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POWER, PRICES, AND INCOMES IN VOTING SYSTEMS†

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PREFACE

One of the tasks of the System and Decision Sciences Area in 1976 has been the investigation of problems in "fair division". In a general sense, the problem is how to divide and distribute various goods (or bads) equitably among competing agents in a system. A particularly important aspect of this question is the institutions through which distributional decisions are made. In particular, what are the consequences of different divisions of decision-making powers? This paper addresses the problem of measuring the relative effectiveness of agents in organizations where decisions are taken by vote. The results have application to the estimation of inequalities in, and the equity of, various distributions of decision-making authority.

SUMMARY

One of the important aspects of the structure of decision-making institutions is the implication this structure has for the interdependencies among agents and their relative abilities to influence choices and outcomes, i.e. their "effectiveness". Various measures of interdependency and effectiveness have been proposed, notably by Shapley and Shubik, Banzhaf, and Coleman. In this paper a new approach is obtained by proposing a kind of "currency" in which structural influence can be traded; this enables one to apply economic concepts and show that in general a trading equilibrium exists whose properties have a natural interpretation for measuring the relative effectiveness of the various decision-making agents.

The "value" or "worth" of a man is, as of all other things, his price, that is to say, so much as would be given for the use of his power.

Thomas Hobbes
Leviathan, Pt. I, Ch. 10

Power, Prices, and Incomes In Voting Systems

INTRODUCTION

The bribing of legislatures and other decision-making bodies for the furtherance of special interests has a long history that doubtless has not ceased to the present day. One of the most notorious alleged cases of bribery was reported by the Scotsman George Lockhart in his *Memoirs of the Affairs of Scotland* [4]. Lockhart charged that the Treaty of Union, which created Great Britain in 1707, was achieved by the Queen's Minister selectively bribing members of the Scottish Parliament. The most astonishing aspect of Lockhart's account is that he published, in 1714, a list of the members bribed and the prices paid. The list contains 32 names, including the Duke of Queensberry, who allegedly got £12325 0s 0d, the Earl of Marchmont £1104 15s 7d, the Marquis of Tweeddale £1000 0s 0d, and so on, the lowest man going for £11 2s 0d. All of those allegedly bribed except one (the Duke of Atholl) voted for the Union. Lockhart concludes bitterly, "It is abundantly disgraceful to be ... a contributor to the misery and ruin of one's native country; but for persons of quality and distinction to sell, and even at so mean a price, themselves and their posterity is so scandalous and infamous, that such persons must be contemptible in the sight of those that bought them ... ".

Whether Lockhart's numbers are accurate or not, they provoke a general question: in the bribing of legislatures or other voting bodies where different members have different degrees of influence, what prices will the various members command? The answer would appear to depend on two factors: the minimum price a voter is willing to accept under any circumstances, and his "worth" to the one who is buying his influence (i.e., his *power*).

In recent years, various numerical measures of power have been proposed -- notably, those of Shapley-Shubik [8], Banzhaf [1] and Coleman [3]. Each of these measures is ultimately based on the idea that a voter is powerful insofar as he can change the outcome by changing his vote. The view we shall take here is that it is not enough that a voter be *able* to change the outcome: he must have an incentive to do so. Thus, if we are able to identify the equilibrium *prices* that a lobbyist, for example, would pay for the members' votes, we would have a measure of their relative power in the Hobbesian sense. In the next sections we shall develop a concept of equilibrium prices and incomes for arbitrary voting games, and compare the results with the Shapley-Shubik and Banzhaf measures.

VOTING GAMES

A *simple game*, or in this context, a *voting game* $G = (N,\mathcal{S})$, is a finite set N of *players*, together with a collection $\mathcal{S}$ of subsets of N called *winning coalitions* which satisfy

(1)
$$\phi \notin \mathcal{S},$$
$$S \in \mathcal{S} \quad \text{and} \quad S \subseteq T \Rightarrow T \in \mathcal{S} \quad .$$

The interpretation of G is the following: if S is precisely the set of players voting *for* a given measure, then the measure will pass if and only if $S \in \mathcal{S}$. The pair (i,S) is *critical* if

$i \in S \in \mathcal{S}$ and $S - \{i\} \notin \mathcal{S}$. Where $c_i(G)$ denotes the number of critical pairs containing i, the *Banzhaf power* [1] of player i is defined to be

$$\beta_i(G) = \frac{c_i(G)}{\sum_{k \in N} c_k(G)} \quad ,$$

i.e., *the relative number of times player* i *is critical.* In particular, if a player is never critical then he has no power (such a player is called a *dummy*). While this seems natural, there seems to be no immediate reason for asserting in general that a player's power is *proportional* to the number of times he is critical. First, in any given situation several players may be critical, hence no one of them has unilateral control over the outcome. Second, there is no apparent incentive for a player in a winning coalition to change his vote and make the measure fail (unless we suppose that each voter's objective is merely the capricious demonstration of his influence, irrespective of his actual preferences). The Coleman measures of power [3] are similar to Banzhaf's but they make a distinction between the power to pass a measure and the power to block it, a valuable distinction that will be discussed later on.

The Shapley-Shubik measure may be defined in the following way. Let all the players line up in a row $i_1, i_2, \ldots, i_n$ (all orderings being equiprobable). Player i_k is *pivotal* if k is the first index for which $\{i_1, i_2, \ldots, i_k\} \in \mathcal{S}$. Thus the pivotal player is the one who putatively gets credit for having passed the measure. The *Shapley-Shubik power* of player i, $\sigma_i(G)$, is defined to be the probability that i is pivotal.

The Shapley-Shubik value is a particular case of a more general value defined by Shapley [7] for simple games. For a more detailed discussion of the various measures and their applications, see Brams [2] and Lucas [5].

EQUILIBRIUM PRICES

In this section we introduce a model of political power in which the players receive the benefits of their power in terms of money payments. To paraphrase Hobbes, the power of a voter will be measured by the amount someone would pay for the use of it. We therefore introduce into the political arena a *lobbyist*, who is assumed to have a large quantity of funds at his disposal. We shall further assume that in the given voting game G a bill (or succession of bills) is introduced that the lobbyist wants passed. The lobbyist desires simply to pass the bills at least cost. The objective of each player is to maximize his "bribe" income. In general we may expect that the more power a player has, the higher the price he will command and the greater the income he will receive. The problem is to find the prices and incomes of the various players.

Let $p_i \geqq 0$ be the *price* of player i, $i \in N$. $\underset{\sim}{p}(S) = \sum_{i \in S} p_i$ is the *cost* of bribing the subset $S \subseteq N$. We shall assume that the lobbyist is a "price-taker", that is, the players announce their prices $\underset{\sim}{p}$ and then the lobbyist bribes some *least-cost* winning set S. A *payment schedule* for the lobbyist is therefore a function f which for any price vector $\underset{\sim}{p}$ gives a set $f(\underset{\sim}{p}) = S \in \mathcal{S}$ satisfying $\underset{\sim}{p}(S) \leqq \underset{\sim}{p}(S')$ for all $S' \in \mathcal{S}$.

Given f, an n-person game is defined on the set N of players in which each player i quotes a price p_i and gets a payoff p_i if $i \varepsilon f(\underset{\sim}{p})$, and zero otherwise. However, in general, the only equilibrium prices $\underset{\sim}{p}$ for such a game result in bribes of zero. Specifically, suppose the underlying voting game (N,S) has no *veto player*, that is, no player i who is in every winning set. Let $\underset{\sim}{p}$ be equilibrium for the given f, $S^* = f(\underset{\sim}{p})$. If i is in *every* minimum cost winning set, then $i \varepsilon S^*$ and i can raise his price by ε and still be certain of being bribed, so that $\underset{\sim}{p}$ would not be in equilibrium. Therefore for every $i \varepsilon S^*$ there exists a minimum cost winning set S_i such that $i \not\varepsilon S_i$. For any $j \varepsilon S_i$, if $p_j > 0$ then j could lower his price by ε and be certain of being bribed. Hence $\underset{\sim}{p}(S_i) = 0$, so $\underset{\sim}{p}(S^*) = 0$ (since it is a minimum), showing that for every $k \varepsilon N$ k either is not bribed, or is bribed with a price of zero.

Such a state of affairs is unrealistic, however, because in general no player will accept arbitrarily small bribes -- if for no other reason than that accepting bribes involves certain *risks*, not to mention time spent negotiating, and so forth. Thus we shall assume that there is a positive minimum price p_i^0, or *floor price*, (a datum of the problem) that player i will accept for a bribe. The price vector $\underset{\sim}{p}$ is said to be *feasible* if $\underset{\sim}{p} \geqq \underset{\sim}{p}^0$.

When is $\underset{\sim}{p}$ in equilibrium? Various concepts of equilibrium for n-person games have been proposed. A feasible $\underset{\sim}{p}$ is in "equilibrium" in the usual sense if no player i can change his price and do better. More generally, if no *set* of players can change their prices and *each* do better, i.e., if there is no set $C \subsetneq N$, $C \neq \phi$, and feasible $\underset{\sim}{p}'$ such that $p_i = p_i'$ for all $i \not\varepsilon C$, and

$$C \subseteq f(\underset{\sim}{p}') \quad ,$$

$$p_j' > p_j \text{ for all } j \in C \cap f(\underset{\sim}{p}) \quad ,$$

then $\underset{\sim}{p}$ is said to be in "strong equilibrium" [6]. Strong equilibria are very special, and few n-person games have them.

An even stronger concept of equilibrium results if we suppose that players cooperate in naming their prices and agree to compensate each other afterwards.

We say that $\underset{\sim}{p} \geq \underset{\sim}{p}^0$ is a *collective equilibrium* (for a given f) if there exists no set $C \subseteq N$, $C \neq \phi$, and feasible prices $\underset{\sim}{p}' \neq \underset{\sim}{p}$ where $p_i' = p_i$ for all $i \notin C$, such that

(2) $\quad C \subseteq f(\underset{\sim}{p}')$

and

(3) C's collective payoff can be distributed such that every player in C receives at least his minimum price and is strictly better off than before.

Given (2), the latter means that there is some $\{d_i\}_{i\varepsilon C}$ satisfying $\sum_{i\varepsilon C} d_i = \sum_{i\varepsilon C} p_i'$, where $d_i \geq p_i^0 > 0$ for all $i \varepsilon C$ and $d_i > p_i$ for all $i \varepsilon C \cap f(\underset{\sim}{p})$; equivalently,

$$\sum_{i\varepsilon C} (p_i' - p_i^0) > \sum_{i\varepsilon C\cap f(\underset{\sim}{p})} (p_i - p_i^0) \quad \text{when} \quad C \cap f(\underset{\sim}{p}) \neq \phi .$$

Every collective equilibrium is, in particular, a strong equilibrium.

Example 1. Consider the voting game on seven players 1,2,3, 4,5,6,7 defined by the following four minimal winning sets:

$$\{1,2\},\ \{1,3,4\},\ \{2,3,4,5\},\ \{3,4,5,6,7\}.$$

Let $\underset{\sim}{p}^0 = \underset{\sim}{1}$ (i.e., all players have equal floor prices) and let f be such that $f(\underset{\sim}{p}) = \{1,2\}$ whenever $\{1,2\}$ is one of *several* minimum cost sets.

Consider the vector $\underset{\sim}{p} = (3,2,1,1,1,1,1)$. Evidently $f(\underset{\sim}{p}) = \{1,2\}$, and no players can (feasibly) lower their prices and improve their positions, no matter what the other players do. Suppose on the other hand that C is a group of players each of whom raises his price, and that each does better than before. If the new price vector is $\underset{\sim}{p}'$, then this implies $C \subseteq f(\underset{\sim}{p}') = T$. Since only the players in C raise their prices, all others remaining the same, any minimum cost winning set S under $\underset{\sim}{p}$ must contain C, since otherwise it would cost less under $\underset{\sim}{p}'$ than does T. Therefore C is contained in every $\underset{\sim}{p}$-minimum cost set; but the intersection of these is empty. Thus $\underset{\sim}{p}$ is a strong equilibrium. We may in fact conclude that $\underset{\sim}{p}$ is a collective equilibrium (and the unique one) with the help of a result which follows Lemma 1.

First we need the following definitions. We say that $\underset{\sim}{p} \geqq \underset{\sim}{p}^0$ is a *canonical equilibrium for* f if it is a collective equilibrium for f and $p_i = p_i^0$ for all $i \notin f(\underset{\sim}{p})$. In other words, a canonical equilibrium is a collective equilibrium in which every player who is *not* bribed is at his floor price. Indeed, at equilibrium there can be no advantage for a nonbribed player to quote more than his floor price, for by quoting his floor price he may at least be *competitive* in the sense that he could be a member of some least cost winning set. In fact, any collective equilibrium is just a canonical equilibrium in which some players who are not bribed quote unrealistically high prices (Lemma 2 below).

For any given floor prices $\underset{\sim}{p}^0 > 0$ define

$$S^0 = \{S \in \mathcal{S} : \underset{\sim}{p}^0(S) = \min_{S' \in \mathcal{S}} \underset{\sim}{p}^0(S')\}$$

$$N^0 = \bigcap_{S \in \mathcal{S}^0} S \quad .$$

The members of $\mathcal{S}^0$ are called *critical sets*, and the members of N^0 the *critical players*.

Lemma 1. A price vector $\underset{\sim}{p}$ is a canonical equilibrium (for some f) if and only if

(4) $\underset{\sim}{p} \geqq \underset{\sim}{p}^0$ and $p_i = p_i^0$ for all $i \notin N^0$,

(5) for some $S^0 \in \mathcal{S}^0$, $\underset{\sim}{p}(S^0) \leqq \underset{\sim}{p}(S)$ for all $S \in \mathcal{S}$,

(6) $\sum_{i \in N^0} p_i$ is maximum over all $\underset{\sim}{p}$ satisfying (4) and (5).

Notice that "for some" in (5) is equivalent to "for every", given (4).

Proof.

Let $\underset{\sim}{p}$ be any feasible price vector such that for some
(7) $\underset{\sim}{p}$-minimum cost winning set T, $p_i = p_i^0$ for all $i \notin T$.
We claim (4) holds and $T \in \mathcal{S}^0$.

Indeed, for any $S \in \mathcal{S}$,

$$0 \leqq \underset{\sim}{p}(S) - \underset{\sim}{p}(T) = \underset{\sim}{p}^0(S) + \sum_{i \in T \cap S} (p_i - p_i^0) - \left[\underset{\sim}{p}^0(T) + \sum_{i \in T} (p_i - p_i^0)\right]$$

$$= \underset{\sim}{p}^0(S) - \underset{\sim}{p}^0(T) - \sum_{i \in T-S} (p_i - p_i^0) \quad .$$

But for any $S^0 \varepsilon \mathcal{S}^0$, $p^0(S^0) - p^0(T) - \sum_{i\varepsilon T-S^0}(p_i - p_i^0) \leqq 0$, hence by the above for any $S^0 \varepsilon \mathcal{S}^0$, $\underset{\sim}{p}^0(S^0) = \underset{\sim}{p}^0(T)$ and $p_i = p_i^0$ for all $i\varepsilon T - S^0$. Therefore (4) holds, and $T \varepsilon \mathcal{S}^0$.

Now let $\underset{\sim}{p}$ be a canonical equilibrium for some f. Then (7) holds for $T = f(\underset{\sim}{p})$, hence $\underset{\sim}{p}$ satisfies (4) and (5). Suppose that there is some other $\underset{\sim}{p}'$ satisfying (4) and (5), say $S' \varepsilon \mathcal{S}^0$ minimizes $\underset{\sim}{p}'(S)$, and that $\sum_{i\varepsilon N^0} p_i' > \sum_{i\varepsilon N^0} p_i$. Let $C = \{i \varepsilon N : p_i' > p_i^0\} \subseteq N^0$, and define $p_i'' = p_i' - \varepsilon$ for all $i \varepsilon C$ and $p_i'' = p_i' = p_i^0$ for $i \not\in C$. For sufficiently small $\varepsilon > 0$ we have $\underset{\sim}{p}''$ feasible, $p_i'' = p_i = p_i^0$ for all $i \not\in N^0$, and

(8) $$\sum_{i\varepsilon N^0} p_i'' > \sum_{i\varepsilon N^0} p_i \quad ;$$

moreover S' minimizes $\underset{\sim}{p}''(S)$ over all $S \varepsilon \mathcal{S}$.

For *any* set $S^* \varepsilon \mathcal{S}$ minimizing $\underset{\sim}{p}''(S)$ we have

$$0 = \underset{\sim}{p}''(S^*) - \underset{\sim}{p}''(S') = \underset{\sim}{p}'(S^*) - |S^* \cap C|\varepsilon - [\underset{\sim}{p}'(S') - |C|\varepsilon]$$

$$= [\underset{\sim}{p}'(S^*) - \underset{\sim}{p}'(S')] + |C - S^*|\varepsilon \geqq 0 \quad ;$$

hence $C \subseteq S^*$, and $\underset{\sim}{p}'(S^*) = \underset{\sim}{p}'(S')$. But $\underset{\sim}{p}'$ differs from $\underset{\sim}{p}^0$ only on C, and $C \subseteq S^* \cap S'$; hence

$$\underset{\sim}{p}^0(S^*) = \underset{\sim}{p}^0(S') \quad ,$$

that is, $S^* \varepsilon \mathcal{S}^0$. In particular $f(\underset{\sim}{p}'')$ minimizes $\underset{\sim}{p}''(S)$, so $f(\underset{\sim}{p}'') \varepsilon \mathcal{S}^0$, hence $N^0 \subseteq f(\underset{\sim}{p}'')$. But then by (8) N^0 is a subset collectively better off under $\underset{\sim}{p}''$ than under $\underset{\sim}{p}$, contradicting the assumption that $\underset{\sim}{p}$ is a canonical equilibrium.

Conversely, let $\underset{\sim}{p}$ satisfy (4) - (6); in particular S^0 minimizes $\underset{\sim}{p}(S)$ for *every* $S^0 \varepsilon \mathcal{S}^0$. Let f be *any* payment schedule such that $f(\tilde{p}) = S^* \varepsilon \mathcal{S}^0$. Suppose, by way of contradiction, that $\tilde{p}$ is not a canonical equilibrium for this f. Then there is a nonempty subset C of voters and a feasible $\tilde{p}'$ such that $p_i' = p_i$ for all $i \notin C$, $C \subseteq f(\tilde{p}')$, and either

$$\text{(i)} \quad \sum_{i\varepsilon C} p_i' - p_i^0 > \sum_{i\varepsilon C\cap S^0} p_i - p_i^0$$

or

$$\text{(ii)} \quad C \cap S^* = \phi \quad .$$

Letting $S' = f(\tilde{p}')$ we have, for *any* member $S^0 \varepsilon \mathcal{S}^0$,

$$0 \leqq \tilde{p}'(S^0) - \tilde{p}'(S') = \tilde{p}(S^0) + \sum_{i\varepsilon C\cap S^0} (p_i' - p_i) - \left[\tilde{p}(S') + \sum_{i\varepsilon C} (p_i' - p_i)\right]$$

$$= \tilde{p}(S^0) - \tilde{p}(S') - \sum_{i\varepsilon C-S^0} (p_i' - p_i)$$

$$= \tilde{p}(S^0) - \tilde{p}(S') - \sum_{i\varepsilon C-S^0} (p_i' - p_i^0) \leqq 0 \quad .$$

Thus,

$$(9) \qquad \tilde{p}(S^0) = \tilde{p}(S') \text{ and } p_i' = p_i^0 \text{ for all } i \varepsilon C - S^0 \text{ and all } S^0 \varepsilon \mathcal{S}^0,$$

from which it follows that $p_i' = p_i^0$ for all $i \notin N^0$ and $\tilde{p}'$ satisfies (4). Hence $\tilde{p}'$ and $\tilde{p}$ differ only on the set $C \cap N^0$. Therefore by (9),

$$\tilde{p}'(S') = \tilde{p}(S') + \sum_{C\cap N^0} (p_i' - p_i) = \tilde{p}(S^0) + \sum_{C\cap N^0} (p_i' - p_i) = \tilde{p}'(S^0),$$

so $\tilde{p}'$ satisfies (5) also. Finally, it is clear that $C \cap S^* = \phi$ (case (ii) above) cannot hold, since then $\tilde{p}' = \tilde{p}$. Hence by the above remarks and (i) we have

$$\sum_{i\varepsilon C\cap N^0} p_i' > \sum_{i\varepsilon C\cap N^0} p_i \quad ,$$

and

$$\sum_{i \in N^0} p_i' > \sum_{i \in N^0} p_i ,$$

contradicting the choice of $\underset{\sim}{p}$. □

In the proof of the converse above we saw that if we choose *any* payment schedule f such that $f(\underset{\sim}{p}) \in S^0$, then $\underset{\sim}{p}$ is a canonical equilibrium for this f. By this and (7) we have the following.

Corollary 1. A canonical equilibrium $\underset{\sim}{p}$ is a canonical equilibrium for f if and only if $f(\underset{\sim}{p}) \in S^0$.

Referring to Example 1, we see that players 1 and 2 are the only ones who could be above their floor prices in a canonical equilibrium. Moreover, among all $\underset{\sim}{p}$ of form $(p_1, p_2, 1, 1, 1, 1, 1)$ such that $p_1 + p_2$ is the *minimum* cost of a winning set, $p_1 + p_2 = 5$ is maximum; hence by the Corollary, (3,2,1,1,1,1,1) is a canonical equilibrium for the f defined earlier, and in fact it is the only one.

Lemma 2. If $\underset{\sim}{p}$ is a collective equilibrium for f then $\hat{\underset{\sim}{p}}$ is a canonical equilibrium for f, where $\hat{p}_i = p_i^0$ for $i \notin f(\underset{\sim}{p})$, $\hat{p}_i = p_i$ for $i \in f(\underset{\sim}{p})$.

Proof. Let $\underset{\sim}{p}$, $\hat{\underset{\sim}{p}}$ be as above, and let $f(\underset{\sim}{p}) = S^*$. Suppose that for some $S \in \mathcal{S}$, $\hat{\underset{\sim}{p}}(S) < \hat{\underset{\sim}{p}}(S^*)$, and we will derive a contradiction.

Let $\hat{\mathcal{S}} = \{S \in \mathcal{S} : \hat{p}(S) = \min = \alpha\}$ and for each $S \in \hat{\mathcal{S}}$ let $C_S = \{i \in S - S^* : p_i > p_i^0\}$. Then $C_S \neq \phi$. Let C_T be a minimal element of the family $\{C_S : S \in \hat{\mathcal{S}}\}$, and define $\underset{\sim}{q}$ by

$$q_i = p_i \quad \text{if} \quad i \notin C_T ,$$

$$q_i = p_i^0 \quad \text{if} \quad i \in C_T .$$

Evidently, $q(T) = \hat{p}(T) = \alpha$, and since $q \geqq \hat{p}$, $\alpha = q(T) = \min_{S \varepsilon S} q(S)$. If T' is *any* winning set such that $q(T') = \alpha$, then $q \geqq \hat{p}$ implies $\hat{p}(T') = \alpha$, i.e. $T' \varepsilon \hat{S}$ and $C_{T'} \subseteq C_T$. By choice of C_T, $C_{T'} = C_T$ and so $C_T \subseteq T'$. In particular, $C_T \subseteq f(q)$. But then under q, every player $i \varepsilon C_T$ gets a payoff of $p_i^0 > 0$ whereas i got nothing under p. Since in going from p to q only the members of C_T changed prices, this contradicts the assumption that p is a collective equilibrium.

Therefore

(10) S^* minimizes $\hat{p}(S)$ over all $S \varepsilon S$.

Then, as in the derivation of (7), we conclude that

(11) $S^* \varepsilon S^0$ and $\hat{p}_i > p_i^0$ implies $i \varepsilon \bigcap_{S \varepsilon S^0} S = N^0$.

In particular, $\hat{p}$ satisfies (4) and (5) of Lemma 1. Therefore, if $\hat{p}$ is *not* a canonical equilibrium there must exist a feasible r, differing from $\hat{p}$ *only* on N^0 and such that

(12) every $S \varepsilon S^0$ minimizes $r(S)$ and $\sum_{i \varepsilon N^0} r_i > \sum_{i \varepsilon N^0} \hat{p}_i = \sum_{i \varepsilon N^0} p_i$.

By subtracting off a small $\varepsilon > 0$ from every r_i, $i \varepsilon N^r = \{i \varepsilon N^0: r_i > p_i^0\}$, we see that r can actually be chosen so that it is feasible, (12) holds, and *every* $S \varepsilon S$ minimizing $r(S)$ contains N^r, that is

(13) $S \varepsilon S$ minimizes $r(S)$ if and only if $S \varepsilon S^0$.

Now define q such that $q_i = r_i$ for $i \varepsilon N^0$, $q_i = p_i$ for $i \not\varepsilon N^0$. For any $S \varepsilon S$, $q(S) \geqq r(S)$ while $q(S^*) = r(S^*)$; hence for $\bar{S} = f(q)$ we have $r(\bar{S}) \leqq q(\bar{S}) \leqq q(S^*) = r(S^*)$, implying that $\bar{S}$ minimizes

$\underset{\sim}{r}(S)$, hence contains N^0. But N^0 is better off under $\underset{\sim}{q}$ than under $\underset{\sim}{p}$, a contradiction. Hence $\hat{\underset{\sim}{p}}$ is a canonical equilibrium. □

Lemma 2 shows that any non-canonical collective equilibrium is just an inessential variant of some canonical equilibrium, and Lemma 1 tells us how to recognize the latter. The problem is then to determine when a canonical equilibrium exists.

Clearly one situation in which it cannot exist is if the voting game contains a veto player, for the price of any such player can increase without bound, and there will be no finite maximum in (6). It turns out that this is the only exception; if there are *no* veto players then the game always has a collective equilibrium.

To see this, consider condition (5) of Lemma 1:

$$\underset{\sim}{p}(S^0) \leqq \underset{\sim}{p}(S) \text{ for all } S \in \mathcal{S} \text{ and } S^0 \in \mathcal{S}$$

is equivalent to

$$\underset{\sim}{p}^0(S^0) + \sum_{i \in S^0 - S} (p_i - p_i^0) \leqq \underset{\sim}{p}^0(S) + \sum_{i \in S - S^0} (p_i - p_i^0) \quad ,$$

which in view of (4) is the same as

$$(14) \qquad \sum_{i \in N^0 - S} (p_i - p_i^0) \leqq \underset{\sim}{p}^0(S) - \underset{\sim}{p}^0(S^0) \quad \text{for all } S \in \mathcal{S} \quad .$$

If $N^0 = \phi$, Lemma 1 implies $\underset{\sim}{p}^0$ is the unique canonical equilibrium. Otherwise, let $\underset{\sim}{A}$ be the (0,1)-incidence matrix whose columns are indexed by the players $i \in N^0$, and whose rows are indexed by the distinct, nonempty sets $N^0 - S$, $S \in \mathcal{S}$. For each row index $T = N^0 - S$ let $b_T = \underset{\sim}{p}^0(S) - \underset{\sim}{p}^0(S^0)$ (this is independent of S^0), and let $\underset{\sim}{b}$ be the column vector of such b_T's. Finally, let $\underset{\sim}{\pi}$ be $\underset{\sim}{p} - \underset{\sim}{p}^0$ restricted to the components $i \in N^0$. Then (14) is equivalent to

(15) $\quad A\pi \leqq b \quad , \quad \pi \geqq 0 \quad .$

A vector p satisfies (4) and (5) of Lemma 1 if and only if it is obtained from such a π by letting $p_i = \pi_i + p_i^0$ for $i \in N^0$, $p_i = p_i^0$ for $i \notin N^0$. Therefore, by Lemma 1, the set of all canonical equilibria for a given p^0 is obtained as the set of the p's corresponding to the optimal solutions of the linear program

(16)
$$\max\ 1 \cdot \pi$$
$$A \cdot \pi \leqq b, \quad \pi \geqq 0 \quad .$$

The dual of (16) is

(17)
$$\min\ b \cdot y$$
$$yA \geqq 1, \quad y \geqq 0 \quad .$$

If A has no zero columns, i.e. if there are no veto players, then $y = 1$ is a feasible solution to (17). Since $b \geqq 0$, $\pi = 0$ is always a feasible solution to (16), and we obtain the following.

Theorem 1. For any voting game without veto players and floor prices $p^0 > 0$ *there exists a canonical equilibrium* p*, and in general* p *is unique.*

Example 2. Consider the voting game on eleven players $\{1,2, \ldots ,11\}$ given by the minimal winning sets:

$\{1,2,3\}$ $\qquad \{1\} \cup T_5$

$\{1,2\} \cup T_2$ $\qquad \{2\} \cup T_6$

$\{1,3\} \cup T_4$ $\qquad \{3,4,5,6,7,8,9,10,11\}$,

$\{2,3\} \cup T_5$

where for each k, T_k ranges over all k-subsets of $\{4,5,6,7,8,9,10,11\}$.

Take $p^0 = 1$; then $\{1,2,3\} = N^0$ is the unique critical set. The linear program (16) is

$$\max \pi_1 + \pi_2 + \pi_3$$

subject to $\pi_1, \pi_2, \pi_3 \geqq 0$ and

$$\begin{aligned}
\pi_1 &\leqq 4 \\
\pi_2 &\leqq 3 \\
\pi_3 &\leqq 1 \\
\pi_1 + \pi_2 &\leqq 6 \\
\pi_1 + \pi_3 &\leqq 4 \\
\pi_2 + \pi_3 &\leqq 3 \quad .
\end{aligned}$$

Here the first row is obtained from (14) by setting $S = \{2,3\} \cup T_5$ for some T_5, and so forth. The unique optimal solution is $\bar{\underset{\sim}{\pi}} = (3\tfrac{1}{2}, 2\tfrac{1}{2}, \tfrac{1}{2})$, so the unique canonical equilibrium is $\bar{\underset{\sim}{p}} = (4\tfrac{1}{2}, 3\tfrac{1}{2}, 1\tfrac{1}{2}, 1, 1, 1, 1, 1, 1, 1, 1)$.

POWER AND INCOME

The power of a player in a voting game is clearly related to his price, but as we have seen, some players may charge high prices and get nothing. The ultimate test is not what the player *asks* but what he *gets*. Given any payment schedule f and prices $\underset{\sim}{p}$ we define the *income* of player i to be

$$g_i(\underset{\sim}{p}, f) = \begin{cases} p_i & \text{if } i \in f(\underset{\sim}{p}) \\ 0 & \text{if } i \notin f(\underset{\sim}{p}) \end{cases} \quad .$$

For a non-canonical collective equilibrium the income to each player is the same as for the corresponding canonical equilibrium, and in any event the establishment of a non-canonical equilibrium is improbable in the context of the problem. Hence we assume that, given any payment schedule f, the players will arrive at some canonical equilibrium prices $\underset{\sim}{p}$ for f, and any such pair $(\underset{\sim}{p}, f)$ will be called an *equilibrium pair*.

Let G = (N,S) be any voting game without veto players. For computing the relative power of the players we assume that, *a priori*, there is no difference in their minimum prices, that is $p_1^0 = p_2^0 = \ldots = p_n^0 = \alpha > 0$ for some constant α. (Later we will consider other possibilities.) It is easy to see that $\underset{\sim}{p}$ is a canonical equilibrium for $\underset{\sim}{p}^0$ if and only if $\alpha\underset{\sim}{p}$ is a canonical equilibrium for $\alpha\underset{\sim}{p}^0 (\alpha > 0)$; hence the choice of α is immaterial.

We define the *passing income* (or *passing power*) of player i, ψ_i, to be his expected income over all equilibrium pairs $(\underset{\sim}{p},f)$ (with $\underset{\sim}{p}^0$ as above), normalized so that the total power is 1.

If $(\underset{\sim}{p},f)$ is an equilibrium pair then $f(\underset{\sim}{p}) \in S^0$, and for any payment schedule g which satisfies $g(\underset{\sim}{p}) \in S^0$, $(\underset{\sim}{p},g)$ is also an equilibrium pair. Therefore if (as is normally the case) $\underset{\sim}{p}$ is unique, then letting $s^0 = |S^0|$, $s_i^0 = |\{S \in S^0 : i \in S\}|$ we have

$$(17) \qquad \psi_i = \frac{p_i s_i^0}{s^0} \Big/ \sum_{j=1}^{n} \frac{p_j s_j^0}{s^0} = p_i s_i^0 \Big/ \sum_{j=1}^{n} p_j s_j^0 \quad .$$

If there is more than one canonical equilibrium, then the set $\underset{\sim}{P}$ of all of them forms a convex set, and the $\underset{\sim}{p}$ of formula (17) is taken to be the *centroid* of $\underset{\sim}{P}$.

Example 3 (The U.S. Federal Game). The members of the United State House of Representatives and the Senate, together with the Vice-President and the President, are players in the voting game G described schematically by the minimal winning sets of type:

{218R,50S,V,P}, {218R,51S,P} and {290R,67S}, where R,S,V,P has the obvious interpretation. The unique equilibrium passing prices for $\underset{\sim}{p}^0 = \underset{\sim}{1}$ are found by inspection to be $p_R = 1$, $p_S = 1$, $p_V = 1$, $p_P = 88$. This solution can also be arrived at by the following heuristic reasoning. At their floor prices, none of R, S, and V is critical -- that is, the lobbyist can always bribe at no extra cost a set excluding any such player --and hence they will never have an incentive to charge more. The President, on the other hand, will charge just enough so that the lobbyist is indifferent between bribing him and bribing some substitute set of players. The least number of players that can be substituted for the President is 88; hence his price. This illustrates a general principle that will be established for weighted voting games in the next section.

Associated with any voting game $G = (N, \mathcal{S})$ is the *complementary game* $\overline{G} = (N, \overline{\mathcal{S}})$, defined by $S \in \overline{\mathcal{S}}$ if and only if $N - S \notin \mathcal{S}$. In G a winning coalition is one that is able to *pass* a measure whereas in $\overline{G}$ a winning coalition is one that is able to *block* a measure. In general, the canonical equilibrium prices for G and $\overline{G}$ will be different.

For the U.S. Federal case the complementary game is described by the minimal winning sets:

$$\{146R,P\}, \quad \{34S,P\}, \quad \{51S\}, \quad \{50S,V\}, \quad \{218R\} .$$

The unique canonical equilibrium prices are seen to be $p_R = p_S = p_V = 1$, $p_P = 17$.

The *blocking income* (or *blocking power*), $\overline{\Psi}_i$, of player i is defined to be his passing income relative to the game $\overline{G}$. The distinction between blocking and passing incomes is a valuable one. However, for comparison with the Banzhaf and Shapley-Shubik indices, it is useful to consider the players' expected income with respect to passing and blocking together. Define the *income* (or *power*) of player i, Ψ_i, to be his total expected income (relative to $\underset{\sim}{p}^0 = \underset{\sim}{1}$), normalized so that the total is 1:

$$\Psi_i = \frac{p_i s_i^0/s^0 + \bar{p}_i \bar{s}_i^0/\bar{s}^0}{\sum_{j=1}^{n} (p_j s_j^0/s^0 + \bar{p}_j \bar{s}_j^0/\bar{s}^0)} \tag{18}$$

where $\underset{\sim}{p}$, $\underset{\sim}{\bar{p}}$ are the centroid canonical equilibria for G and $\bar{G}$ respectively and s^0, $\bar{s}^0$, etc., have the obvious interpretation.

The passing and blocking incomes for the U.S. Federal Game are given in Table 1, and the incomes are compared with the Shapley-Shubik and Banzhaf values in Table 2.

Table 1. Incomes for the U.S. Federal Game.

Player	Passing Income	Blocking Income
R	.00140	0.
S	.00141	.00667
V	.00141	0.
P	.24654	.33333

Table 2. Power Measures for the U.S. Federal Game.

Player	Income	Banzhaf Value	Shapley-Shubik Value
R	.00123	.00149	.00097
S	.00207	.00310	.00414
V	.00124	.00310	.00265
P	.25735	.03893	.16314

For *a priori* computations of power it was assumed that the floor prices of all players were equal. This is in keeping with the notion that a floor price represents the minimum payment commensurate with the act of accepting a bribe at all, which *a priori* is not different for the different players. Another interpretation is that the floor price represents some kind of minimum *expectation;* it could then be argued that the more powerful players will naturally have higher expectations, and therefore higher floor prices. If we follow this idea to its conclusion, we might indeed assert that the equilibrium prices, once established, become the new floor prices. Does this lead to a kind of "second order" equilibrium? The answer is easily seen to be *no*, since if $\underset{\sim}{p}$ is a collective equilibrium for initial floor prices $\underset{\sim}{p}^0$, then in particular there is no player contained in $\bigcap_{\underset{\sim}{p}(S)=\min} S$, since any such player could raise his price further. Therefore if $\underset{\sim}{p}$ is taken to be the new floor prices, then there are no critical players; hence the floor prices themselves constitute the unique canonical equilibrium. Therefore no new equilibria are obtained.

WEIGHTED VOTING GAMES

A voting game $G = (N,\mathcal{S})$, $N = \{1,2,\ldots,n\}$, is representable as a *weighted* voting game if there are numbers q; $w_1,w_2,\ldots,w_n$ such that $S \in \mathcal{S}$ if and only if $\sum_{i \in S} w_i \geq q$. q is called the *quota*, the w_i's are called the *weights*.

Example 4. The County of Nassau in New York State has a County Board of Supervisors consisting of six members, one for each municipality in the County. As of 1971 the members' votes were weighted as shown in Table 3 with a majority of 63 of 115 required to pass a measure.

Table 3. Weights for the Nassau County Board of Supervisors (1971).

Municipality	Weight
Hempstead No. 1 (H1)	31
Hempstead No. 2 (H2)	31
Oyster Bay (OB)	28
North Hempstead (NH)	21
Glen Cove (GC)	2
Long Beach (LB)	2

For equal floor prices, $\underset{\sim}{p}^0 = \underset{\sim}{1}$, the critical sets are

$$\{H1, H2, OB\}$$
$$\{H1, H2, NH\}$$
$$\{H1, H2, GC\}$$
$$\{H1, H2, LB\}$$
$$\{H1, OB, NH\}$$
$$\{H2, OB, NH\} \quad .$$

Since no player is critical, $\underset{\sim}{p}^0 = 1$ is the unique canonical equilibrium. For the complementary game 53 votes are required to block, and for $\underset{\sim}{p}^0 = \underset{\sim}{1}$ the critical sets are $\{H1, H2\}$, $\{H1, OB\}$, and $\{H2, OB\}$. Again, no player is critical. The resulting expected incomes (normalized) are compared in Table 4 with the Banzhaf and Shapley-Shubik values.

Table 4. Power Measures for the Nassau County Board of Supervisors.

Municipality	Income	Banzhaf Value	Shapley-Shubik Value
H1	.300	.278	.283
H2	.300	.278	.283
OB	.233	.204	.217
NH	.100	.130	.117
GC	.033	.056	.050
LB	.033	.056	.050

In the U.S. Federal Game we noticed that the price of the President can be interpreted as a kind of marginal rate of substitution of other players for the President. In the above example the prices also have this interpretation. More precisely, given any voting game G = (N,S) without veto players, and $\underset{\sim}{p}^0 = \underset{\sim}{1}$, define v_i, for each $i \in N$, to be the cardinality of the smallest winning set not containing i, and similarly define u_i to be the cardinality of the smallest winning set containing i. We call $r_i = v_i - u_i + 1$ the *integral substitution rate* for i. Notice that in Examples 3 and 4 the canonical equilibrium prices equal the integral substitution rates or 1, whichever is larger. Although this result does not hold for all voting games, it is "approximately" true for all weighted voting games (in the sense of Theorem 2 below); moreover it often holds in practice -- e.g. the U.S. Federal Game, which is not representable as a weighted voting game.

Theorem 2. Let G *be a weighted voting game with player set* $\{1,2,\ldots,n\} = N$ *and representation* $(q;w_1,w_2,\ldots,w_n)$ *where* $w_1 \geqq w_2 \geqq \cdots \geqq w_n$ *and* $\sum_{i=2}^{n} w_i \geqq q$. *Let* $\underset{\sim}{p}^0 = \underset{\sim}{1}$.

(i) *For every canonical equilibrium* $\underset{\sim}{p}$ *either* $p_i = 1$ *or* $p_i > 1$ *and* $r_i - 1 \leqq p_i \leqq r_i$.

(ii) *There exists some canonical equilibrium* $\underset{\sim}{\bar{p}}$ *such that* $\bar{p}_1 \geqq \bar{p}_2 \geqq \cdots \geqq \bar{p}_n$.

Proof. By the hypotheses that $w_1 \geqq w_2 \geqq \ldots \geqq w_n$ and $\sum_{i=2}^{n} w_i \geqq q$, G has no veto players. With $\underset{\sim}{p}^0 = \underset{\sim}{1}$, let N^0 be the set of critical players, $\underset{\sim}{p}$ a canonical equilibrium price vector, and $\pi_j = p_j - p_j^0$, $j \varepsilon N^0$. To prove (i) it suffices to show that $r_j - 2 \leqq \pi_j \leqq r_j - 1$ for any $j \varepsilon N^0$. Let S^0 be any critical set (i.e. minimum cardinality winning set). For each $j \varepsilon N^0$, let S^j be a minimum cardinality set in S *not* containing j. Then the $(N^0 - S^j)$-row of the linear program (16) states that

$$\sum_{i \varepsilon N^0 - S^j} \pi_i \leqq b_{N^0 - S^j} = |S^j| - |S^0| = r_j - 1 \quad ,$$

so $\pi \geqq 0$ implies

(19) $$\pi_j \leqq r_j - 1 \quad .$$

We note that (19) holds for any voting game G without veto players. With G as hypothesized we now show that

$$r_j - 2 \leqq \pi_j \quad \text{for all} \quad j \varepsilon N^0 \quad .$$

Let k be the least integer such that $q \leqq \sum_{1 \leqq i \leqq k} w_i$. Then $S^0 = \{1,2,\ldots,k\}$ is a minimum cardinality winning set. For any $C \subseteq N^0$, $C \neq \phi$, such that $N - C \varepsilon S$, let r_C be the least integer ℓ such that $q \leqq \sum_{\substack{1 \leqq i \leqq k \\ i \notin C}} w_i + \sum_{k+1 \leqq i \leqq k+\ell} w_i$. Then $r_C = |S| - |S^0 - C|$ where S is a minimum cardinality winning set such that $S \cap C = \phi$. Evidently $r_{\{i\}} = r_i$ for all $i \varepsilon S^0$.

We claim that whenever r_C is defined and $|C| \geq 2$,

(20) $$r_C \geqq r_{C-\{j\}} + r_{\{j\}} - 1 \text{ for every } j \in C.$$

By definition,

(21) $$\sum_{k<i<k+r_{C-\{j\}}} w_i + \sum_{\substack{1 \leqq i \leqq k \\ i \notin C-\{j\}}} w_i < q$$

and

(22) $$\sum_{k<i \leqq k+r_C} w_i + \sum_{\substack{1 \leqq i \leqq k \\ i \notin C}} w_i \geqq q \quad ;$$

hence subtracting (21) from (22),

$$w_j < \sum_{k+r_{C-\{j\}} \leqq i \leqq k+r_C} w_i \quad .$$

Letting $\ell = r_C - r_{C-\{j\}} + 1$ it follows from the fact that the w_i's are nonincreasing that

$$w_j < \sum_{k+1 \leqq i \leqq k+\ell} w_i \quad .$$

Hence

(23) $$\sum_{\substack{1 \leqq i \leqq k \\ i \neq j}} w_i + \sum_{k+1 \leqq i \leqq k+\ell} w_i \geqq q$$

and

$$r_{\{j\}} \leqq r_C - r_{C-\{j\}} + 1 \quad ,$$

proving (20).

Now fix $j \in N^0 \subseteq S^0$.

If $\underset{\sim}{y}$ is a dual optimal solution to (17), then $\underset{\sim}{y}A \geqq \underset{\sim}{1}$ implies that for any given $j \in N^0$ there exists an $S^* \in \mathcal{S}$ such that

$j \in C^* = N^0 - S^*$ and $y_{C^*} > 0$; hence by complementary slackness

(24) $$\sum_{i \in C^*} \pi_i = b_{C^*} = |S^*| - |S^0| .$$

On the other hand, for any $S \in \mathcal{S}$ such that $S \cap C^* = \phi$ we have (by the feasibility of $\underset{\sim}{\pi}$)

$$\sum_{i \in C^*} \pi_i \leqq |S| - |S^0| ;$$

hence S^* is a *minimum cardinality* $S \in \mathcal{S}$ such that $S \cap C^* = \phi$, whence

$$|S^*| - |S^0 - C^*| = r_{C^*} ,$$

and

(25) $$\sum_{i \in C^*} \pi_i = r_{C^*} - |C^*| .$$

If $C^* = \{j\}$, then we have $\pi_j = r_{\{j\}} - 1$ and we are done.

If $\{j\} \subsetneqq C^*$, then $\underset{\sim}{\pi}$ feasible implies

$$\sum_{i \in C^* - \{j\}} \pi_i \leqq |S| - |S^0| \text{ for all } S \in \mathcal{S} \text{ s.t. } S \cap (C^* - \{j\}) = \phi ,$$

in particular,

(26) $$\sum_{i \in C^* - \{j\}} \pi_i \leqq r_{C^* - \{j\}} - |C^* - \{j\}| .$$

From (20), (25) and (26) it follows that

$$\pi_j \geqq r_{\{j\}} - 2 = r_j - 2 ,$$

and since j was arbitrary in N^0, statement (i) is proved.

Now suppose there is no monotone canonical equilibrium $\underset{\sim}{p}$ in the sense of statement (ii) of the theorem. For any canonical equilibrium $\underset{\sim}{p}$ a pair (i,j) is *bad* if $i < j$ and $p_i < p_j$. Let $\bar{\underset{\sim}{p}}$ be a canonical equilibrium having the smallest number of bad pairs. We may choose i and j such that $i < j$, $\bar{p}_i < \bar{p}_j$, and $\bar{p}_h \geqq \bar{p}_j$ for $h \leq i-1$, $\bar{p}_k \leqq \bar{p}_i$ for $k \geqq j+1$.

Let $\varepsilon = (\bar{p}_j - \bar{p}_i)/2$ and define

$$p_i^* = \bar{p}_i + \varepsilon$$
$$p_j^* = \bar{p}_j - \varepsilon$$
$$p_k^* = \bar{p}_k \text{ for } k \neq i,j \quad .$$

We claim that $\underset{\sim}{p}^*$ has fewer bad pairs than $\bar{\underset{\sim}{p}}$. Indeed, $\underset{\sim}{p}^*$ can introduce no *new* bad pair (i,h) where $i < h$. On the other hand, if $h < i$ then by choice of (i,j) $\bar{p}_h \geqq \bar{p}_j = \bar{p}_i + 2\varepsilon$, so $p_h^* = \bar{p}_h > \bar{p}_i + \varepsilon = p_i^*$ and (h,i) is not a bad pair for $\underset{\sim}{p}^*$. Similarly we show that $\underset{\sim}{p}^*$ introduces no new bad pairs involving j. Since (i,j) is not a bad pair for $\underset{\sim}{p}^*$, $\underset{\sim}{p}^*$ has strictly fewer bad pairs than $\bar{\underset{\sim}{p}}$. Let $\bar{\pi}_k = \bar{p}_k - 1$ for $k \varepsilon N^0$; then $\bar{\underset{\sim}{\pi}}$ is optimal for (16). Moreover, since $\bar{\pi}_j > \bar{\pi}_i \geqq 1$, we must have $j \varepsilon N^0$. We claim that $\underset{\sim}{\pi}^*$ is feasible for (16), where $\pi_k^* = p_k^* - 1$ for $k \varepsilon N^0$.

Indeed, for fixed $S^0 \varepsilon \mathcal{S}^0$ and any $S \varepsilon \mathcal{S}$ such that $N^0 - S \neq \phi$,

$$(27) \qquad \sum_{k \varepsilon N^0 - S} \pi_k^* \leqq \sum_{k \varepsilon N^0 - S} \bar{\pi}_k \leqq |S| - |S^0| \quad ,$$

unless $i \varepsilon N^0 - S$ and $j \not\varepsilon N^0 - S$; that is, unless $j \varepsilon S$ and $i \not\varepsilon S$. In this case, since $w_i \geqq w_j$, it follows that $S' = S \cup \{i\} - \{j\} \varepsilon \mathcal{S}$. Since $\bar{\underset{\sim}{\pi}}$ is feasible we have

$$\sum_{k \varepsilon N^0 - S'} \bar{\pi}_k \leqq |S'| - |S^0| \quad ,$$

that is,

$$\sum_{k\in N^0-S'} \bar{\pi}_k = \sum_{k\in N^0-S} \bar{\pi}_k + \bar{\pi}_j - \bar{\pi}_i = \sum_{k\in N^0-S} \bar{\pi}_k + 2\varepsilon \leqq |S'| - |S^0| = |S| - |S^0|$$

and so

$$\sum_{k\in N^0-S} \pi_k^* = \sum_{k\in N^0-S} \bar{\pi}_k + \varepsilon < |S| - |S^0| \quad ,$$

showing that (27) holds in any case. Thus $\underset{\sim}{\pi}^*$ is feasible, and since $\underset{\sim}{1} \cdot \underset{\sim}{\pi}^* = \underset{\sim}{1} \cdot \underset{\sim}{\bar{\pi}}$, $\underset{\sim}{\pi}^*$ is also optimal; hence by Lemma 1 $\underset{\sim}{p}^*$ is a canonical equilibrium with fewer bad pairs than $\underset{\sim}{\bar{p}}$, a contradiction. □

Example 5. Let G be the weighted voting game with representation $\underset{\sim}{w} = (12,6,5,4,4,4,4,1,1)$, $q = 23$. With $\underset{\sim}{p}^0 = \underset{\sim}{1}$ the unique critical set consists of the first three players. The associated linear program is

$$\begin{aligned}
&\max\ \pi_1 + \pi_2 + \pi_3 \\
&\text{subject to } \pi_1, \pi_2, \pi_3 \geqq 0 \text{ and} \\
&\pi_1 \leqq 2 \\
&\pi_2 \leqq 1 \\
&\pi_3 \leqq 1 \\
&\pi_1 + \pi_2 \leqq 4 \\
&\pi_1 + \pi_3 \leqq 3 \\
&\pi_2 + \pi_3 \leqq 1
\end{aligned}$$

Then $\underset{\sim}{\pi}^1 = (2,0,1)$ is an optimal solution that is non-monotone in the players' weights. Moreover it may be observed that there is no alternate representation of the game by different weights in which player 3 has a weight equal to or greater than that of player 2, because players {2,4,5,6,7,8} constitute a winning set, whereas

{3,4,5,6,7,8} do not.

The voting game of Example 2 may actually be represented as a weighted voting game with weights $\underset{\sim}{w} = (13,10,4,3,3,3,3,3,3,3,3)$ and quota $q = 27$. As shown before, for $\underset{\sim}{p}^0 = \underset{\sim}{1}$ the unique canonical equilibrium prices are $(4\frac{1}{2},3\frac{1}{2},1\frac{1}{2},1,1,1,1,1,1,1,1)$, which satisfy the conditions of Theorem 2 but are not integer.

INCORRUPTIBLES

So far it has been assumed that every player's vote can be bought, and, moreover, that each player threatens to vote contrary to the lobbvist's wishes unless he is bribed. The more general viewpoint may be adopted that for any issue which the lobbyist supports there will be a certain subset of players who support the measure and therefore do not need to be bribed, whereas there are other players who cannot be bought at any price (*incorruptibles*). This situation can be handled by a simple modification of the fore-going ideas. Given game $G = (N,\mathcal{S})$ we assume that before bribing begins, a certain set $A \subseteq N$ of players announce their position in favor, that another set $B \subseteq N - A$ of players are irrevocably opposed, and that the remainder, $N - (A \cup B)$, are merely waiting to be bribed. In effect, the players in A voluntarily accept a bribe price of 0, while those irrevocably opposed have a floor price of $+\infty$. If A wins, or if $N - B$ loses, then the lobbyist has nothing to do. Otherwise, the lobbyist behaves as if the game were

$$G' = \{N - (A \cup B), \mathcal{S}'\} \quad ,$$

where $\mathcal{S}' = \{S \subseteq (N - (A \cup B)) : S \cup A \in \mathcal{S}\}$ and prices and incomes are determined accordingly.

REFERENCES

[1] Banzhaf, John F. III, Weighted Voting Doesn't Work: A Mathematical Analysis, *Rutgers Law Review*, 31 (1965), 317-343.

[2] Brams, Steven J., *Game Theory and Politics*, Free Press, New York, 1975.

[3] Coleman, James J., Control of Collectivities and the Power of a Collectivity to Act, in Bernhardt Lieberman, ed., *Social Choice*, Gordon and Breach, New York, 1971.

[4] Lockhart, George, *Memoirs of the Affairs of Scotland*, London, 1714.

[5] Lucas, William F., Measuring Power in Weighted Voting Systems, *Case Studies in Applied Mathematics*, CUPM Report of the Mathematical Association of America, 1976.

[6] Owen, Guillermo, *Game Theory*, W.B. Saunders Company, Philadelphia, Penn., 1968.

[7] Shapley, Lloyd S., A Value for n-Person Games, in *Annals of Mathematics Studies*, No. 28: *Contributions to the Theory of Games*, Vol. II, H.W. Kuhn and A.W. Tucker, eds., Princeton University Press, Princeton, N.J., 1953, 307-317.

[8] Shapley, Lloyd S., and Martin Shubik, A Method for Evaluating the Distribution of Power in a Committee System, *American Political Science Review*, 48 (1954), 787-792.