International Institute for
Applied Systems Analysis
Schlossplatz 1
A-2361 Laxenburg, Austria

Tel: +43 2236 807 342
Fax: +43 2236 71313
E-mail: publications@iiasa.ac.at
Web: www.iiasa.ac.at

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Optimization of R&D Investment under Technology Spillovers: A Model and a Case Study (Sony Corporation)

Klavdia Izmodenova-Matrossova (izmklav@rmbc.ru)
Alexander Tarasyev (tam@imm.uran.ru)
Chihiro Watanabe (chihiro@me.titech.ac.jp)

Approved by

Arkadii Kryazhimskii (kryazhim@aha.ru)
Project Leader, Dynamic Systems

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Contents

Abstract

This work is devoted to characterizing an optimal R&D investment policy for a growing economy taking into account the phenomenon of technology spillovers. We focus on the issue of a reasonable balance between domestic technologies and assimilated technology spillovers. Both factors require R&D expenditures inducing decrease in production rate in the short run. The efficiency of the utilization of spillover technologies depends on the firm's assimilation capacity. The assimilation capacity is a function of the level of the technology stock and ability to maximize the benefits of a learning exercise and, consequently, of the level of accumulated R&D expenditures. The domestic technology stock supposes high inputs into scientific, technological and production research. In the long run R&D investment leads to increase of sales and production diversity. We also take into account a nonlinear effect of the influence of technology intensity on growth in production rate.

The model is applied on a company level. We identify model's parameters using real data series (in particular, for the Sony Corporation).

Using dynamic optimality principles the corresponding model is analyzed and the optimal level for the R&D intensity is constructed. The uniqueness of the optimal solution is stated and properties of optimal regimes are explored.

Background

The paper deals with classical problems of economic growth and optimal allocation of resources (see [Arrow, 1985], [Arrow, Kurz, 1970], [Grossman, Helpman, 1991]). The analysis refers to the endogenous growth theory [Grossman, Helpman, 1991], in particular economies' utility functions are defined as the discounted integrated consumption indices of the logarithmic type. A generalized endogenous growth model for economies with absorptive capacities was analyzed in [Borisov, Hutschenreiter, Kryazhimskii, 1999] where the asymptotic behavior of knowledge-exchanging economies was investigated.

The type of the growth dynamics under consideration was studied in [Watanabe, 1992]. For the description of interactions between technology spillovers and indigenous technologies we use econometric constructions of this paper.

Also we apply basic elements of the model proposed in [Tarasyev, Watanabe, 2001], [Watanabe, 1992] dealing with the structure of production, technological change and the rate of growth of total factor productivity.

Our analysis of the spillover effect is based on a modification of the nonlinear model elaborated in [A.Tarasyev, C.Watanabe, 2001].

About the Authors

Klavdia Izmodenova-Matrossova
Faculty of Mathematics and Mechanics
Moscow State University
Vorobyevy Gory, 119899 Moscow, Russia

Alexander Tarasyev
Head of Dynamic Optimization Sector
Institute of Mathematics and Mechanics
of the Ural Branch of the Russian Academy of Sciences
ul. S. Kovalevskoi 16, 620219 Ekaterinburg, Russia
and
Dynamic Systems Project
International Institute for Applied Systems Analysis
A-2361 Laxenburg, Austria

Chihiro Watanabe
Department of Industrial Engineering & Management
Head of Laboratory
Tokyo Institute of Technology, Tokyo, Japan
and
Senior Advisor to the Director
International Institute for Applied Systems Analysis
A-2361 Laxenburg, Austria

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1 Introduction

For a technological firm, the problem of optimal R&D investment consists in finding a policy, which maximizes an appropriately chosen utility function. A strong difficulty in choosing optimal R&D policy arises due to the mutually contradicting trends – growth and decline – in interaction between production and technology. On the one hand, investments in R&D generate new sales, on the other hand they redistribute resources between production and the technology stock and, thus, introduce a risky factor into the process of technology innovation.

The assimilation of technologies produced externally (the technology spillover effect) is able to improve the performance of the firm through increasing the technology stock of the firm. It has been widely discussed (see, e.g. [13]) that the firm's assimilation capacity plays a significant role in governing the R&D intensity, technology stock, and production. This effect provides a serious motivation for analyzing the development of the assimilation capacity in the context of dynamic interactions between the technology stock, sales, and R&D intensity.

Our research adjoins classical studies on economic growth and optimal allocation of resources [1], [2], [4]-[8], [10]. Unlike the model described in [6], which treats the dynamics of the knowledge stock as a function of the price for the technology output, we deal with a dynamics which describes the growth of sales due to R&D investments.

The model describes the behavior of a firm in an economy sector. The firm's outputs are production, y, and the domestic technology stock, T_d, which is included in the potential spillover pool for other companies. Production is measured in terms of sales. The direct interaction between the technology stock and firm's production is described in terms of R&D expenditures. A control parameter is the R&D intensity, i.e. the share of revenues, which is spent into R&D (R&D expenditures/production).

The firm has the assimilation capacity. The firm's accumulated technology stock, T, consists of indigenous technologies, T_d, and assimilated technologies, zT_s, generated by other firms. It is assumed that the results of research activities of neighboring firms are assimilated at very low prices; the model supposes that the prices are zero. The total technology stock of all firms forms the "potential spillover pool" T_s.

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2 System Dynamics

The suggested model of a firm which describes dynamic interactions between production, technology stock and R&D investments uses the following variables:

t – time;

$y = y(t)$ – production;

$T_d = T_d(t)$ – stock of domestic (indigenous) technologies;

$T = T(t)$ – total technology stock of the firm;

$T_s = T_s(t)$ – spillover pool (exogenous technologies);

$r_d = r_d(t)$ – R&D intensity;

$z = z(t)$ – assimilation capacity;

$\dot{y}(t)/y(t)$ – production rate;

$\dot{T}_d(t) = r_d(t)y(t)$ – marginal change of domestic technologies $T_d(t)$ caused by R&D expenditures;

$\dot{T}_s(t)$ – marginal change of spillover pool caused by total R&D expenditures of other firms at the technology market;

T/y – technology intensity;

y/T – productivity of technology;

y/T_d – productivity of domestic technology;

$\psi_1 = \psi_1(t)$ – the "shadow price" of production $y(t)$;

$\psi_2 = \psi_2(t)$ – the "shadow price" of domestic technologies $T_d(t)$;

$\psi_1 y$ – the "cost" of production;

$\psi_2 T_d$ – the "cost" of domestic technologies;

$n = n(t)$ – measure of invented products.

To define the dynamics of production, we use the equation obtained via differentiating a Cobb-Douglas type production function (see [10], [11]):

$$\frac{\dot{y}(t)}{y(t)} = f_1(t) + f_2 \cdot \left(\frac{T(t)}{y(t)}\right)^{\gamma} - g_d(t)r_d(t), \tag{1}$$

where function $f_1(t)$ represents a non-R&D contribution and $g_d(t)$ is the discounted marginal productivity of domestic technology. The negative sign in front of the net contribution of technological investments $(-g_d(t)r_d(t))$ shows that in the short-run spending into the domestic R&D prevails upon the rate of returns due to domestic technologies and, therefore, provides a risky factor of technological investments. Furthermore, in (1) parameter γ is an elasticity of technology to production $(0 \leq \gamma \leq 1)$, and parameter f_2 is a scale coefficient $(f_2 > 0)$. We assume that the following inequality is valid:

$$g_d(t) = p_d(t) - q_d(t) > 0.$$

Here $p_d(t)$ describes the decrease in production due to domestic R&D expenditures, and marginal productivity of domestic technologies, $q_d(t)$, shows the increase of the R&D knowledge stock. A procedure of measuring the discounted marginal productivity of technology is described in [15].

The technology stock, $T(t)$, is expressed through the domestic technologies $T_d(t)$, and assimilated spillover technologies, $z(t)T_s(t)$, as follows:

$$T(t) = T_d(t) + z(t)T_s(t). \tag{2}$$

This structure of the technology stock $T(t)$ is justified by the empirical analysis which shows that this assimilation capacity approach is statistically extremely significant (see [12]).

The next equation in the model's dynamics describes the evolution of the domestic technology stock:

$$\dot{T}_d(t) = u(t) = r_d(t)y(t). \tag{3}$$

Here $u = u(t) = r_d(t)y(t))$ (or, equivalently, R&D intensity $r_d = r_d(t))$ is a control parameter which is responsible for the current change in technology stock $T_d(t)$.

In line with the previous approaches [13], the assimilation capacity $z(t)$ is modeled as

$$z(t) = \frac{1}{1 + \frac{\dot{T}_s(t)}{T_s(t)} / \frac{\dot{T}_d(t)}{T_d(t)}} \frac{T_d(t)}{T_s(t)}. \tag{4}$$

Equation (4) suggests that the assimilation capacity $z(t)$ is proportional to the ratio of the indigenous technologies and the potential spillovers pool.

Introducing notations

$$\xi = \dot{T}_d/T_d, \qquad \omega = \dot{T}_s/T_s \tag{5}$$

for the technology rates and linearizing formula (4) with respect to the domestic technology rate ξ we get the following approximate expression for the assimilation capacity :

$$z = \frac{\xi}{\omega} \frac{T_d}{T_s} = \frac{\dot{T}_d}{\dot{T}_s}. \tag{6}$$

Taking into account formula (6), we represent the technology stock T through a linear approximation :

$$T = T_d + \frac{\dot{T}_d}{\dot{T}_s} T_s = T_d + \frac{u}{\omega} \tag{7}$$

Combining formulas (1)-(7), we obtain a system of three differential equations describing the distribution of resources between the productivity rate $\dot{y}(t)/y(t)$ and investment $r_d(t)$ into the domestic technology $T_d(t)$.

Production $y(t)$ and the domestic technology stock $T_d(t)$ are the phase parameters in the model. The R&D change $u = r_d y$ (or, equivalently, R&D intensity r_d) is the control parameter. From the economic point of view it is clear that $r_d = r_d(t)$ is bounded from above:

$$r_d(t) \le r_d^u < +\infty. \tag{8}$$

We also assume that functions $f_1(t)$ and $g_d(t)$ are continuous and bounded ($f_1(t) > 0$, $0 \le g_d(t) \le 1$).

3 Utility Function

Now we formulate the firm's goal and define its long-run profit arising due to R&D investments. We consider the firm's utility function (see [1]) :

$$W_{t_0} = \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \ln D(t) dt. \tag{9}$$

Here $D(t)$ is a consumption index representing the utility of products (technologies) at time t, ρ is the discount rate, t is the running time, and t_0 is the fixed initial time.

If one assumes a constant elasticity of substitution between every two products, the form of the consumption index $D(t)$ is specified as follows (here we refer to the CES type of demand function [5]),

$$D(t) = \left(\sum_{j=1}^{n} x_j^{\alpha}(t) \right)^{1/\alpha}, \qquad n = n(t). \tag{10}$$

Here j is the current index of innovative goods, $x_j(t)$ is consumption in brand with index j, $n(t)$ is the number of available varieties at time t. The elasticity of substitution between any two products, e, is defined through a parameter α as

$$e = \frac{1}{1-\alpha} > 1.$$

The utility function is transformed into an expression depending on production, the technology stock and R&D investment. Similarly to [5] we assume that quantities $x_j(t)$ are equal for each index j, thus,

$$x_j(t) = \frac{y(t)}{n(t)}. \tag{11}$$

The quantity of innovative products $n(t)$ depends on the accumulated R&D investment, $T(t)$, and the rate of change in technology, $u(t)$, through the relations

$$n = n(t) = bT^{\beta_1}(t)u^{\beta_2}(t), \qquad u(t) = r_d(t)y(t). \tag{12}$$

Here β_1 and β_2, respectively, are elasticities of the technology stock $T(t)$ and technology change $u(t)$ with respect to the index of innovative products $n = n(t)$. Formulas (11), (12) imply that innovation depends upon the forefront R&D activities demonstrated by the domestic technology change $u(t)$ and upon the accumulation of the past R&D activities and technology spillovers, which are represented by the technology stock $T(t)$.

Combining equations (10)-(12), one finds that

$$D(t) = \left[\sum_{j=1}^{n} \left(\frac{y(t)}{n(t)} \right)^{\alpha} \right]^{1/\alpha} = \frac{y(t)}{n(t)} n^{1/\alpha}(t) = y(t)n^{\frac{1-\alpha}{\alpha}}(t). \tag{13}$$

The substitution of (13) into (9) leads to the following formulas for the utility function:

$$W_{t_0} = \int_{t_0}^{\infty} e^{-\rho(t-t_0)}[\ln y(t) + a_1 \ln T(t) + a_2 \ln u(t)) + A \ln b]dt =$$

$$= \int_{t_0}^{\infty} e^{-\rho(t-t_0)}[\ln y(t) + a_1 \ln(T_d(t) + \frac{u(t)}{\omega}) +$$

$$+ a_2 \ln u(t)]dt + \int_{t_0}^{\infty} e^{-\rho(t-t_0)} A \ln b) dt$$

where

$$a_1 = A\beta_1, \qquad a_2 = A\beta_2, \qquad A = \frac{1-\alpha}{\alpha}.$$

The second integral does not depend on $y(t)$, $T_d(t)$, $T_s(t)$, and $u(t)$. Therefore, it does not influence on the choice of optimal investment. Hence, we can consider the equivalent utility function

$$U_{t_0} = \int_{t_0}^{\infty} e^{-\rho(t-t_0)}[\ln y(t) + +a_1 \ln(T_d(t) + \frac{u(t)}{\omega}) + a_2 \ln u(t)]dt. \quad (14)$$

The structure of the utility function U_{t_0} (14) shows that the investors are interested in growth of production $y(t)$ as well as in growth of the domestic technology stock $T_d(t)$, its current change $u(t)$, and assimilated technologies $z(t)T_s(t) = u(t)/\omega$.

The logarithmic terms in U_{t_0} (14) imply that production $y(t)$, the technology stock $T(t)$, the marginal change of the domestic technology $u(t)$, and the R&D intensity $r_d(t) = u(t)/y(t)$ are strictly positive; moreover, we assume that these values are strictly separated from zero:

$$0 < y^l \le y(t), \quad 0 < T^l \le T(t), \quad 0 < r_d^l \le r_d(t), \quad 0 < r_d^l y(t) \le u(t). \quad (15)$$

Combining the upper and lower bounds (8) and (15), we get upper and lower bounds for the R&D intensity $r_d(t)$:

$$0 < r_d^l \le r_d(t) \le r_d^u < +\infty. \quad (16)$$

4 Optimization Problem

We consider the following problem of optimal control, Problem (P): find the R&D intensity $r_d^*(t)$, which maximizes the utility function (14)

$$U_{t_0} = \int_{t_0}^{\infty} e^{-\rho(t-t_0)}[\ln y(t) + a_1 \ln(T_d(t) + \frac{r_d(t)y(t)}{\omega}) +$$

$$+a_2 \ln r_d(t)y(t)]dt = \int_{t_0}^{\infty} e^{-\rho(t-t_0)}[(1+a_2)\ln y(t) +$$

$$+a_1 \ln(T_d(t) + \frac{r_d(t)y(t)}{\omega}) + a_2 \ln r_d(t)]dt \quad (17)$$

provided the dynamics is described by

$$\frac{\dot{y}(t)}{y(t)} = f_1 + f_2 \left(\frac{T(t)}{y(t)}\right)^{\gamma} - g_d(t)r_d(t), \qquad T(t) = T_d(t) + \frac{r_d(t)y(t)}{\omega},$$

$$\dot{T}_d(t) = u(t) = r_d(t)y(t),$$

with constraints

$$0 < r_d^l \le r_d(t) \le r_d^u < +\infty,$$

and initial conditions

$$y(t_0) = y^0, \qquad T_d(t_0) = T_d^0, \qquad T_s(t_0) = T_s^0.$$

The main difference of Problem (P) from classical problems of optimal control ([9]) consists in the unboundedness of its time interval. Generally the application of the Pontryagin maximum principle in the case of infinite time horizon is less effective when in the case of finite time horizon (see [3]). In the numerical approximation solution of Problem (P) one can restrict the time horizon to a large but finite interval. Therefore, we formulate the problem for a finite time horizon $[t_0, \theta]$.

For the utility function (14) we have

$$U_t = U_{t_0}^{\theta} + U_{\theta},$$

where

$$U_{t_0}^{\theta} = \int_{t_0}^{\theta} e^{-\rho(t-t_0)}[(1+a_2)\ln y(t) + \\ + a_1 \ln T(t) + a_2 \ln r_d(t)]dt, \tag{18}$$

and U_θ is the approximation error.

We will estimate U_θ from above by a small parameter $\varepsilon = \varepsilon(\theta)$. Let us start with estimating the integrand. We make a natural assumption that $T(t) < y(t)$. Due to (1) we have

$$\dot{y}(t) < My(t),$$

where

$$M = f_1 + f_2 - \overline{g_d} r_d^l, \qquad \overline{g_d} = sup g_d(t).$$

Thus,

$$y < y(\theta)e^{M(t-\theta)} = y(t_0)e^{M(\theta-t_0)}e^{M(t-\theta)}.$$

Substituting this estimate into the integral U_θ, we obtain the following relation

$$U_\theta < \int_{\theta}^{\infty} e^{-\rho(t-t_0)}[(1+a_1+a_2)(\ln y(\theta) + M(t-\theta)) + a_2 \ln r_d^u]dt =$$

$$= [(1+a_1+a_2)(\ln y(\theta) - M\theta) + a_2 \ln r_d^u] \int_{\theta}^{\infty} e^{-\rho(t-t_0)}dt +$$

$$+ M(1+a_1+a_2)\int_{\theta}^{\infty} te^{-\rho(t-t_0)}dt =$$

$$= \frac{1}{\rho}((1+a_1+a_2)\ln y(\theta) + a_2 \ln r_d^u + \frac{M}{\rho}(1+a_1+a_2))e^{-\rho(\theta-t_0)} =$$

$$= \frac{1}{\rho}((1+a_1+a_2)(\ln y(t_0) + M(\theta - t_0)) + \\ + a_2 \ln r_d^u + \frac{M}{\rho}(1+a_1+a_2))e^{-\rho(\theta-t_0)}. \tag{19}$$

The last expression tends to zero when θ tends to infinity, and can be bounded from above by the accuracy estimate $\varepsilon = \varepsilon(\theta)$. Thus, we get the uniform convergence of the indefinite integral (14).

Let us denote by (P_1) the optimal control problem with the utility function (18) on the finite horizon instead of utility (14) on the infinite horizon. Then Problem (P_1) is a classical optimal control problem with the free right end point on the fixed time interval $[t_0, \theta]$, and the Pontryagin maximum principle [9] is a necessary optimality condition in this problem.

The Hamiltonian for Problem (P_1) has the form

$$H(t,y,T_d,r_d,\psi_1,\psi_2)=e^{-\rho(t-t_0)}((1+a_2)\ln y+$$
$$+a_1\ln(T_d+\frac{r_d y}{\omega})+a_2\ln r_d)+$$
$$+\psi_1(f_1y+f_2(T_d+\frac{r_d y}{\omega})^{\gamma}y^{(1-\gamma)}-g_d r_d y)+\psi_2 r_d y. \qquad (20)$$

Taking into account formula (3) for technology change $\dot{T}_d(t)=u(t)$, we get the following presentation of the Hamiltonian H through the control parameter $u=u(t)$

$$H(t,y,T_d,u,\psi_1,\psi_2)=e^{-\rho(t-t_0)}(\ln y+$$
$$+a_1\ln(T_d+\frac{u}{\omega})+a_2\ln u)+$$
$$+\psi_1(f_1y+f_2(T_d+\frac{u}{\omega})^{\gamma}y^{(1-\gamma)}-g_d u)+\psi_2 u, \qquad (21)$$

The maximum function for the Hamiltonian (21) has the following form:

$$\hat{H}(t,y,T_d,\psi_1,\psi_2)=\sup_{r_d\in[r_d^l,r_d^u]}H(t,y,T_d,r_d,\psi_1,\psi_2). \qquad (22)$$

If $(y(t),T_d(t),r_d(t))$ is a control process, $(\psi_1(t),\psi_2(t))$ is the pair of adjoint variables, then at any time t the Hamiltonian H (20) describes the current flow of utility from all sources.

For the adjoint variables ψ_1 and ψ_2 interpretable as "shadow prices" of production y and domestic technologies T_d, respectively, we have the following dynamics

$$\dot{\psi}_1(t)=-\frac{\partial H}{\partial y}=$$
$$=-e^{-\rho(t-t_0)}(\frac{(1+a_2)}{y(t)}+\frac{a_1}{(T_d(t)+r_d(t)y(t)/\omega)}\frac{r_d}{\omega})-$$
$$-\psi_1(t)(f_1(t)+f_2(1-\gamma)\left(\frac{T_d(t)+r_d(t)y(t)/\omega}{y(t)}\right)^{\gamma}+$$
$$+f_2\gamma\left(\frac{T_d(t)+r_d(t)y(t)/\omega}{y(t)}\right)^{\gamma-1}\frac{r_d}{\omega}-g_d(t)r_d(t))-\psi_2(t)r_d(t),$$
$$\dot{\psi}_2(t)=-\frac{\partial H}{\partial T_d}=$$
$$=-e^{-\rho(t-t_0)}\frac{a_1}{(T_d(t)+r_d(t)y(t)/\omega)}-$$
$$-\psi_1(t)f_2\gamma\left(\frac{T_d(t)+r_d(t)y(t)/\omega}{y(t)}\right)^{\gamma-1}. \qquad (23)$$

Prices ψ_1 and ψ_2 measure the marginal contribution of y and T_d to the utility function (18).

For the finite time horizon $[t_0,\theta]$ $(t_0\le\theta<+\infty)$ the following transversality conditions are valid:

$$\psi_i(\theta)=0, \qquad i=1,2. \qquad (24)$$

It is easy to see that due to (23)-(24) the following result holds:

Lemma 1. *The solution $(\psi_1(t),\psi_2(t))$ of the system (23) subject to the dynamics (1)-(5) and restrictions (16) satisfies the inequalities:*

$$\psi_i(t)>0, \qquad t\in[t_0,\theta], \qquad i=1,2. \qquad (25)$$

Thus the Pontryagin maximum principle [9] for Problem (P_1) can be formulated as follows:

Theorem 1. *Let $(y^*(t), T_d^*(t), r_d^*(t))$ be an optimal control process in Problem (P_1). Then there exists a pair $(\psi_1(t), \psi_2(t))$ of adjoint variables such that $(\psi_1(t), \psi_2(t))$ is a solution of adjoint system (23), taken along the optimal control process $(y^*(t), T_d^*(t), r_d^*(t))$;*

the maximum condition holds:

$$H(t, y^*(t), T_d^*(t), r_d^*(t), \psi_1(t), \psi_2(t)) \overset{a.e.}{=} \\ \hat{H}(t, y^*(t), T_d^*(t), \psi_1(t), \psi_2(t)); \tag{26}$$

the transversality condition (24) takes place;

and, moreover, both functions $\psi_1(t)$, $\psi_2(t)$ are strictly positive (25).

Let us assume that the maximality condition of the Pontryagin maximum principle holds for $t \in [t_0, \theta]$:

$$H(t, y^*(t), T_d^*(t), r_d^*(t), \psi_1(t), \psi_2(t)) = \\ = \hat{H}(t, y^*(t), T_d^*(t), \psi_1(t), \psi_2(t)). \tag{27}$$

That is

$$e^{-\rho(t-t_0)}((1+a_2)\ln y^*(t) + a_1 \ln(T_d^*(t) + \frac{r_d^*(t)y^*(t)}{\omega}) + \\ +a_2 \ln r_d^*(t)) + \psi_1(t)(f_1 y^*(t) + f_2(T_d^*(t) + \frac{r_d^*(t)y^*(t)}{\omega})^\gamma y^*(t)^{(1-\gamma)} - \\ -g_d(t) r_d^*(t) y^*(t)) + \psi_2(t) r_d^*(t) y^*(t) = \\ = \hat{H}(t, y^*(t), T_d^*(t), \psi_1(t), \psi_2(t)), \tag{28}$$

where the admissible triple $(y^*(t), T_d^*(t), r_d^*(t))$ satisfies the conditions of the Pontryagin maximum principle (see [9]), and $T^*(t) = T_d^*(t) + r_d^*(t)y^*(t)/\omega$. Thus, if the maximized Hamiltonian $\hat{H}(t,\ y^*(t),\ T_d^*(t),\ \psi_1(t),\ \psi_2(t))$ is differentiable in y, T_d at $y^*(t)$, $T_d^*(t)$ then the adjoint equation (23) can be rewritten in the form

$$\dot{\psi}(t) = -\frac{\partial \hat{H}(y^*(t), T_d^*(t), \psi(t))}{\partial(y, T_d)}, \tag{29}$$

where $\psi(t) = (\psi_1(t), \psi_2(t))$.

Proposition 1. *The maximized Hamiltonian $\hat{H}$ is a continuously differentiable and strictly concave function in y and T_d for any $t \in [t_0, \theta]$.*

Proof. 1. First, let us show that the Hamiltonian H (21) is a twice continuously differentiable and strictly concave function of variables y, T_d, and u for any $t \in [t_0, \theta]$, $\psi_1(t) > 0$, $\psi_2(t) > 0$.

Twice differentiability of Hamiltonian H (21) in variables y, T_d, and u follows from it's structure: the logarithmic and power functions are twice continuously differentiable.

To prove the strict concavity of the Hamiltonian $\hat{H}$ in y and T_d, let us show first that the matrix of second derivatives

$$J = \frac{\partial^2 H(y, T_d, u, \psi_1, \psi_2)}{\partial(y, T_d, u)^2} \tag{30}$$

is negative definite, i.e.

$$\Delta J \Delta^T < 0, \qquad \text{for all} \qquad \Delta = (\Delta y, \Delta T_d, \Delta u) \neq 0.$$

To show this we use the Sylvester's criterion. Let us calculate the first derivatives of the Hamiltonian H (21) with respect to variables y, T_d, and u

$$\frac{\partial H}{\partial y} = e^{-\rho(t-t_0)}\frac{1}{y} + \psi_1(f_1 + f_2(1-\gamma)(T_d + \frac{u}{\omega})^{\gamma} y^{-\gamma}), \tag{31}$$

$$\frac{\partial H}{\partial T_d} = e^{-\rho(t-t_0)}\frac{a_1}{(T_d + u/\omega)} + \psi_1 f_2 \gamma (T_d + \frac{u}{\omega})^{(\gamma-1)} y^{(1-\gamma)}, \tag{32}$$

$$\frac{\partial H}{\partial u} = e^{-\rho(t-t_0)}\left(\frac{a_2}{u} + \frac{a_1}{(T_d + u/\omega)}\frac{1}{\omega}\right) + \\ + \psi_1 f_2 \gamma (T_d + \frac{u}{\omega})^{(\gamma-1)} \frac{1}{\omega} y^{(1-\gamma)} - \psi_1 g_d + \psi_2. \tag{33}$$

We calculate now second derivatives of the Hamiltonian H (21)

$$\frac{\partial^2 H}{\partial y^2} = -e^{-\rho(t-t_0)}\frac{1}{y^2} - \psi_1 f_2 \gamma (1-\gamma)(T_d + \frac{u}{\omega})^{\gamma} y^{-(1+\gamma)} < 0, \tag{34}$$

$$\frac{\partial^2 H}{\partial T_d^2} = -e^{-\rho(t-t_0)}\frac{a_1}{(T_d + u/\omega)^2} - \\ -\psi_1 f_2 \gamma (1-\gamma)(T_d + \frac{u}{\omega})^{(\gamma-2)} y^{(1-\gamma)} < 0, \tag{35}$$

$$\frac{\partial^2 H}{\partial u^2} = -e^{-\rho(t-t_0)}(\frac{a_2}{u^2} + \frac{a_1}{(T_d + u/\omega)^2}\frac{1}{\omega^2}) - \\ -\psi_1 f_2 \gamma (1-\gamma)(T_d + \frac{u}{\omega})^{(\gamma-2)} \frac{1}{\omega^2} y^{(1-\gamma)} < 0, \tag{36}$$

$$\frac{\partial^2 H}{\partial y \partial T_d} = \psi_1 f_2 \gamma (1-\gamma)(T_d + \frac{u}{\omega})^{(\gamma-1)} y^{-\gamma}, \tag{37}$$

$$\frac{\partial^2 H}{\partial y \partial u} = \psi_1 f_2 \gamma (1-\gamma)(T_d + \frac{u}{\omega})^{(\gamma-1)} \frac{1}{\omega} y^{-\gamma}, \tag{38}$$

$$\frac{\partial^2 H}{\partial T_d \partial u} = -e^{-\rho(t-t_0)}\frac{a_1}{(T_d + u/\omega)^2}\frac{1}{\omega} - \\ -\psi_1 f_2 \gamma (1-\gamma)(T_d + \frac{u}{\omega})^{(\gamma-2)} \frac{1}{\omega} y^{(1-\gamma)}. \tag{39}$$

According to the Sylvester's criterion in the case of 3×3 symmetric matrix J (30) to justify its negative definiteness and, hence, to check the strict concavity of the Hamiltonian H (21) in variables y, T_d, and u we should verify the following inequalities:

$$\Delta_1^{(1)} = \frac{\partial^2 H}{\partial y^2} < 0, \quad \Delta_1^{(2)} = \frac{\partial^2 H}{\partial T_d^2} < 0, \quad \Delta_1^{(3)} = \frac{\partial^2 H}{\partial u^2} < 0, \tag{40}$$

$$\Delta_2 = \frac{\partial^2 H}{\partial y^2}\frac{\partial^2 H}{\partial T_d^2} - \left(\frac{\partial^2 H}{\partial y \partial T_d}\right)^2 > 0, \tag{41}$$

$$\Delta_3 = \frac{\partial^2 H}{\partial u^2}\left(\frac{\partial^2 H}{\partial y^2}\frac{\partial^2 H}{\partial T_d^2} - \left(\frac{\partial^2 H}{\partial y \partial T_d}\right)^2\right) -$$
$$-\frac{\partial^2 H}{\partial y^2}\left(\frac{\partial^2 H}{\partial T_d \partial u}\right)^2 - \frac{\partial^2 H}{\partial T_d^2}\left(\frac{\partial^2 H}{\partial y \partial u}\right)^2 +$$
$$+2\frac{\partial^2 H}{\partial y \partial T_d}\frac{\partial^2 H}{\partial y \partial u}\frac{\partial^2 H}{\partial T_d \partial u} < 0. \tag{42}$$

The determinants $\Delta_1^{(i)}$ (40) of the first order are negative

$$\Delta_1^{(i)} < 0, \qquad i = 1, 2, 3 \tag{43}$$

due to inequalities (34)-(36).

The determinant Δ_2 (41) due to reduction of similar terms with positive and negative signs in multiplication of formulas (34)-(35), (37) is presented by relation

$$\Delta_2 = e^{-2\rho(t-t_0)}\frac{1}{y^2}\frac{a_1}{(T_d + u/\omega)^2} +$$
$$+e^{-\rho(t-t_0)}(1 + a_1)\psi_1 f_2 \gamma(1-\gamma)(T_d + \frac{u}{\omega})^{(\gamma-2)} y^{-(1+\gamma)} > 0, \tag{44}$$

and is evidently positive.

In calculation of the determinant Δ_3 (42) all positive terms in multiplication of formulas (34)-(39) are compensated by negative terms and the final relation has the negative sign

$$\Delta_3 = -e^{-\rho(t-t_0)}\frac{a_2}{u^2}[e^{-2\rho(t-t_0)}\frac{1}{y^2}\frac{a_1}{(T_d + u/\omega)^2} +$$
$$+e^{-\rho(t-t_0)}(1 + a_1)\psi_1 f_2 \gamma(1-\gamma)(T_d + \frac{u}{\omega})^{(\gamma-2)} y^{-(1+\gamma)}] < 0. \tag{45}$$

We completely prove that the Hamiltonian H (21) is a strictly concave function in variables y, T_d, and u.

2. Let us prove now the following result for the concave Hamiltonian H (21).

Lemma 2. *Let function $H = H(y, T_d, u) : (0, +\infty) \times (0, +\infty)\ (0, +\infty) \to R$ be twice continuously differentiable and strictly concave in y, T_d, and u. Assume that $u^0 = u^0(y, T_d)$ delivers maximum to $H(y, T_d, u)$ in u. Then the composite function*

$$F(y, T_d) = H(y, T_d, u^0(y, T_d)) \tag{46}$$

is strictly concave in y, T_d.

Proof. Assuming the existence of the maximum point $u^0 = u^0(y, T_d)$ for function $u \to H(y, T_d, u)$ we have the uniqueness of this maximum due to strict concavity of function $H(y, T_d, u)$ in u. This maximum point $u^0 = u^0(y, T_d)$ is a solution of the necessary maximum conditions

$$\frac{\partial H}{\partial u}(y, T_d, u) = 0. \tag{47}$$

Since $\partial^2 H(y, T_d, u)/\partial u^2 < 0$ (36), then $\partial H(y, T_d, u)/\partial u$ is a strictly monotonic function and according to the implicit function theorem there exists the unique solution $u^0 = u^0(y, T_d)$ of equation (47). This solution is differentiable and its derivatives are defined by relation

$$\frac{\partial u^0}{\partial y} = -\frac{\partial^2 H}{\partial y \partial u}\Big/\frac{\partial^2 H}{\partial u^2}, \qquad \frac{\partial u^0}{\partial T_d} = -\frac{\partial^2 H}{\partial T_d \partial u}\Big/\frac{\partial^2 H}{\partial u^2}. \tag{48}$$

Let us consider the composite function $F(y, T_d) = H(y, T_d, u^0(y, T_d))$ and show that it is strictly concave. For this purpose we use the Sylvester's criterion. Let us calculate the matrix of second derivatives of function $F(y, T_d)$ and prove that it is negative definite. The first derivatives of function $F(y, T_d)$ are calculated according to the rule of differentiation of composite functions and taking into account the necessary maximum conditions (47)

$$\frac{\partial F}{\partial y} = \frac{\partial H}{\partial y} + \frac{\partial H}{\partial u}\frac{\partial u^0}{\partial y} = \frac{\partial H}{\partial y}(y, T_d, u^0(y, T_d)), \tag{49}$$

$$\frac{\partial F}{\partial T_d} = \frac{\partial H}{\partial T_d} + \frac{\partial H}{\partial u}\frac{\partial u^0}{\partial T_d} = \frac{\partial H}{\partial T_d}(y, T_d, u^0(y, T_d)). \tag{50}$$

Second derivatives of function $F(y, T_d)$ are calculated as follows:

$$\frac{\partial^2 F}{\partial y^2} = \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial y \partial u}\frac{\partial u^0}{\partial y} = \frac{\partial^2 H}{\partial y^2} - \left(\frac{\partial^2 H}{\partial y \partial u}\right)^2 / \frac{\partial^2 H}{\partial u^2} =$$
$$= \left(\frac{\partial^2 H}{\partial y^2}\frac{\partial^2 H}{\partial u^2} - \left(\frac{\partial^2 H}{\partial y \partial u}\right)^2\right) / \frac{\partial^2 H}{\partial u^2} < 0, \tag{51}$$

the sign in (51) is negative since due to strict concavity of function $H(y, T_d, u)$ the numerator is positive and the denominator is negative;

$$\frac{\partial^2 F}{\partial T_d^2} = \frac{\partial^2 H}{\partial T_d^2} + \frac{\partial^2 H}{\partial T_d \partial u}\frac{\partial u^0}{\partial T_d} = \frac{\partial^2 H}{\partial T_d^2} - \left(\frac{\partial^2 H}{\partial T_d \partial u}\right)^2 / \frac{\partial^2 H}{\partial u^2} =$$
$$= \left(\frac{\partial^2 H}{\partial T_d^2}\frac{\partial^2 H}{\partial u^2} - \left(\frac{\partial^2 H}{\partial T_d \partial u}\right)^2\right) / \frac{\partial^2 H}{\partial u^2} < 0, \tag{52}$$

the sign in (52) is negative since due to strict concavity of function $H(y, T_d, u)$ the numerator is positive and the denominator is negative;

$$\frac{\partial^2 F}{\partial y \partial T_d} = \frac{\partial^2 H}{\partial y \partial T_d} + \frac{\partial^2 H}{\partial y \partial u}\frac{\partial u^0}{\partial T_d} = \frac{\partial^2 H}{\partial y \partial T_d} - \frac{\partial^2 H}{\partial y \partial u}\frac{\partial^2 H}{\partial T_d \partial u} / \frac{\partial^2 H}{\partial u^2} =$$
$$= \left(\frac{\partial^2 H}{\partial y \partial T_d}\frac{\partial^2 H}{\partial u^2} - \frac{\partial^2 H}{\partial y \partial u}\frac{\partial^2 H}{\partial T_d \partial u}\right) / \frac{\partial^2 H}{\partial u^2}. \tag{53}$$

Let us calculate the determinant of the matrix of second derivatives for function $F(y, T_d)$:

$$\Delta_F = \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 F}{\partial T_d^2} - \left(\frac{\partial^2 F}{\partial y \partial T_d}\right)^2 =$$
$$= [\left(\frac{\partial^2 H}{\partial y^2}\frac{\partial^2 H}{\partial u^2} - \left(\frac{\partial^2 H}{\partial y \partial u}\right)^2\right)\left(\frac{\partial^2 H}{\partial T_d^2}\frac{\partial^2 H}{\partial u^2} - \left(\frac{\partial^2 H}{\partial T_d \partial u}\right)^2\right) -$$
$$- \left(\frac{\partial^2 H}{\partial y \partial T_d}\frac{\partial^2 H}{\partial u^2} - \frac{\partial^2 H}{\partial y \partial u}\frac{\partial^2 H}{\partial T_d \partial u}\right)^2] / \left(\frac{\partial^2 H}{\partial u^2}\right)^2 =$$
$$= [\frac{\partial^2 H}{\partial u^2}\left(\frac{\partial^2 H}{\partial y^2}\frac{\partial^2 H}{\partial T_d^2} - \left(\frac{\partial^2 H}{\partial y \partial T_d}\right)^2\right) -$$
$$- \frac{\partial^2 H}{\partial T_d \partial u}\left(\frac{\partial^2 H}{\partial y^2}\frac{\partial^2 H}{\partial T_d \partial u} - \frac{\partial^2 H}{\partial y \partial T_d}\frac{\partial^2 H}{\partial y \partial u}\right) +$$

$$+\frac{\partial^2 H}{\partial y \partial u}\left(\frac{\partial^2 H}{\partial y \partial T_d}\frac{\partial^2 H}{\partial T_d \partial u} - \frac{\partial^2 H}{\partial T_d^2}\frac{\partial^2 H}{\partial y \partial u}\right)]/\frac{\partial^2 H}{\partial u^2} =$$

$$= \Delta_3 / \frac{\partial^2 H}{\partial u^2} > 0, \tag{54}$$

the sign in (54) is positive since due to strict concavity of function $H(y, T_d, u)$ the numerator Δ_3 is negative and the denominator is negative.

Since the determinants (51), (52), and (54) alternate the sign from minus to plus, then according to the Sylvester's criterion the matrix of second derivatives for function $F(y, T_d)$ is negative definite and, hence, function $F(y, T_d)$ is strictly concave in variables y, T_d. Lemma 2 is proved.

3. Let us consider now restrictions on control parameter r_d (16) and, hence, on control parameters u

$$r_d^l y \leq u \leq r_d^u y. \tag{55}$$

We choose the lower $u = r_d^l y$ and upper $u = r_d^u y$ bounds for control parameter u and substitute them to the Hamiltonian (21). We obtain two composite functions

$$G(y, T_d) = H(y, T_d, ry), \qquad r = r_d^l, \qquad r = r_d^u, \quad r > 0. \tag{56}$$

Let us prove that composite functions $G(y, T_d)$ (56) are strictly concave. For this purpose we use the Sylvester's criterion and show that the matrix of second derivatives of function $G(y, T_d)$ is negative definite.

We calculate the first derivatives of function $G(y, T_d)$

$$\frac{\partial G}{\partial y} = \frac{\partial H}{\partial y} + \frac{\partial H}{\partial u} r, \qquad \frac{\partial G}{\partial T_d} = \frac{\partial H}{\partial T_d}. \tag{57}$$

Second derivatives of function $G(y, T_d)$ have the following form

$$\frac{\partial^2 G}{\partial y^2} = \frac{\partial^2 H}{\partial y^2} + 2r\frac{\partial^2 H}{\partial y \partial u} + r^2 \frac{\partial^2 H}{\partial u^2}, \tag{58}$$

$$\frac{\partial^2 G}{\partial T_d^2} = \frac{\partial^2 H}{\partial T_d^2}, \tag{59}$$

$$\frac{\partial^2 G}{\partial y \partial T_d} = \frac{\partial^2 H}{\partial y \partial T_d} + \frac{\partial^2 H}{\partial T_d \partial u} r. \tag{60}$$

Due to the Sylvester's criterion for the matrix of second derivatives of the Hamiltonian $H(y, T_d, u)$ (21) we have in particular the following inequalities for the determinant of the first and second orders

$$\frac{\partial^2 H}{\partial y^2} < 0, \qquad \frac{\partial^2 H}{\partial T_d^2} < 0, \qquad \frac{\partial^2 H}{\partial u^2} < 0,$$

$$\frac{\partial^2 H}{\partial y^2}\frac{\partial^2 H}{\partial u^2} - \left(\frac{\partial^2 H}{\partial y \partial u}\right)^2 > 0,$$

which in turn imply the estimate

$$\left|\frac{\partial^2 H}{\partial y \partial u}\right| < \left|\frac{\partial^2 H}{\partial y^2}\right|^{1/2} \left|\frac{\partial^2 H}{\partial u^2}\right|^{1/2}. \tag{61}$$

The minors of the first order for the matrix of second derivatives of the composite function $G(y, T_d)$ (56) can be estimated as follows

$$\begin{aligned}
&\frac{\partial^2 G}{\partial y^2} = \frac{\partial^2 H}{\partial y^2} + 2r\frac{\partial^2 H}{\partial y \partial u} + r^2\frac{\partial^2 H}{\partial u^2} \le \\
&\le \frac{\partial^2 H}{\partial y^2} + 2r\left|\frac{\partial^2 H}{\partial y \partial u}\right| + r^2\frac{\partial^2 H}{\partial u^2} < \\
&< \frac{\partial^2 H}{\partial y^2} + 2r\left|\frac{\partial^2 H}{\partial y^2}\right|^{1/2}\left|\frac{\partial^2 H}{\partial u^2}\right|^{1/2} + r^2\frac{\partial^2 H}{\partial u^2} = \\
&= -\left(\left|\frac{\partial^2 H}{\partial y^2}\right| - 2r\left|\frac{\partial^2 H}{\partial y^2}\right|^{1/2}\left|\frac{\partial^2 H}{\partial u^2}\right|^{1/2} + r^2\left|\frac{\partial^2 H}{\partial u^2}\right|\right) = \\
&= \left(\left|\frac{\partial^2 H}{\partial y^2}\right| - r\left|\frac{\partial^2 H}{\partial u^2}\right|\right)^2 \le 0,
\end{aligned} \tag{62}$$

$$\frac{\partial^2 G}{\partial T_d^2} = \frac{\partial^2 H}{\partial T_d^2} < 0. \tag{63}$$

Let us calculate the determinant of the second order for the matrix of second derivatives of function $G(y, T_d)$

$$\begin{aligned}
\Delta_G &= \frac{\partial^2 G}{\partial y^2}\frac{\partial^2 G}{\partial T_d^2} - \left(\frac{\partial^2 G}{\partial y \partial T_d}\right)^2 = \\
&= \left(\frac{\partial^2 H}{\partial y^2} + 2r\frac{\partial^2 H}{\partial y \partial u} + r^2\frac{\partial^2 H}{\partial u^2}\right)\frac{\partial^2 H}{\partial T_d^2} - \\
&- \left(\frac{\partial^2 H}{\partial y \partial T_d} + r\frac{\partial^2 H}{\partial T_d \partial u}\right)^2 = \\
&= \left(\frac{\partial^2 H}{\partial y^2}\frac{\partial^2 H}{\partial T_d^2} - \left(\frac{\partial^2 H}{\partial y \partial T_d}\right)^2\right) + \\
&+2r\left(\frac{\partial^2 H}{\partial y \partial u}\frac{\partial^2 H}{\partial T_d^2} - \frac{\partial^2 H}{\partial y \partial T_d}\frac{\partial^2 H}{\partial T_d \partial u}\right) + \\
&+r^2\left(\frac{\partial^2 H}{\partial u^2}\frac{\partial^2 H}{\partial T_d^2} - \left(\frac{\partial^2 H}{\partial T_d \partial u}\right)^2\right) = \\
&= D_1 + 2rD_2 + r^2 D_3,
\end{aligned} \tag{64}$$

where D_1, D_2, D_3 are the corresponding minors of the matrix of second derivatives for the Hamiltonian $H(y, T_d, u)$ (21).

Let us show that for the minors D_1, D_2, D_3 the following inequality takes place

$$D_1 D_3 - D_2^2 > 0. \tag{65}$$

Really, we have the chain of relations

$$\begin{aligned}
&D_1 D_3 - D_2^2 = \\
&= \left(\frac{\partial^2 H}{\partial y^2}\frac{\partial^2 H}{\partial T_d^2} - \left(\frac{\partial^2 H}{\partial y \partial T_d}\right)^2\right)\left(\frac{\partial^2 H}{\partial u^2}\frac{\partial^2 H}{\partial T_d^2} - \left(\frac{\partial^2 H}{\partial T_d \partial u}\right)^2\right) -
\end{aligned}$$

$$
\begin{aligned}
&-\left(\frac{\partial^2 H}{\partial y \partial u}\frac{\partial^2 H}{\partial T_d^2}-\frac{\partial^2 H}{\partial y \partial T_d}\frac{\partial^2 H}{\partial T_d \partial u}\right)^2 = \\
&=\frac{\partial^2 H}{\partial T_d^2}[\left(\frac{\partial^2 H}{\partial y^2}\frac{\partial^2 H}{\partial u^2}-\left(\frac{\partial^2 H}{\partial y \partial u}\right)^2\right)\frac{\partial^2 H}{\partial T_d^2} - \\
&-\left(\frac{\partial^2 H}{\partial y^2}\frac{\partial^2 H}{\partial T_d \partial u}-\frac{\partial^2 H}{\partial y \partial T_d}\frac{\partial^2 H}{\partial y \partial u}\right)\frac{\partial^2 H}{\partial T_d \partial u} - \\
&-\left(\frac{\partial^2 H}{\partial u^2}\frac{\partial^2 H}{\partial y \partial T_d}-\frac{\partial^2 H}{\partial y \partial u}\frac{\partial^2 H}{\partial T_d \partial u}\right)\frac{\partial^2 H}{\partial y \partial T_d}] = \\
&=\frac{\partial^2 H}{\partial T_d^2}\Delta_3 > 0,
\end{aligned}
\tag{66}
$$

the sign is positive since both multipliers (see (43), (45)) are negative.

Basing on relation (65) one can evaluate the sign of the determinant Δ_G (64)

$$
\begin{aligned}
&\Delta_G = D_1 + 2rD_2 + r^2 D_3 \geq \\
&\geq D_1 - 2r|D_2| + r^2 D_3 > D_1 - 2rD_1^{1/2}D_3^{1/2} + r^2 D_3 = \\
&= (D_1^{1/2} - rD_3^{1/2})^2 \geq 0.
\end{aligned}
\tag{67}
$$

Thus, minors of the first order (62), (63), and the second order (67) alternate signs starting from minus and, hence, according to the Sylvester's criterion the matrix of second derivatives for the composite function $G(y, T_d)$ (56) is negative definite, and the composite function $G(y, T_d)$ is strictly concave.

4. To get the maximized Hamiltonian $\hat{H}(y, T_d)$ (22) we paste strictly concave functions $F(y, T_d)$ (46), $G(y, T_d)$ (56). Let us show that sewing of these functions is continuously differentiable. To this end it is necessary to calculate partial derivatives of functions $F(y, T_d)$, $G(y, T_d)$ and verify that these derivatives are equal to each other at points of sewing of these functions. Points of sewing (y^s, T_d^s) of functions $F(y, T_d)$, $G(y, T_d)$ are defined by relations

$$
F(y^s, T_d^s) = H(y^s, T_d^s, u^0(y^s, T_d^s)) = H(y^s, T_d^s, ry^s) = G(y^s, T_d^s), \tag{68}
$$

or, equivalently, by relation

$$
u^0(y^s, T_d^s) = ry^s. \tag{69}
$$

Let us calculate partial derivatives of functions $F(y, T_d)$, $G(y, T_d)$ at points (y^s, T_d^s) of sewing of these functions

$$
\begin{aligned}
&\frac{\partial F}{\partial y}(y^s, T_d^s) = \frac{\partial H}{\partial y}(y^s, T_d^s, u^0(y^s, T_d^s)) + \\
&+\frac{\partial H}{\partial u}(y^s, T_d^s, u^0(y^s, T_d^s))\frac{\partial u^0}{\partial y}(y^s, T_d^s),
\end{aligned}
\tag{70}
$$

$$
\begin{aligned}
&\frac{\partial F}{\partial T_d}(y^s, T_d^s) = \frac{\partial H}{\partial T_d}(y^s, T_d^s, u^0(y^s, T_d^s)) + \\
&+\frac{\partial H}{\partial u}(y^s, T_d^s, u^0(y^s, T_d^s))\frac{\partial u^0}{\partial T_d}(y^s, T_d^s),
\end{aligned}
\tag{71}
$$

$$
\frac{\partial G}{\partial y}(y^s, T_d^s) = \frac{\partial H}{\partial y}(y^s, T_d^s, ry^s) + \frac{\partial H}{\partial u}(y^s, T_d^s, ry^s)r, \tag{72}
$$

$$\frac{\partial G}{\partial T_d}(y^s, T_d^s) = \frac{\partial H}{\partial T_d}(y^s, T_d^s, ry^s). \tag{73}$$

Due to necessary conditions of optimality (47) at points of sewing (69) we have relations

$$\frac{\partial H}{\partial u}(y^s, T_d^s, u^0(y^s, T_d^s)) = \frac{\partial H}{\partial u}(y^s, T_d^s, ry^s) = 0, \tag{74}$$

and, hence, partial derivatives (70), (71) of function $F(y, T_d)$ and partial derivatives (72), (73) of function $G(y, T_D)$ coincide with each other

$$\begin{aligned}\frac{\partial F}{\partial y}(y^s, T_d^s) &= \frac{\partial H}{\partial y}(y^s, T_d^s, u^0(y^s, T_d^s)) = \\ &= \frac{\partial H}{\partial y}(y^s, T_d^s, ry^s) = \frac{\partial G}{\partial y}(y^s, T_d^s)\end{aligned} \tag{75}$$

$$\begin{aligned}\frac{\partial F}{\partial T_d}(y^s, T_d^s) &= \frac{\partial H}{\partial T_d}(y^s, T_d^s, u^0(y^s, T_d^s)) = \\ &= \frac{\partial H}{\partial T_d}(y^s, T_d^s, ry^s) = \frac{\partial G}{\partial T_d}(y^s, T_d^s)\end{aligned} \tag{76}$$

at points of sewing (y^s, T_d^s) (69) of these functions.

Thus, we obtain that the maximized Hamiltonian $\hat{H}$ (22) is continuously differentiable and strictly concave function in (y, T_d). Proposition 1 is proved.

Proposition 2. *Under the conditions of Proposition 1 the Pontryagin maximum principle gives sufficient conditions to find the unique optimal solution in the Problem (P_1).*

Proof. Let $(y, T_d, r_d) = (y(t), T_d(t), r_d(t))$ be an arbitrary admissible process. Denote by x the pair (y, T_d). Due to the strict concavity of $\hat{H}$ in x the following inequality holds:

$$\left\langle \frac{\partial \hat{H}(x^*(t), \psi(t))}{\partial x}, x^*(t) - x(t) \right\rangle < \hat{H}(x^*(t), \psi(t)) - \hat{H}(x(t), \psi(t)), \tag{77}$$

if $x(t) \neq x^*(t)$.

Combining this inequality with condition (28) we obtain that for $t \in [t_0, \theta]$ the following chain of inequalities takes place:

$$\begin{aligned}&\left\langle \dot{\psi}(t), x(t) - x^*(t) \right\rangle < \hat{H}(x^*(t), \psi(t)) - \hat{H}(x(t), \psi(t)) \leq \\ &\leq \langle \psi(t), \dot{x}^*(t) - \dot{x}(t) \rangle + \\ &+ e^{-\rho(t-t_0)}(\ln D(x^*(t), r_d^*(t)) - \ln D(x(t), r_d(t))),\end{aligned}$$

where

$$\begin{aligned}&\ln D(x(t), r_d(t)) = \ln D(y(t), T_d(t), r_d(t)) = \\ &= (1 + a_2)\ln y(t) + a_1 \ln(T_d(t) + \frac{r_d(t)y(t)}{\omega}) + a_2 \ln r_d(t).\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{d}{dt}\langle \psi(t), x(t) - x^*(t) \rangle + e^{-\rho(t-t_0)} \ln D(x(t), r_d(t)) < \\ &< e^{-\rho(t-t_0)} \ln D(x^*(t), r_d^*(t)).\end{aligned}$$

Integrating this inequality over $t \in [t_0, \theta]$, we get

$$\langle \psi(\theta), x(\theta) - x^*(\theta) \rangle + \int_{t_0}^{\theta} e^{-\rho(t-t_0)} \ln D(x(t), r_d(t)) dt <$$

$$< \int_{t_0}^{\theta} e^{-\rho(t-t_0)} \ln D(x^*(t), r_d^*(t)) dt.$$

Taking into account the transversality conditions (24), we obtain

$$\int_{t_0}^{\theta} e^{-\rho(t-t_0)} \ln D(x(t), r_d(t)) dt < \int_{t_0}^{\theta} e^{-\rho(t-t_0)} \ln D(x^*(t), r_d^*(t)) dt.$$

Thus, $(x^*, r_d^*) = (y^*(t), T_d^*(t), r_d^*(t))$ is the unique optimal solution in the Problem (P_1). The proposition is proved.

5 Results of Numerical Simulations (Sony Corporation)

This section presents the results of application of the model to the Japanese Electrical Machinery Sector. We develop simulations on the basis of the described mathematical model and construct numerically optimal solutions. In parallel we carry out the multi-variant scenario analysis and sensitivity analysis.

For numerical analysis of the model we choose the software package "Dynamic Model Optimizer" which has been developed by Ivan Matrossov. Methods of numerical analysis of differential equations and optimal control problems are realized in this package.

5.1 Identification of model's parameters

We focus on the Sony Corporation. The simulation scenarios are calibrated on the real data. The source of data is Tokyo Institute of Technology (Laboratory of Prof. C.Watanabe). We simulate the model in period 1980 - 2020 and compare the optimal trajectories of the model with the real time series in period 1980-2000. In figures given below the horizontal axis displays time and the vertical axis shows values of a presented variable. All variables (except R&D intensity and assimilation capacity) are measured in billion of Yen.

For Sony Corporation we use the following initial values of the main variables in 1980 (in bln Yen):

$y^0 = 427.22$ – production,

$T_d^0 = 157.1$ – domestic technologies, (36)

$T_s^0 = 2482.88$ – spillover pool.

First, by the method of least squares we find linear functions $R_d(t) = 13.4(t - 1980) + 50.7$ $R_s(t) = 125.81(t - 1980) + 723.8$ for the rates of domestic technologies $R_d(t)$ and spillover pool $R_s(t)$ as approximations of the real data (Fig. 1,2, respectively).

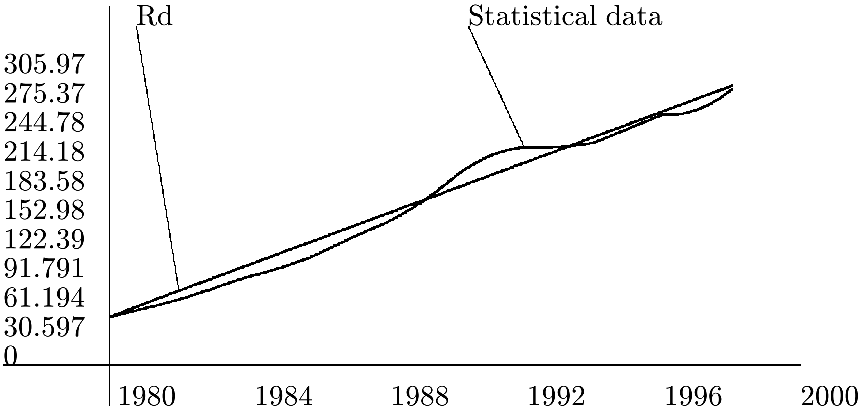

Fig. 1.

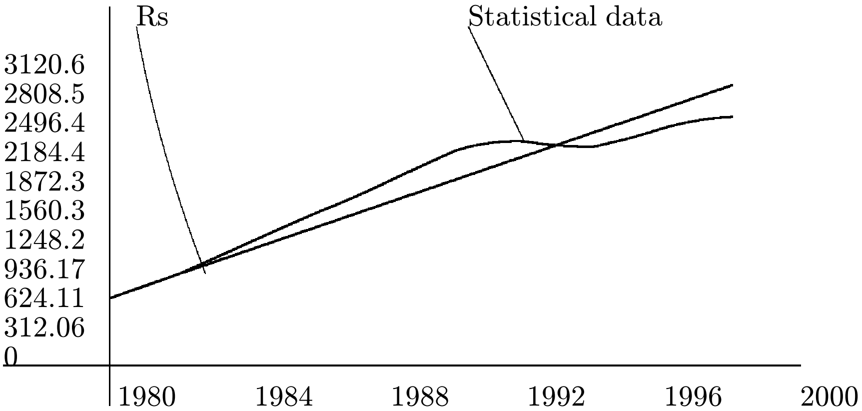

Fig. 2.

Analyzing statistical data we choose function $f_1(t)$ as a smoothed stair (Fig.3) with max = 0.14 and min = 0.07.

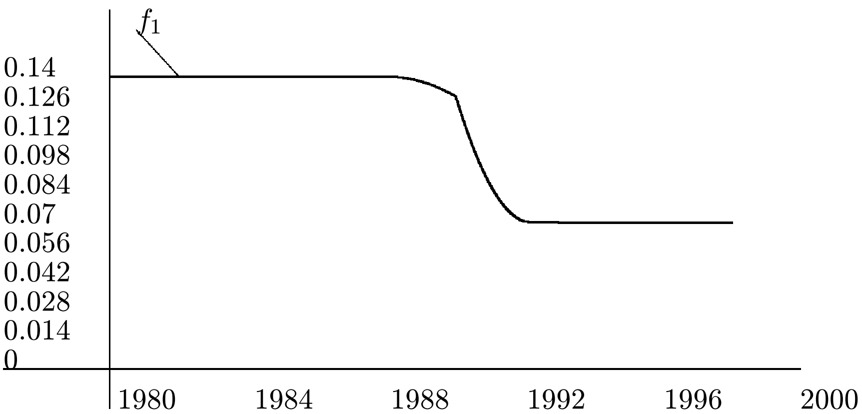

Fig. 3.

The algorithm of identification of other model's parameters consists in the following. We have the data in the period from 1980 till 2000 for the main variables of the model, such as $y^{stat}(t)$, $T_d^{stat}(t)$ T_s^{stat}. Our problem is to find such values of parameters f_2, g_d, γ, that the solution of the system of differential equations (1)-(5) with initial conditions (36) approximates the real data in the best way. In order to do that we solve the following auxiliary optimal control problem: find such values of $f_2 \in [f_2^l, f_2^u]$, $g_d \in [g_d^l, g_d^u]$, $\gamma \in$

$[\gamma^l, \gamma^u]$, which maximize the objective function

$$\int_{1980}^{2000} \left(-(y(t) - y^{stat}(t))^2 - (T_d(t) - T_d^{stat}(t))^2 - (T_s(t) - T_s^{stat}(t))^2 \right) dt,$$

subject to the dynamics (1)-(5).

For solution of this problem we apply the BFG QN algorithm. Let us indicate the values for model's parameters which we choose after the identification of model's parameters: $f_2 = 0.1$, $g_d = 0.65$, $\gamma = 0.77$.

The further analysis of data and statistical results in [12], [14], [15] allow us to find the values of the parameters: $e = 1.69$, $\beta_1 = 0.62$, $\beta_2 = 0.34$.

Under such values of parameters the conditions of proposition 1 is hold and we can use the results of theorems 1 and 2.

Figures 4-7 illustrate two curves, one of which is the solution of Cauchy problem (1)-(5) with the mentioned numerical values of parameters and initial data (36), the second curve is statistical data.

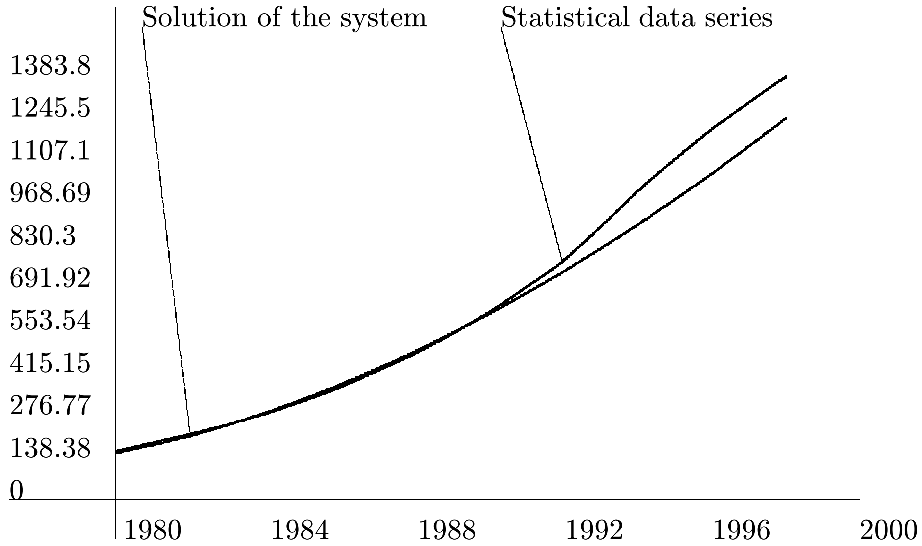

Fig. 4. Trajectories of Domestic Technology Stock T_d.

One can see that trajectories of the simulation model approximate the real time series with a good accuracy.

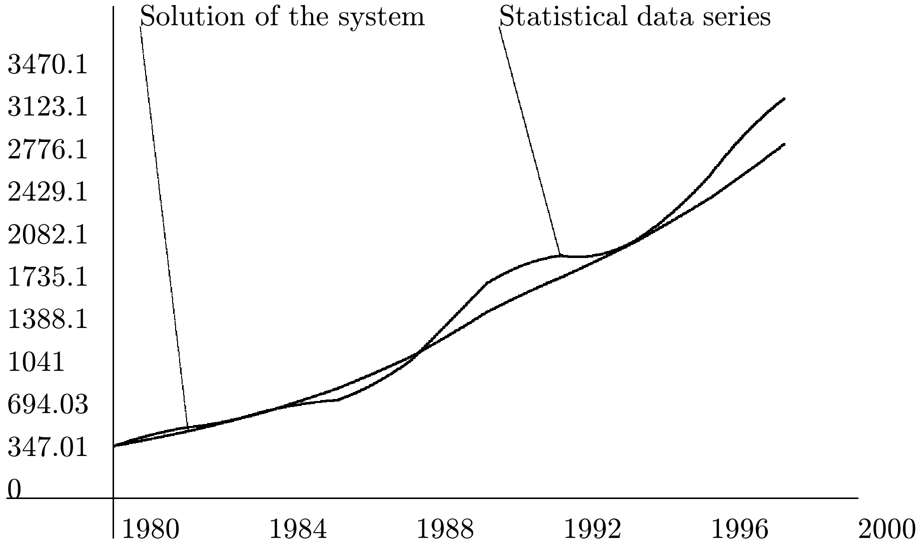

Fig. 5. Trajectories of Production y**.**

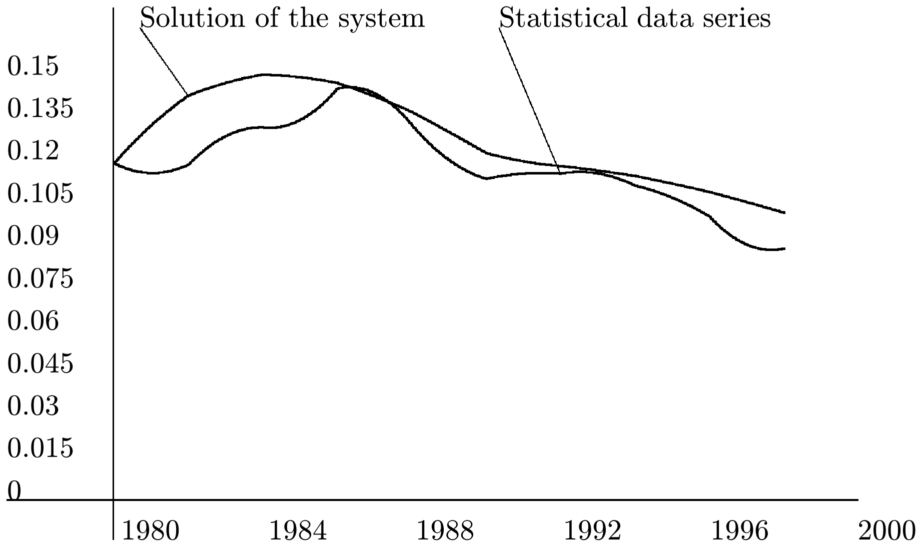

Fig. 6. Trajectories of R&D Intensity r_d**.**

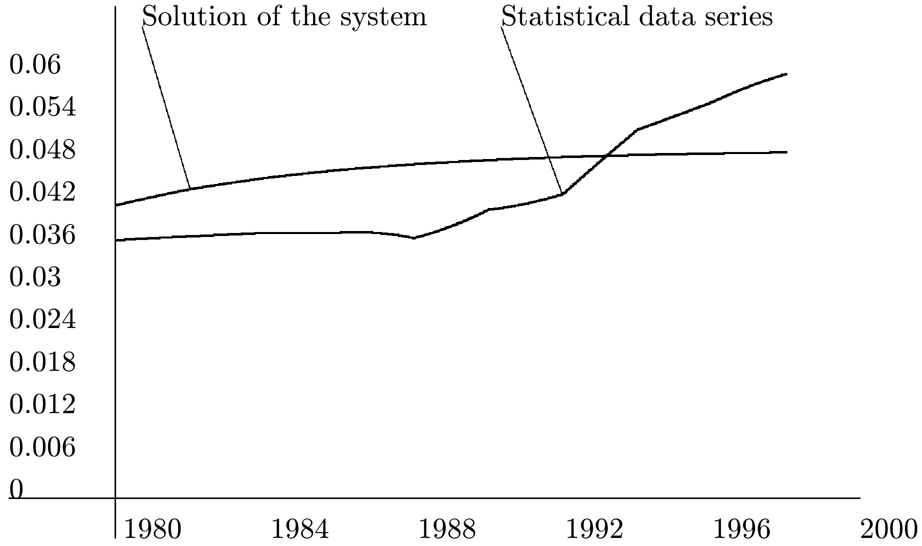

Fig. 7. Trajectories of Assimilation Capacity z.

Fig. 8 shows the interaction between production and R&D intensity in the model for Sony Corporation. Due to the time lag in R&D investments, which equals approximately to 4 years, the R&D intensity increased during 1980-1984, then its stagnation started, whereas production continued to grow. These trends correspond to the so called inertial scenario. For the period 1980 - 1998 the inertial trajectories almost coincide with the empirically observed trajectories.

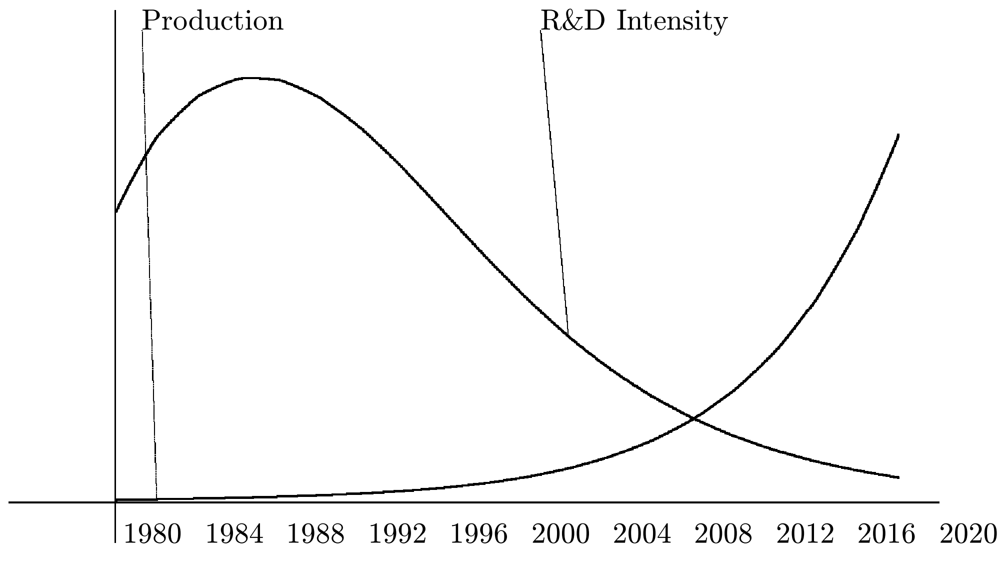

Fig. 8. Trends of production and R&D intensity in inertial scenario.

5.2 Analysis of the impact of technology spillovers

Our next experiment intends to analyze the effect of technology spillovers for Sony Corporation. Two scenarios are considered - in the first case, the firm is able to use technologies developed by other firms, and, in the second case, the firm does not have such possibilities.

Table 1 presents the values of the optimal solutions of the model in two scenarios. Scenario 1 demonstrates the firm's evolution given by the solution of the optimal control

Problem (P_1). Scenario 2 assumes that there is no spillover effect that is the firm cannot utilize spillover technologies (in terms of the model it means that the firm's assimilation capacity is zero, i.e. $z = 0$). Table 1 shows the values of the main firm's parameters in years 1990, 2000, 2020.

Table 1. Simulation - based comparative analysis of the development of the Sony Corp. under scenarios 1 and 2.

Year	***Scenario 1:*** external technologies assimilated (z≠0)	***Scenario 2:*** no assimilation of external technologies (z = 0)	***Variable***
1990	1643.4	1653	***Production***
2000	6218.6	6173.9	
2020	61741	59017	
1990	0.074	0.062	***R&D Intensity***
2000	0.07	0.059	
2020	0.067	0.057	
1990	627.39	407.47	***Technology Stock***
2000	1949	1112.7	
2020	13536	7229.2	
1990	448.57	407.47	***Domestic Technology Stock***
2000	1266.5	1112.7	
2020	8323.8	7229.2	
1990	106.24	103.1	***Objective Function***
2000	181.18	175.62	
2020	262.38	254.24	

One can see that under Scenario 1, all firm's parameters take greater values than under Scenario 2. The simulations show that this trend holds during the entire period 1990-2020 and implies that Scenario 1 provides a higher value of the utility.

This case study demonstrates advantages of utilizing spillover technologies.

5.3 Sensitivity analysis 1

Now we shall estimate the sensitivity of the model with respect to parameter g_d, the discounted marginal productivity of technology, which serves as a risky factor of R&D investments in the model. First (case 1), we take g_d as function of time, which approximates the real data series, $g_d = g_d(t)$ (see Fig. 9). Solving Problem (P_1), we obtain the optimal R&D intensity shown on Fig. 10.

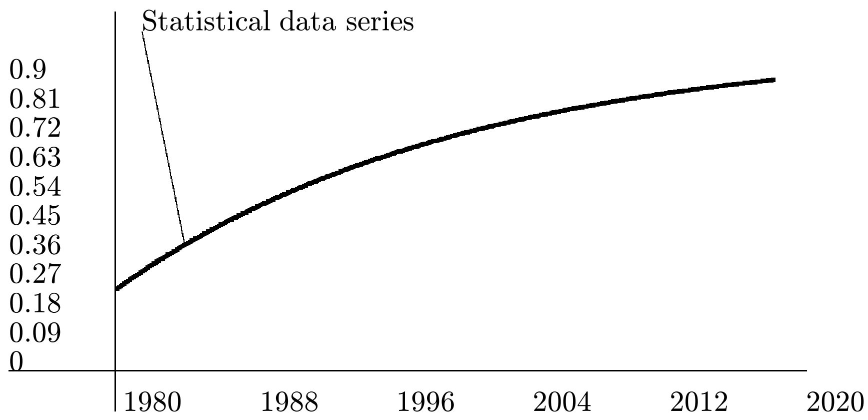

Fig. 9. Discounted marginal productivity of technology.

In the case 2 we deal with the constant average value of g_d for 20 years from 1980 till 2000, i.e.

$$g_d^a = \frac{\int_0^{20} g_d(t)dt}{20} = \frac{\int_0^{20}(1 - 0.75exp(-0.09 * t))dt}{20} \approx 0.65.$$

The simulation result is demonstrated in Fig. 11. In both cases the trends in the optimal R&D intensity differ on the first half of the time interval. Function $g_d(t)$ (Fig.9) increases fastly from 1980 till 2000, thus increasing the uncertainty of R&D investments. This trend explains the decrease of the R&D intensity in this period in the case 1 (Fig. 10) (the investors were probably not willing to spend much into R&D due to the unstable economic situation).

In the case 2 trajectories of the R&D intensity do not have trend to decrease. Table 2 shows the simulated values of the optimal R&D intensity, production, technology stock and objective function for cases 1 and 2.

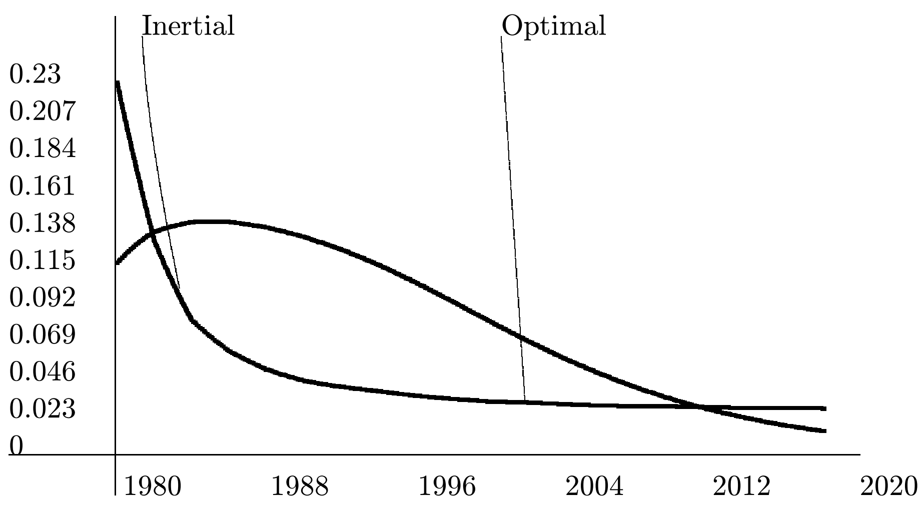

Fig. 10. The optimal R&D intensity in the case 1 ($g_d = g_d(t)$) (Sony Corp.).

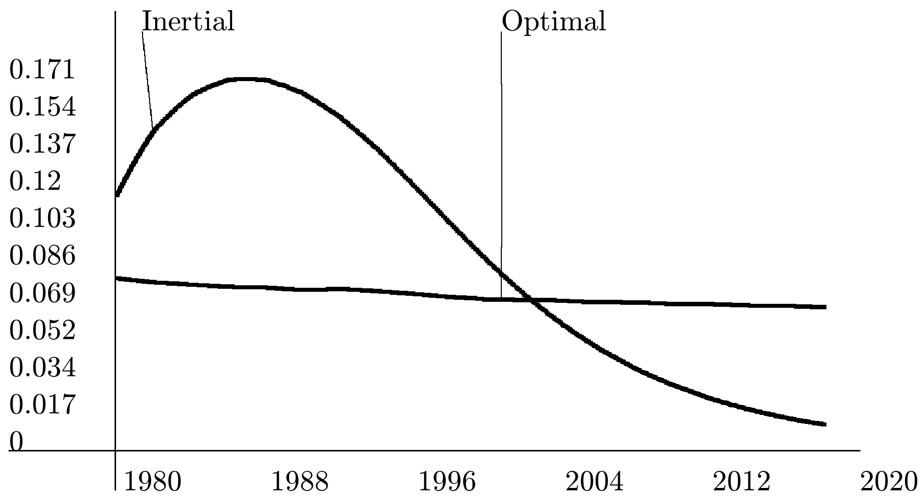

Fig. 11. The optimal R&D intensity in the case 2 ($g_d = const$) (Sony Corp.).

During the first half of the time interval the domestic technology stock and the total technology stock are greater in the case 1 due to a higher level of the optimal initial R&D intensity. Since 1990 the optimal R&D intensity in the case 2 prevails over the case 1 and this fact explains the essential difference in technology stocks in cases 1 and 2. Production function has the opposite trends in cases 1 and 2. Note, that in the case 2 where the risky factor g_d is stable the objective function keeps greater values at the end of the analyzed period (till 2020).

Table 2. Simulation-based comparative analysis of the influence of discounted marginal productivity of technology for the Sony Corporation in cases 1 and 2.

Variable	Case	**1985**	**1990**	**2000**	**2010**	**2020**
R&D Intensity	case 1 case 2	0.0948 0.0765	0.0599 0.0745	0.043 0.07	0.039 0.068	0.037 0.0665
Production	case 1 case 2	809.29 838.71	1626.5 1643.4	6347.5 6218.6	22441 21972	62428 61741
Domestic Tech.Stock	case 1 case 2	341.05 265.51	514.24 448.51	1112.4 1266.5	2583.2 3492.1	5670.2 8323.8
Technology Stock	case 1 case 2	454.66 359.96	685.09 627.39	1699.8 1994.9	4102.4 5623.3	9151.5 13536
Objective Function	case 1 case 2	59.601 57.291	108 106.24	182.94 181.18	232.5 231.87	262.22 262.38

Figures 12 and 13 show the evolution of production function in cases 1 and 2, respectively.

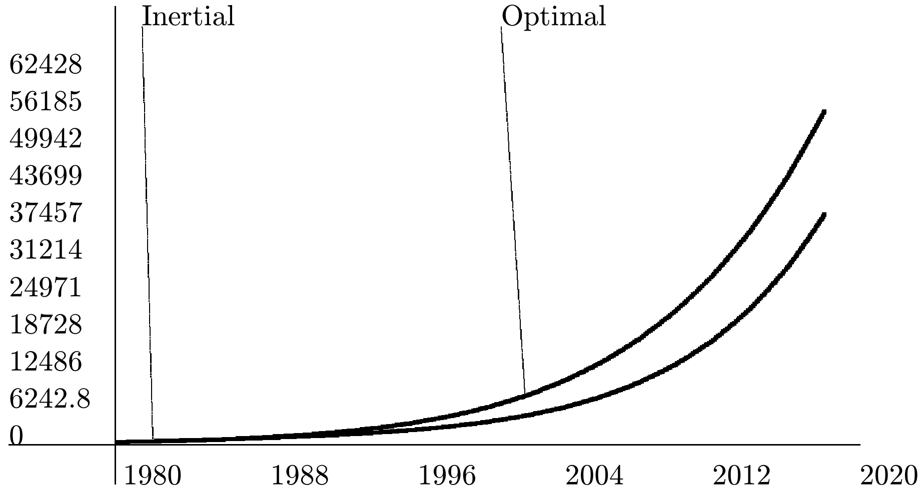

Fig. 12. Optimal production in the case 1 ($g_d = g_d(t)$) (Sony Corp.).

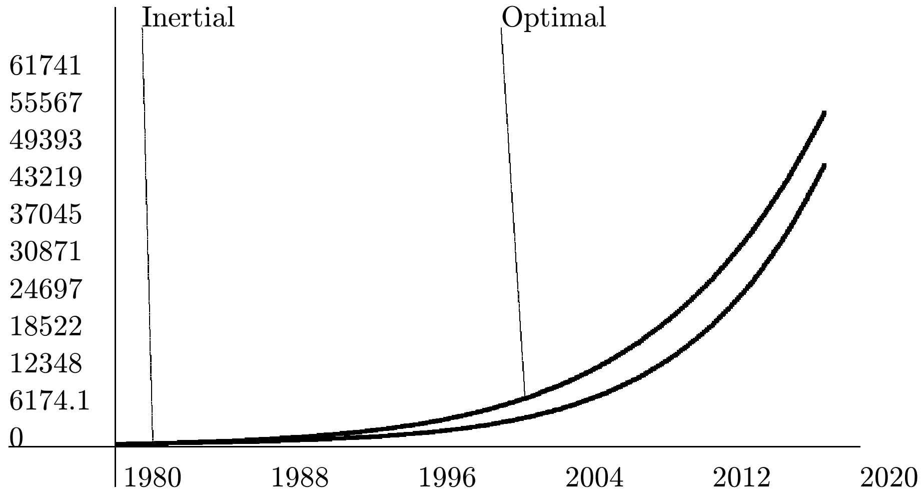

Fig. 13. Optimal production in the case 2 ($g_d = const$) (Sony Corp.).

In the case 2 the technology stock is much higher due to a higher level of the R&D intensity (see Figures 14 and 15).

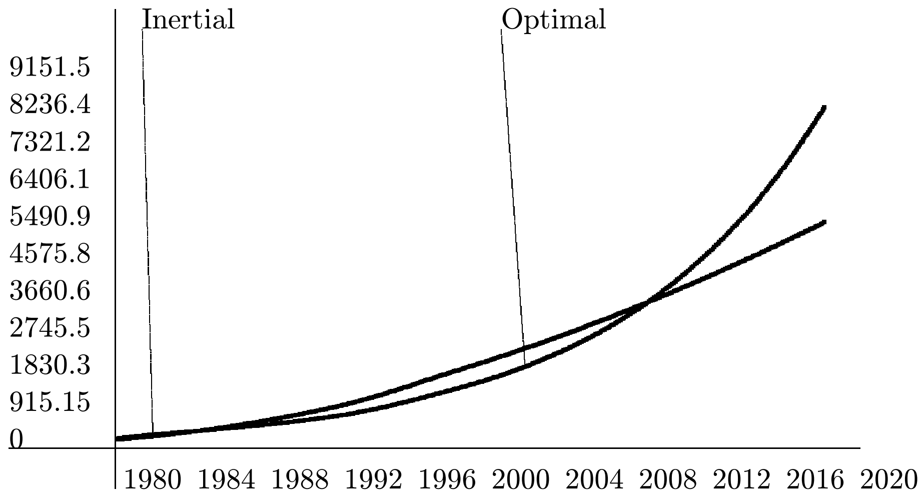

Fig. 14. The optimal technology stock in the case 1 ($g_d = g_d(t)$) (Sony Corp.).

The sensitivity analysis leads to preliminary conclusions that the model is sensitive to the factor g_d, and, therefore, calibration of this factor demands high precision.

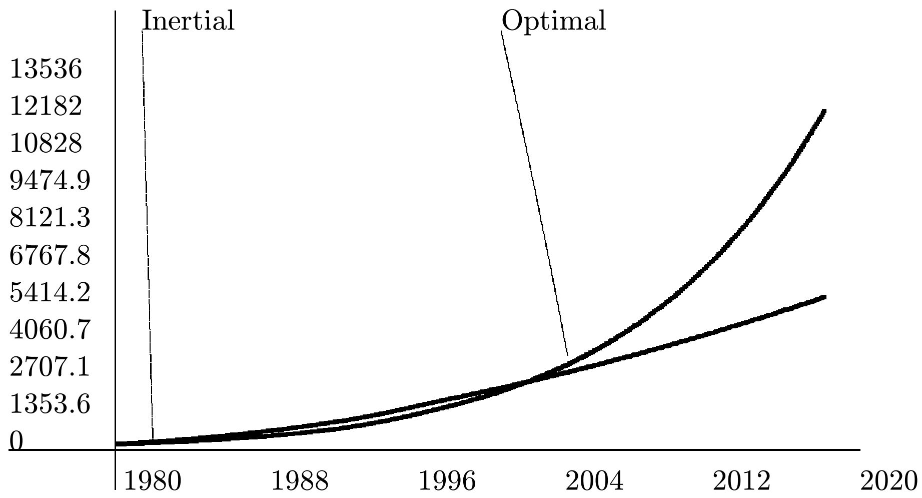

Fig. 15. The optimal technology stock in the case 2 ($g_d = const$) (Sony Corp.).

5.4 Sensitivity analysis 2

Now let us pass to the simulation results with respect to changes in the elasticity of substitution e, which plays an important role in identification and forecast of the optimal R&D intensity. In all previous experiments the elasticity of substitution equals 1.69, which is the average value for the Electric Machinery Industry (see [15]).

The simulations show that trends in the model's trajectories are not influenced by variation of e whereas the values of the parameters depend on e essentially. Table 3 shows the outcomes for Scenario 1: $e = 1.69$ and Scenario 2: $e = 1.02$. One can see that for Scenario 2 the optimal R&D intensity is about 15% higher whereas the optimal production is kept on a lower level. A possible explanation is that in Scenario 2 where products are easily substitutable the firm is not interested in producing huge amounts of "similar" products; its interest is focused primarily on producing new technologies to improve its products. As a result for Scenario 2 we have a very high level of the technology stock.

In 2020 absolute values of the domestic technology stock and the total technology stock in Scenario 1 are greater due to an essential difference in production levels. But if we look at the relative values we see that the technology intensity is higher in Scenario 2 (0.63) than in Scenario 1 (0.21). This result shows that the optimal R&D investment strategies under different economic conditions can alter considerably.

Table 3. Simulation - based comparative analysis of variation of the elasticity of substitution for the Sony Corporation in Scenarios 1 and 2.

Variable	Scenario	1985	1990	2000	2010	2020
R&D Intensity	Scenario 1	0.0765	0.0745	0.07	0.068	0.0665
	Scenario 2	0.163	0.159	0.153	0.151	0.15
Production	Scenario 1	838.71	1643.4	6218.6	21972	61741
	Scenario 2	672.95	1114	3185.6	8691.5	19291
Domestic Technology Stock	Scenario 1	265.51	448.57	1266.5	3492.1	8323.8
	Scenario 2	337.91	598.29	1567.6	3738.5	7679.3
Technology Stock	Scenario 1	359.96	627.39	1994.9	5623.3	13536
	Scenario 2	471.32	837.8	2432.6	5904.3	12227

6 Strategies of Optimal Balance between Domestic Technology and Absorbed Technology

This section is devoted to the question about optimal balance between domestic technology stock and technology spillovers. In order to investigate this problem we make further modification of the model. In this version we introduce the second control parameter r_a, which describes the expenditures for applying to technology spillover. We divide R&D intensity r into two parts: the first part is the share of production which goes to elaboration of domestic technologies, and the second part is directed to the process of technology absorption

$$r(t) = r_d(t) + r_a(t).$$

It involves a modification of the equation for production (1), namely, we add the second negative term in the equation related to the risk of investment into technology absorption $-g_a r_a$. Also we introduce the auxiliary variable T_a - the technology stock used for applying spillover technology. The dynamics for this variable has the same structure like the dynamics of domestic technology stock

$$\dot{T}_a(t) = r_a(t)y(t).$$

Finally we change the equation for assimilation capacity (4) – instead of domestic technology T_d we use the absorbed technology stock T_a.

The optimal control problem is to find such level of control parameters r_d^* and r_a^* which maximize the utility function (18) and subject to the following equations:

$$\frac{\dot{y}(t)}{y(t)} = f_1(t) + f_2\left(\frac{T(t)}{y(t)}\right)^{\gamma} - g_d(t)r_d(t) - g_a(t)r_a(t),$$

$$\dot{T}_d(t) = r_d(t)y(t),$$

$$\dot{T}_a(t) = r_a(t)y(t),$$

$$T(t) = T_d(t) + zT_s(t),$$

$$z(t) = \frac{1}{1 + \frac{\dot{T}_s(t)}{T_s(t)} / \frac{\dot{T}_a(t)}{T_a(t)}} \frac{T_a(t)}{T_s(t)}.$$

We consider the results for two firms from the Japanese Electric Machinery Industry with high (Sony Corporation) and low (Kokusai Company) levels of domestic technology stock and production. The initial stage and parameters of the model for Sony are the same as in Section 5.1. New parameters are the following: $T_a^0 = 100$, $g_a = 0.5$. Let's indicate the initial stage and model's parameters for Kokusai Co.:

$y^0 = 23.98$, $T_d^0 = 6.48$, $T_a^0 = 4$, $T_s^0 = 2622.49$.

$f_1 = 0.14$, $f_2 = 0.05$, $g_d = 0.6$, $g_a = 0.2$, $\gamma = 0.75$,

$e = 1.69$, $\beta_1 = 0.62$, $\beta_2 = 0.34$.

For identification of the model's parameters we use the algorithm described in Section 5.1.

The results of comparison analysis of the optimal strategies for these two firms are the following. For Sony Corp. we obtain that the optimal strategy prescribes to spend a relatively small portion of R&D for utilizing technology spillovers. One can see (Fig. 16) the curves for intensities of the total investment r , of domestic R&D investments r_d and technology absorption r_a. The share of technology absorption in the total investment varies from 0.55 till 0.7, but the difference between intensities of domestic investments and technology absorption is not large.

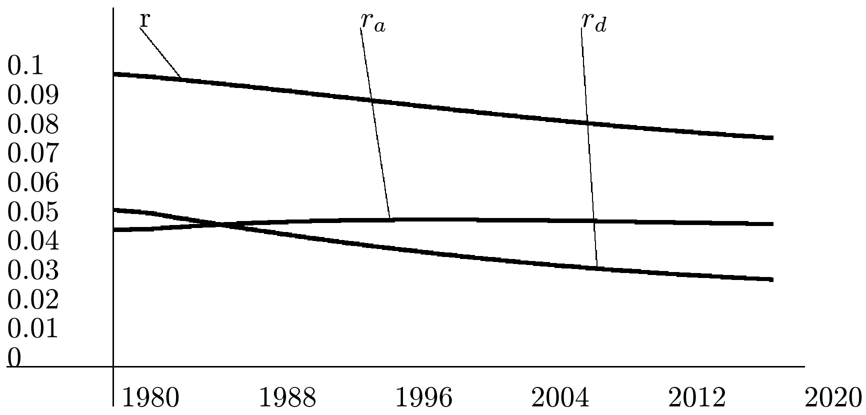

Fig. 16. Trajectories for Sony Corporation.

Optimal policy for Kokusai Company orients on technology absorption and the share of technology absorption in the total investment is about 0.9 - 0.95 (see Fig. 17).

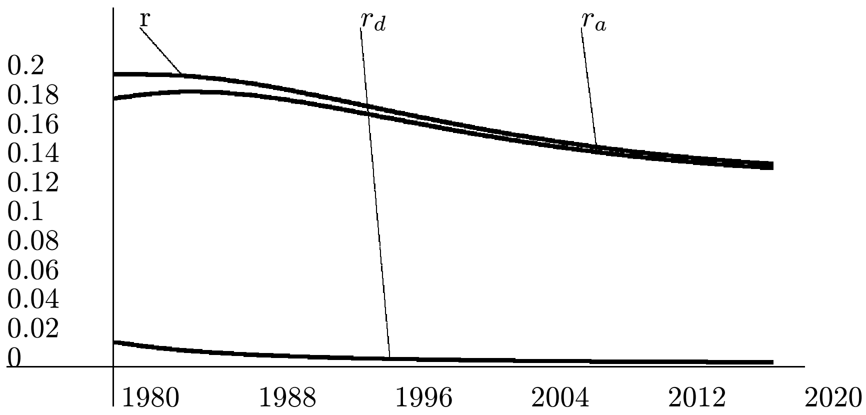

Fig. 17. Trajectories for Kokusai Company.

This investment policy involves the high level of optimal assimilation capacity for this firm. The optimal synthetic absorption is essentially higher than figures of the real absorption policy (see Fig. 18). For Sony Corp. we observe the growth trend of the optimal assimilation capacity but there is no such large difference between the synthetic scenario and the statistical data (see Fig. 19).

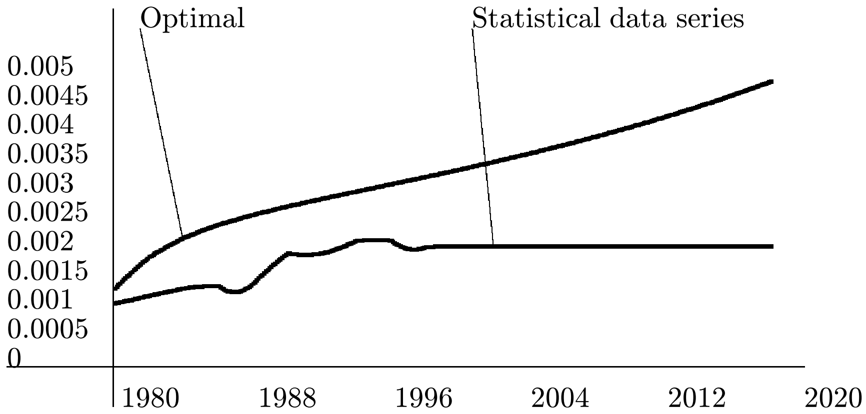

Fig. 18. Trajectories of assimilation capacity for Kokusai Company.

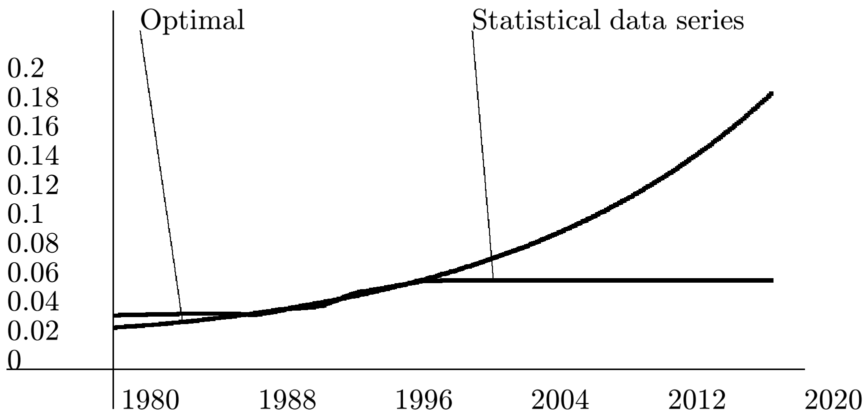

Fig. 19. Trajectories of assimilation capacity for Sony Corporation.

Finally let us analyze the impact of domestic and spillover technologies on the total technology stock in the firm. Two pictures for Sony Corp. (see Fig. 20, 21) show the trajectories of the real process and the optimal synthetic scenario. In the real process the domestic technology T_d has the dominant impact on the total technology stock T (see Fig. 20), while in the optimal synthetic scenario the impact of the assimilated technology T_a and the domestic technology T_d are almost the same (see Fig. 21).

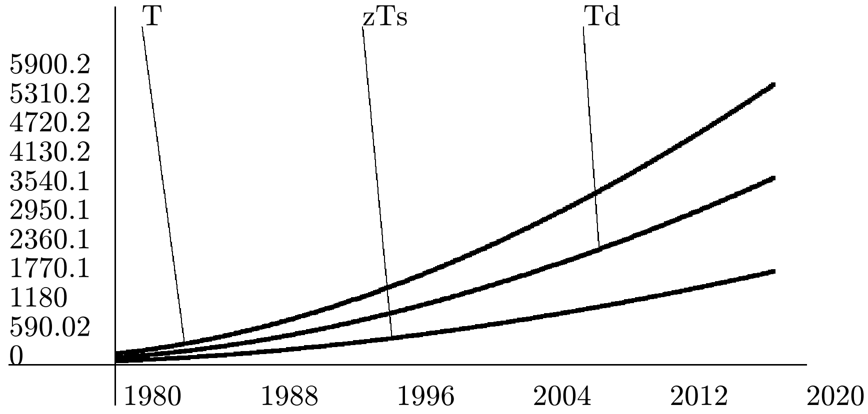

Fig. 20. Trajectories of the real technology stocks for Sony Corporation.

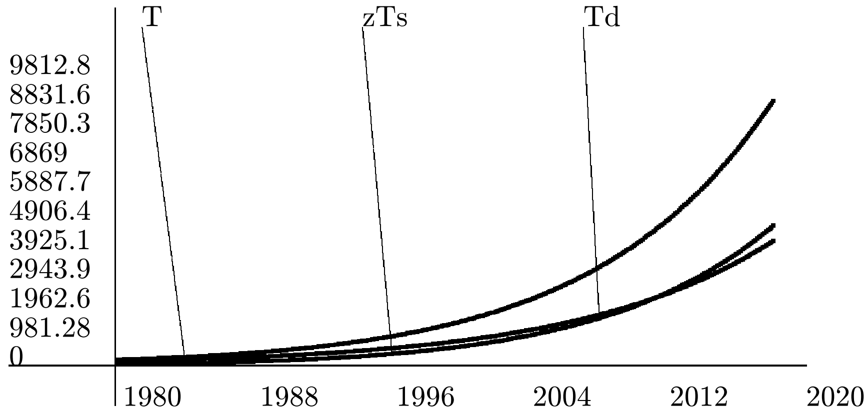

Fig. 21. Trajectories of the optimal technologies stocks for Sony Corporation.

For Kokusai Co. the difference between the domestic technology and assimilated technology is essential in both cases (in the real data series and optimal synthetic scenarios), but in the real process (see Fig. 22) the level of domestic technology is high ($T_d/zT_s = 5/1$), while in the optimal synthetic scenario (see Fig. 23), vice versa, the assimilated technology is larger than domestic technology ($T_d/zT_s = 1/3$). Consequently, there are higher levels of the total technology stock, production and utility function in the optimal synthetic scenario than in the real data.

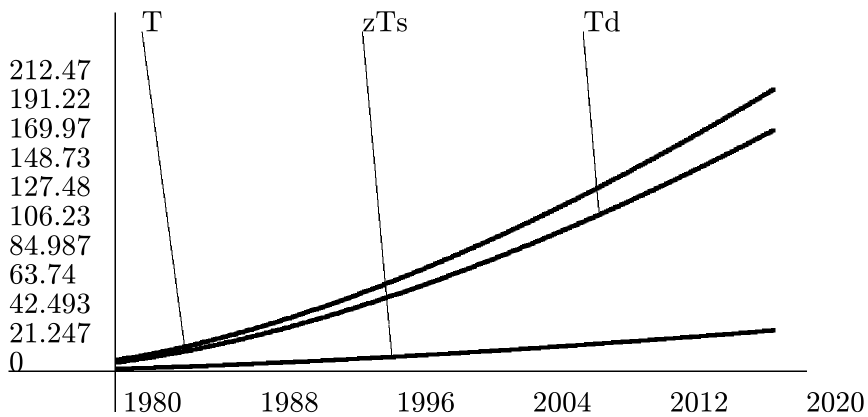

Fig. 22. Trajectories of the real technologies stocks for Kokusai Company.

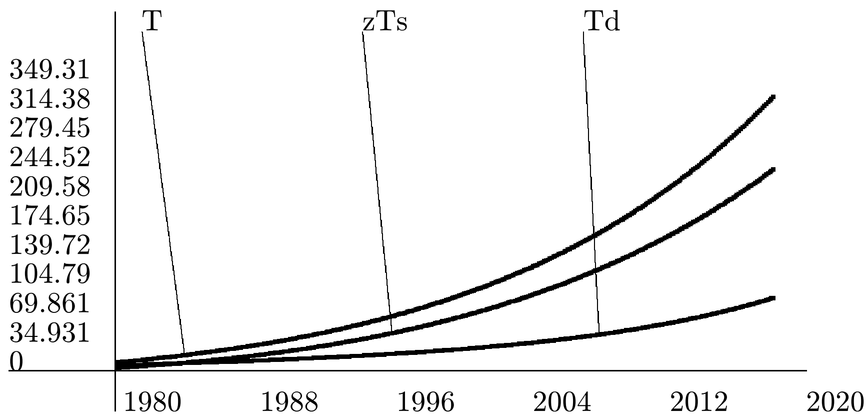

Fig. 23. Trajectories of the optimal technologies stocks for Kokusai Company.

In contrast to the previous version of the model the new modification has two control parameters and allows to investigate the optimal balance between investments into domestic technology and assimilated technology. Solution of the optimal control problem shows that more profitable for a firm to have larger share of investment into absorbed technology in comparison with proportions of the real data on investments, especially, for firms with small high-tech levels.

7 Conclusions

The goal of this paper is to elaborate a dynamical model, which describes impacts of technology stock and technology spillover effect on production of a firm. Dynamics of the model combines the growth trend generated by the exponential term of technology intensity and the decline trend provided by the risky factor of technology innovation. The optimal control problem is formulated in the framework of the model and the maximum principle of Pontryagin is applied to its solution. Due to the uniqueness of the optimal solution we apply numerical methods to design the optimal levels of production, technology stock, R&D intensity, and assimilation capacity.

The sensitivity analysis of the model with respect to discounted marginal productivity of technology and elasticity of substitution is carried out. It is shown that in the case of quickly increasing discounted marginal productivity of technology the optimal R&D intensity reduces swiftly. We demonstrate that if the elasticity of substitution is close to the value 1 then the level of optimal R&D intensity grows rapidly.

The multi-variant scenario analysis of the model for the Sony Corporation is carried out. It demonstrates advantages of efficient utilization of absorbed technology in manufacturing. Computer simulations show that the impact of assimilation capacity is essential.

We compare the optimal trajectories with the real data series in the interval since 1980 till 1999 and show that the real development of the Sony Corporation was close to the optimal synthetic trajectories of the proposed model. We also compare the optimal solutions for two firms from the Electric Machinery Industry of Japan and demonstrate that the optimal strategy is very sensitive to the initial stage of a firm. Our analysis shows that for firms with low level of domestic technology it is optimal to increase the share of investments into technology absorption.

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