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DUAL SYSTEMS OF DYNAMIC LINEAR PROGRAMMING

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PREFACE

Many problems which are of interest to applied IIASA projects can be formulated in the framework of dynamic linear programming (DLP). Examples might be long-range energy, water, etc. supply models, problems of national settlement planning, the choice of an optimal mix of agricultural technologies for the long-range development of a region, manpower planning models, resource allocation models for health planning, etc.

To develop computer methods for solving DLP problems some theoretical background is necessary--first of all the investigation of duality relations for dynamic linear programs. While the primal problem usually seeks to assign optimum values to the control variables so as to maximize outputs, the dual problem seeks to assign marginal values (shadow prices) to the constrained inputs. Thus the dual problem can be a valuable guide in planning systems development.

This paper is one in the IIASA publication series devoted to dynamic linear programming. The paper aims at the systematic presentation of the theoretical properties of DLP and concerns duality relations and optimality conditions which provide a basis for computational methods of DLP as well as for the interpretation of dual problems in practical cases.

ABSTRACT

A pair of dual problems are formulated and the relations between them established. Optimality conditions, including the maximum principle for primal and the minimum principle for dual problems are obtained. Results are first forumlated for a cononical form of dynamic linear programming and then modifications and particular cases are considered.

Dual Systems of Dynamic Linear Programming

INTRODUCTION

At present, the methods and fields of application for (static) linear programming (LP) are clearly delineated and are quite well known ([1,2]). Dynamic problems of LP arise when a program or plan of optimal development of a system is required. Any static problem of LP can, in principle, also admit a dynamic variant, the latter being of growing importance in practice. Thus, dynamic linear programming (DLP) is a new stage of linear programming development.

There is abundant literature concerning DLP problems (see for example [1-8]). But in the majority of cases, dynamical problems are treated by conventional LP methods. Direct application of these to problems of this kind does not usually produce the required result: the LP problems arrived at are so large, that they cannot be solved even by using the most up-to-date computers. Special methods, therefore, are required to solve DLP problems, which take into account their dynamic origin and employ both methods of optimal control theory and linear programming.

The formulation of DLP as an optimal control problem was made in [9-11] and some optimality conditions and duality theorems were obtained there.

The aim of this paper is a systematic presentation of theoretical properties of DLP, concerning duality relations and optimality conditions.

The pair of dual problems are formulated and the relations between them are obtained. From these relations, optimality conditions, including maximum principle for primal and minimum principle for dual problems, are derived and provide a basis for computational methods of DLP. The results are formulated for a canonical form of DLP, then modifications of the canonical form are considered.

DYNAMIC LINEAR PROGRAMS IN CANONICAL FORM

In formulating DLP problems, it is useful to single out:

(i) *state equations* of the systems with the distinct separation of *state* and *control* variables;

(ii) *constraints* imposed on these variables;

(iii) *planning period* T - the number of stages during which the system is considered;

(iv) *performance index* (objective function) which quantifies the quality of a program.

State Equations

These have the form

$$x(t+1) = A(t)x(t) + B(t)u(t) + s(t) \quad , \tag{1}$$

where the vector $x(t) = \{x_1(t),\ldots,x_n(t)\}$ defines the state of the system at stage t in the state space X, which is supposed to be the n-dimensional Euclidean space; the vector $u(t) = \{u_1(t),\ldots,u_r(t)\} \in E^r$ (r-dimensional Euclidean space) specifies the controlling action at stage t; the vector $s(t) = \{s_1(t),\ldots,s_n(t)\}$ defines the external effects on the system (uncontrolled, but known *a priori* in the deterministic models).

Constraints

In rather general form, constraints imposed on the state and control variables may be written as

$$G(t)x(t) + D(t)u(t) \leq f(t) \quad , \tag{2}$$

$$u(t) \geq 0 \quad , \tag{3}$$

where $f(t) = \{f_1(t),\ldots,f_m(t)\}$ is a given vector.

Planning Period

The planning period T is supposed to be fixed. It is also assumed that the initial state of the system is

$$x(0) = x^0 \quad . \tag{4}$$

Performance Index

The performance index (which is to be maximized for certainty) has the form

$$J_1(u) = (a(T),x(T)) + \sum_{t=0}^{T-1} [(a(t),x(t)) + (b(t),u(t))] \quad , \qquad (5)$$

where (a,b) denotes the inner product of the vectors a,b.

Definitions:

(i) The vector sequence $u = \{u(0),\ldots,u(T-1)\}$ is *a control (program)* of the system.

(ii) The vector sequence $x = \{x(0),\ldots,x(T)\}$, which corresponds to control u from the state equations (1) with the initial state x(0), is the *system's trajectory*.

(iii) The process {u,x}, which satisfies all the constraints of the problem (i.e., (1)-(4) in this case), is *feasible*.

(iv) The feasible process $\{u^*,x^*\}$ maximizing the performance index (5) is *optimal*.

Hence, the DLP problem in its canonical form is formulated as follows:

Problem 1: To find a control $u = \{u(0),\ldots,u(T-1)\}$ and a trajectory $x = \{x(0),\ldots,x(T)\}$, satisfying the state equation

$$x(t+1) = A(t)x(t) + B(t)u(t) + s(t)$$

with the initial condition

$$x(0) = x^0$$

and the constraints

$$G(t)x(t) + D(t)u(t) \leq f(t)$$

$$u(t) \geq 0 \quad ,$$

which maximize the performance index

$$J_1(u) = (a(T),x(T)) + \sum_{t=0}^{T-1} [(a(t),x(t)) + (b(t),u(t))] \; .$$

In the above, vectors x^0, s(t), a(t), b(t), f(t) and the matrices A(t), B(t), G(t), D(t) are of dimensions $(1 \times n)$, $(1 \times n)$, $(1 \times n)$, $(1 \times r)$, $(1 \times m)$ and $(n \times n)$, $(n \times r)$, $(m \times n)$, $(m \times r)$, respectively, and are supposed to be given.

Remarks:

(i) For T = 1, Problem 1 becomes a conventional LP problem.

(ii) The choice of a canonical form of the problem is to some extent arbitrary, and various modifications and particular cases of Problem 1 are possible. In the next section some of these will be considered.

MODIFICATIONS OF PROBLEM 1

Some particular cases and modifications of Problem 1 are given in Table 1. In state equations, for example, matrices A, B, and/or s cannot depend on the number of stage t (I.2) or external disturbance s(t) may vanish. Equations (I.3) are obtained, for example, by considering the difference approximation of the continuous analog of Problem 1.

An important class of DLP are systems with delays in state and/or control variables (I.4). Here, $\{n_1,\ldots,n_\nu\}$, $\{m_1,\ldots,m_\mu\}$ are the sets of integers; in particular, when $\{n_1,\ldots,n_\nu\} = \{0\}$, $\{m_1,\ldots,m_\mu\} = \{0\}$ a conventional system is obtained (I.1).

Constraints on the state and control variables can have the form of equalities (II.2) or can be separate ((II.3), (II.4)). These variables can have additional restrictions on their sign ((II.5), (II.6)). In some cases, the constraints should be considered in the summarized form ((II.7) or (II.8)).

It is useful to single out the constraints on the left and/ or right side of the trajectory (the boundary conditions). For example, the left and/or right side of the trajectory can be fixed ((III.1), (III.3)) or free ((III.2), (III.4)).

The number of steps T of the planning period can be fixed (IV.1) or may be defined by some conditions on the terminal state (i.e., (II.3), (II.5) for t = T).

The value of the performance index can depend only on the trajectory $\{x(t)\}$ (V.4) or on the control sequence $\{u(t)\}$ (V.3) or can even be determined only by the terminal state x(T) of the trajectory (V.2).

In connection with Table 1, we shall consider the following modifications of Problem 1 (Table 2).

Problem 1a (with terminal performance index): In this problem, the performance index (V.1) has been changed on (V.2). Besides, $s(t)$ and $f(t)$ vanish for all t.

Problem 1b (with equality constraints): For this problem, the variable constraints are of equality form (II.2).

Problem 1c (without state constraints): For this problem, constraints (II.4), (II.6) are imposed only on control variables. For the linear performance index, the problem is trivial, but it is of significance when the objective function is concave (in particular, non-positive quadratic) [10].

Problem 1d (non-fixed planning period): Here, the number of stages T is not fixed but determined by the condition (III.3): $x(T) = x_T$.

Problem 1e: For this problem, variable constraints are of the form (II.8).

Of course, Tables 1 and 2 do not present all the modifications for Problem 1 and, naturally, Problems 1-1e do not present the total set of the possible DLP problems. (For example, DLP problems with delays (I.4) require special consideration). But on the whole, they characterize the main features of DLP problems.

Any problem stated above can be transferred into the other (excluding Problem 1d with the integer variable T) [10].

For example, let us consider Problems 1 and 1a. Introducing a new variable $x_0(t)$ $(t = 0,\dots,T)$, subject to

$$x_0(t+1) = x_0(t) + (a(t),x(t)) + (b(t),u(t)) \quad ,$$

$$x_0(0) = 0 \quad ,$$

one can see that

$$x_0(T) = \sum_{t=0}^{T-1} [(a(t),x(t)) + (b(t),u(t))] \quad .$$

So Problem 1 will have a form of Problem 1a with the performance index

$$J_1 = (\tilde{a}(T),\tilde{x}(T))$$

and the state equations

$$\tilde{x}(t+1) = \tilde{A}(t)\tilde{x}(t) + \tilde{B}(t)u(t) + s(t) \quad ,$$

where $\tilde{a}(T) = \{1\ a(T)\} \in E^{n+1}$, $\tilde{x}(t) = \{x_0(t)\ x(t)\} \in E^{n+1}$ $(t=1,\ldots,T)$. $\tilde{x}(0) = \{0, x^0(0)\}$,

$$\tilde{A}(t) = \begin{pmatrix} 1 & a(t) \\ 0 & A(t) \end{pmatrix} , \qquad \tilde{B}(t) = \begin{pmatrix} 0 & b(t) \\ 0 & B(t) \end{pmatrix} .$$

If we consider the constraints (II.8) (Problem 1e) and introduce the variables $x_{n+i}(t)$ $(i = 1,\ldots,m)$, subject to state equations

$$x_{n+i}(t+1) = x_{n+i}(t) + [G(t)x(t) + D(t)u(t)]_i , \quad x_{n+i}(0) = 0 \ (i=1,\ldots,m),$$

where $[z]_i$ is the ith component of a vector z, then we obtain Problem 1 with equations

$$\tilde{x}(t+1) = \tilde{A}(t)\tilde{x}(t) + \tilde{B}(t)u(t) + s(t)$$

and only the terminal constraint

$$\tilde{G}(T)\tilde{x}(T) \leq \tilde{f} \quad ,$$

where $\tilde{x} = \{x_1(t),\ldots,x_n(t);\ x_{n+1}(t),\ldots,x_{n+m}(t)\} \in E^{n+m}$,
$\tilde{f} = \{0,\ldots,0,f_1,\ldots,f_m\}$,

$$\tilde{A}(t) = \begin{pmatrix} A(t) & 0 \\ G(t) & I \end{pmatrix} , \quad \tilde{B}(t) = \begin{pmatrix} B(t) & 0 \\ D(t) & 0 \end{pmatrix} , \quad \tilde{G}(T) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} ,$$

where I and 0 are the identity and zero matrices of proper dimensions.

These reasonings show that in many practical cases it is sufficient to investigate only Problem 1.

DUALITY RELATIONS

As we return to Problem 1, we can see that this problem can be considered as a certain "large" LP problem, with constraints on its variables in the form of equalities (1),(4) and inequalities (2),(3). Let us introduce the Lagrange function for Problem 1:

$$L(u,x;\lambda,p) = (a(T),x(T)) + \sum_{t=0}^{T-1} [(a(t),x(t)) + (b(t),u(t)] \tag{6}$$
$$+ \sum_{t=0}^{T-1} (p(t+1),A(t)x(t) + B(t)u(t) + s(t) - x(t+1))$$
$$+ \sum_{t=0}^{T-1} (\lambda(t),f(t) - G(t)x(t) - D(t)u(t)) + (p(0),x^0 - x(0))$$

In the above, $p(t) \in E^n (t = T,\ldots,0)$, $\lambda(t) \in E^m$, $\lambda_i(t) \geq 0$ $(i = 1,\ldots,m;\ t = T-1,\ldots,0)$ are the Lagrange multipliers for the constraints (1), (4), and (2) respectively.

Employing the Lagrange function (6) the following subproblems are considered:

$$\sup_{\substack{x \\ u\geq 0}} \inf_{\substack{p \\ \lambda\geq 0}} L(u,x;\lambda,p) = \omega_1 \quad , \tag{7}$$

$$\inf_{\substack{p \\ \lambda\geq 0}} \sup_{\substack{x \\ u\geq 0}} L(u,x;\lambda,p) = \omega_2 \quad . \tag{8}$$

The problems (7), (8) will be studied separately. It is assumed that an optimal process (solution) of the original Problem 1 exists and is denoted by $\{u^*,x^*\}$.

Lemma 4.1. Any solution $\{u^*,x^*\}$ of Problem 1 is also a solution of (7) with $J_1(u^*) = \omega_1$.

If $\omega_1 > -\infty$, then any solution of (7) is a solution of Problem 1; otherwise the system of constraints (1)-(4) is inconsistent.

The proof, being a standard one in mathematical programming, is omitted here (see, for example, [12]).

Now let us rewrite the Lagrange function in the "dual" form

$$L(\lambda,p;u,x) = (a(T) - p(T),x(T)) + \sum_{t=T-1}^{0} ((p(t+1)A(t) - \lambda(t)G(t) + a(t)$$

$$- p(t),x(t)) + \sum_{t=T-1}^{0} (p(t+1)B(t) - \lambda(t)D(T) + b(t),u(t))$$

$$+ \sum_{t=T-1}^{0} [(p(t+1),s(t)) + (\lambda(t),f(t))] + (p(0),x^0) \quad ,$$

and consider the following dual problem.

Problem 2: To find a dual control $\lambda = \{\lambda(T-1),\ldots,\lambda(0)\}$ and a dual trajectory $p = \{p(T),\ldots,p(0)\}$ such that they satisfy the co-state equations

$$p(t) = A^T(t)p(t+1) - G^T(t)\lambda(t) + a(t) \; (t = T-1,\ldots,0) \tag{9}$$

with the boundary condition

$$p(T) = a(T) \tag{10}$$

and constraints

$$-B^T(t)p(t+1) + D^T(t)\lambda(t) \geq b(t) \tag{11}$$

$$\lambda(t) \geq 0 \quad , \tag{12}$$

which minimize the dual performance index

$$J_2(\lambda) = (p(0),x^0) + \sum_{t=0}^{T-1} [(p(t+1),s(t) + (\lambda(t),f(t))] \quad . \tag{13}$$

Here transposition is denoted by T.

We shall call Problems 1 and 2 a pair of dual problems. It should be noted that dual Problem 2, as well as primary Problem 1, is a control problem, in which the variable $\lambda(t)$ specifies the dual controlling action at the state t, the variable $p(t)$ is the dual state (co-state) at the stage t; in the dual problem, time is taken in the reversed direction: $t = T-1,\ldots,1,0$. .

So the following definitions are natural:

(i) the system, which is described by co-state equations (9),(10), is the *dual (conjugate) system* (to (1),(4));

(ii) the vector sequence $\lambda = \{\lambda(T-1),\dots,\lambda(0)\}$ is a *dual control*; the vector sequence $p(t) = \{p(T),\dots,p(0)\}$ is a *dual (conjugate) trajectory*;

(iii) the process $\{\lambda,p\}$, which satisfies all the constraints of Problem 2 (i.e. (9)-(12)), is *dual feasible*;

(iv) the feasible process $\{\lambda^*,p^*\}$, minimizing the performance index (13), is *dual optimal*.

The following proposition is proved in a similar manner to Lemma 4.1.

Lemma 4.2. Any solution $\{\lambda^*,p^*\}$ *of Problem 2 is also a solution of the problem (8) with* $J_2(\lambda^*) = \omega_2$.

If $\omega_2 < \infty$ *then any solution of (8) is a solution of Problem 2; otherwise the system of constraints (9)-(12) is inconsistent.*

Now we shall consider the relations between the dual Problems 1 and 2.

First of all, the following assertion results from Lemmas 4.1 and 4.2 and the known relation of the games theory [13]:

Theorem 4.1. For any controls u *and* λ *of the primary and dual Problems 1 and 2, the inequality* $J_1(u) \le J_2(\lambda)$ *is held, where the values of* $J_1(u)$ *and* $J_2(\lambda)$ *are computed from (5) and (13), by using (1),(4) and (9),(10).*

The following assertions show that for optimal controls u^* and λ^*, the inequality of Theorem 4.1 becomes an equality.

Lemma 4.3. (cf. [14]). The necessary and sufficient condition that $\{u^* \ge 0, x^*\}$ *and* $\{\lambda^* \ge 0, p^*\}$ *be the optimal processes for the dual Problems 1 and 2 is that* $\{u^*,x^*;\lambda^*,p^*\}$ *be a saddle point for the Lagrange function (6), that is*

$$L(u^*,x^*;\lambda,p) \ge L(u^*,x^*;\lambda^*,p^*) \ge L(u,x;\lambda^*,p^*) \quad .$$

If $\{u^*,x^*\}$ *and* $\{\lambda^*,p^*\}$ *are optimal, then* $L(u^*,x^*;\lambda^*p^*)$ *is the optimal value of the performance indexes of dual Problems 1 and 2.*

Theorem 4.2. (Duality Theorem.) If one of the dual Problems 1 and 2 has an optimal control, then the other has an optimal control as well, and the values of the performance indexes of the primary and dual Problems 1 and 2 are equal:

$$J_1(u^*) = J_2(\lambda^*) \quad .$$

If the performance index either of Problem 1 or 2 is unbounded (for Problem 1 from above and for Problem 2 from below), then the other problem has no feasible control.

The proof of Theorem 4.2 can be obtained in many ways. In particular, one can apply the duality theory of "static" LP ([1, 14]) to Problem 1, regarding it as a static LP problem with constraints on the variables u(t) and x(t), both in the form of equalities (1),(4) and inequalitites (2),(3), or, using the dynamic programming approach, one can reduce Problem 1 to a recurrent sequence of static linear programming problems and apply the duality theorem to them successively.

From the basic dual Theorem 4.2, the next optimality and existence conditions follow for Problems 1 and 2.

Theorem 4.3. A feasible control u^* is optimal if and only if there is a feasible λ^* with $J_2(\lambda^*) = J_1(u^*)$. A feasible control λ^* is optimal if and only if there is a feasible primary control u^* with $J_1(u^*) = J_2(\lambda^*)$.

Theorem 4.4. (Existence Theorem.) A necessary and sufficient condition that one (and thus both) of the dual Problems 1 and 2 have optimal controls is that both have feasible controls.

One can see that the above theorems are similar to their static analogs ([1,12,14]). But in the dynamic case, the assertions stated below are more important, because they realize, in a sense, a decomposition of the problem. They permit the effective optimality conditions for the problems being obtained.

OPTIMALITY CONDITIONS

Let us introduce the Hamilton function

$$H_1(p(t+1),u(t)) = (b(t),u(t)) + (p(t+1),B(t)u(t)) \qquad (14)$$

for the primary Problem 1 and

$$H_2(x(t),\lambda(t)) = (\lambda(t),f(t)) - (\lambda(t),G(t)x(t)) \qquad (15)$$

for the dual Problem 2.

Lemma 5.1. For any feasible controls u and λ the equality

$$J_1(u) - J_2(\lambda) = \sum_{t=0}^{T-1} [H_1(p(t+1),u(t)) - H_2(x(t),\lambda(t))]$$

is valid.

Proof: Let us consider the difference

$$J_1(u) - J_2(\lambda) = (a(T),x(T)) + \sum_{t=0}^{T-1} [(a(t),x(t)) + (b(t),u(t))]$$

$$- \sum_{t=0}^{T-1} [(p(t+1),s(t)) + (\lambda(t),f(t))] - (p(0),x^0) \quad .$$

Inserting the value x(T), defined by the primary system (1), when t = T - 1, and using the definitions of the dual system (9) and Hamilton functions (14), (15), one can obtain

$$\begin{aligned}
J_1(u) - J_2(\lambda) &= (a(T),A(T-1)x(T-1)) \\
&\quad + (a(T),B(T-1)u(T-1)) + (a(T),s(T-1)) \\
&\quad + (a(T-1),x(T-1)) + (b(T-1),u(T-1)) \\
&\quad + \sum_{t=0}^{T-2} \Big[(a(t),x(t)) + (b(t),u(t))\Big] \\
&\quad - (p(T),s(T-1)) - (\lambda(T-1),f(T-1)) \\
&\quad - \sum_{t=0}^{T-2} \Big[(p(t+1),s(t)) + (\lambda(t),f(t))\Big] \\
&\quad - (p(0),x^0) \\
&= H_1(p(T),u(T-1)) - H_2(x(T-1),\lambda(T-1)) \\
&\quad + (p(T-1),x(T-1)) \\
&\quad + \sum_{t=0}^{T-2} \Big[(a(t),x(t)) + (b(t),u(t))\Big] \\
&\quad - \sum_{t=0}^{T-2} \Big[(p(t+1),s(t)) + (\lambda(t),f(t))\Big] \\
&\quad - (p(0),x^0) = \dots \\
&= \sum_{t=0}^{T-1} \Big[H_1(p(t+1),u(t)) - H_2(x(t),\lambda(t))\Big] \quad .
\end{aligned}$$

In the section on modifications of Problem 1, the relations between the objective functions of the primary and dual problems, which characterize the problem as a whole, were established. Now "local" duality theorems will be obtained, which establish relations between the Hamilton functions of these problems. For simplicity of statements, it is assumed that Problem 1 (and, hence, Problem 2) has a solution.

Lemma 5.2. For any feasible process $\{u,x\}$ *and* $\{\lambda,p\}$ *the following inequalities hold:*

$$H_1(p(t+1),u(t)) \leq H_2(x(t),\lambda(t))(t = 0,\ldots,T-1) \quad . \; .$$

Proof: One can obtain successively from equations (14), (12), (2), (15), (11), and (3)

$$H_1(p(t+1),u(t)) = (b(t),u(t)) + (p(t+1),B(t)u(t)) \leq (b(t),u(t))$$

$$+ (p(t+1),B(t)u(t)) + (\lambda(t),f(t) - G(t)x(t) - D(t)u(t))$$

$$= H_2(x(t),\lambda(t)) + (p(t+1)B(t) - \lambda(t)D(t) + b(t),u(t))$$

$$\leq H_2(x(t),\lambda(t)) \quad .$$

It should be noted that the statement of Theorem 4.1 also follows from Lemmas 5.1 and 5.2 for any feasible process $\{u,x\}$, $\{\lambda,p\}$.

Theorem 5.1. ("Local" Duality Theorem.) For any feasible processes $\{u^*,x^*\}$ *of the primal and* $\{\lambda^*,p^*\}$ *of the dual to be optimal it is necessary and sufficient that the values of Hamilton functions are equal:*

$$H_1(p^*(t+1),u^*(t)) = H_2(x^*(t),\lambda^*(t))(t = 0,\ldots,T-1) \quad . \; .$$

Proof: One obtains from duality Theorem 5.1 and Lemma 5.1, that for optimal processes of dual Problems 1 and 2 the equality

$$\sum_{t=0}^{T-1} H_1(p^*(t+1),u^*(t)) = \sum_{t=0}^{T-1} H_2(x^*(t),\lambda^*(t)) \tag{16}$$

is valid. Hence and from Lemma 5.2, it follows that the values of the Hamiltonians must be equal for $t = 0,1,\ldots,T-1$ in case of optimal processes $\{u^*,x^*\}$ and $\{\lambda^*,p^*\}$.

Indeed, let us assume that it is not so, that is, let for some $0 \leq t_0 \leq T - 1$:

$$H_1(p^*(t_0+1),u^*(t_0)) \leq H_2(x^*(t_0),\lambda^*(t_0)) \quad .$$

This, however, is inconsistent with the equality (16). The contradiction completes the proof of the theorem.

Considering the proof of Lemma 5.2 and the equality (16), it is not difficult to obtain that for optimality of feasible $\{u^*,x^*\}$ and $\{x^*,p^*\}$ it is necessary and sufficient that the following conditions must be satisfied:

$$(\lambda^*(t), f(t) - G(t)x^*(t) - D(t)u^*(t)) = 0$$

$$(p^*(t+1)B(t) - \lambda^*(t)D(t) + b(t), u^*(t)) = 0$$

$$(t = 0,\ldots,T-1) \quad . \; .$$

Thus we can introduce the pairs of *dual constraints:*

$$[f(t) - G(t)x(t) - D(t)u(t)]_i \geq 0 \quad \text{and} \quad \lambda_i(t) \geq 0$$

or

$$[p(t+1)B(t) - \lambda(t)D(t) + b(t)]_j \leq 0 \quad \text{and} \quad u_j(t) \geq 0$$

where variables $\{u(t),x(t)\}$ and $\{\lambda(t),p(t+1)\}$ satisfy primal (1) and dual (9) state equations with boundary constraints (4) and (10).

From the above equalities and the definitions of dual constraints one can obtain in the usual way the following "differential" optimality conditions for Problems 1 and 2 (cf. [1,2,14]).

Lemma 5.3. If both Problems 1 and 2 have feasible controls, then they have optimal controls u^*,λ^*, such that:

if u^* satisfies a constraint as an equation, then λ^* satisfies the dual constraint as a strict inequality;

if λ^* *satisfies a constraint as an equation, then* u^* *satisfies the dual constraint as a strict inequality.*

Lemma 5.4. If both Problems 1 and 2 are feasible, then for any i *either*

$$[G(t)x^*(t) + D(t)u^*(t)]_i < f_i(t)$$

for some optimal u^* *and*

$$\lambda_i^*(t) = 0$$

for every optimal λ^*;
or

$$[G(t)x^*(t) + D(t)u^*(t)]_i = f_i(t)$$

for every optimal u^* *and*

$$\lambda_i^*(t) > 0$$

for some optimal λ^*.

For any j *either*

$$[-B^T(t)p^*(t+1) + D^T(t)\lambda^*(t)]_j > b_j(t)$$

for some optimal λ^* *and*

$$u_j^*(t) = 0$$

for every optimal u^*;
or

$$[-B^T(t)p^*(t+1) + D^T(t)\lambda^*(t)]_j = b_j(t)$$

for every optimal λ^* *and*

$$u_j^*(t) > 0$$

for some optimal u^*.

The conditions stated in Lemmas 5.3 and 5.4 are similar to the complementary slackness relations in linear programming ([1, 2,14]). From these lemmas the known Kuhn-Tucker optimality conditions easily follow for Problems 1 and 2. As the assertions of the lemmas are not only necessary but also sufficient, it is not difficult to see that in order to investigate a pair of dual dynamic Problems 1 and 2 it is sufficient to consider a pair of dual "local" (static) problems of LP:

$$\begin{aligned} &\max H_1(p(t+1),u(t)) \\ &G(t)x(t) + D(t)u(t) \le f(t) \\ &\qquad\qquad u(t) \ge 0 \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\min H_2(x(t),\lambda(t)) \\ &-B^T(t)p(t+1) + D^T(t)\lambda(t) \ge b(t) \\ &\qquad\qquad \lambda(t) \ge 0 \end{aligned} \tag{18}$$

interrelated by the primary (1) and dual (9) state equations with boundary conditions (4) and (10).

So, any of the "static" duality relations or LP optimality conditions ([1,2,12,14]) for the pair of the dual LP problems (17) and (18) linked by the state equations (1),(4) and (9),(10) determine the corresponding optimality conditions for the pair of the dual DLP Problems 1 and 2. Such conditions have been formulated above; in a similar manner the following important optimality conditions are obtained.

Theorem 5.2. (Maximum principle for primary Problem 1.) For a control u^* to be optimal in the primary Problem 1, it is necessary and sufficient that there exists a feasible process $\{\lambda^*,p^*\}$ of the dual Problem 2, such that for $t = 0,1,\ldots,T-1$ the equality

$$\max H_1(p^*(t+1),u(t)) = H_1(p^*(t+1),u^*(t))$$

holds, where the maximum is taken over all $u(t)$, satisfying the constraints (2),(3), and $\lambda^*(t)$ is the optimal dual variable in the LP problem (18).

Theorem 5.3. (Minimum principle for dual Problem 2.) For a control λ^* to be optimal in the dual Problem 2, it is necessary *and sufficient that there exists a feasible process* $\{u^*, x^*\}$ *of the primary Problem 1, such that for* $t = 0,1,\ldots,T-1$ *the equality*

$$\min H_2(x^*(t),\lambda(t)) = H_2(x^*(t),\lambda^*(t))$$

holds, where the minimum is taken over all $\lambda(t)$*, satisfying the constraints (9),(10), and* $u^*(t)$ *is the optimal primary variable in the LP problem (17).*

These theorems can also be obtained by using the corresponding optimality conditions for discrete control systems [10].

CONCLUSION

We can summarize the optimality conditions for the dual DLP problems stated above as follows.

1. The values of performance indexes for the pair of dual problems are equal: $J_1(u^*) = J_2(\lambda^*)$ (Theorem 4.2).

2. The processes $\{u^*,x^*\}$, $\{\lambda^*,p^*\}$ are the saddle point for the Lagrange function (6) (Lemma 4.3).

3. The dual controls $\{u^*,\lambda^*\}$ are linked by the conditions of complementary slackness: $u_i^*(t) = 0$, if the dual constraint is a strict inequality, and the dual constraint is an equality, if $u_i^*(t) > 0$; $\lambda_j^*(t) = 0$, if the primary constraint is a strict inequality, and the primary constraint is an equality, if $\lambda_j^*(t) > 0$ (Lemmas 5.3 and 5.4).

4. The values of Hamilton functions for the pair of dual problems are equal: $H_1(p^*(t+1),u^*(t)) = H_2(x^*(t),\lambda^*(t))$ (Theorems 5.2 and 5.3).

5. The Hamilton functions achieve their extreme values for controls u^* and λ^* (Theorems 5.2 and 5.3).

6. The pair $\{u^*(t),x^*(t)\}$, $\{\lambda^*(t),p^*(t+1)\}$ is a saddle point for the "local" Lagrange function: $L(t) = (b(t),u(t)) + (p(t+1)B(t),u(t)) + (\lambda(t),f(t)) + (\lambda(t),G(t)x(t)) + (\lambda(t),D(t)u(t))$ of the problems (17) and (18).

In all conditions above, the variables $\{u(t),x(t)\}$, $\{x(t),p(t+1)\}$ are supposed to be connected by the primary ((1),(4)) and the dual ((9),(10)) state equations.

One can see that the conditions 1-3 are of "global" nature, while the conditions 4-6 are of "local", decomposable nature, and reduce solving of "global" dual Problems 1 and 2 to successive solution of "local" dual problems (17) and (18), linked by the primary ((1),(4)) and dual ((9),(10)) state equations.

For transition from the primary problem to the dual one, Table 3 can be used. Duals for the Problems 1-1e are given in Table 2.

In Table 2 the dual Problems 1(2) and 1a(2a) are fully symmetrical. The Problem 1c without state constraints has no dual control and completely decomposed into T unlinked static LP problems. The coincidence condition of the Hamiltonians (condition 4) for the Problems 1e and 2e is the following:

$$\sum_{t=0}^{T-1} H_1(p(t+1),u(t)) = H_2(\lambda) \quad .$$

The duality relations stated above and the resulting optimality conditions have a clear economic interpretation (partly given in [11] and expected to be considered in a separate paper). These conditions provide a basis for the construction of numerical methods, but analysis of such methods is outside the framework of the present paper.

Table 1.

I. State Equations

(I.1) $x(t+1) = A(t)x(t) + B(t)u(t) + s(t)$

(I.2) $x(t+1) = Ax(t) + Bu(t) + s$

(I.3) $x(t+1) = x(t) + A(t)x(t) + B(t)u(t) + s(t)$

(I.4) $x(t+1) = \sum_{i=1}^{\nu} A(t-n_i)x(t-n_i) + \sum_{j=1}^{\mu} B(t-m_j)u(t-m_j)$

II. Constraints

(II.1) $G(t)x(t) + D(t)u(t) \leq f(t)$

(II.2) $G(t)x(t) + D(t)u(t) = f(t)$

(II.3) $G(t)x(t) \leq f^{(1)}(t)$

(II.4) $D(t)u(t) \leq f^{(2)}(t)$

(II.5) $x(t) \geq 0$

(II.6) $u(t) \geq 0$

(II.7) $\sum_{\tau=0}^{t-1} [G(\tau)x(\tau) + D(\tau)u(\tau)] \leq f(t) \qquad (t = 1,\ldots,T)$

(II.8) $\sum_{t=0}^{T-1} [G(t)x(t) + D(t)u(t)] \leq f$

III. Boundary Conditions

(III.1) $x(0) = x^0$

(III.2) $x(0)$ is free

(III.3) $x(T) = x_T$

(III.4) $x(T)$ is free

Table 1 Cont.

IV. Planning Period

(IV.1) T is fixed

(IV.2) T is free

V. Performance Indexes

(V.1) $J_1(u) = (a(T),x(T)) + \sum_{t=0}^{T-1} [(a(T),x(t)) + (b(t),u(t))]$

(V.2) $J_1(u) = (a(T),x(T))$

(V.3) $a(t) = 0 \quad (t = 0,...,T)$

(V.4) $b(t) = 0 \quad (t = 0,...,T-1)$

Table 2.

Primal	Dual
Problems 1(2)	
State Equations	
$x(t+1) = A(t)x(t) + B(t)u(t) + s(t)$	$p(t) = A^T(t)p(t+1) - G^T(t)\lambda(t) + a(t)$
$x(0) = x^0 \quad (t = 0,\ldots,T-1)$	$p(t) = a(T) \quad (t = T-1,\ldots,1,0)$
Constraints	
$G(t)x(t) + D(t)u(t) \leq f(t)$	$-B^T(t)p(t+1) + D^T(t)\lambda(t) \geq b(t)$
$u(t) \geq 0$	$\lambda(t) \geq 0$
Performance Indexes	
$J_1 = (a(T),x(T)) + \sum_{t=0}^{T-1} [(a(t),x(t)) + (b(t),u(t))]$	$J_2 = (p(0),x^0) + \sum_{t=0}^{T-1} [(p(t+1),s(t)) + (\lambda(t),f(t))]$
Hamilton Functions	
$H_1 = (b(t),u(t)) + (p(t+1)B(t),u(t))$	$H_2 = (\lambda(t),f(t)) - (\lambda(t),G(t)x(t))$
Problems 1a(2a)	
State Equations	
$x(t+1) = A(t)x(t) + B(t)u(t)$	$p(t) = A^T(t)p(t+1) - G^T(t)\lambda(t)$
$x(0) = x^0$	$p(T) = a(T)$

Table 2 cont.

Primal	Dual
Constraints	
$G(t)x(t) + D(t)u(t) \leq 0$	$-B^T(t)p(t+1) + D^T(t)\lambda(t) \geq 0$
$u(t) \geq 0$	$\lambda(t) \geq 0$
Performance Indexes	
$J_1 = (a(T), x(T))$	$J_2 = (p(0), x^0)$
Hamilton Functions	
$H_1 = (p(t+1)B(t), u(t))$	$H_2 = -(\lambda(t), G(t)x(t))$

<u>Problems 1b(2b)</u>

Primal	Dual
State Equations	
$x(t+1) = A(t)x(t) + B(t)u(t) + s(t)$	$p(t+1) = A^T(t)p(t+1) - G^T(t)\lambda(t) + a(t)$
$x(0) = x^0$	$p(T) = a(T)$
Constraints	
$G(t)x(t) + D(t)u(t) = f(t)$	$-B^T(t)p(t+1) + D^T(t)\lambda(t) \geq b(t)$
$u(t) \geq 0$	
Performance Indexes	
$J_1 = (a(T),x(T)) + \sum_{t=0}^{T-1} [(a(t),x(t)) + (b(t),u(t))]$	$J_2 = (p(0),x^0) + \sum_{t=0}^{T-1} [(p(t+1),s(t)) + (\lambda(t),f(t))]$

Table 2 cont.

Primal	Dual

Hamilton Functions

$H_1 = (b(t),u(t)) + (p(t+1)B(t),u(t))$ $\qquad$ $H_2 = (\lambda(t),f(t)) - (\lambda(t),G(t)x(t))$

<u>Problems 1c(2c)</u>

State Equations

$x(t+1) = A(t)x(t) + B(t)u(t) + s(t)$ $\qquad$ $p(t) = A^T(t)p(t+1) + a(t)$

$x(0) = x^0$ $\qquad$ $p(T) = a(T)$

Constraints

$D(t)u(t) \leq f(t)$ $\qquad$ $-B^T(t)p(t+1) + D^T(t)\lambda(t) \geq b(t)$

$u(t) \geq 0$ $\qquad$ $\lambda(t) \geq 0$

Performance Indexes

$J_1 = (a(T),x(T)) + \sum_{t=0}^{T-1} [(a(t),x(t)) + (b(t),u(t))]$ $\qquad$ $J_2 = (p(0),x^0) + \sum_{t=0}^{T-1} [(p(t+1),s(t)) + (\lambda(t),f(t))]$

Hamilton Functions

$H_1 = (p(t+1),B(t)u(t))$ $\qquad$ $H_2 = (\lambda(t),f(t))$

<u>Problems 1d(2d)</u>

State Equations

$x(t+1) = A(t)x(t) + B(t)u(t) + s(t)$ $\qquad$ $p(t) = A^T(t)p(t+1) - G^T(t)\lambda(t) + a(t)$

$x(0) = x^0 \quad x(T) = x_T$ $\qquad$ $p(T) + \lambda(T) = a(T)$

Table 2 cont.

Primal	Dual
Constraints	
$G(t)x(t) + D(t)u(t) \leq f(t)$	$-B^T(t)p(t+1) + D^T(t)\lambda(t) \geq b(t)$
$u(t) \geq 0$	$\lambda(t) \geq 0$
Performances Indexes	
$J_1 = (a(T),x(T)) + \sum_{t=0}^{T-1} [(a(t),x(t)) + (b(t),u(t))]$	$J_2 = (p(0),x^0) + \sum_{t=0}^{T-1} [(p(t+1),s(t)) + (\lambda(t),f(t))] + (\lambda(T),x_T)$
Hamilton Functions	
$H_1 = (b(t),u(t)) + (p(t+1),B(t)u(t))$	$H_2(t) = (\lambda(t),f(t)) - (\lambda(t),G(t)x(t))$
	$H_2(T) = (\lambda(T),x_T)$

<u>Problems 1e(2e)</u>

Primal	Dual
State Equations	
$x(t+1) = A(t)x(t) + B(t)u(t)$	$p(t) = A^T(t)p(t+1) - G^T(t)\lambda$
$x(0) = x^0$	$p(T) = a(T)$
Constraints	
$\sum_{t=0}^{T-1} [G(t)x(t) + D(t)u(t)] \leq f$	$-B^T(t)p(t+1) + D^T(t)\lambda \geq 0$
$u(t) \geq 0$	$\lambda \geq 0$
Performance Indexes	
$J_1 = (a(T),x(T))$	$J_2 = (p(0),x^0) + (\lambda,f)$
Hamilton Functions	
$H_1 = (p(t+1)B(t),u(t))$	$H_2 = (\lambda,f) - (\lambda, \sum_{t=0}^{T-1} G(t)x(t))$

Table 3.

	Primal	Dual
State variable	$x(t)$	$p(t+1)$
Control variable	$u(t)$	$\lambda(t)$
State transform matrix	$A(t)$	$A^T(t)$
Control transform matrix	$B(t)$	$-G^T(t)$
State constraint matrix	$G(t)$	$-B^T(t)$
Control constraint matrix	$D(t)$	$D^T(t)$
Constraint vector	$f(t)$	$b(t)$
External disturbance	$s(t)$	$a(t)$
Price state vector	$a(t)$	$s(t)$
Price control vector	$b(t)$	$f(t)$
Boundary condition	x^0	$a(T)$
Time	$\{0,1,\ldots,T-1\}$	$\{T-1,\ldots,1,0\}$

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