag{1.4}$$

can also be given inaccurately. It is clear that in order to guarantee that every admissible system of the form (1.3) (1.4) is stabilized (in the sence of (1.2)), one should impose further constraints on the functions g and initial states (1.4). We describe such constraints in section 7.

In sections 2 – 5 we pose the stabilization problem in a general form, formulate basic assumptions and describe our solution method originating from theory of guaranteed control (Krasovskii and Subbotin, 1974). In section 6 we show that models of the type (1.3), (1.4) are included in the class of general control systems, introduced in section 2 (in particular, $u(t)$ in (1.3) acts as the derivative $\dot{w}(t)$ of a control $w(t)$ used in the general setting). In section 7 we apply the method to solving the above outlined problem of stabilization of the amount of carbon in the atmoshpere. The desired carbon concentration stabilization strategies update $u(t)$ using data on the current values of atmospheric carbon, $x(t)$.

2 Stabilization problem: an introduction

We consider n-dimensional dynamical control systems of the form

$$\dot{x}(t) = f(t, x(t), w(t), \dot{w}(t)), \tag{2.1}$$

$$x(0) = x^0; \tag{2.2}$$

here $t \geq 0$ is the running time, $x(t) \in R^n$ is the state of the system at time t, $w(t)$ and $\dot{w}(t)$ are the values of a 1-dimensional control and its derivative at time t, respectively, and x^0 is the system's state at time 0. In what follows, we identify a system (2.1), (2.2) with the pair (f, x^0) where $f : [0, \infty) \times R^n \times R^1 \times R^1 \mapsto R^n$ and $x^0 \in R^n$; we call f and x^0 the system's *dynamics* and *initial state*, respectively.

Somewhat nontraditionally for control theory, we assume that the admissible control functions (admissible controls) are smooth enough. Two key features of our setting are the following: every admissible control w has a limit at infinity,

$$\lim_{t\to\infty} w(t) = \bar{w}, \tag{2.3}$$

and the system's dynamics f has an essential limit depending on $\bar{w}$ (2.3):

$$\lim_{t\to\infty} \mathrm{vraimax}_{\tau\geq t}|f(t, x, w(t), \dot{w}(t)) - \bar{f}(x, \bar{w})| = 0. \tag{2.4}$$

Note that under a natural assumption that f is continuous and autonomous (i.e., $f(t, x, w, u) = f(x, w, u)$), (2.4) is ensured if each admissible control w satisfies

$$\lim_{t\to\infty} \mathrm{vraisup}_{\tau\geq t}|\dot{w}(\tau)| = 0. \tag{2.5}$$

Therefore, we include the latter requirement in our definition of the admissible controls w. We fix a nonempty set $\mathcal{W}$ of real functions w on $[0, \infty)$ such that w is absolutely continuous on every finite subinterval of $[0, \infty)$, (2.5) holds and (2.3) holds for some real $\bar{w}$. We call each $w \in \mathcal{W}$ an *admissible control* and denote by $\bar{w}$ its *limit value* defined by (2.3). We set

$$\bar{W} = \{\bar{w} : w \in \mathcal{W}\}. \tag{2.6}$$

For every $t > 0$ and every $w \in \mathcal{W}$ we denote by $\mathcal{W}(t, w)$ the set of all admissible controls v such that $v(\tau) = w(\tau)$ for all $\tau \in [0, t]$; we call $\mathcal{W}(t, w)$ the *set of extensions of w beyond t*; for $t = 0$ the set of *initial extensions* is identified with $\mathcal{W}$ and does not depend on a $w \in \mathcal{W}$.

Given a system (f, x^0) and an admissible control w, every (Caratheodory) solution x to (2.1), defined on $[0, \infty)$ and satisfying (2.2) is called a *trajectory* of (f, x^0) corresponding to w. The controller starts the control process from the initial state x^0 at time 0.

We suppose that the "real" system is not known to the controller. Instead, the controller is given a set $\mathcal{S}$ of "admissible" systems containing the real one; $\mathcal{S}$ is minimal in the sense that all systems in $\mathcal{S}$ are equally admissible to be the real one. The stabilization problem we deal with requires to construct an admissible control that brings the kth coordinate of the state vector $x(t)$ of the (uncertain) real system to a prescribed value as $t \to \infty$. It is allowed to update controls using observations of the current values of the state coordinate $x_k(t)$. With no loss of generality we set $k = n$. In what follows, we denote by $\hat{x}_n$ the prescribed limit value for the nth coordinate of the state vector. Thus, a sought admissible control should ensure $x_n(t) \to \hat{x}_n$ as $t \to \infty$.

The issue of stabilization of dynamical systems with uncertainties arises in many applications including engineering, economy and ecological management (see, e.g., Lurye, 1959; Aiserman and Gantmakher, 1963; Emelyanov, 1967; Leitmann and Wan, 1977; Coreless and Leitmann, 1985; Lee and Leitmann, 1994). Usual stabilization techniques are based on the design of appropriate Lyapunov functions including Lyapunov vector functions (see, e.g., Ledyaev and Sontag, 1999; Clarke, et. al., 2000; Matrosov, 2001; Bobylyov, et. al., 2002). A general approach to constructing feedbacks that stibilize systems with uncertain dynamics has been elaborated in Krasovskii and Subbotin, 1974 within the framework of theory of closed-loop differential games.

The setting considered in the present paper deals with nonstandard smooth controls which have limits at infinity, and also assumes a limited information on the current states. These features create serious difficulties in using the Lyapunov approach. We take into account specific properties of the systems considered, including (2.4) (another key property is outlined in the next paragraph), and solve the problem using a relevant asymptotic version of the Krasovskii extremal shift feedback principle (Krasovskii and Subbotin, 1974).

The proposed solution method employs a stabilization property of admissible controls (see assumption A8 in section 4): a trajectory x of an $(f, x^0) \in \mathcal{S}$ corresponding to a $w \in \mathcal{W}$ converges, as time goes to infinity, to a rest point $\bar{x}$ of the "limit dynamics" $\bar{f}$ (see (2.4)); moreover, $\bar{x}$ is determined by the limit value $\bar{w}$ for w, i.e., $\bar{f}(\bar{x}, \bar{w}) = 0$. Assuming for a moment that the stabilization propety holds true, we notice that the contoller's task is equivalent to the formation of an admissible control w such that the rest point $\bar{x}$ for the limit $\bar{f}$ of the real system's dynamics f, which is determined by $\bar{w}$, has the prescribed value $\hat{x}_n$ as its nth coordinate: $\bar{x}_n = \hat{x}_n$. If at some point in time the controller finds that the latter equality is inconsistent with the current admissible control, he/she decides to choose another extension of the current admissible control so as to change the trajectory and make the equality hold. Within this pattern, major technical tasks are obviously to identify a signal on the inconsistency of the current admissible control and to choose its new extension upon the receipt of the inconsistency signal. In section 5 we fulfill these tasks using additional assumptions given in section 4.

Thus, a control strategy consists in step-by-step updating the extensions of current admissible controls at appropriate "switching" times $t_1, t_2, \ldots$. In the rest of this section we describe the implementation of a control strategy informaly. At the initial time 0 the controller selects an initial admissible control w_0 and estimates a set $\bar{W}_0 \subset \bar{W}$ of the limit values $\bar{w}$ of "inconsistent" admissible controls w that are unable to solve the stabilization problem. The motion of the real system starts under w_0 and goes along a trajectory x_0. At each time $t \geq 0$ the controller observes $x_0(t)_n$ and decides if w_0 must be switched to another extension, w_1. If the controller's decides to switch at a time t_0^*, he/she fixes a delay $\delta(t_0^*, w_0) \geq 0$ for the switch and switches the admissible control w_0 to w_1 at the time

$$t_1 = \delta(t_0^*, w_0).$$

The controller decides to switch upon the receipt of a signal on the inconsistency of the current admissible control w_0. Generally, an inconsistency signal can appear as a result of an analysis of w_0 and the entire past of the observed (nth) coordinate of the current trajectory x_0. In the present study we use a simpler pattern assuming that the inconsistency signal appears if $s(t, 0, w_0, x_0(t)_n, x_0(0)_n, \bar{W}_0) = 1$ where s is a 1/0-valued (logical) function of "informational parameters": the current time t, the initial time 0, the current admissible control w_0, the current and initial values of the observed state coordinate, $x_0(t)_n$ and $x_0(0)_n$, and the initial set estimate $\bar{W}_0$ of inconsistent limit values of admissible controls. The time t_0^*, at which the controller decides to switch, is set to be the time of the first occurrence of the inconsistency signal:

$$t_0^* = \inf T_0$$

where

$$T_0 = \{t \geq 0 : s(t, 0, w_0, x_0(t)_n, x_0(t_0)_n, \bar{W}_0) = 1\}.$$

Recall that the receipt of the inconsistency signal implies that $\bar{w}_0$ is inconsistent in the sense that $\bar{x}$ determined by $\bar{f}(\bar{x}, \bar{w}_0) = 0$ where $\bar{f}$ is the limit dynamics for the real system (f, x^0) does not satisfy the criterion $\bar{x}_n = \hat{x}_n$. This allows the controller to extend the initial set $\bar{W}_0$ of inconsistent limit values of admissible controls to a new set $\bar{W}_1$ by adding $\bar{w}_0$. Generally, $\bar{W}_1$ can contain other extra elements. We assume that the controller forms $\bar{W}_1$ as a function of the current informational parameters:

$$\bar{W}_1 = I(t_0^*, 0, w_0, x_0(t)_n, x_0(0)_n, \bar{W}_0).$$

If the decision on a switch is made and the time t_1 for the switch is fixed, the controller chooses w_1 as a function of the updated informational parameters:

$$w_1 = E(t_1, 0, w_0, x_0(t)_n, x_0(0)_n, \bar{W}_1).$$

This completes the first step of the control process (w_0 is never changed if $s(t, 0, \bar{w}_0, x_0(t)_n, x_0(0)_n, \bar{W}_0) = 0$ holds for all $t \geq 0$).

The performance of m steps of the control process results in the formation of admissible controls $w_0, w_1, \ldots, w_m$ switched on sequentially at times $0, t_1, \ldots, t_m$ and a set estimate $\bar{W}_m$ of inconsistent limit values of admissible controls. On $[t_i, t_{i+1})$ the real system goes along a trajectory x_i corresponding to w_i ($i = 0, 1, \ldots, m-1$). At each time $t \geq t_m$ the controller observes $x_m(t)_n$ and decides if w_m must be switched to another extension, w_{m+1}. If the controller's decides to switch at a time t_m^*, he/she fixes a delay $\delta(t_m^*, w_m) \geq 0$ for the switch and switches w_m to w_{m+1} at the time

$$t_{m+1} = t_m^* + \delta(t_m^*, w_m). \tag{2.7}$$

The controller decides to switch upon the receipt of an inconsistency signal appearing if $s(t, t_m, w_m, x_m(t)_n, x_m(t_m)_n, \bar{W}_m) = 1$; here s is a 1/0-valued function of informational parameters including the current time t, the latest switching time t_m, w_m, $x_m(t)_n$, the value of the observed state coordinate at the latest switching time, $x_m(t_m)_n$, and the current set estimate $\bar{W}_m$ of inconsistent limit values of admissible controls. The time t_m^*, at which the controller decides to switch, is set to be the time of the first occurrence of the inconsistency signal:

$$t_m^* = \inf T_m \tag{2.8}$$

where

$$T_m = \{t \geq t_m : s(t, t_m, w_m, x_m(t)_n, x_m(t_m)_n, \bar{W}_m) = 1\}. \tag{2.9}$$

The receipt of the inconsistency signal implies that the limit value $\bar{w}_m$ is inconsistent; therefore the controller can include $\bar{w}_m$ in the set of inconsistent limit values and replace $\bar{W}_m$ by a new set $\bar{W}_{m+1}$. Generally, $\bar{W}_{m+1}$ can also contain elements different from $\bar{w}_m$. The controller forms $\bar{W}_{m+1}$ as a function of the current collection of informational parameters:

$$\bar{W}_{m+1} = I(t_m^*, t_m, w_m, x_m(t)_n, x_m(t_m)_n, \bar{W}_m). \tag{2.10}$$

If the decision on a switch is made and the time t_{m+1} for the switch is fixed, the controller chooses w_{m+1} as a function of the updated informational parameters:

$$w_{m+1} = E(t_{m+1}, t_m, w_m, x_m(t)_n, x_m(t_m)_n, \bar{W}_{m+1}). \tag{2.11}$$

This completes step $m+1$ of the control process; this step never terminates and w_m is never changed if $s(t, w_m, x_m(t)_n, x_m(t_m)_n, \bar{W}_m) = 0$ for all $t \geq t_m$.

3 Problem formulation

In this section we transform the above informal description of a control process to strict definitions of a control strategy and corresponding trajectories and give an accurate formulation of the stabilization problem.

We identify a *collection of informational parameters* with a 6-tuple $(t, \tau, \omega, \xi, \zeta, \bar{V})$ where $t \geq \tau \geq 0$, $\omega \in \mathcal{W}$, ξ and ζ are reals and $\bar{V}$ is a subset of the set $\bar{W}$ of the limit values for all admissible controls. In practice we deal with special collections of informational parameters, described in the previous paragraph; therefore in what follows, avoiding unnecessary formalism in notations, we denote by $(t, t_m, w_m, x(t)_n, x(t_m)_n, \bar{W}_m)$, by $(t^*_m, t_m, w_m, x(t)_n, x(t_m)_n, \bar{W}_m)$, or by $(t, t_m, w_m, x(t)_n, x(t_m)_n, \bar{W}_{m+1})$ $(m = 0, 1, \ldots)$ an arbitrary collection of informational parameters. We denote by P the set of all collections of informational parameters.

Any function s mapping P into $\{0, 1\}$ will be called an *inconsistency signal map.* Any function $\delta : [0, \infty) \times \mathcal{W} \mapsto [0, \infty)$ such that $\liminf_{t \to \infty} \inf_{w \in \mathcal{W}} \delta(t, w) > 0$ will be called a *delay map.* Any function I mapping P into the set of all subsets of $\mathcal{W}$ will be called an *inconsistency estimate map.* A function $E : P \mapsto \mathcal{W}$ will be called an *extension map* if for every $(t, t_m, w_m, x_m(t)_n, x_m(t_m)_n, \bar{W}_{m+1}) \in P$ the admissible control w_{m+1} given by (2.11) belongs to $\mathcal{W}(t, w_m)$.

We define a *control strategy* to be a 6-tuple $R = (w_0, \bar{W}_0, s, \delta, I, E)$ where w_0 is an admissible control, $\bar{W}_0$ is a subset of reals, δ is a delay map, s is an inconsistency signal map, I is an inconsistency estimate map and E is an extension map.

We use the notions of a control flow and a trajectory flow to define system's trajectories corresponsing to a given control strategy.

We define an *infinite control flow* to be an arbitrary sequence $(t_m, w_m)_{m=0}^{\infty}$ such that $0 = t_0$, $w_0 \in \mathcal{W}$, $t_{m+1} > t_m$ $(m = 0, 1, \ldots)$, $\lim_{m \to \infty} t_m = \infty$ and $w_{m+1} \in \mathcal{W}(t_{m+1}, w_m)$ for all $m = 0, 1, \ldots$; we call t_m $(m = 0, 1, \ldots)$ the *switching times* for $(t_m, w_m)_{m=0}^{\infty}$. If $(t_m, w_m)_{m=0}^{\infty}$ is an infinite control flow, the (unique) admissible control w such that $w(t) = w_m(t)$ for $t \in [t_m, t_{m+1})$ $(m = 0, 1, \ldots)$ is called the *composition* of $(t_m, w_m)_{m=0}^{\infty}$.

We define a *trajectory flow* for a system $(f, x^0) \in \mathcal{S}$, corresponding to an infinite control flow $(t_m, w_m)_{m=0}^{\infty}$, to be a sequence $(t_m, x_m)_{m=0}^{\infty}$ such that for each $m = 0, 1, \ldots$ x_m is a trajectory of (f, x^0) corresponding to w_m and for each $m = 0, 1, \ldots$ the restrictions of x_{m+1} and x_m to $[0, t_m]$ coincide. If $(t_m, x_m)_{m=0}^{\infty}$ is a trajectory flow for a system $(f, x^0) \in \mathcal{S}$, corresponding to an infinite control flow $(t_m, w_m)_{m=0}^{\infty}$, then the (unique) function $x : [0, \infty) \mapsto R^n$ such that $x(t) = x_m(t)$ for $t \in [t_m, t_{m+1})$ $(m = 0, 1, \ldots)$ is called the *composition* of $(t_m, x_m)_{m=0}^{\infty}$.

Remark 3.1 Clearly, if x is the composition of a trajectory flow $(t_m, x_m)_{m=0}^{\infty}$ for an $(f, x^0) \in \mathcal{S}$, corresponding to an infinite control flow $(t_m, w_m)_{m=0}^{\infty}$, then x is a trajectory of (f, x^0), corresponding to the composition of $(t_m, w_m)_{m=0}^{\infty}$.

Similiarly we define finite control flows and corresponding trajectory flows. A *finite control flow* is an arbitrary sequence $(t_m, w_m)_{m=0}^{r}$ such that $0 = t_0$, $w_0 \in \mathcal{W}$, $t_{m+1} > t_m$ $(m = 0, 1, \ldots, r-1)$ and $w_{m+1} \in \mathcal{W}(t_{m+1}, w_m)$ for all $m = 0, 1, \ldots, r-1)$; we call t_m $(i = 1, \ldots, m)$ the *switching times* for $(t_m, w_m)_{m=0}^{r}$. If $(t_m, w_m)_{m=0}^{r}$ is a finite control flow, the (unique) admissible control w such that $w(t) = w_m(t)$ for $t \in [t_m, t_{m+1})$ $(m = 0, \ldots, r-1)$ and $w(t) = w_r(t)$ for $t \geq t_r$ is called the *composition* of $(t_m, w_m)_{m=0}^{r}$.

A *trajectory flow* for a system $(f, x^0) \in \mathcal{S}$, corresponding to a finite control flow $(t_i, w_m)_{m=0}^{r}$, is a sequence $(t_m, x_m)_{m=0}^{r}$ such that for each $m = 0, 1, \ldots, r$ x_m is a trajectory of (f, x^0) corresponding to w_m and for each $m = 0, 1, \ldots, r-1$ the restrictions of x_{m+1} and x_m to $[0, t_m]$ coincide. If $(t_m, x_m)_{m=0}^r$ is a trajectory flow for a system $(f, x^0) \in \mathcal{S}$, corresponding to a finite control flow $(t_m, w_m)_{m=0}^r$, then the (unique) function $x : [0, \infty) \mapsto R^n$ such that $x(t) = x_m(t)$ for $t \in [t_m, t_{m+1})$ $(m = 0, \ldots, r-1)$ and $x(t) = x_r(t)$ for $t \geq t_r$ is called the *composition* of $(t_m, x_m)_{m=0}^r$.

Remark 3.2 If x is the composition of a finite trajectory flow $(t_m, x_m)_{m=0}^r$ for an $(f, x^0) \in \mathcal{S}$, corresponding to a finite control flow $(t_m, w_m)_{m=0}^r$, then x is a trajectory of (f, x^0), corresponding to the composition of $(t_m, w_m)_{m=0}^r$.

Given a control strategy $R = (w_0, \bar{W}_0, \delta, s, T, I, E)$ and a system $(f, x^0) \in \mathcal{S}$ we define an *infinite processing flow* for (f, x^0) under R to be a sequence $(t_m, w_m, x_m, \bar{W}_m)_{m=0}^\infty$ such that

(i) $(t_m, w_m)_{m=0}^\infty$ is an infinite control flow,

(ii) $(t_m, x_m)_{m=0}^\infty$ is a trajectory flow for (f, x^0), corresponding to $(t_m, w_m)_{m=0}^\infty$,

(iii) for each $m = 0, 1 \ldots$ the set T_m (2.9) is nonempty and t_{m+1} is defined by (2.7) with t_m^* given by (2.8),

(iv) for each $m = 0, 1 \ldots$ the relations (2.10) and (2.11) hold.

Note that in the above definition the requirement that the set T_m (2.9) is nonempty is a formal interpretation of the fact that the inconsistency signal appears at some $t \geq t_m$, and hence there is a need to find a new extension w_{m+1} for the current admissible control w_m.

We denote by $\mathcal{P}_\infty(f, x^0, S)$ the set of all infinite processing flows for a system $(f, x^0) \in \mathcal{S}$ under a control strategy R. We call a function $x : [0, \infty) \mapsto R^n$ an *infinite-step trajectory* of a system $(f, x^0) \in \mathcal{S}$ under a control strategy R if x is the composition of the trajectory flow $(t_m, x_m)_{m=0}^\infty$ for some $(t_m, w_m, x_m, \bar{W}_m)_{m=0}^\infty \in \mathcal{P}_\infty(f, x^0, S)$.

We also need to introduce finite-step versions of the above definitions. We define a *finite processing flow* for a system $(f, x^0) \in \mathcal{S}$ under a control strategy $R = (w_0, \bar{W}_0, s, I, E)$ to be a finite family $(t_m, w_m, x_m, \bar{W}_m)_{m=0}^r$ such that

(i) $(t_m, w_m)_{m=0}^r$ is a finite control flow,

(ii) $(t_m, x_m)_{m=0}^r$ is a trajectory flow for (f, x^0), corresponding to $(t_m, w_m)_{m=0}^r$,

(iii) for each $m = 0, 1 \ldots, r-1$, T_m (2.9) is nonempty and t_{m+1} is defined by (2.7) with t_m^* given by (2.8),

(iv) T_r is empty,

(v) for each $m = 0, 1 \ldots, r-1$ the relations (2.10) and (2.11) hold.

The single element differing the latter definition from the definition of an infinite processing flow is (iv), which is a formal description of the fact that the inconsistency signal never appears after t_r, and the admissible control w_r is therefore never changed after t_r.

We denote by $\mathcal{P}(f, x^0, S)$ the set of all finite processing flows for a system $(f, x^0) \in \mathcal{S}$ under a control strategy R. We call a function $x : [0, \infty) \mapsto R^n$ a *finite-step trajectory* of a system $(f, x^0) \in \mathcal{S}$ under a control strategy R if x is the composition of the trajectory flow $(t_m, x_m)_{m=0}^r$ for some $(t_m, w_m, x_m, \bar{W}_m)_{m=0}^r \in \mathcal{P}(f, x^0, S)$. Infinite-step and finite-step trajectories of a system $(f, x^0) \in \mathcal{S}$ under a control strategy R will be called *trajectories* of (f, x^0) under R.

An accurate formulation of the *stabilization problem* under consideration is as follows: find a control strategy R such that for every system $(f, x^0) \in \mathcal{S}$ and for every trajectory x of (f, x^0) under R it holds that $\lim_{t \to \infty} x(t)_n = \hat{x}_n$. A control strategy R that solves the stabilization problem will be called a *stabilization strategy.* Let us stress that a sought stabilization strategy R is by definition robust with respect to systems in $\mathcal{S}$: no matter which system is chosen in $\mathcal{S}$, R brings its nth state coordinate to the prescribed limit $\hat{x}_n$

as time approaches infinity. An important practical implication is that R stabilizes the uncertain real system provided there is a guarantee that it lies in $\mathcal{S}$.

4 Assumptions

Our basic assumptions on the set $\mathcal{S}$ of admissible systems are the following.

A1. For every system $(f, x^0) \in \mathcal{S}$ the system's dynamics f is continuous.

A2. For every $(f, x^0) \in \mathcal{S}$ and every $w \in \mathcal{W}$ there exists a trajectory of (f, x^0), corresponding to w.

Theorem 4.1 *Let assumptions* A1 *and* A2 *be fulfilled. Then for every control strategy* R *and every system* $(f, x^0) \in \mathcal{S}$ *there is a trajectory of* (f, x^0) *under* R.

Theorem 4.1 follows straightforwardly from the definition of a trajectory of an admissible system under a control strategy and Remarks 3.1 and 3.2.

Let us introduce a set $X \subset R^n$ such that for every $w \in W$, every $(f, x^0) \in \mathcal{S}$ and every trajectory x of (f, x^0), corresponding to w, it holds that $x(t) \in X$ for all $t \geq 0$.

A3. For every $(f, x^0) \in \mathcal{S}$ there is a continuius function $\bar{f} : R^n \times R^1 \mapsto R^n$ such that for each $w \in \mathcal{W}$ and each $x \in X$ one has (2.4); we call $\bar{f}$ the *limit dynamics* for (f, x^0).

Remark 4.1 Note that if for every $(f, x^0) \in \mathcal{S}$ the dynamics f is stationary (i.e., $f(t, x, w, u) = f(x, w, u)$), A3 follows straightforwardly from the continuity of f (see assumption A1) and (2.5).

For every $(f, x^0) \in \mathcal{S}$ the limit dynamics $\bar{f}$ is unique; this follows from the continuity of f. Given a $(f, x^0) \in \mathcal{S}$ and an $\eta \in \bar{W}$ (see (2.6)), the equation $\bar{f}(\bar{x}, \eta) = 0$ determines the *rest points* of the limit dynamics $\bar{f}$ for the parameter η. We make the following assumption.

A4. For every $(f, x^0) \in \mathcal{S}$ and every $\eta \in \bar{W}$, the set X contains the unique rest point of $\bar{f}$ for the parameter η.

By A4 for every $(f, x^0) \in \mathcal{S}$ there is the unique function $\bar{x}(\cdot\,|f, x^0) : \bar{W} \mapsto X$ such that for every $\eta \in \bar{W}$ the value $\bar{x}(\eta\,|f, x^0)$ is the rest point of $\bar{f}$ for the parameter η, contained in X; we call $\bar{x}(\cdot\,|f, x^0)$ the *rest point map* for the system (f, x^0).

Remark 4.2 Under assumpion A3 and A4 the rest point map $\bar{x}(\cdot\,|f, x^0)$ is continuous for every $(f, x^0) \in \mathcal{S}$.

Now for every $(f, x^0) \in \mathcal{S}$ we define the *nth coordinate projection* of the rest point map $\bar{x}(\cdot\,|f, x^0)$; to be the function $\bar{x}_n(\cdot\,|f, x^0) : \bar{W} \mapsto R^1$ whose value, for any $\eta \in \bar{W}$, equals the nth coordinate of $\bar{x}(\eta\,|f, x^0)$, i.e., $\bar{x}_n(\eta\,|f, x^0) = (\bar{x}(\eta\,|f, x^0))_n$.

Remark 4.3 Under assumpions A3 and A4 the coninuity of $\bar{x}(\cdot\,|f, x^0)$ (see Remark 4.2) implies that $\bar{x}_n(\cdot\,|f, x^0)$ is continuous.

Assumptions A5, A6, A7 and A8 are key for our solution method.

A5. There is a nonempty bounded interval $[w^-, w^+] \subset \bar{W}$ such that for every $(f, x^0) \in \mathcal{S}$ one can find an $\eta \in [w^-, w^+]$ satisfying $\bar{x}_n(\eta\,|f, x^0) = \hat{x}_n$, and for every $(f, x^0) \in \mathcal{S}$ the nth coordinate projection $\bar{x}_n(\cdot\,|f, x^0)$ of the rest point map is increasing on $[w^-, w^+]$.

Remark 4.4 If all $\bar{x}_n(\cdot\ |f,x^0)$ are decreasing, assumption A5 is fulfilled after the state transformation $x \mapsto -x$.

Note that if assumption A5 is fulfilled, then for every $(f,x^0) \in \mathcal{S}$ the equation $\bar{x}_n(\eta|f,x^0) = \hat{x}_n$ has the unique sulution in $[w^-,w^+]$; we denote this solution by $\hat{w}(f,x^0)$ and call it the *target limit value* for admissible controls for (f,x^0).

To formulate assumptions A6, A7 and A8 we need several further definitions. We suppose that for every $t > 0$ and every $w \in \mathcal{W}$ a nonempty set $\hat{\mathcal{W}}(t,w) \subset \mathcal{W}(t,w)$ of *operative extensions* (of w beyond t) is selected; for $t = 0$ the set of *initial* operative extensions is a given nonempty subset $\hat{\mathcal{W}}_0$ of $\mathcal{W}$, not depending on w. We also assume that

$$\bar{v} \in [w^-,w^+] \quad \text{for all} \quad v \in \hat{\mathcal{W}}_0, \tag{4.1}$$

$$\bar{v} \in [w^-,w^+] \quad \text{for all} \quad v \in \hat{\mathcal{W}}(t,w),\ t \geq 0,\ w \in \mathcal{W}. \tag{4.2}$$

Substantially, the sets of operative extensions represent stocks of the extensions of the admissible controls, that are used in stabilization strategies.

Let us fix a delay map δ_0 such that $t \mapsto \delta_0(t,w)$ is increasing for every $w \in \mathcal{W}$ and

$$\lim_{t\to\infty} \inf_{w\in\mathcal{W}} \delta_0(t,w) = \infty; \tag{4.3}$$

we call δ_0 the *operative* delay map. An infinite control flow $(t_m,w_m)_{m=0}^{\infty}$ will be said to be *operative* if $w_0 \in \hat{\mathcal{W}}_0$, $t_{m+1} \geq t_m + \delta_0(t_m,w_m)$ and $w_{m+1} \in \hat{\mathcal{W}}(t_{m+1},w_m)$ $(m = 0,1,\ldots)$; similarly, a finite control flow $(t_m,w_m)_{m=0}^{r}$ will be said to be *operative* if $w_0 \in \hat{\mathcal{W}}_0$, $t_{m+1} \geq t_m + \delta_0(t_m,w_m)$ $(m = 0,1,\ldots,r-1)$ and $w_{m+1} \in \hat{\mathcal{W}}(t_{m+1},w_m)$ $(m = 0,1,\ldots,r-1)$.

Remark 4.5 If $(t_m,w_m)_{m=0}^{\infty}$ is an infinite operative control flow, then, in view of (4.3), $\lim_{m\to\infty}(t_{m+1} - t_m) = \infty$.

The "limit controllability " assumption A6 states that the interval $[w^-,w^+]$ is covered by the limit values for the operative extensions of the final control in an arbitrary finite operative control flow.

A6. For every finite operative control flow $(t_i,w_i)_{i=0}^{m}$, every $t_{m+1} \geq t_m + \delta_0(t_m,w_m)$ and every $\eta \in [w^-,w^+]$ there exists a $w_{m+1} \in \hat{\mathcal{W}}(t_{m+1},w_m)$ such that $\bar{w}_{m+1} = \eta$.

The "uniform limit continuity" assumption A7 requires that a switch from w_m to w_{m+1} within a finite operative control flow $(t_i,w_i)_{m=0}^{m+1}$ implies a small change in trajectories provided the limit values $\bar{w}_m$ and $\bar{w}_{m+1}$ are close to each other, and the distance between the trajectories is estimated from above uniformly with respect to the operative control flows, systems and trajectories.

A7. There is a positive-valued function ω on $[0,\infty)$ such that

(i) $\lim_{\mu\to 0}\omega(\mu) = 0$, and

(ii) for every finite operative control flow $(t_i,w_i)_{i=0}^{m+1}$, every $(f,x^0) \in \mathcal{S}$ and every trajectory flow $(t_i,x_i)_{m=0}^{m+1}$ for (f,x^0), corresponsing to $(t_i,w_i)_{m=0}^{m+1}$, it holds that

$$|x_m(t)_n - x_{m+1}(t)_n| \leq \omega(|\bar{w}_m - \bar{w}_{m+1}|) \tag{4.4}$$

for all $t \geq t_{m+1}$.

The "uniform convergence" assumption A8 states that the composition w of any finite operative control flow brings the trajectories of every system $(f,x^0) \in \mathcal{S}$ to the rest point of the limit dynamics $\bar{f}$, corresponding to the limit value $\bar{w}$ for w, and the rate of convergence is uniform with respect to the operative control flows, systems and trajectories.

A8. There are real-valued functions ν^- and ν^+ on $[0,\infty)$ such that

(i) $\lim_{r\to\infty}\nu^-(r) = \lim_{r\to\infty}\nu^+(r) = 0$, and

(ii) for every finite operative control flow $(t_i, w_i)_{i=0}^m$, every $(f, x^0) \in \mathcal{S}$ and every trajectory flow $(t_i, x_i)_{i=0}^m$ of (f, x^0), corresponding to $(t_i, w_i)_{i=0}^m$, it holds that

$$\nu^-(t - t_m) \le x_m(t)_n - \bar{x}_n(\bar{w}_m | f, x^0) \le \nu^+(t - t_m) \tag{4.5}$$

for all $t \ge t_m$.

5 Stabilization strategy

In this section we construct a stabilization strategy under assumptions A1 – A8. Intending to use assumptions A6 – A8 involving operative control flows, we restrict our analysis to a class of control strategies that produce operative control flows only. We call a control strategy $R = (w_0, \bar{W}_0, \delta, s, I, E)$ *operative* if $\delta = \delta_0$ and

$$w_{m+1} = E(t_{m+1}, t_m, w_m, x(t)_n, x(t_m)_n, \bar{W}_{m+1}) \in \hat{W}(t_{m+1}, w_m) \tag{5.1}$$

for every $(t_{m+1}, t_m, w_m, x(t)_n, x(t_m)_n, \bar{W}_{m+1}) \in P$.

Lemma 5.1 *Let $R = (w_0, \bar{W}_0, \delta_0, s, I, E)$ be an operative control strategy, $(f, x^0) \in \mathcal{S}$ and $(t_m, w_m, x_m, \bar{W}_m)_{m=0}^\infty$ be an infinite processing flow for (f, x^0) under R. Then the infinite control flow $(t_m, w_m)_{m=0}^\infty$ is operative and*

$$\lim_{m\to\infty}(t_{m+1} - t_m) = \infty. \tag{5.2}$$

Proof. By the definition of an infinite processing flow for (f, x^0) under R, for each $m = 0, 1 \ldots$ the set $T_m = \{t \ge t_m : s(t, t_m, w_m, x_m(t)_n, x_m(t_m)_n, \bar{W}_m) = 1\}$ is nonempty and $t_{m+1} = t_m^* + \delta_0(t, w_m)$ where $t_m^* = \inf T_m$. By the definition of the operative delay map the function $t \mapsto \delta_0(t, w_m)$ is increasing; therefore, $t_{m+1} \ge t_m + \delta_0(t_m, w_m)$ $(m = 0, 1, \ldots)$. Taking into account (5.1), we find that the infinite control flow $(t_m, w_m)_{m=0}^\infty$ is operative. Using Remark 4.5, we get (5.2).

Similarly, we prove the following.

Lemma 5.2 *Let $R = (w_0, \bar{W}_0, \delta_0, s, I, E)$ be an operative control strategy, $(f, x^0) \in \mathcal{S}$ and $(t_m, w_m, x_m, \bar{W}_m)_{m=0}^r$ be a finite processing flow for (f, x^0) under R. Then the finite control flow $(t_m, w_m)_{m=0}^\infty$ is operative.*

Our next observation (based on assumptions A3, A4, A7 and A8) is essentially the following. If a processing flow $(t_m, w_m, x_m, \bar{W}_m)_{m=0}^\infty$ generated by an operative control strategy is such that the limit values $\bar{w}_m$ of the current (operative) extensions w_m tend to the target limit value $\hat{w}(f, x^0)$, then the corresponding trajectory x of the system (f, x^0) is stabilized: $x(t)_n \to \hat{x}$ as $t \to \infty$. The next lemma provides an accurate formulation of this result and suggests an estimate for the rate of convergence.

Lemma 5.3 *Let*

(i) *assumptions* A3, A4, A7 *and* A8 *be fulfilled,*

(ii) *$R = (w_0, \bar{W}_0, \delta_0, s, I, E)$ be an operative control strategy and $(t_m, w_m, x_m, \bar{W}_m)_{m=0}^\infty$ be an infinite processing flow for an $(f, x^0) \in \mathcal{S}$ under R,*

(iii)

$$\lim_{m\to\infty}\bar{w}_m = \hat{w}(f, x^0). \tag{5.3}$$

Then the composition x of the trajectory flow $(t_m, x_m)_{m=0}^{\infty}$ satisfies

$$\beta^-(t) \le x(t)_n - \hat{x}_n \le \beta^+(t) \quad \textit{for all} \quad t \ge 0 \tag{5.4}$$

where

$$\lim_{t\to\infty} \beta^-(t) = \lim_{t\to\infty} \beta^+(t) = 0; \tag{5.5}$$

moreover,

$$\beta^-(t) \;=\; \begin{cases} \nu^-(t) - |\hat{x}_n - \xi_0| & \textit{if} \quad t \in [0, t_1), \\ \inf_{m\ge k} \sigma_m^- & \textit{if} \quad t \ge t_{k+1}, \ k = 0, 1, \ldots, \end{cases} \tag{5.6}$$

$$\beta^+(t) \;=\; \begin{cases} \nu^+(t) + |\hat{x}_n - \xi_0| & \textit{if} \quad t \in [0, t_1), \\ \sup_{m\ge k} \sigma_m^+ & \textit{if} \quad t \ge t_{k+1}, \ k = 0, 1, \ldots, \end{cases} \tag{5.7}$$

where

$$\sigma_m^- \;=\; \inf_{t\ge t_{m+1}} \nu^-(t - t_m) - \omega(|\bar{w}_{m+1} - \bar{w}_m|) - |\hat{x}_n - \xi_m|, \tag{5.8}$$

$$\sigma_m^+ \;=\; \sup_{t\ge t_{m+1}} \nu^+(t - t_m) + \omega(|\bar{w}_{m+1} - \bar{w}_m|) + |\hat{x}_n - \xi_m|, \tag{5.9}$$

$$\xi_m = \bar{x}_n(\bar{w}_m | f, x^0). \tag{5.10}$$

Proof. Let us use notation (5.10). The continuity of $\bar{x}_n(\cdot\ | f, x^0)$ (see Remark 4.3) and (5.3) yield

$$\lim_{m\to\infty} \xi_m = \bar{x}_n(\hat{w}(f, x^0) | f, x^0) = \hat{x}_n \tag{5.11}$$

(the latter equality holds by the definition of the target limit value $\hat{w}(f, x^0)$). By the definition of an infinite precessing flow the trajectory flow $(t_m, x_m)_{m=0}^{\infty}$ corresponds to the infinite control flow $(t_m, w_m)_{m=0}^{\infty}$. Therefore for any $m = 0, 1, \ldots$ the trajectory flow $(t_i, x_i)_{i=0}^{m}$ corresponds to the finite control flow $(t_i, w_i)_{i=0}^{m}$. Then by assumption A8

$$\nu^-(t - t_m) \le x_m(t)_n - \xi_m \le \nu^+(t - t_m) \tag{5.12}$$

for all $t \ge t_m$ and by assumption A7

$$|x_{m+1}(t)_n - x_m(t)_n| \le \omega(|\bar{w}_{m+1} - \bar{w}_m|)$$

for all $t \ge t_m$. Hence, for all $t \ge t_{m+1}$

$$\nu^-(t - t_m) - \omega(|\bar{w}_{m+1} - \bar{w}_m|) \le |x_{m+1}(t)_n - \xi_m| \le \nu^+(t - t_m) + \omega(|\bar{w}_{m+1} - \bar{w}_m|).$$

Therefore

$$\sigma_m^- \le x_{m+1}(t)_n - \hat{x}_n \le \sigma_m^+ \quad \text{for all} \quad t \ge t_{m+1} \tag{5.13}$$

where σ_m^- and σ_m^+ are given in (5.8) and (5.9). For the composition x of the trajectory flow $(t_m, x_m)_{m=0}^{\infty}$ we have $x(t) = x_m(t)$ for all $t \in [t_m, t_{m+1}]$. Then (5.13) gives us

$$\inf_{m\ge k} \sigma_m^- \le x(t)_n - \hat{x}_n \le \sup_{m\ge k} \sigma_m^+ \quad \text{for all} \quad t \ge t_{k+1}.$$

Due to the arbitrarity of k for all $t \ge t_{k+1}$ we have (5.4) with $\beta^-(t)$ and $\beta^+(t)$ given by (5.6) and (5.7) (see the expressions for $t \ge t_{k+1}$). For $t \in [0, t_1)$ (5.4) holds due to (5.12) where we set $m = 0$ (see the expressions for $t \in [0, t_1)$ in (5.6) and (5.7)). By (5.3) and by Assumption A8 $\omega(|\bar{w}_{m+1} - \bar{w}_m|) \to 0$ as $m \to \infty$. By Lemma 5.1 we have (5.2). Assumption A8 and (5.2) imply that $\min_{t \geq t_{m+1}} \nu^-(t-t_m) \to 0$ as $m \to \infty$. These relations and (5.11) yield that $\sigma_m^- \to 0$ as $m \to \infty$ (see (5.8)). Similarly we get that $\sigma_m^+ \to 0$ as $m \to \infty$. Hence, for $\beta^-(t)$ and $\beta^+(t)$ (see (5.6) and (5.7)) we have (5.5). The lemma is proved.

A similar result holds for finite processing flows.

Lemma 5.4 *Let*

(i) *assumptions* A3, A4, A7 *and* A8 *be fulfilled,*

(ii) $R = (w_0, \bar{W}_0, \delta_0, s, I, E)$ *be an operative control strategy and* $(t_m, w_m, x_m, \bar{W}_m)_{m=0}^r$ *be a finite processing flow for an* $(f, x^0) \in \mathcal{S}$ *under* R,

(iii)

$$\bar{w}_r = \hat{w}(f, x^0). \tag{5.14}$$

Then the composition x *of the trajectory flow* $(t_m, x_m)_{m=0}^r$ *satisfies*

$$\beta^-(t) \leq x(t)_n - \hat{x}_n \leq \beta^+(t) \quad \textit{for all} \quad t \geq 0$$

where

$$\lim_{t\to\infty} \beta^-(t) = \lim_{t\to\infty} \beta^+(t) = 0;$$

moreover,

$$\beta^-(t) = \begin{cases} \nu^-(t) + |\hat{x}_n - \xi_0| & \textit{if} \;\; t \in [0, t_1), \\ \min\{\inf_{r-2 \geq m \geq k} \sigma_m^-, \; \nu^-(t - t_r)\} & \textit{if} \;\; t \in [t_{k+1}, t_{k+2}], \; k \leq r-2, \\ \nu^-(t - t_r) & \textit{if} \;\; t \geq t_r, \end{cases}$$

$$\beta^+(t) = \begin{cases} \nu^+(t) + |\hat{x}_n - \xi_0| & \textit{if} \;\; t \in [0, t_1), \\ \max\{\sup_{r-2 \geq m \geq k} \sigma_m^+, \; \nu^+(t - t_r)\} & \textit{if} \;\; t \in [t_{k+1}, t_{k+2}], \; k \leq r-2, \\ \nu^+(t - t_r) & \textit{if} \;\; t \geq t_r \end{cases}$$

with σ_m^-, σ_m^+ *and* ξ_m *defined by* (5.8), (5.9), (5.10).

We omit the proof which is similar to the proof of Lemma 5.3.

Lemma 5.3 is more informative than Lemma 5.4, since it adresses a "generic" case where a processing flow corresponding to a control strategy is infinite. In Lemma 5.3 a key condition for the stabilization relations (5.4) is the convergence (5.3) of the limit values for the current admissible controls to the target limit value $\hat{w}(f, x^0)$. Lemma 5.4 deals with an "exceptional" case where $\hat{w}(f, x^0)$ is (apparently) "found" by the limit value for the admissible control at some finite step of the control process (see (5.14)). Wishing to use these conditions to stabilize the (real) system, the controller faces a nontrivial task: to ensure the convergence (5.3) (or the precise equality (5.14)) without knowing the system (f, x^0) that is actually regulated.

We approach a solution using an appropriate asymptotics for the lower estimates $\bar{W}_m$ of the set of all inconsistent limit values $\bar{w}$ for admissible controls; the latter set consists obviously of all $\bar{w} \neq \hat{w}(f, x^0)$. Step by step the estimates $\bar{W}_m$ are extended so that eventually they cover the entire interval $[w^-, w^+]$ of "meaningful" (see assumption A5) limit values for admissible controls – except for the target limit value $\hat{w}(f, x^0)$. Such remarkable asymptotics of $\bar{W}_m$ emerges thanks to assumptions A8 and A5, which allow the controller to register the inconsistency of the limit value $\bar{w}_m$ of a current (operative) extension w_m with an immediate identification of which of the "inconsistency" inequalities, $\bar{w}_m < \hat{w}(f, x^0)$ or $\bar{w}_m > \hat{w}(f, x^0)$, holds actually.

An accurate formulation is the following.

Lemma 5.5 *Let*

(i) *assumptions* A3, A4, A5 *and* A8 *be fulfilled,*

(ii) $(t_i, w_i)_{i=0}^m$ *be a finite operative control flow,* $(f, x^0) \in \mathcal{S}$*,* $(t_i, x_i)_{i=0}^m$ *be a trajectory flow for* (f, x^0)*, corresponding to* $(t_i, w_i)_{i=0}^m$*, and* ν^- *and* ν^+ *be the functions defined in assumption* A8.

Then for every $t \geq t_m$ *the inequality*

$$x_m(t)_n - \hat{x}_n < \nu^-(t - t_m) \tag{5.15}$$

implies

$$\bar{w}_m < \hat{w}(f, x^0), \tag{5.16}$$

and the inequality

$$x(t)_n - \hat{x}_n > \nu^+(t - t_m) \tag{5.17}$$

implies

$$\bar{w}_m > \hat{w}(f, x^0). \tag{5.18}$$

Proof. Let $t \geq t_m$ and (5.15) hold. By the definition of the target limit value $\hat{w}(f, x^0)$ we have $\hat{x}_n = \bar{x}_n(\hat{w}(f, x^0)|f, x^0)$. Therefore, (5.15) is equivalent to

$$x_m(t)_n - \bar{x}_n(\hat{w}(f, x^0)|f, x^0) < \nu^-(t - t_m).$$

By assumption A8

$$\nu^-(t - t_m) \leq x_m(t)_n - \bar{x}_n(\bar{w}_m|f, x^0).$$

Hence,

$$x_m(t)_n - \bar{x}_n(\hat{w}(f, x^0)|f, x^0) < x_m(t)_n - \bar{x}_n(\bar{w}_m|f, x^0),$$

or

$$\bar{x}_n(\hat{w}(f, x^0)|f, x^0) > \bar{x}_n(\bar{w}_m|f, x^0). \tag{5.19}$$

By assumption A5 $\bar{x}_n(\cdot\,|f, x^0)$ is increasing on $[w^-, w^+]$. Furthermore, $\hat{w}(f, x^0) \in [w^-, w^+]$ and $\bar{w}_m \in [w^-, w^+]$ since the finite control flow $(t_i, w_i)_{i=0}^m$ is operative (see (4.1) and (4.2)). Consequently, (5.19) yields (5.16). We showed that (5.15) implies (5.16). Similarly, we show that (5.17) implies (5.18). The lemma is proved.

In what follows, the common notation $|d_1, d_2|$ is used for an open, closed or half-open interval of reals with the endpoints d_1 and d_2.

Lemma 5.5 can be interpreted as follows: if in step $m + 1$ of the control process at some point in time, t, the observed state $x_m(t)_n$ of the real system satisfies (5.15) or (5.17), then the limit value $\bar{w}_m$ of the current admissible control w_m is inconsistent; moreover, (5.15) implies that the entire interval $[w^-, \bar{w}_m]$ is inconsistent (in the sense that it comprises inconsistent limit values only), whereas (5.17) implies that the entire interval $[\bar{w}_m, w^+]$ is inconsistent. This gives us a clear idea of the construction of an inconsistency signal map s and an inconsistency estimate map I in a desired (operative) control strategy $R = (w_0, \bar{W}_0, \delta_0, s, I, E)$. Indeed, we see that the inconsistency signal map s should produce an inconsistency signal (i.e., take value 1) if either (5.15) or (5.17) is registered; and the inconsistency estimate map I should define the new set estimate $\bar{W}_{m+1}$ as the union of $\bar{W}_m$ and one of the intervals $[w^-, \bar{w}_m]$ and $[\bar{w}_m, w^+]$, depending on which of the inconsistency inequalities, (5.15) or (5.17), is registered.

Thus, it is reasonable to set

$$s(t, t_m, w_m, x_m(t)_n, x_m(t_m)_n, \bar{W}_m) = \begin{cases} 1 & \text{if} \quad x_m(t)_n - \hat{x}_n < \nu^-(t - t_m), \\ 1 & \text{if} \quad x_m(t)_n - \hat{x}_n > \nu^+(t - t_m), \\ 0 & \text{otherwise}, \end{cases} \tag{5.20}$$

and

$$\begin{aligned} \bar{W}_{m+1} &= I(t^*_m, t_m, w_m, x_m(t)_n, x_m(t_m)_n, \bar{W}_m) \\ &= \begin{cases} \bar{W}_m \cup [w^-, \bar{w}_m] & \text{if} \quad x_m(t^*_m)_n - \hat{x}_n \le \nu^-(t - t_m), \\ \bar{W}_m \cup [\bar{w}_m, w^+] & \text{if} \quad x_m(t^*_m)_n - \hat{x}_n \ge \nu^+(t - t_m), \\ \bar{W}_m & \text{otherwise} \end{cases} \end{aligned} \tag{5.21}$$

(in the latter formula the last line is just a formality; it corresponds to the case where the inconsistency signal is not produced and, consequently, there is no need to update $\bar{W}_m$).

If the controller uses (5.20) and (5.21), then each step m, in which the current limit value $\bar{w}_m$ is identified (via s) as inconsistent, results in an essential extension of the current estimate $\bar{W}_m$: one of the “solid” intervals, $[w^-, \bar{w}_m]$ or $[\bar{w}_m, w^+]$, is added to $\bar{W}_m$. In this situation, the current (upper) estimate for the set of “consistent” limit values, i.e., the complement $[w^-, w^+] \setminus \bar{W}_m$, is necessarily an interval $|v^-_m, v^+_m| \subset [w^-, w^+]$ containing the target value $\hat{w}(f, x^0)$. Now suppose that the extension map E places the new limit value $\bar{w}_{m+1}$ in the middle of the new “consistency” interval $|v^-_{m+1}, v^+_{m+1}|$:

$$\bar{w}_{m+1} = \frac{v^-_{m+1} + v^+_{m+1}}{2}. \tag{5.22}$$

Then in step $m+1$ the “consistency” interval $|v^-_{m+2}, v^+_{m+2}|$ is two times shorter than $|v^-_{m+1}, v^+_{m+1}|$ (unless step $m+1$ terminates the control process, implying $\bar{w}_{m+1} = \hat{w}(f, x^0)$). As a result, $|v^-_m, v^+_m|$ shrinks gradually to $\hat{w}(f, x^0)$. This pattern can be viewed as an asymptotic version of the Krasovskii extremal shift feedback principle, known in differential games (see Krasovskii and Subbotin, 1974). Together with the convergence properties stated in Lemmas 5.3 5.4, it leads us to a solution of the stabilization problem.

In the rest of this section we implement the above informal argument rigorously. We call an operative control strategy $R = (w_0, \bar{W}_0, \delta_0, s, I, E)$ a *target identification* strategy if w_0 is an operative extension ($w_0 \in \hat{\mathcal{W}}_0$);

$$\bar{W}_0 = (-\infty, \infty) \setminus [w^-, w^+]; \tag{5.23}$$

the inconsistency signal map s is given by (5.20); the inconsistency estimate map I is given (5.21); and the extension map E is such that $w_{m+1} = E(t_{m+1}, t_m, w_m, x_m(t)_n, x_m(t_m)_n, \bar{W}_{m+1})$ satisfies (5.22) provided

$$[w^-, w^+] \setminus \bar{W}_{m+1} = |v^-_{m+1}, v^+_{m+1}|. \tag{5.24}$$

Note that the definition of E is correct thanks to assumption A6. We aslo note that generally an extension w_{m+1} satisfying (5.22) is not unique, therefore a target identification strategy is defined not uniquely.

Our main statement is the following.

Theorem 5.1 *Let assumptions* A1 – A8 *be fulfilled. Then*

1) *every target identification strategy* $R = (w_0, \bar{W}_0, s, I, E)$ *is a stabilization strategy;*

2) *if the set* $\{\bar{x}_n(\cdot|f,x^0) : (f,x^0) \in \mathcal{S}\}$ *of the nth coordinate rest point maps is uniformly continuous on* $[w^-, w^+]$, *i.e.,*

$$\lim_{\varepsilon\to 0} \psi(\varepsilon) = 0 \tag{5.25}$$

where

$$\psi(\varepsilon) = \sup\{|\bar{x}_n(\eta_1|f,x^0) - \bar{x}_n(\eta_2|f,x^0)| : \eta_1,\eta_2 \in [w^-,w^+],\ |\eta_1-\eta_2| \le \varepsilon,\ (f,x^0)\in\mathcal{S}\},$$

then there exist real functions $\bar{\beta}^-$ *and* $\bar{\beta}^+$ *on* $[0,\infty)$ *such that*

$$\lim_{\varepsilon\to 0} \bar{\beta}^-(t) = \lim_{\varepsilon\to 0} \bar{\beta}^+(t) = 0 \tag{5.26}$$

and for every system $(f,x^0)\in\mathcal{S}$ *and every trajectory* x *of* (f,x^0) *under the control strategy* R, *it holds that*

$$\bar{\beta}^-(t) \le x(t)_n - \hat{x}_n \le \bar{\beta}^+(t) \quad \textit{for all} \quad t\ge 0; \tag{5.27}$$

moreover,

$$\bar{\beta}^-(t) \;=\; \begin{cases} \nu^-(t) - |\hat{x}_n - \xi_0| & \textit{if}\ \ t\in[0,t_1), \\ \inf_{m\ge k} \bar{\sigma}_m^- & \textit{if}\ \ t \ge t_{k+1},\ k=0,1,\ldots, \end{cases} \tag{5.28}$$

$$\bar{\beta}^+(t) \;=\; \begin{cases} \nu^+(t) + |\hat{x}_n - \xi_0| & \textit{if}\ \ t\in[0,t_1), \\ \sup_{m\ge k} \bar{\sigma}_m^+ & \textit{if}\ \ t \ge t_{k+1},\ k=0,1,\ldots, \end{cases} \tag{5.29}$$

where

$$\bar{\sigma}_m^- \;=\; \inf_{t\ge t^0_{m+1}} \nu^-(t-t^0_m) - \omega\left(\frac{w^+-w^-}{2^{m-2}}\right) - \rho\left(\frac{w^+-w^-}{2^{m-1}}\right), \tag{5.30}$$

$$\bar{\sigma}_m^+ \;=\; \sup_{t\ge t^0_{m+1}} \nu^+(t-t^0_m) + \omega\left(\frac{w^+-w^-}{2^{m-2}}\right) + \rho\left(\frac{w^+-w^-}{2^{m-1}}\right), \tag{5.31}$$

$$t^0_0 = 0,\quad t^0_{m+1} = t^0_m + \inf\{\delta_0(t,w) : t\ge t^0_m,\ w\in W\} \tag{5.32}$$

$$(m=1,2,\ldots).$$

Proof. Let (f,x^0) be an arbitrary system in $\mathcal{S}$ and x be an arbitrary trajectory of (f,x^0) under the control strategy R.

In order to prove statement 1 we must show that

$$\lim_{t\to\infty} x(t)_n = \hat{x}_n. \tag{5.33}$$

Suppose x is an infinite-step trajectory. By definition there exists an infinite processing flow $(t_m, w_m, x_m, \bar{W}_m)_{m=0}^\infty$ for (f,x^0) under R such that x is the composition of the trajectory flow $(t_m, x_m)_{m=0}^\infty$, i.e.,

$$x(t) = x_m(t) \quad \text{for all} \quad t\in[0,t_m] \tag{5.34}$$

for each $m=0,1,\ldots$. By the definition of an infinite processing flow the trajectory flow $(t_m,x_m)_{m=0}^\infty$ correspons to the control flow $(t_m,w_m)_{m=0}^\infty$; for each $m=0,1\ldots$ the set

$$T_m = \{t\ge t_m : s(t,t_m,w_m,x_m(t)_n,x_m(t_m)_n,\bar{W}_m) = 1\} \tag{5.35}$$

is nonempty; $t_{m+1} = t_m^* + \delta(t_m^*, w_m)$, where $t_m^* = \inf T_m$; and for each $m = 0, 1 \ldots$

$$\bar{W}_{m+1} = I(t_m^*, t_m, w_m, x_m(t)_n, x_m(t_m)_n, \bar{W}_m),$$

$$w_{m+1} = E(t_{m+1}, t_m, w_m, x_m(t)_n, x_m(t_m)_n, \bar{W}_{m+1}). \tag{5.36}$$

The former relation and (5.21) yield

$$\bar{W}_{m+1} = \begin{cases} \bar{W}_m \cup [w^-, \bar{w}_m] & \text{if} \quad x_m(t_m^*)_n - \hat{x}_n \leq \nu^-(t_m^* - t_m), \\ \bar{W}_m \cup [\bar{w}_m, w^+] & \text{if} \quad x_m(t_m^*)_n - \hat{x}_n \geq \nu^+(t_m^* - t_m), \\ \bar{W}_m & \text{otherwise.} \end{cases} \tag{5.37}$$

For each $m = 0, 1 \ldots$ the fact that the set T_m (5.35) is nonempty and the definition of the inconsistency signal map s (see (5.20)) yield that for every $t \in T_m$ we have either $x_m(t)_n - \hat{x}_n < \nu^-(t - t_m)$ or $x_m(t)_n - \hat{x}_n > \nu^+(t - t_m)$. By Lemma 5.5 the former inequality and the latter inequality imply, respectively, the inequalities $\bar{w}_m < \hat{w}(f, x^0)$, $\bar{w}_m > \hat{w}(f, x^0)$. Therefore, for $t_m^* = \inf T_m$ we have one of two cases:

$$x_m(t_m^*)_n - \hat{x}_n \leq \nu^-(t_m^* - t_m), \quad \bar{w}_m < \hat{w}(f, x^0), \tag{5.38}$$

$$x_m(t_m^*)_n - \hat{x}_n \geq \nu^-(t_m^* - t_m), \quad \bar{w}_m > \hat{w}(f, x^0) \tag{5.39}$$

(Lemma 5.5 is applicable since, as noticed above, the finite control flow $(t_i, w_i)_{i=0}^m$ is operative).

Basing on this, we will now show that for every $m = 0, 1, \ldots$ the set $[w^-, w^+] \setminus \bar{W}_{m+1}$ is an interval $|v_{m+1}^-, v_{m+1}^+|$ containing the target limit value $\hat{w}(f, x^0)$. We use induction. As noted above, for $m = 0$ we have either (5.38) or (5.39). Let (5.38) hold with $m = 0$. Since $w_0 \in \hat{\mathcal{W}}_0$, we have $\bar{w}_0 \in [w^-, w^+]$ (see (4.1)). Then by (5.37) and (5.23)

$$\bar{W}_1 = \bar{W}_0 \cup [w^-, \bar{w}_0] = (-\infty, \infty) \setminus [\bar{w}_0, w^+],$$

and clearly $\hat{w}(f, x^0) \notin \bar{W}_1$; hence, $[w^-, w^+] \setminus \bar{W}_1 = |v_1^-, v_1^+| = (\bar{w}_0, w^+]$ and $\hat{w}(f, x^0) \in |v_1^-, v_1^+|$. Similarly, if (5.39) holds with $m = 0$, we state that $[w^-, w^+] \setminus \bar{W}_1 = |v_1^-, v_1^+| = [w^-, \bar{w}_0)$ and $\hat{w}(f, x^0) \in |v_1^-, v_1^+|$. Now assume that

$$\hat{w}(f, x^0) \in |v_m^-, v_m^+| = [w^-, w^+] \setminus \bar{W}_m \tag{5.40}$$

for some $m = 0, 1, \ldots$. As we noticed earlier, we have (5.38) or (5.39). Suppose (5.38) holds. Due to (5.37)

$$\bar{W}_{m+1} = \bar{W}_m \cup [w^-, \bar{w}_m] = \big[(-\infty, \infty) \setminus |v_m^-, v_m^+|\big] \cup [w^-, \bar{w}_m].$$

By (5.40) and (5.38) $w^- \leq \bar{w}_m < \hat{w}(f, x^0) \leq v_m^+ \leq w^+$. Therefore

$$\hat{w}(f, x^0) \in [w^-, w^+] \setminus \bar{W}_{m+1} = |v_{m+1}^-, v_{m+1}^+| = |\max\{v_m^-, \bar{w}_m\}, v_m^+|. \tag{5.41}$$

Similarly, if (5.39) holds, we state that

$$\hat{w}(f, x^0) \in [w^-, w^+] \setminus \bar{W}_{m+1} = |v_{m+1}^-, v_{m+1}^+| = |v_m^-, \min\{v_m^+, \bar{w}_m|. \tag{5.42}$$

This completes the proof by induction. Moreover, we stated that for each $m = 0, 1, \ldots$ one of the relations (5.41) and (5.42) holds, implying, in particular, (5.24).

Recalling the definition of the extension map E and taking into account (5.24), we find that for every $m = 0, 1 \ldots$ the limit value $\bar{w}_{m+1}$ for w_{m+1} (5.36) is given by (5.22). Now replacing m by $m+1$ in (5.41) and (5.42), we get that one of the following relations holds:

$$\hat{w}(f, x^0) \in |v_{m+2}^-, v_{m+2}^+| = |\max\{v_{m+1}^-, \bar{w}_{m+1}\}, v_{m+1}^+|, \tag{5.43}$$

$$\hat{w}(f,x^0) \in |v^-_{m+2}, v^+_{m+2}| = |v^-_{m+1}, \min\{v^-_{m+1}, \bar{w}_{m+1}\}|. \tag{5.44}$$

Suppose (5.43) is satisfied. Taking into account (5.22), we get

$$|v^-_{m+2}, v^+_{m+2}| = \left| v^-_{m+1}, \frac{v^-_{m+1} + v^+_{m+1}}{2} \right|;$$

hence,

$$v^+_{m+2} - v^-_{m+2} = \frac{v^+_{m+1} - v^-_{m+1}}{2}. \tag{5.45}$$

Similarly, we obtain (5.45) if we assume that (5.44) holds. We stated that for every $m = 0, 1, \ldots$ (5.45) holds. Therefore,

$$v^+_{m+1} - v^-_{m+1} \le \frac{w^+ - w^-}{2^m} \quad \text{for all} \quad m = 0, 1, \ldots.$$

Hence, in view of $\hat{w}(f,x^0) \in |v^-_{m+1}, v^+_{m+1}|$ (see (5.41) and (5.42)) and (5.22),

$$|\bar{w}_{m+1} - \hat{w}(f,x^0)| \le \frac{w^+ - w^-}{2^m} \quad \text{for all} \quad m = 0, 1, \ldots. \tag{5.46}$$

Now we make use of Lemma 5.3. Assumptions (i) and (ii) of Lemma 5.3 are satisfied trivially. Above we noticed that the infinite control flow $(t_m, w_m)_{m=0}^{\infty}$ is operative. Furthermore, by (5.46) $\lim_{m\to\infty} \bar{w}_m = \hat{w}(f,x^0)$. Thus, assumption (iii) of Lemma 5.3 is fulfilled, too. Applying Lemma 5.3, we find that the trajectory x being the composition of the trajectory flow $(t_m, x_m)_{m=0}^{\infty}$ satisfies

$$\beta^-(t) \le x(t)_n - \hat{x}_n \le \beta^+(t) \quad \text{for all} \quad t \ge 0 \tag{5.47}$$

where $\lim_{t\to\infty} \beta^-(t) = \lim_{t\to\infty} \beta^+(t) = 0$. We get (5.33) and thus complete the proof of statement 1.

Let us prove statement 2. Suppose (5.25) holds. Taking into account that $\nu^-(r), \nu^+(r) \to 0$ as $r \to \infty$ (see assumption A8) $t^0_{m+1} - t^0_m \to \infty$ as $m \to \infty$ (see (4.3) and $\omega(\mu) \to 0$ as $\mu \to 0$ (see assumption A7), we easily find that for $\bar{\beta}^-$ and $\bar{\beta}^+$ given by (5.28) and (5.28) the limit relations (5.26) hold true.

Let us show the estimates (5.27). Consider the estimates (5.47). By Lemma 5.3 $\beta^-(t)$ and $\beta^+(t)$ (see (5.47)) are given by (5.7), (5.7) where σ^-_m and σ^+_m are defined in (5.8), (5.9), (5.10). By (5.46)

$$|\bar{w}_{m+1} - \bar{w}_m| \le \frac{w^+ - w^-}{2^{m-2}};$$

hence, in view of (5.8) and (5.9)

$$\begin{aligned} \sigma^-_m &\le \inf_{t \ge t_{m+1}} \nu^-(t - t_m) - \omega\left(\frac{w^+ - w^-}{2^{m-1}}\right) - |\hat{x}_n - \xi_m|, \\ \sigma^+_m &\ge \sup_{t \ge t_{m+1}} \nu^+(t - t_m) + \omega\left(\frac{w^+ - w^-}{2^{m-1}}\right) + |\hat{x}_n - \xi_m|. \end{aligned}$$

One can easily state that

$$\sigma^-_m \ge \bar{\sigma}^-_m, \quad \sigma^+_m \le \bar{\sigma}^+_m \tag{5.48}$$

where $\bar{\sigma}^-_m$ and $\bar{\sigma}^+_m$ are given by (5.30) and (5.31). Indeed, comparing σ^-_m with $\bar{\sigma}^-_m$ and σ^+_m with $\bar{\sigma}^+_m$, we see that for (5.48) it is sufficient to show that

$$\inf_{t \ge t_{m+1}} \nu^-(t - t_m) \ge \inf_{t \ge t^0_{m+1}} \nu^-(t - t^0_m), \qquad \sup_{t \ge t_{m+1}} \nu^-(t - t_m) \le \sup_{t \ge t^0_{m+1}} \nu^-(t - t^0_m), \tag{5.49}$$

and

$$|\hat{x}_n - \xi_m| \le \psi\left(\frac{w^+ - w^-}{2^{m-1}}\right). \tag{5.50}$$

Since $t_{i+1} \ge t_i + \delta_0(t_i, w_i)$ $(i = 0, 1, \ldots)$, we have $t_m \ge t_m^0$, $t_{m+1} \ge t_{m+1}^0$ (see (5.32)). Therefore, (5.49) holds true. Noticing that $\xi_m = \bar{x}_n(\bar{w}_m|f, x^0)$ (see (5.10)) and $\hat{x}_n = \bar{x}_n(\hat{w}(f, x^0)|f, x^0)$ and taking into account the definition of $\psi(\varepsilon)$ and (5.46), we get (5.50). Thus, (5.48) is stated. Now, comparing β^- (5.6) with $\bar{\beta}^-$ (5.28) and β^+ (5.7) with $\bar{\beta}^+$ (5.29), we find that $\beta_m^- \ge \bar{\beta}_m^-$ and $\beta_m^+ \le \bar{\beta}_m^+$. Hence, (5.47) implies (5.27).

The case where x is a finite-step trajectory of (f, x^0) under the strategy R is treated similarly; Lemma 5.4 is used instead of Lemma 5.3.

The proof is completed.

6 Stabilization of balance processes

In this section we consider an application of Theorem 5.1 to control systems of the form presented earlier in the context of the problem of stabilization of atmospheric carbon (see section 1):

$$\begin{aligned} \dot{x}(t) &= \varphi(t) + cu(t) + g(x(t), y(t)), \\ \dot{y}(t) &= -g(x(t), y(t)), \end{aligned} \tag{6.1}$$

$$x(0) = x^0, \quad y(0) = y^0; \tag{6.2}$$

here $x(t), y(t) \in R^n$ represent the system's state; $u(t)$ is a 1-dimensional control parameter; $c \in R^n$, the functions $\varphi : [0, \infty) \mapsto R^n$ and $g : R^n \times R^n \mapsto R^n$ are continuous. For a system of form (6.1), (6.2) we have

$$x(t) + y(t) = x^0 + y^0 + cw(t) + \Phi(t) \tag{6.3}$$

where

$$w(t) = \int_0^t u(\tau) d\tau, \tag{6.4}$$

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau. \tag{6.5}$$

If $x(t)$ and $y(t)$ represent the amounts of a certain substance in two compartments of a physical system, (6.3) gives us a balance equation showing that the total amount of the substance in the system remains constant unless extra amounts of the substance enter one of the compartments with a nonzero rate $\varphi(t) + u(t)$.

Following the notations of sections 2 – 5, we treat functions w (6.4) as controls, and write $\dot{w}(t)$ instead of $u(t)$. We fix a bounded interval $[w_0^-, w_0^+]$ containing 0 and a bounded interval $[w_1^-, w_1^+]$ and make the following assumption.

A9. The set $\mathcal{W}$ of all admissible controls consists of all real functions $w : [0, \infty) \mapsto [w_0^-, w_0^+]$ such that w is absolutely continuous on every bounded subinterval of $[0, \infty)$, $w(0) = 0$, the limit relation (2.3) holds for some real $\bar{w}$, and $\dot{w}$ takes values in $[w_1^-, w_1^+]$.

We assume that there is a "real" system of the form (6.1), (6.2), and the controller needs to design an admissible control w that brings the nth coordinate, $x_n(t)$, of the state component $x(t)$ of the "real" system to the prescribed value $\hat{x}_n$ as $t \to \infty$. The real system is not known to the controller; instead the controller is given a class $\mathcal{B}$ of systems of the form (6.1), (6.2), which contains the real one. When forming a desired admissible control the controller is allowed to observe the current values $x_n(t)$.

This stabilization problem is identical to the one considered in sections 2 – 5; the two problems differ in the form of the control systems only (compare (6.1), (6.2) and (2.1), (2.2)). However, this difference is merely formal. Using (6.3), we easily reduce a system of the form (6.1), (6.2) to a system of the form (2.1), (2.2), and thus come to the stabilization problem posed and analyzed in sections 2 – 5. Namely, expressing $y(t)$ from (6.3) and substituting into the first equation in (6.1), we get

$$\dot{x}(t) = f(t, x(t), y(t), w(t), \dot{w}(t)),$$

$$x(0) = x^0$$

where

$$f(t, x, w, v) = \varphi(t) + cv + g(x, -x + x^0 + y^0 + cw + \Phi(t)). \tag{6.6}$$

A formal (obvious) statement is as follows.

Lemma 6.1 *Let $c \in R^n$, $\varphi : [0, \infty) \mapsto R^n$ and $g : [0, \infty) \times R^n \times R^n \mapsto R^n$ be continuos, $x^0, y^0 \in R^n$, $f : [0, \infty) \times R^n \times R^n \mapsto R^n$ be given by* (6.6)*, $w \in \mathcal{W}$ and $u = \dot{w}$. Then a function $(x(\cdot), y(\cdot)) : [0, \infty) \mapsto R^n \times R^n$ is a (Caratheodory) solution to* (6.1), (6.2) *if and only if $x(\cdot)$ is a trajectory of the system (f, x^0), corresponding to w, and* (6.3) *holds for all $t \geq 0$.*

Now we specify the class $\mathcal{B}$ of amissible systems of the form (6.1), (6.2). We fix a continuous function $\varphi : [0, \infty) \mapsto R^n$ such that

$$\lim_{t \to \infty} \varphi(t) = 0 \tag{6.7}$$

and the function Φ given by (6.5) satisfies

$$\lim_{t \to \infty} \Phi(t) = \bar{\Phi} \tag{6.8}$$

with some $\bar{\Phi} \in R^n$. We also fix $K_0 \geq 0$, $K \geq 0$, $a_1 > 0$, $b_1 > 0$, $c_1 > 0$, and nonempty bounded sets X^0 and Y^0 in R^n. We identify $\mathcal{B}$ with a subset of the set of all 4-tuples (c, g, x^0, y^0), where $c \in R^n$ $g : R^n \times R^n \mapsto R^n$ and $x^0, y^0 \in R^n$, such that

(i)

$$|c| \leq c_1; \tag{6.9}$$

(ii)

$$x^0 \in X^0, \ y^0 \in Y^0; \tag{6.10}$$

(iii)

$$g(0, 0) = 0, \quad |g(0, y)| \leq K_0(1 + |y|) \quad \text{for all} \quad y \in R^n; \tag{6.11}$$

(iv) g is continuously differentiable and

$$\left| \frac{\partial g(x, y)}{\partial y} \right| \leq K \quad \text{for all} \quad x, y \in R^n, \tag{6.12}$$

(v) for all $x, y \in R^n$ the matrix

$$G(x, y) = \frac{\partial g(x, y)}{\partial x} - \frac{\partial g(x, y)}{\partial y} \tag{6.13}$$

is nondegenerate and

$$\left(G^{-1}(x, y) \frac{\partial g(x, y)}{\partial y} c \right)_n \leq 0 \tag{6.14}$$

(here $\partial g(x,y)/\partial x$ and $\partial g(x,y)/\partial y$ are the Jacobi matricies for the maps $x \mapsto g(x,y)$ and $y \mapsto g(x,y)$, respectively; we also recall that z_n is the nth coordinate of a $z \in R^n$);

(vi) the matrix functions $(x,y) \mapsto \partial g(x,y)/\partial y$ and $(x,y) \mapsto G^{-1}(x,y)$ are Lipschitz continuous on $R^n \times R^n$;

(vii)

$$\langle x^1 - x^2, g(x^1,y) - g(x^2,y)\rangle \le -a_1|x^1 - x^2|^2 \quad \text{for all} \quad x^1, x^2 \in R^n,\ y \in R^n \tag{6.15}$$

and

$$b_1|y^1 - y^2|^2 \le \langle y^1 - y^2, g(x,y^1) - g(x,y^2)\rangle \quad \text{for all} \quad x \in R^n,\ y^1, y^2 \in R^n; \tag{6.16}$$

here and in what follows, $|\cdot|$ and $\langle\cdot,\cdot\rangle$ are, respectively, the Euclinean norm and scalar product in R^n.

In this section we assume the following.

A10. The class $\mathcal{S}$ of admissible systems is the set of all (f, x^0) (of the form (2.1), (2.2)) such that f is given by (6.6) for some $(c, g, x^0, y^0) \in \mathcal{B}$.

Under this assumption, we study the stabilization problem formulated in section 2; the latter problem represents formally the above outlined stabilization problem for an uncertain system of the form (6.1), (6.2).

Technically, our goal is to state that assumptions A1 – A8 are fulfilled.

Lemma 6.2 *Let assumptions* A9 *and* A10 *be fulfilled. Then assumptions* A1 *and* A2 *are fulfilled. Moreover, there is a bounded set* $X \subset R^n$ *such that for every system* $(f, x^0) \in \mathcal{S}$ *and every admissible control* w *the trajectory of* (f, x^0)*, corresponding to* w*, takes values in* X*.*

Proof. Consider assumption A1. The continuity of φ and g for every $(c, g, x^0, y^0) \in \mathcal{B}$ yields that f (see (6.6)) is continuous for every $(f, x^0) \in \mathcal{S}$. Assumption A1 is fulfilled.

Consider assumption A2. Let $(f, x^0) \in \mathcal{S}$. Then f is given by (6.6) where $(c, g, x^0, y^0) \in \mathcal{B}$. Take a $w \in \mathcal{W}$. For all $x^1, x^2 \in R^n$ and almost all $t \ge 0$

$$\Delta(x^1, x^2, t) = \langle x^1 - x^2, f(t, x^1, w(t), \dot w(t)) - f(t, x^2, w(t), \dot w(t))\rangle \tag{6.17}$$

is transformed as follows:

$$\begin{aligned}
\Delta(x^1, x^2, t) &= \langle x^1 - x^2, g(x^1, -x^1 + z) - g(x^2, -x^2 + z)\rangle \\
&= \langle x^1 - x^2, g(x^1, -x^1 + z) - g(x^2, -x^1 + z)\rangle + \\
&\quad\ \langle x^1 - x^2, g(x^2, -x^1 + z) - g(x^2, -x^2 + z)\rangle \\
&= \langle x^1 - x^2, g(x^1, -x^1 + z) - g(x^2, -x^1 + z)\rangle - \\
&\quad\ \langle -x^1 - (-x^2), g(x^2, -x^1 + z) - g(x^2, -x^2 + z)\rangle
\end{aligned} \tag{6.18}$$

where $z = x^0 + y^0 + cw(t) + \Phi(t)$. Hence, by (6.15) and (6.16)

$$\Delta(x^1, x^2, t) \le -(a_1 + b_1)|x^1 - x^2|^2. \tag{6.19}$$

Then, in view of (6.17), (6.6) and (6.11), for all $x \in R^n$ and almost all $t \ge 0$

$$\begin{aligned}
\Delta(x, 0, t) &= \langle x, f(t, x, w(t), \dot w(t))\rangle - \\
&\quad\ \langle x, \varphi(t) + c\dot w(t) + g(0, x^0 + y^0 + cw(t) + \Phi(t))\rangle \\
&\le -(a_1 + b_1)|x|^2.
\end{aligned} \tag{6.20}$$

The continuity of φ and (6.7) imply that φ is bounded. By assumption A9 w and $\dot{w}$ take values in the bounded intervals $[w_0^-, w_0^+]$ and $[w_1^-, w_1^+]$, respectively. Therefore, due to (6.11) and due to the boundedness of X^0 and Y^0

$$|\varphi(t) + c\dot{w}(t) + g(0, x^0 + y^0 + cw(t) + \Phi(t))| \leq C \quad \text{for all} \quad t \geq 0$$

with some $C \geq 0$ not depending on (f, x^0) and on w. Now (6.20) yields that for all $x \in R^n$ and almost all $t \geq 0$

$$\langle x, f(t, x, w(t), \dot{w}(t)) \rangle \leq K_1|x| - (a_1 + b_1)|x|^2$$

with some $K_1 \geq 0$ not depending on (f, x^0) and on w. The latter condition guarantees that every solution to the differential equation (2.1) is extendable to $[0, \infty)$, which proves assumption A2. Moreover, taking into account that X^0 is bounded, we find that there is a bounded $X \subset R^n$. such that every solution to (2.1), (2.2) takes values in X. The proof is completed.

Assumptions A3 and A4 involve a set X containing the values of the trajectories of all systems $(f, x^0) \in \mathcal{S}$. In this section we assume that X is a bounded set defined in Lemma 6.2.

Lemma 6.3 *Let assumption* A10 *be fulfilled. Then assumptions* A3 *and* A4 *are fulfilled; moreover, for every system* $(f, x^0) \in \mathcal{S}$ *the limit dynamics* $\bar{f}$ *is given by*

$$\bar{f}(x, \eta) = g(x, -x + x^0 + y^0 + c\eta + \bar{\Phi}) \tag{6.21}$$

where $(c, g, x^0, y^0) \in \mathcal{B}$ *is such that* f *satisfies* (6.6).

Proof. Consider assumption A3. Let $(f, x^0) \in \mathcal{S}$ and f be given by (6.6) where $(c, g, x^0, y^0) \in \mathcal{B}$. Let $w \in \mathcal{W}$. The form of f (see (6.6)) and relations (6.7), (2.5) and (6.8) show that for every $x \in R^n$

$$\lim_{t \to \infty} \text{vraimax}_{\tau \geq t} |f(t, x, w(t), \dot{w}(t)) - \bar{f}(x, \bar{w})| = 0$$

where $\bar{f}$ is given by (6.21). Assumption A3 is fulfilled.

Consider assumption A4. Let $(f, x^0) \in \mathcal{S}$, f be given by (6.6) with $(c, g, x^0, y^0) \in \mathcal{B}$, and $\eta \in \bar{W}$. By (6.21) $\bar{f}(x, \eta) = g(x, -x + z)$ where $z = x^0 + y^0 + c\eta + \bar{\Phi}$. Thus, we must show that the equation

$$g(x, -x + z) = 0 \tag{6.22}$$

has the unique solution in R^n. Suppose the differential equation

$$\left(\frac{\partial g(\xi(\lambda), -\xi(\lambda) + \lambda z)}{\partial x} - \frac{\partial g(\xi(\lambda), -\xi(\lambda) + \lambda z)}{\partial y} \right) \frac{d\xi(\lambda)}{d\lambda} + \frac{\partial g(\xi(\lambda), -\xi(\lambda) + \lambda z)}{\partial y} z = 0$$

or, equivalently,

$$G(\xi(\lambda), -\xi(\lambda) + \lambda z) \frac{d\xi(\lambda)}{d\lambda} + \frac{\partial g(\xi(\lambda), -\xi(\lambda) + \lambda z)}{\partial y} z = 0 \tag{6.23}$$

(see (6.13)) has a solution ξ on $[0, 1]$, which satisfies the initial condintion $\xi(0) = 0$. Then, clearly, for all $\lambda \in [0, 1]$

$$g(\xi(\lambda), -\xi(\lambda) + \lambda z) = g(\xi(0), -\xi(0)) = g(0, 0) = 0$$

(see (6.11)); in particular, $\xi(1)$ solves (6.22). Rewriting (6.23) as

$$\frac{d\xi(\lambda)}{d\lambda} = -G^{-1}(\xi(\lambda), -\xi(\lambda)+\lambda z)\frac{\partial g(\xi(\lambda), -\xi(\lambda)+\lambda z)}{\partial y} z$$

and taking into account that the matrix functions $(x, y) \mapsto \partial g(x, y)/\partial y$ and $(x, y) \mapsto G^{-1}(x, y)$ are Lipschitz continuous, we find that the desired solution $\xi(\cdot)$ exists indeed. Thus, the equation (6.22) has a solution. Suppose there are two different solutions to (6.22), x^1 and x^2. Then

$$D = \langle x^1 - x^2, g(x^1, -x^1 + z) - g(x^2, -x^2 + z)\rangle = 0.$$

On the other hand, similarly to (6.18), (6.19) we get

$$\begin{aligned} D &= \langle x^1 - x^2, g(x^1, -x^1 + z) - g(x^2, -x^1 + z)\rangle - \\ &\quad \langle -x^1 - (-x^2), g(x^2, -x^1 + z) - g(x^2, -x^2 + z)\rangle \\ &\leq -(a_1 + b_1)|x^1 - x^2|^2 < 0 \end{aligned}$$

(see (6.15) and (6.16)). The contradiction shows that the solution to (6.22) is unique. Assumption A4 is filfilled. The lemma is proved.

By Lemma 6.3 for every $(f, x^0) \in \mathcal{S}$ the rest point map $\bar{x}(\cdot|f, x^0)$ and its nth coordinate projection $\bar{x}_n(\cdot|f, x^0)$ are defined on $\bar{W}$.

Let us fix a nonempty closed interval $[w^-, w^+] \subset [w_0^-, w_0^+]$.

Lemma 6.4 *Let assumption* A10 *be fulfilled and*

$$\bar{x}_n(w^-|f, x^0) \leq \hat{x}_n, \quad \bar{x}_n(w^+|f, x^0) \geq \hat{x}_n \tag{6.24}$$

for every $(f, x^0) \in \mathcal{S}$. *Then for the interval* $[w^-, w^+]$ *assumption* A5 *is fulfilled.*

Proof. Let $(f, x^0) \in \mathcal{S}$ and f be given by (6.6) where $(c, g, x^0, y^0) \in \mathcal{B}$. In view of (6.24) it is sufficient to show that $\bar{x}_n(\cdot|f, x^0)$ is increasing on $[w^-, w^+]$. For brievity we write $\bar{x}_n(\cdot)$ and $\bar{x}(\cdot)$ instead of $\bar{x}_n(\cdot|f, x^0)$ and $\bar{x}(\cdot|f, x^0)$, respectively. For every $\eta \in [w^-, w^+]$ we have $\bar{f}(\bar{x}(\eta), \eta) = 0$ where $\bar{f}$ is the limit dynamics for (f, x^0) By Lemma 6.3 $\bar{f}$ is given by (6.21). Therefore, for every $\eta \in [w^-, w^+]$

$$g(\bar{x}(\eta), -\bar{x}(\eta) + c\eta + z) = 0$$

where $z = x^0 + y^0 + \bar{\Phi}$. The differentation yields that for all $\eta \in [w^-, w^+]$

$$G(\bar{x}(\eta), -\bar{x}(\eta) + c\eta + z)\frac{d\bar{x}(\eta)}{d\eta} + \frac{\partial g(\bar{x}(\eta), -\bar{x}(\eta) + c\eta + z)}{\partial y} c = 0$$

(see (6.13)); hence,

$$\frac{d\bar{x}_n(\eta)}{d\eta} = \left(-G^{-1}(\bar{x}(\eta), -\bar{x}(\eta) + c\eta + z)\frac{\partial g(\bar{x}(\eta), -\bar{x}(\eta) + c\eta + z)}{\partial y} c\right)_n.$$

By assumption the right hand side is nonnegative (see (6.14)). Therefore, $\bar{x}_n(\cdot)$ is increasing on $[w^-, w^+]$. The lemma is proved.

Now we turn to assumptions A6 – A8 invloving operative control flows and specify the definition of the sets of operative extensions. Let us suppose that every switch of an admissible control is "penalized" by a new constraint on the subsequent admissible controls. Namely, let us fix a positive-valued decreasing function γ on $[0,\infty)$ such that

$$\lim_{t\to\infty} \gamma(t) = 0. \tag{6.25}$$

We suppose that if at some time t_{i+1} the current admissible control w_i is switched to a $w_{i+1} \in \mathcal{W}(t,w_i)$, then necessarily $|\dot{w}_{i+1}(\tau)| \leq \gamma(t_{i+1})$ for almost all $\tau \geq t_{i+1}$. Recall that the target identification strategies (we are going to apply) are operative, i.e., in every switch they generate some $w_{i+1} \in \hat{\mathcal{W}}(t,w_i)$ (see (5.1)). Therefore, in order to satisfy the "switch penalization" constraint automaticaly, we assume that if $v \in \hat{\mathcal{W}}(t,w)$, then $|\dot{v}(\tau)| \leq \gamma(t)$ for almost all $\tau \geq t$. Further details in the definition of the sets of operative extensions are technical and intend to meet assumptions A6 – A8.

An admissible control w will be said to be *stabilized* if there is a $\tau \geq 0$ such that $w(t) = \bar{w}$ for all $t \geq \tau$; τ will be called a *stabilization time* for w. We fix a positive-valued decreasing function γ_0 on $[0,\infty)$ such that $\gamma_0(t) \leq \gamma(t)$ for all $t \geq 0$ and assume the following.

A11. For every $w \in \mathcal{W}$ and every $t > 0$ the set $\hat{\mathcal{W}}(t,w)$ of operative extensions of w beyond t consists of all $v \in \mathcal{W}(t,w)$ such that

(i) v is stabilized,

(ii) $\bar{v} \in [w^-, w^+]$,

(iii) $\dot{v}(\tau)$ is either nonnegative for almost all $\tau \geq 0$ or nonpositive for almost all $\tau \geq 0$,

(iv) $\gamma_0(t) \leq |\dot{v}(\xi)| \leq \gamma(t)$ for almost all $\xi \in [t,\tau]$ where τ is the minimum stabilization time for v.

The set $\hat{\mathcal{W}}_0$ of initial operative extensions consists of all $v \in \mathcal{W}$ satisfying (i) – (iv).

Lemma 6.5 *Let assumptions* A9 *and* A11 *be fulfilled,* $(t_i, w_i)_{m=0}^m$ *be a finite operative control flow, and* τ_k *be the minimum stabilization time for* w_i $(i = 0,1,\ldots,m)$. *Then for all* $i = 0,1,\ldots,m$

$$\tau_i \leq t_i + \frac{\bar{d}}{\gamma_0(t_i)} \tag{6.26}$$

where

$$\bar{d} = \max\{w^+, 0\} - \min\{w^-, 0\}. \tag{6.27}$$

Proof. Indeed, by assumption A11 $\dot{w}_0$ is either positive-valued or negative-valued on $[0,\tau_0]$. Hence, $w_0(t)$ is located between $w_0(0) = 0$ (see assumption A9) and $w_0(\tau_0) = \bar{w}_0 \in [w^-, w^+]$ for all $t \geq t_k$. Consequently,

$$w_0(t) \in [\min\{w^-, 0\}, \max\{w^+, 0\}] \quad \text{for all} \quad t \geq 0. \tag{6.28}$$

Since $|\dot{w}_0(\tau)| \geq \gamma_0(t)$ for almost all $t \in [0,\tau_0]$ (see assumption A11, (iv))

$$|\bar{w}_0| = |\bar{w}_0 - w_0(0)| = |w_0(\tau_0) - w_0(0)| \geq \gamma_0(\tau_0).$$

By (6.28) the left-hand side does not exceed $\bar{d}$ (see (6.27)). Then (6.26) holds for $i = 0$. Similarly, using induction, we state (6.26) for $i = 1,\ldots,m$. The proof is completed.

Our final assumption (needed to prove the validity of assumption A8 – see Lemma 6.9 below) concerns the operative delay map δ_0.

A12. For every stabilized $w \in \mathcal{W}$ and every $t > 0$ it holds that $t + \delta_0(t,w) \geq \tau$ where τ is the minimum stabilization time for w.

Lemma 6.6 *Let assumptions* A9 – A11 *be fulfilled. Then*
1) *assumption* A6 *is fulfilled,*
2) *assumption* A7 *is fulfilled with*

$$\omega(\mu) = \frac{c_2 + K}{a_1 + b_1}\mu.$$

Proof. Assumption A6 is fulfilled obviously (see (ii) in assumption A11). Let us prove statement 2. Let $(t_m, w_m)_{m=0}^{m+1}$ be a finite operative control flow, $(f, x^0) \in \mathcal{S}$ have the form (6.6) for some $(c, g, x^0, y^0) \in \mathcal{B}$ and $(t_i, x_i)_{i=0}^{m+1}$ be a trajectory flow $(t_i, x_i)_{i=0}^{m+1}$ for (f, x^0), corresponsing to $(t_i, w_i)_{i=0}^{m+1}$. We must show that

$$|x_{m+1}(t) - x_m(t)| \le \frac{c_2 + K}{a_1 + b_1}|\bar{w}_{m+1} - \bar{w}_m| \tag{6.29}$$

for all $t \ge t_{m+1}$. For almost all $t \ge t_{m+1}$ we have (see (6.6))

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|x_{m+1}(t) - x_m(t)|^2 &= \langle x_{m+1}(t) - x_m(t), \dot{x}_{m+1}(t) - \dot{x}_m(t)\rangle \\ &= \langle x_{m+1}(t) - x_m(t), c(\dot{w}_{m+1}(t) - \dot{w}_m(t))\rangle + \\ &\quad \langle x_{m+1}(t) - x_m(t), \Delta g_m(t)\rangle \end{aligned} \tag{6.30}$$

where

$$\Delta g_m(t) = g(x_{m+1}(t), -x_{m+1}(t) + z_{m+1}(t)) - g(x_m(t), -x_m(t) + z_m(t))$$

$$\begin{aligned} z_{m+1}(t) &= x^0 + y^0 + cw_{m+1}(t) + \Phi(t), \\ z_m(t) &= x^0 + y^0 + cw_m(t) + \Phi(t). \end{aligned}$$

We set

$$\Delta g_m(t) = \Delta^1 g_m(t) + \Delta^2 g_m(t)$$

with

$$\begin{aligned} \Delta^1 g_m(t) &= g(x_{m+1}(t), -x_{m+1}(t) + z_{m+1}(t)) - g(x_m(t), -x_m(t) + z_{m+1}(t)), \\ \Delta^2 g_m(t) &= g(x_m(t), -x_m(t) + z_{m+1}(t)) - g(x_m(t), -x_m(t) + z_m(t)). \end{aligned}$$

Using (6.15) and (6.16), similarly to (6.18), (6.19) we get

$$\langle x_{m+1}(t) - x_m(t), \Delta^1 g_m(t)\rangle \le -(a_1 + b_1)|x_{m+1}(t) - x_m(t)|^2.$$

In view of (6.12) and (6.9)

$$\begin{aligned} \langle x_{m+1}(t) - x_m(t), \Delta^2 g_m(t)\rangle &\le |x_{m+1}(t) - x_m(t)||\Delta^2 g_m(t)| \\ &\le K|x_{m+1}(t) - x_m(t)||z_{m+1}(t)) - z_m(t))| \\ &\le K|x_{m+1}(t) - x_m(t)|c_1|w_{m+1}(t) - w_m(t)|. \end{aligned}$$

Now (6.30) yields

$$\frac{1}{2}\frac{d}{dt}|x_{m+1}(t) - x_m(t)|^2 \le |x_{m+1}(t) - x_m(t)|h_m(t)$$

where

$$h_m(t) = c_1|\dot{w}_{m+1}(t) - \dot{w}_m(t)| - (a_1+b_1)|x_{m+1}(t) - x_m(t)| + Kc|w_{m+1}(t) - w_m(t)|.$$

Hence, for almost all $t \geq t_{m+1}$ such that $|x_{m+1}(t) - x_m(t)| > 0$

$$\frac{d}{dt}|x_{m+1}(t) - x_m(t)| = \frac{1}{2|x_{m+1}(t) - x_m(t)|}\frac{d}{dt}|x_{m+1}(t) - x_m(t)|^2 \leq h_m(t).$$

Recalling that

$$w_{m+1}(t_{m+1}) = w_m(t_{m+1}) \tag{6.31}$$

and using the form of $h_m(t)$, we find that for all $t \geq t_{m+1}$

$$|x_{m+1}(t) - x_m(t)| \leq I_m^1(t) + I_m^2(t) \tag{6.32}$$

where

$$I_m^1(t) = \int_{t_{m+1}}^t e^{-(a_1+b_1)(t-\tau)}c_1|\dot{w}_{m+1}(\tau) - \dot{w}_m(\tau)|d\tau, \tag{6.33}$$

$$I_m^2(t) = \int_{t_{m+1}}^t e^{-(a_1+b_1)(t-\tau)}K|w_{m+1}(\tau) - w_m(\tau)|d\tau. \tag{6.34}$$

Now we use assumptions A11 and A12 to estimate the integrals (6.33) and (6.34). By the definition of an operative control flow $t_{m+1} \geq t_m + \delta_0(w_m, t_m)$ and by assumption A12 $t_m + \delta_0(w_m, t_m) \geq \xi_m$ where ξ_m is a stabilization time for w_m. Therefore,

$$w_m(\tau) = \bar{w}_m \quad \text{for all} \quad \tau \geq t_{m+1} \tag{6.35}$$

and $|\dot{w}_{m+1}(\tau) - \dot{w}_m(\tau)| = |\dot{w}_{m+1}(\tau)|$ for almost all $\tau \geq t_{m+1}$. By assumption A11 (see (iii)) $\dot{w}_{m+1}(\tau)$ is either nonnegative for almost all $\tau \geq 0$ or nonpositive for almost all $\tau \geq 0$. Then for

$$\zeta(t) = \int_{t_{m+1}}^t |\dot{w}_{m+1}(\tau)|d\tau$$

we have

$$\zeta(t) = |w_{m+1}(t) - w_{m+1}(t_{m+1})| \leq |\bar{w}_{m+1} - w_{m+1}(t_{m+1})| = |\bar{w}_{m+1} - \bar{w}_m|; \tag{6.36}$$

the latter equality follows from (6.35) and (6.31). Intergrating by parts in (6.33), we get

$$\begin{aligned} I_m^1(t) &= c_1 e^{-(a_1+b_1)t}\left[\frac{1}{a_1+b_1}e^{(a_1+b_1)t}\zeta(t) - \frac{1}{a_1+b_1}\int_{t_{m+1}}^t e^{(a_1+b_1)\tau}\zeta(\tau)d\tau\right] \\ &\leq \frac{c_1}{a_1+b_1}\zeta(t) \leq \frac{c_1}{a_1+b_1}|\bar{w}_{m+1} - \bar{w}_m|. \end{aligned} \tag{6.37}$$

Consider the integral $I_m^2(t)$ (6.34). In view of (6.35) and (6.31) for all $\tau \geq t_{m+1}$

$$|w_{m+1}(\tau) - w_m(\tau)| = |w_{m+1}(\tau) - w_m(t_{m+1})| = \zeta(\tau) \leq |\bar{w}_{m+1} - \bar{w}_m|$$

(see (6.36)). Therefore,

$$I_m^2(t) \leq K\int_{t_{m+1}}^t e^{-(a_1+b_1)(t-\tau)}|\bar{w}_{m+1} - \bar{w}_m|d\tau \leq \frac{K}{a_1+b_1}|\bar{w}_{m+1} - \bar{w}_m|. \tag{6.38}$$

Now (6.32) (6.37) and (6.38) yield (6.29). Statement 2 is proved.

To prove the validity of assumption A8 we need two technical statements formulated below as Lemmas 6.7 and 6.8. We set

$$\zeta(p,q)=\left\{\begin{array}{ll}1 & \text{if} \quad p\in[q,q+\bar{d}/\gamma_0(q)],\\ e^{-(a_1+b_1)(p-q-\bar{d}/\gamma_0(q))} & \text{if} \quad p>q+\bar{d}/\gamma_0(q)\end{array}\right. \qquad (p\ge q\ge 0), \tag{6.39}$$

where $\bar{d}$ is defined in (6.27), and

$$\bar{\zeta}(r)=\sup\{\gamma(q)\zeta(p,q):q\ge 0,\ p=q+r\} \quad (r\ge 0). \tag{6.40}$$

Lemma 6.7 *It holds that*

$$\lim_{r\to 0}\bar{\zeta}(r)=0. \tag{6.41}$$

Proof. Take an arbitrary $\varepsilon>0$. For every $r\ge 0$ let $q(r)\ge 0$ be such that

$$\gamma(q(r))\zeta(q(r)+r,q(r))\ge\bar{\zeta}(r)-\frac{\varepsilon}{2}. \tag{6.42}$$

Let

$$t(\varepsilon)=\sup\left\{q\ge 0:\gamma(q)\ge\frac{\varepsilon}{2}\right\}$$

Since γ is decreasing, $\gamma(q)\le\varepsilon/2$ for all $q\ge t(\varepsilon)$. Taking into account $\zeta(p(r),q(r))\le 1$ (see (6.39)) and (6.42), we find that

$$\bar{\zeta}(r)\le\gamma(q(r))\zeta(q(r)+r,q(r))+\frac{\varepsilon}{2}\le\gamma(q(r))+\frac{\varepsilon}{2}\le\varepsilon \quad \text{if} \quad q(r)\ge t(\varepsilon). \tag{6.43}$$

Consider the case where $q(r)<t(\varepsilon)$. Since γ_0 is decreasing,

$$\gamma_0(q(r))\ge\gamma_0(t(\varepsilon)) \quad \text{if} \quad q(r)<t(\varepsilon). \tag{6.44}$$

Let $\rho(\varepsilon)\ge\bar{d}/\gamma_0(t(\varepsilon))$ be such that

$$\gamma(0)e^{-(a_1+b_1)(\rho(\varepsilon)-\bar{d}/\gamma_0(t(\varepsilon)))}\le\frac{\varepsilon}{2}$$

Then for all $r\ge\rho(\varepsilon)$ such that $q(r)<t(\varepsilon)$, in view of (6.44), we have (see also (6.39))

$$\begin{array}{rcl}\gamma(q(r))\zeta(q(r)+r,q(r)) & \le & \gamma(q(r))e^{-(a_1+b_1)(r-\bar{d}/\gamma_0(q(r))}\\ & \le & \gamma(0)e^{-(a_1+b_1)(\rho(\varepsilon)-\bar{d}/\gamma_0(t(\varepsilon))}\le\frac{\varepsilon}{2}\end{array}$$

By (6.42) the left hand side is not smaller than $\bar{\zeta}(r)-\varepsilon/2$. Therefore,

$$\bar{\zeta}(r)\le\varepsilon \quad \text{if} \quad q(r)<t(\varepsilon) \quad \text{and} \quad r\ge\rho(\varepsilon).$$

Combining with (6.43), we get $\bar{\zeta}(r)\le\varepsilon$ for all $r\ge\rho(\varepsilon)$. This proves the limit relation (6.41). The lemma is proved.

We set

$$\lambda(r)=\sup_{q\ge 0}\int_0^r e^{-(a_1+b_1)(r-\rho)}|\Phi(q+\rho)-\bar{\Phi}|d\rho \quad (r\ge 0). \tag{6.45}$$

Lemma 6.8 *It holds that*

$$\lim_{t\to\infty}\lambda(r)=0. \tag{6.46}$$

Proof. Take an arbitrary $\varepsilon > 0$. Let $\rho_0(\varepsilon) \geq 0$ be such that $|\Phi(\rho) - \bar{\Phi}| \leq (a_1 + b_1)\varepsilon/2$ for all $\rho \geq \rho_0(\varepsilon) \geq 0$. Then

$$|\Phi(q+\rho) - \bar{\Phi}| \leq \frac{(a_1+b_1)\varepsilon}{2} \quad \text{for all} \quad q \geq 0 \quad \text{and all} \quad \rho \geq \rho_0(\varepsilon) \geq 0.$$

Denote

$$C = \sup\{|\Phi(\rho) - \bar{\Phi}| : \rho \geq 0\}.$$

For every $\rho \geq \rho_0(\varepsilon)$ and every $q \geq 0$

$$\begin{aligned} h(r,q) &= \int_0^r e^{-(a_1+b_1)(r-\rho)}|\Phi(q+\rho) - \bar{\Phi}|d\rho \\ &\leq C\int_0^{\rho_0(\varepsilon)} e^{-(a_1+b_1)(r-\rho)}d\rho + \frac{(a_1+b_1)\varepsilon}{2}\int_{\rho_0(\varepsilon)}^r e^{-(a_1+b_1)(r-\rho)}d\rho \\ &= \frac{C}{a_1+b_1}\left(e^{-(a_1+b_1)(r-\rho_0(\varepsilon))} - e^{-(a_1+b_1)r}\right) + \\ &\quad \frac{(a_1+b_1)\varepsilon}{2}\frac{1}{a_1+b_1}\left(1 - e^{-(a_1+b_1)(r-\rho_0(\varepsilon))}\right) \\ &\leq \frac{C}{a_1+b_1}e^{-(a_1+b_1)(r-\rho_0(\varepsilon))} + \frac{\varepsilon}{2}. \end{aligned}$$

Let $\rho_1(\varepsilon) \geq \rho_0(\varepsilon)$ be such that

$$\frac{C}{a_1+b_1}e^{-(a_1+b_1)(\rho_1(\varepsilon))-\rho_0(\varepsilon))} \leq \frac{\varepsilon}{2}.$$

Then $h(r,q) \leq \varepsilon$ for all $\rho \geq \rho_1(\varepsilon)$ and all $q \geq 0$. Since $\lambda(r) = \sup_{q\geq 0} h(r,q)$ (see (6.45), $\lambda(r) \leq \varepsilon$ for all $\rho \geq \rho_1(\varepsilon)$. This proves (6.46). The lemma is proved.

We are ready to state the validity of assumption A8.

Lemma 6.9 *Let assumptions* A9 – A12 *be fulfilled. Then assumption* A8 *is fulfilled with*

$$\nu^+(r) = -\nu^-(r) = e^{-(a_1+b_1)(r)}|X| + \left(\frac{c_1}{a_1+b_1} + \frac{Kc_1}{(a_1+b_1)^2}\right)\bar{\zeta}(r) + K\lambda(r) \qquad (6.47)$$

where $|X|$ is the diameter of the set $X \subset R^n$ defined in Lemma 6.2 ($|X| = \sup\{|y_1 - y_2| : y_1, y_2 \in X\}$), *$\bar{\zeta}$ is defined in* (6.40), (6.39) *and λ is defined in* (6.45).

Proof. First of all, Lemmas 6.7 and 6.8 show that $\lim_{r\to\infty}\nu^-(r) = \lim_{r\to\infty}\nu^+(r) = 0$. Let $(t_i, w_i)_{m=0}^m$ be a finite operative control flow, $(f, x^0) \in \mathcal{S}$ have the form (6.6) with some $(c, g, x^0, y^0) \in \mathcal{B}$ and $(t_i, x_i)_{i=0}^m$ be a trajectory flow $(t_i, x_i)_{m=0}^m$ for (f, x^0), corresponsing to $(t_i, w_i)_{i=0}^m$. Denote $\bar{x}_m = \bar{x}(\bar{w}_m|f, x^0)$. We will state that

$$\begin{aligned} |x_m(t) - \bar{x}_m| \;&\leq\; e^{-(a_1+b_1)(t-t_m)}|x_m(t_m) - \bar{x}_m| + \\ &\quad \frac{c_1}{a_1+b_1}\bar{\zeta}(t-t_m) + \frac{Kc_1}{(a_1+b_1)^2}\bar{\zeta}(t-t_m) + K\lambda(t-t_m) \end{aligned} \qquad (6.48)$$

for all $t \geq t_m$. Assume (6.48) holds for all $t \geq t_m$. Note that the right hand side in (6.48) tends to 0 as $t \to \infty$. Therefore, considering the left hand side in (6.48) and recalling that by Lemma 6.2 $x_m(t) \in X$ for all $t \geq t_m$, we find that $\bar{x}_m$ lies in the closure of X. Hence, $|x_m(t) - \bar{x}_m| \leq |X|$ for all $t \geq t_m$. Consequently, in (6.48) in the first term on the right $|x_m(t) - \bar{x}_m|$ can be replaced by $|X|$. Then $|x_m(t) - \bar{x}_m| \leq \nu^+(t - t_m)$ $(t \geq t_m)$ with ν^+

given in (6.47). This completes the proof (see assumption A8). Thus, our goal is to state that (6.48) holds for all $t \geq t_m$.

By Lemma 6.3 and by the definition of $\bar{x}_m$

$$\bar{f}(\bar{x}_m, \bar{w}_m) = g(\bar{x}_m, -\bar{x}_m + \bar{z}_m)$$

where

$$\bar{z}_m = x^0 + y^0 + c\bar{w}_m + \bar{\Phi}$$

For almost all $t \geq t_m$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|x_m(t) - \bar{x}_m|^2 &= \langle x_m(t) - \bar{x}_m(t), \dot{x}_m(t)\rangle \\ &= \langle x_m(t) - \bar{x}_m, c\dot{w}_m(t)\rangle + \langle x_m(t) - \bar{x}_m, \Delta g_m(t)\rangle \end{aligned} \tag{6.49}$$

where

$$\Delta g_m(t) = g(x_m(t), -x_m(t) + z_m(t)) = g(x_m(t), -x_m(t) + z_m(t)) - g(\bar{x}_m, -\bar{x}_m + \bar{z}_m),$$

$$z_m(t) = x^0 + y^0 + cw_m(t) + \Phi(t).$$

We have

$$\Delta g_m(t) = \Delta^1 g_m(t) + \Delta^2 g_m(t)$$

where

$$\begin{aligned} \Delta^1 g_m(t) &= g(x_m(t), -x_m(t) + z_m(t)) - g(\bar{x}_m, -\bar{x}_m + z_m(t)), \\ \Delta^2 g_m(t) &= g(\bar{x}_m, -\bar{x}_m + z_m(t)) - g(\bar{x}_m, -\bar{x}_m + \bar{z}_m). \end{aligned}$$

Using (6.15) and (6.16), similarly to (6.18), (6.19) we get

$$\langle x_m(t) - \bar{x}_m, \Delta^1 g_m(t)\rangle \leq -(a_1 + b_1)|x_m(t) - \bar{x}_m|^2,$$

and using (6.12) and (6.9), we find that

$$\begin{aligned} \langle x_m(t) - \bar{x}_m, \Delta^2 g_m(t)\rangle &\leq K|x_m(t) - \bar{x}_m||z_m(t) - \bar{z}_m| \\ &\leq K|x_m(t) - \bar{x}_m|(c_1|w_m(t) - \bar{w}_m| + |\Phi(t) - \bar{\Phi}|). \end{aligned}$$

Now (6.49) yields

$$\frac{1}{2}\frac{d}{dt}|x_m(t) - x_m|^2 \leq |x_m(t) - \bar{x}_m|h_m(t)$$

where

$$h_m(t) = c_1|\dot{w}_m(t)| - (a_1 + b_1)|x_m(t) - \bar{x}_m| + K(c_1|w_m(t) - \bar{w}_m| + |\Phi(t) - \bar{\Phi}|).$$

Hence, for almost all $t \geq t_m$ such that $|x_m(t) - \bar{w}| > 0$

$$\frac{d}{dt}|x_m(t) - \bar{x}_m| = \frac{1}{2|x_m(t) - \bar{x}_m|}\frac{d}{dt}|x_m(t) - \bar{x}_m|^2 \leq h_m(t).$$

Using the form of $h_m(t)$, we find that for all $t \geq t_m$

$$|x_m(t) - \bar{x}_m| \leq e^{-(a_1+b_1)(t-t_m)}|x_m(t_m) - \bar{x}_m| + I_m^1(t) + I_m^2(t) + I_m^3(t) \tag{6.50}$$

where

$$I_m^1(t) = \int_{t_m}^t e^{-(a_1+b_1)(t-\tau)} c_1 |\dot{w}_m(\tau)| d\tau, \tag{6.51}$$

$$I_m^2(t) = \int_{t_m}^t e^{-(a_1+b_1)(t-\tau)} K c_1 |w_m(\tau) - \bar{w}_m| d\tau, \tag{6.52}$$

$$I_m^3(t) = \int_{t_m}^t e^{-(a_1+b_1)(t-\tau)} K |\Phi(\tau) - \bar{\Phi}|) d\tau. \tag{6.53}$$

Let us estimate the integrals (6.51) – (6.53) from above. Consider the integral $I_m^1(t)$ (6.51). Let τ_m be the minimum stabilization time for w_m. Then $\dot{w}_m(\tau) = 0$ for almost all $\tau \geq \tau_m$ and $|\dot{w}_m(\tau)| \leq \gamma(t_m)$ for almost all $\tau \in [t_m, \tau_m]$ (see Assumtion A11, (iv)). Therefore, for $t \in [t_m, \tau_m]$

$$I_m^1(t) \leq \frac{c_1}{a_1 + b_1} \gamma(t_m)$$

and for $t \geq \tau_m$

$$\begin{aligned} I_m^1(t) &\leq \frac{c_1}{a_1 + b_1} \gamma(t_m) \left(e^{-(a_1+b_1)(t-\tau_m)} - e^{-(a_1+b_1)(t-t_m)} \right) \\ &\leq \frac{c_1}{a_1 + b_1} \gamma(t_m) e^{-(a_1+b_1)(t-\tau_m)}. \end{aligned}$$

By Lemma 6.5)

$$\tau_m \leq t_m + \frac{\bar{d}}{\gamma_0(t_m)}. \tag{6.54}$$

Therefore,

$$I_m^1(t) \leq \frac{c_1}{a_1 + b_1} \gamma(t_m) \zeta(t, t_m) \leq \frac{c_1}{a_1 + b_1} \bar{\zeta}(t - t_m) \tag{6.55}$$

where ζ and $\bar{\zeta}$ are defined in (6.39) and (6.40). Consider the integral $I_m^2(t)$ (6.52). Since

$$|w_m(\tau) - \bar{w}_m| \leq \gamma(t_m)(\tau_m - \tau) \quad \text{for} \quad \tau \in [t_m, \tau_m],$$

$$|w_m(\tau) - \bar{w}_m| = 0 \quad \text{for} \quad \tau \geq \tau_m,$$

for all $t \in [t_m, \tau_m]$

$$\begin{aligned} I_m^2(t) &\leq e^{-(a_1+b_1)t} K c_1 \gamma(t_m) \int_{t_m}^t e^{(a_1+b_1)\tau} (\tau_m - \tau) d\tau \\ &= K c_1 \gamma(t_m) e^{-(a_1+b_1)t} \left[\int_{t_m}^t e^{(a_1+b_1)\tau} \tau_m d\tau - \int_{t_m}^t e^{(a_1+b_1)\tau} \tau d\tau \right] \\ &\leq K c_1 \gamma(t_m) e^{-(a_1+b_1)t} \frac{1}{a_1 + b_1} \left(e^{(a_1+b_1)t} t - e^{(a_1+b_1)t_m} t \right) - \\ &\quad K c_1 \gamma(t_m) e^{-(a_1+b_1)t} \frac{1}{a_1 + b_1} \left(e^{(a_1+b_1)t} t - e^{(a_1+b_1)t_m} t_m \right) + \\ &\quad K c_1 \gamma(t_m) e^{-(a_1+b_1)t} \frac{1}{(a_1 + b_1)^2} \left(e^{(a_1+b_1)t} - e^{(a_1+b_1)t_m} \right) \\ &= K c_1 \gamma(t_m) e^{-(a_1+b_1)t} \frac{1}{a_1 + b_1} \left(-e^{(a_1+b_1)t_m} t + e^{(a_1+b_1)t_m} t_m \right) + \\ &\quad K c_1 \gamma(t_m) e^{-(a_1+b_1)t} \frac{1}{(a_1 + b_1)^2} \left(e^{(a_1+b_1)t} - e^{(a_1+b_1)t_m} \right) \\ &\leq K c_1 \gamma(t_m) e^{-(a_1+b_1)t} \frac{1}{(a_1 + b_1)^2} e^{(a_1+b_1)t} \\ &= \frac{K c_1}{(a_1 + b_1)^2} \gamma(t_m) \end{aligned}$$

and for all $t \geq \tau_m$

$$\begin{aligned} I_m^2(t) &= e^{-(a_1+b_1)(t-\tau_m)} \int_{t_m}^{\tau_m} e^{-(a_1+b_1)(\tau_m-\tau)} Kc_1 |w_m(\tau) - \bar{w}_m| d\tau \\ &= e^{-(a_1+b_1)(t-\tau_m)} I_m^2(\tau_m) \\ &\leq \frac{Kc_1}{(a_1+b_1)^2} \gamma(t_m) e^{-(a_1+b_1)(t-\tau_m)}. \end{aligned}$$

Using (6.54), we find that

$$I_m^2(t) \leq \frac{Kc_1}{(a_1+b_1)^2} \gamma(t_m) \zeta(t, t_m) \leq \frac{Kc_1}{(a_1+b_1)^2} \bar{\zeta}(t - t_m) \tag{6.56}$$

where ζ and $\bar{\zeta}$ are defined in (6.39) and (6.40). Finally, for the integral $I_m^3(t)$ (6.53) we have

$$I_m^3(t) = K \int_0^{t-t_m} e^{-(a_1+b_1)(t-t_m-\rho)} |\Phi(t_m+\rho) - \bar{\Phi}|) d\rho \leq K\lambda(t - t_m) \tag{6.57}$$

where λ is defined in (6.45). Now the estimates (6.50) (6.55), (6.56) and (6.57) yield the desired inequality (6.48). The lemma is proved.

Lemmas 6.2, 6.3, 6.4, 6.6, 6.9 and Theorem 5.1 lead to our final statement on the stabilization of an uncertain system of the form (6.1), (6.2).

Theorem 6.1 *Let assumptions* A9 – A12 *be fulfilled and* (6.24) *hold for every* $(f, x^0) \in \mathcal{S}$. *Then every target identification strategy is a stabilization strategy.*

Proof. By Lemmas 6.2, 6.3, 6.4, 6.6 and 6.9 assumptions A1 – A8 are fulfilled. Then by Theorem 5.1 every target identification strategy is a stabilization strategy.

Remark 6.1 The definition of the class $\mathcal{B}$ of adimissible systems of the form (6.1), (6.2) is general enough. It particular it is not implied that the set $\{\bar{x}_n(\cdot|f, x^0) : (f, x^0) \in \mathcal{S}\}$ of the nth coordinate rest point maps for systems $(f, x^0) \in \mathcal{S}$ (see (6.6) and assumption A10) is uniformly continuous on $[w^-, w^+]$. However, if one restricts $\mathcal{B}$ so that the latter uniform continuity property is satisfied, then statement 2 of Theorem 5.1 is applicable and explicit estimates for the distance of the current value $x(t)_n$ to the target point $\hat{x}_n$ are guaranteed in advance.

7 Example: stabilization of atmospheric carbon

Let us come back to the problem of stabilization of carbon in the atmosphere (see section 1). In section 1 we suggested to view this problem as a problem of stabilization of an uncertain system of the form (1.3), (1.4), or, equivalently, (6.1), (6.2). Here, we treat this problem using the formal setting described in section 6. The single specification is that the dimension n of the state variables x and y is 1.

Let us provide a reasonable interpretation for assumption A11. Consider the basic emission scenario φ. We assume that φ is decreasing, $\varphi(t) > 0$ for all $t \geq 0$ and $\lim_{t\to\infty} \varphi(t) = 0$. Recall that the derivatives $u(t) = \dot{w}(t)$ of admissible controls w represent "correction" emissions modifying the basic emission scenario $\varphi(t)$. The latter is corrected sequentially by $u_0(t) = \dot{w}_0(t)$, $u_1(t) = \dot{w}_1(t), \ldots$ switched on at times $0, t_1, \ldots$ as determined by a chosen operative control strategy. Clearly, it is advisable to use correction emissions $u_i(t)$ that are considerably smaller than the basic emissions $\varphi(t)$. In this context, we assume that every new correction emission scenario $u_{i+1}(t)$ to be held during some period starting from t_{i+1}, should not exceed, in absolute values, a given fraction of the basic-scenario emission planned for the switching time: $|u_{i+1}(\tau)| \le \gamma(t_{i+1})$ $(\tau \ge t_{i+1})$ where

$$\gamma(t) = \alpha\varphi(t) \tag{7.1}$$

and α lies between 0 and 1. This implies that assumption A11 is fulfilled with γ given by (7.1) (to meet assumption A11 entirely, we suppose in addition that $|u_{i+1}(\tau)| \ge \gamma_0(t_{i+1})$ $(\tau \ge t_{i+1})$ where γ_0 is a decreasing function on $[0, \infty)$ such that $\gamma_0(t) \le \gamma(t)$ for all $t \ge 0$).

Let us summarize. We suppose the following: the dimension n of the state variables in systems $(c, g, x^0, y) \in \mathcal{B}$ is 1; the class $\mathcal{W}$ of admissible controls satisfies assumption A9; the class $\mathcal{S}$ of admissible (1-dimensional) systems (f, x^0) of the form (2.1), (2.2) (related to the class $\mathcal{B}$) satisfies assumption A10; the classes $\hat{\mathcal{W}}(t, w)$ of operative extensions of admissible controls w satisfy assumption A11; and the operative delay map δ_0 satisfies assumption A12. Note that the original linear carbon cycle models given by the first two equations in (1.1) fall in $\mathcal{B}$ with a certain range of parameters α_1, α_2.

In order to apply the stabilization Theorem 6.1 one should state the (implicitly defined) inequalities (6.24) for every system $(f, x^0) \in \mathcal{S}$. Here, we provide simple conditions sufficient for (6.24). The conditions require that the interval $[w^-, w^+]$ is sufficiently large and contains 0 in the interior.

Taking into account that $n = 1$, we identify $\hat{x}$ with $\hat{x}_n$, and $\bar{x}(\cdot|f, x^0)$ with $\bar{x}_n(\cdot|f, x^0)$ $((f, x^0) \in \mathcal{S})$. Then (6.24) reads

$$\bar{x}(w^-|f, x^0) \le \hat{x}, \quad \bar{x}(w^+|f, x^0) \ge \hat{x}. \tag{7.2}$$

Lemma 7.1 *Let $n = 1$, assumption* A10 *be fulfilled, $w^- < 0 < w^+$ and the inequalities*

$$g(\hat{x}, -\hat{x} + x^0 + y^0 + \bar{\Phi}) + \frac{b_1}{c} w^- \le 0, \tag{7.3}$$

$$g(\hat{x}, -\hat{x} + x^0 + y^0 + \bar{\Phi}) + \frac{b_1}{c} w^+ \ge 0 \tag{7.4}$$

hold for all $(c, g, x^0, y^0) \in \mathcal{B}$. Then (7.2) *(or, equivalently,* (6.24)*) holds for all $(f, x^0) \in \mathcal{S}$.*

Proof. Take an $(f, x^0) \in \mathcal{S}$ with f given by (6.6) for some $(c, g, x^0, y^0) \in \mathcal{B}$. Using the form of the limit dynamics $\bar{f}$ (see Lemma 6.3), we find that

$$\bar{f}(\hat{x}, w^-) = g(\hat{x}, -\hat{x} + z + cw^-) = g(\hat{x}, -\hat{x} + z) - \Delta g \tag{7.5}$$

where $z = x^0 + y^0 + \bar{\Phi}$ and

$$\Delta g = g(\hat{x}, -\hat{x} + z) - g(\hat{x}, -\hat{x} + z + cw^-)$$

Due to (6.16)

$$b_1(w^-)^2 \le -cw^- \Delta g = c|w^-|\Delta g.$$

Hence,

$$-\Delta g \le -\frac{b_1}{c}|w^-| = \frac{b_1}{c} w^-.$$

Now in view of (7.5) and (7.3) we get

$$\bar{f}(\hat{x}, w^-) = g(\hat{x}, -\hat{x} + z + cw^-) \le g(\hat{x}, -\hat{x} + z) + \frac{b_1}{c} w^- \le 0.$$

Denote $\bar{x} = \bar{x}(w^-|f, x^0)$. Recall that by the definition of the rest point map $\bar{x}(\cdot|f, x^0)$ we have

$$\bar{f}(\bar{x}, w^-) = 0. \tag{7.6}$$

Suppose the first inequality in (7.2) is violated, i.e., $\bar{x} > \hat{x}$. By (6.15) and (6.16) the function $x \mapsto g(x, -x + z + cw^-)$ is strictly decreasing. Therefore,

$$\bar{f}(\bar{x}, w^-) = g(\bar{x}, -\bar{x} + z + cw^-) < g(\hat{x}, -\hat{x} + z + cw^-) = \bar{f}(\bar{x}, w^-) \leq 0$$

which contradicts (7.6). The contradiction proves the first inequality in (7.2). In a similar manner we use the inequalities $w^+ > 0$ and (7.4) to prove the second inequality in (7.2). The lemma is proved.

Theorem 6.1 and Lemma 7.1 yield the following statement.

Theorem 7.1 *Let $n = 1$, assumptions* A9 – A12 *be fulfilled, $w^- < 0 < w^+$ and (7.3) and (7.4) hold for all $(c, g, x^0, y^0) \in \mathcal{B}$. Then every target identification strategy is a stabilization strategy.*

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