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Dynamic Optimization of R&D Intensity under the Effect of Technology Assimilation: Econometric Identification for Japan's Automotive Industry

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Abstract

This paper introduces a dynamic model of optimization of R&D intensity under the effect of technology assimilation. The model involves R&D investment, technology stock, production, and technology productivity as main variables. The model characterizes the "growth" and "decline" trends that describe interaction between R&D investment and transformation process of production factors. The technology stock is constructed as a function of indigenous and exogenous technology stocks and their growth rates. The research focuses on the issue of a reasonable balance between the indigenous technology stock and assimilated technology flow. The maximum principle of Pontryagin is applied to construct an optimal R&D investment policy. The existence and uniqueness result for the saddle-type equilibrium is obtained. The optimal solution is constructed analytically and its properties are investigated. The model is calibrated on the aggregate data of Japan's automotive industry over the period 1982-2000.

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1. Introduction

The paper is devoted to characterizing the impact of technology assimilation on optimization of R&D investment policy for a growing economy. The focus is on the issue of a reasonable balance between the indigenous technology stock and assimilated technology flow. Such statement is closely connected with the problem of optimal allocation of resources (Arrow and Kurz, 1970 [2]; Arrow, 1985 [3]; and Leitmann and Lee, 1999 [17]).

The efficiency of utilization of technology depends on an assimilation capacity of an economy to absorb the exogenous technology stock from the global market place. It is assumed in this paper that the assimilation capacity is conditioned by the development of the world market technology stock and the ability to maximize benefits of a learning exercise. Consequently, the assimilation capacity is a function of the level of the indigenous technology stock and the assimilated spillover technology, and the growth rates of these parameters.

To date, a number of studies have analyzed the measurement of technology formation and its stock as well as an expected return of R&D investment. Scherer (1965 [23], 1983 [24]), Hall et al. (1988 [10]), Hall, Griliches and Hausman (1983 [11], 1986 [12], and 1984 [13]), Pakes and Griliches (1984 [18]), and Acs and Audretsch (1989 [1]) have thoroughly analyzed the effects of R&D investment on technology stock formation and productivity growth. In this paper we combine an econometric procedure for identification of the assimilation capacity with dynamic optimization of R&D investment policy.

The proposed model includes the growth and decline trends of R&D investment. The growth of the indigenous technology stock requires R&D expenditures inducing decrease in production rate in the short run. In the long run, R&D investment leads to increase of sales and production diversity. The dynamic model includes an integral utility function that correlates accumulative R&D investment and production diversity. The endogenous growth theory (Grossman and Helpman, 1991 [8]) is referred here as a tool for studying control models of optimal resources allocation with the utility functions of the logarithmic type. The discounted utility functions with the consumption index of the logarithmic type and equal elasticity of substitution of invented products have been used also in the papers (Tarasyev and Watanabe, 2001[a] [27], 2001[b] [28]; Watanabe et al., 2001 [32]; Tarasyev et al., 2002 [29]; Reshmin et al., 2002 [21]; and Izmodenova-Matrossova et al. (2003 [15]).

The problem is to find an optimal R&D investment policy that maximizes the utility function in presence of "growth" and "decline" trends in dynamics of R&D investment and production. The optimal control problem for trajectories of technology growth under the technology assimilation effect is analyzed and the main qualitative features of optimal trajectories are characterized basing on concavity properties of the Hamiltonian function for the corresponding dynamic system of techno-economic growth. The impact of technology assimilation on the optimal R&D level is revealed in formulas of the Pontryagin's maximum principle (Pontryagin et al., 1962 [19]). The existence and uniqueness result for equilibrium of the corresponding Hamiltonian system of differential equations is proved. The Hamiltonian system is linearized around the equilibrium point, and eigenvalues and eigenvectors of the Jacobi matrix are estimated. This standard analysis demonstrates the saddle type of the equilibrium point. The optimal trajectories are constructed as paths leading the system to equilibrium.

The synthetic trends of optimal trajectories reflect properly the real economic tendencies of technology development. This conclusion is confirmed by econometric analysis of the real data on Japan's automotive industry over the period 1982-2000. The calibration procedure employing elements of the sensitivity analysis adjusts the model to the qualitative trends in the empirical time series for technology, production, technology productivity, and R&D intensity.

Section 2 presents the design of the techno-production model and empirical measurements of the technology stock and its dependency on the spillover technology. Section 3 analyzes the optimal trajectory of R&D investment on the basis of the theory of optimal control and its application to economic models (Pontryagin et al., 1962 [19], Krasovskii, A.N., Krasovskii, N.N., 1995 [16], Subbotin, 1995 [25], Intriligator, 1971 [14]; Watanabe, 1992 [30]; Borisov et al., 1999 [4]; Crandall and Lions, 1983 [5]; Dolcetta, 1983 [6]; Feichtinger and Wirl, 2000 [7]; and Tarasyev, 1999 [26]). Section 4 summarizes new findings and policy implications.

2. Technology Stock and Its Dependency on Spillover Technology

2.1. Basic Parameters

For constructing a dynamic model of interaction between the domestic technology stock and the spillover technology the following basic variables are used:

T_d - domestic technology stock;

$\Delta T_d = \frac{dT_d}{dt}$ - change in the domestic technology stock;

$\xi = \frac{\Delta T_d}{T_d}$ - the rate of the domestic technology stock;

T_s - technology spillover pool;

$\Delta T_s = \frac{dT_s}{dt}$ - change in the technology spillover pool;

$\omega = \frac{\Delta T_s}{T_s}$ - the rate of the technology spillover pool;

z - coefficient of the assimilation capacity (assimilation capacity);

T - gross technology stock.

2.1.1. Technology Stock

Following the model of the technological knowledge stock (technology stock) by Pakes and Griliches (1984 [18]) we describe the increase in technology stock $\dot{T}_{i,t}$ in industry i at time t by the regression equation

$$\dot{T}_{i,t} = \hat{a} + \hat{b} \cdot t + \sum_{\tau=0}^{l} \theta_\tau \cdot r_{i,t-\tau} + e_{i,t} = f_i(t, r_{i,t-\tau}) . \qquad (1)$$

Here $\hat{a}$ is a constant; $\hat{b}$ is the time coefficient of regression; t is the time trend effect; θ_τ are weights of the lagged variables; $r_{i,t}$ is R&D investment of industry i at time t; τ is the time-lag between R&D investment and its commercialization; and $e_{i,t}$ is a disturbance term. It is assumed that the time lag τ between R&D investment and its commercialization varies in the interval 0 - l years, $l \geq 0$. Usually the period of delay equals to 5 years, $l = 4$.

Taking into account the time lag and the obsolescence effect in the R&D investment process the domestic technology stock can be measured as follows (see, for example, Watanabe, 2000 [31]):

$$T_{i,t} = r_{t-m} + (1 - \rho) \cdot T_{i,t-1} . \qquad (2)$$

Here r_{t-m} is R&D investment at time $t - m$; m is the time-lag between R&D investment and commercialization; ρ is the obsolescence rate of technology.

In this stage, a dynamic autoregressive geometric distributed-lag (AGDL) model for the domestic technology T_d is constructed. Let us note that the lagged variables should be included into the model explicitly (Gujarati, 1995 [9]; Pyndick and Rubinfeld, 1991 [20]) due to a substantial period of time that may pass between the economic decision-making period and the final impact on a change in R&D investment as a policy variable.

Let us introduce a postulate that R&D investment r_t, as well as the accumulative technology stock $T_{d,t-1}$ of the previous year, significantly contribute into determination of the level of the domestic technology stock. Basing on this postulate and taking into account equation (1) one can describe the domestic technology stock $T_{d,t}$ by the following relations

$$T_{d,t} = g\left(\dot{T}_t, T_{d,t-1}\right) = g(f(t, r_{t-\tau}), T_{d,t-1}) . \tag{3}$$

Let us fix actual 3 years time-lag ($l = 2$, $\tau = 0,1,2$) in Japan's manufacturing industry (Watanabe, 2000 [31]). We specify equation (3) in the following form

$$T_{d,t} = \alpha_0 + \alpha_1 \cdot t + \alpha_2 \cdot \left(r_t + \theta \cdot r_{t-1} + \theta^2 \cdot r_{t-2}\right) + \alpha_3 \cdot (1-\rho) \cdot T_{d,t-1} + \varepsilon_t =$$

$$= \alpha_0 + \alpha_1 \cdot t + \alpha_2 \cdot \sum_{\tau=0}^{2} \theta^\tau \cdot r_{t-\tau} + \alpha_3 \cdot (1-\rho) \cdot T_{d,t-1} + \varepsilon_t , \tag{4}$$

$\theta > 0$, $0 < \rho < 1$, $\alpha_2 \geq 0$, $\alpha_3 \geq 0$.

Here $T_{d,t}$ is the indigenous technology stock at time t; t is the time trend; r_t is R&D investment at time t; θ is the weight coefficient for the lagged variables; τ is the time lag between R&D investment and commercialization; ρ is the obsolescence coefficient of the technology stock $T_{d,t-1}$; α_0 is the intercept term; α_2 is the calibration coefficient for the lagged variables; α_1 and α_3 are regression coefficients of explanatory variables; and ε_t describes disturbances. Let us note that the weight coefficient θ is a nonstandard regression parameter introduced to describe the net effect of R&D investment. The calibration procedure for this model is described in detail in the paper by Pyndick and Rubinfeld, 1991 [20].

In this model, R&D investment in respective years contributes distinctly to formation of the domestic technology stock in accordance with its weights θ^τ. The total contribution of R&D investment within time t - τ should be greater than the obsolete part of the technology stock of the previous year in order to maintain the steady growth of the domestic technology stock. Therefore, it is assumed that parameters of equation (4) satisfy the following relation

$$\alpha_2 \cdot \sum_{\tau=0}^{2} \theta^\tau \cdot r_{t-\tau} - \alpha_3 \cdot \rho \cdot T_{d,t-1} > 0 . \tag{5}$$

Application of constraint (5) implies the following requirement:

(a) the weights of the geometric lagged explanatory variables θ^τ are positive, decline in time, and never become zero.

Besides, in the econometric model (4) the standard assumptions on disturbances are introduced:

(b) the disturbance term ε_t is normally distributed, independent of variables $r_{t-\tau}$ and $T_{d,t-1}$, and neither serially correlated nor heteroscedastic.

2.1.2. Spillover Technology and Assimilation Capacity

Technology has some peculiar properties as an economic commodity that bear on its role in the growth process (Romer, 1990 [22]). The *partial nonexcludability* of technology suggests that industrial R&D may generate *technology spillover*. That means: (*i*) firms can acquire information created by others without paying for that information in a market transaction, and (*ii*) the creators, or current owners of the information have no effective sources under the present prevailing legislation to protect this information in the case if other firms acquire it and utilize.

Basing on this postulate one can introduce into the model the technology spillover pool T_s which consists of technologies generated by other firms and available at the market place.

To describe the technology spillover pool T_s let us use a modified Cobb-Douglas type function which includes the lagged variables of the net value of R&D funds

$$T_{s,t} = A \cdot e^{\delta \cdot t} \prod_{\tau=0}^{2} (\tilde{r}_{t-\tau})^{\eta \cdot \phi(\tau)} (T_{r,t-1})^{\zeta \cdot (1-\rho)} (\mathrm{Im}_t)^{\sigma}, \quad \phi(\tau) = \phi^{\tau}, \tag{6}$$

where A is a scale factor; $\tilde{r}_{t-\tau}$ is the net value of R&D funds received and paid to outside at time t - τ; $T_{r,t-1}$ is the technology stock generated by R&D investment $\tilde{r}$ at time t - 1; Im_t is the technology import at time t; $\imath$ is the coefficient of the lagged variable; δ, ζ and c are the regression coefficients of explanatory variables; ϕ^{τ} are the weights of the lagged variables; and τ is the time-lag of R&D investment and commercialization.

In the following stage, the assimilation capacity z is measured according to the econometric model proposed in the paper by Watanabe et al., 2001 [33]

$$z = \frac{T_d / T_s}{1 + \dfrac{\Delta T_s / T_s}{\Delta T_d / T_d}} . \tag{7}$$

Introducing notations

$$\xi = \frac{\Delta T_d}{T_d} > 0, \quad \omega = \frac{\Delta T_s}{T_s} > 0 \tag{8}$$

for the rates of the domestic technology stock, T_d, and the technology spillover pool, T_s, one can get the following presentation of the assimilation capacity

$$z = z(\xi) = \frac{1}{1 + \dfrac{\omega}{\xi}} \cdot \frac{T_d}{T_s} = \frac{\xi}{\xi + \omega} \cdot \frac{T_d}{T_s} . \tag{9}$$

Then the gross technology stock T at time t is defined as the total sum of the domestic technology stock and the assimilated spillover technology

$$T_t = T_{d,t} + z_t \cdot T_{s,t} . \tag{10}$$

Linearization by the Taylor expansion of the assimilation capacity z with respect to the change rate of the domestic technology ξ around the fixed rate $\xi_0 > 0$ provides the following approximation

$$z(\xi) \approx z(\xi_0) + \frac{dz}{d\xi}(\xi_0) \cdot (\xi - \xi_0) ,$$

where

$$z(\xi_0) = \frac{\xi_0}{\xi_0 + \omega} \cdot \frac{T_d}{T_s} , \tag{11}$$

$$\frac{dz(\xi_0)}{d\xi} = \frac{\omega}{(\xi_0 + \omega)^2} \cdot \frac{T_d}{T_s} . \tag{12}$$

Hence, the assimilation capacity z can be approximated by the following equation

$$z = z(\xi) = \left(\frac{\xi_0}{(\xi_0 + \omega)} + \frac{\omega}{(\xi_0 + \omega)^2} \cdot (\xi - \xi_0) \right) \cdot \frac{T_d}{T_s} =$$

$$
= \left(\frac{\xi_0}{(\xi_0 + \omega)} - \frac{\omega \cdot \xi_0}{(\xi_0 + \omega)^2} + \frac{\omega \cdot \xi}{(\xi_0 + \omega)^2} \right) \cdot \frac{T_d}{T_s} =
$$

$$
= \left(\frac{\xi_0^2}{(\xi_0 + \omega)^2} + \frac{\omega \cdot \xi}{(\xi_0 + \omega)^2} \right) \cdot \frac{T_d}{T_s} . \tag{13}
$$

Thus, the gross technology stock T after linearization of the assimilation capacity z can be presented as follows:

$$
T = T(\xi) = T_d + \left(\frac{\xi_0^2}{(\xi_0 + \omega)^2} + \frac{\omega \cdot \xi}{(\xi_0 + \omega)^2} \right) \cdot \frac{T_d}{T_s} \cdot T_s =
$$

$$
= T_d + \left(\frac{\xi_0^2}{(\xi_0 + \omega)^2} + \frac{\omega \cdot \Delta T_d / T_d}{(\xi_0 + \omega)^2} \right) \cdot T_d =
$$

$$
= \left(1 + \frac{\xi_0^2}{(\xi_0 + \omega)^2} \right) \cdot T_d + \frac{\omega}{(\xi_0 + \omega)^2} \cdot \Delta T_d . \tag{14}
$$

Introducing notations for coefficients

$$
\mu = 1 + \frac{\xi_0^2}{(\xi_0 + \omega)^2} \geq 1 , \qquad \nu = \frac{a}{(\xi_0 + \omega)^2} \leq \frac{1}{\omega} \tag{15}
$$

one can obtain the following presentation for the gross technology stock

$$
T = \mu \cdot T_d + \nu \cdot \Delta T_d . \tag{16}
$$

2.2. Empirical Analysis

2.2.1. Indigenous Technology

By means of regression analysis applied to the period 1982-2000 one can identify the model coefficients (4) for the domestic technology stock

$$
\underset{(97.99)}{T_{d,t} = 772160} + \underset{(8.25)}{1.73} \sum_{\tau=0}^{2} \underset{(1.88)}{(1.48)^{\tau}} \cdot r_{t-\tau} + \underset{(4.48)}{4.07} \cdot (1-\rho) \cdot T_{d,t-1} , \tag{17}
$$

$adj.R^2 = 0.981, \qquad DW = 1.34 .$

Here the symbol $adj.R^2$ denotes the adjusted coefficient of determination, the symbol DW denotes the Durbin-Watson test statistic, and figures in the brackets denote the Student's t-statistic of the corresponding regression coefficients. The value of the obsolescence coefficient ρ is identified at the level $\rho = 0.105$ in the paper by Watanabe, 2000 [31], on the basis of the 10.5 years data for the actual obsolescence rates of technology in the Japanese manufacturing industry.

The statistical result in equation (17) demonstrates that all identified coefficients are statistically significant. The corresponding data is given Appendix A.1. which describes trends in R&D expenditure and technology import in Japan's Automotive Industry at current prices.

2.2.2. Technology Spillover Pool

Using the similar regression analysis over the same period one can obtain the model coefficients (6) for the technology spillover pool

$$T_{s,t} = e^{-4.88 \cdot t} \prod_{\tau=0}^{2} (\tilde{r}_{t-\tau})^{0.99 \cdot \phi(\tau)} (T_{r,t-1})^{0.67 \cdot (1-\rho)} (\mathrm{Im}_t)^{0.06}, \quad \phi(\tau) = (0.97)^{\tau}, \tag{18}$$

(-16.13) (6.69) (6.78) (3.90) (8.64)

$adj.R^2 = 0.990$, $DW = 1.26$.

Equation (18) also demonstrates statistical significance with respect to all identified coefficients.

Conceptually, the technology spillover pool can be decomposed into three components as illustrated in Figure 1: the net value of R&D funds received and paid to outside and the accumulative technology stock generated from it; the technology import; and the time trend effect of the economy.

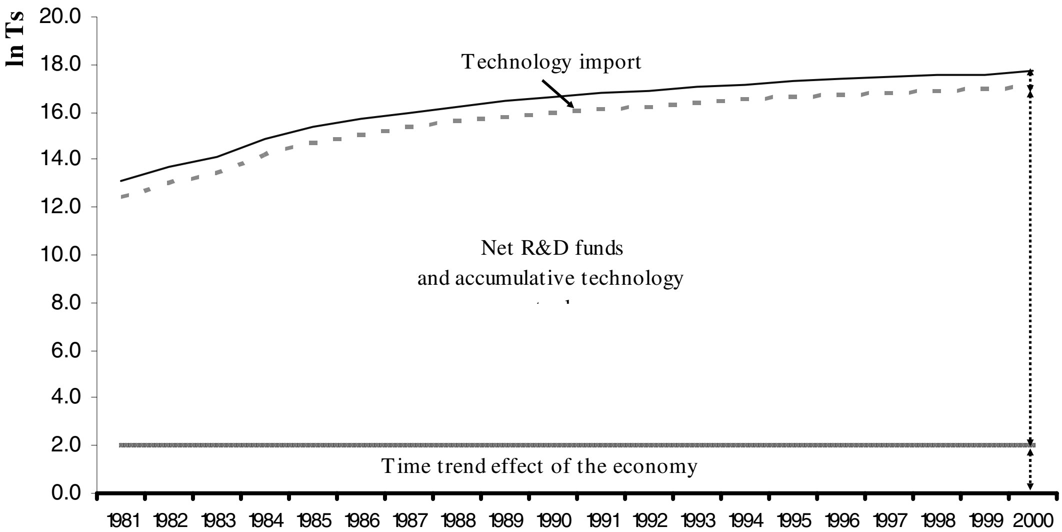

Figure 1. Econometric Trajectory of Technology Spillover Pool.

Let us note that the time trend effect of the economy, $A \cdot e^{\delta t}$, fluctuates quite small. Thus, when its value is compared to the contribution value of net R&D funds, $\prod_{\tau=0}^{2} (\tilde{r}_{t-\tau})^{\eta \cdot \phi(\tau)}$, and accumulative technology stock generated by R&D investment $\tilde{r}$ at time (t-1), $(T_{r,t-1})^{\varsigma \cdot (1-\rho)}$, or technology import, $(\mathrm{Im}_t)^{\sigma}$, it can be depicted linearly as it is shown on Fig. 1.

2.2.3. Assimilation Capacity

Let us introduce dummy variables, D_i, $i = 1,2,3$, into equation (13) for the assimilation capacity, z. The dummy variables, D_i $i = 1,2,3$, describe restructuring of the time series trends and correspond to the periods before, during and after the bursting of the bubble economy in the Japanese manufacturing industry, respectively:

$D_1 = 1$ in the period 1982-1986, $D_1 = 0$ in other years;

$D_2 = 1$ in the period 1987-1990 , $D_2 = 0$ in other years;

$D_3 = 1$ in the period 1991-2000, $D_3 = 0$ in other years. (19)

One can consider the following model for identification of the rate of the indigenous technology stock, ξ_0,

$$z = z(\xi) = D_1 \cdot \left(\frac{\xi_0^2}{(\xi_0 + \omega)^2} + \frac{\omega \cdot \xi}{(\xi_0 + \omega)^2} \right) \cdot \frac{T_d}{T_s} + D_2 \cdot \left(\frac{\xi_0^2}{(\xi_0 + \omega)^2} + \frac{\omega \cdot \xi}{(\xi_0 + \omega)^2} \right) \cdot \frac{T_d}{T_s} +$$

$$+ D_3 \cdot \left(\frac{\xi_0^2}{(\xi_0 + \omega)^2} + \frac{\omega \cdot \xi}{(\xi_0 + \omega)^2} \right) \cdot \frac{T_d}{T_s} \tag{20}$$

By means of the nonlinear regression analysis (software SPSS 10.0J) the initial rate of the indigenous technology stock is identified as follows:

$\xi_0 = 0.14$, $\quad a = 0.24 \quad$ $adj.R^2 = 0.970$, $\quad DW = 2.66$.

(14.71) (7.14)

The regression model (19)-(20) demonstrates that the rate coefficient ξ_0 is statistically significant and proves that the linear approximation (13) fits well to the data time series and properly substitutes the nonlinear model (7) for the assimilation capacity z.

2.2.4. Gross Technology Stock

On the basis of prior econometric analysis one can identify trajectories of the indigenous technology stock and the technology spillover pool for Japan's automotive industry over the last two decades.

Basing on the econometric measurements of the domestic technology stock, T_d (17), the spillover technology pool, T_s (18), the change rate of the indigenous technology stock, ξ_0 (19), one can identify the gross technology stock T by the following equation

$$T = D_1 \cdot (\mu \cdot T_d + \nu \cdot \Delta T_d) + D_2 \cdot (\mu \cdot T_d + \nu \cdot \Delta T_d) + D_3 \cdot (\mu \cdot T_d + \nu \cdot \Delta T_d) \tag{21}$$

with dummy variables D_i, $i = 1,2,3$, as described in (19).

Figure 2 demonstrates good coincidence of the linearized model (21) with the nonlinear model (8), (9) for the gross technology stock T. The trajectory T^{nl} of the nonlinear model (8), (9) is depicted by the solid line, and the trajectory T^{l} of the linearized model (21) is shown by the dashed line.

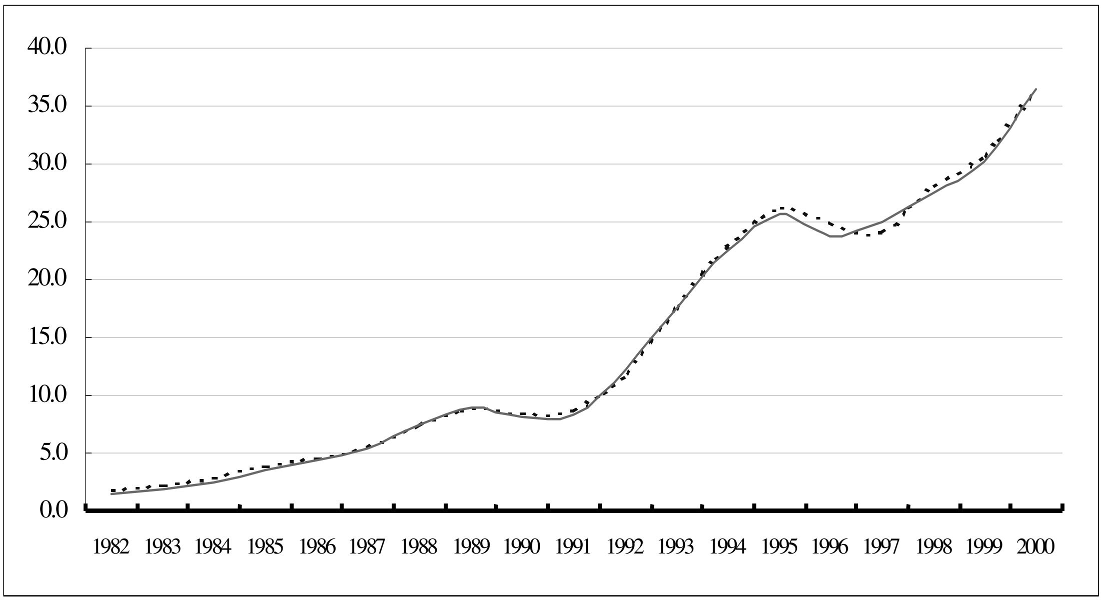

Figure 2. Trajectories of the gross technology stock of Japan's Automotive Industry (*1982-2000*) in the nonlinear and linearized models – *trillion Yen in 1995 fixed prices.*

This good coincidence of two trajectories can be demonstrated numerically if one construct a regression of the trajectory T^{nl} of the nonlinear model (8), (9) on the trajectory T^{l} of the linearized model (21). The numerical results of this regression can be presented by the following figures

$$\ln T^{nl} = \underset{(-6.09)}{-0.82} + \underset{(126.19)}{1.05} \cdot \ln T^{l}, \quad adj.R^2 = 0.999, \quad DW = 1.89, \tag{22}$$

which show good numerical fitness of trajectory T^{l} to trajectory T^{nl}.

Figure 3 depicts the growth trends in development trajectories of the gross technology stock, T, the indigenous technology stock, T_d, and the assimilated spillover technology, $z \cdot T_s$, in Japan's automotive industry over the period 1982-2000. The gross technology stock, T, is depicted by the solid line with dot markers, the domestic technology stock, T_d, is presented by the solid line, and the assimilated spillover technology, $z \cdot T_s$, is shown in the dashed line.

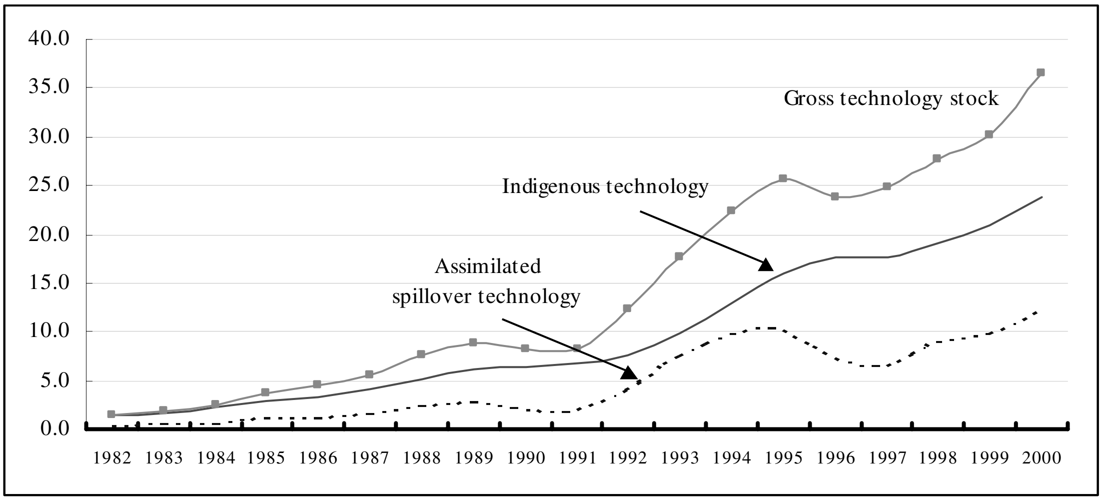

Figure 3. Growth trends of technology trajectories in Japan automotive industry

(1982-2000) – trillion Yen in 1995 fixed prices.

Figure 3 demonstrates significant growth of the gross technology stock in Japan's automotive industry in the 1990s corresponding to the period after the bursting of the bubble economy in 1991.

3. Dynamic Optimality of R&D Intensity

3.1. Utility Function

According to Grossman and Helpman (1991 [8]) for determining the optimal trajectory of the gross technology stock one can use the utility function J represented by an integral with a discount rate λ

$$J = \int_{t_0}^{+\infty} e^{-\lambda(t-t_0)} \log D(t)dt . \qquad (23)$$

Here $D(t)$ represents a consumption index at time t; time t varies on the infinite horizon, $t \in [t_0, +\infty)$; t_0 is the initial time.

Let us assume for the consumption index D a specification that imposes a constant and equal elasticity of substitution between any pair of products including the new invented products generated by R&D investments

$$D = d \cdot \left[\int_0^n x^{\pi}(j)dj \right]^{1/\pi} . \qquad (24)$$

Here d is a scale factor; $n = n(t)$ is the amount of innovative goods; j is the current index of innovative goods, $0 \le j \le n$; $x = x(j)$ is the quantity of production of the brand with index j; π is the parameter of elasticity for variety of products, $0 < \pi < 1$; ε is the elasticity of substitution between two innovative goods,

$$\varepsilon = \frac{1}{1 - \pi} > 1, \tag{25}$$

or, equivalently,

$$\pi = 1 - \frac{1}{\varepsilon}.$$

Introducing notation $y = y(t)$ for production of innovative goods and assuming that quantities $x = x(j)$ are equal for each index j, $0 \le j \le n$, one can get the following relation

$$x = \frac{y(t)}{n(t)}.$$

Hence, the consumption index D can be presented by the formula

$$D = D(t) = d \cdot \left[\left(\frac{y(t)}{n(t)} \right)^{\pi} \cdot n(t) \right]^{1/\pi} = d \cdot y(t) \cdot (n(t))^{(1-\pi)/\pi}. \tag{26}$$

Let us assume that the number of innovative products, n, depends on the gross technology stock, T, and on the change, $u = \Delta T_d$, in the indigenous technology stock, T_d, according to the regression equation (see Watanabe, 2000 [31])

$$n = n(t) = c \cdot e^{\chi t} \cdot T^{\beta_1} \cdot (\Delta T_d)^{\beta_2} = c \cdot e^{\chi t} \cdot T^{\beta_1} \cdot u^{\beta_2}. \tag{27}$$

Here c is the scale factor; χ is the coefficient of the time trend; β_i, $i = 1,2$, are the regression coefficients of explanatory variables.

Substituting formulas (26), (27) for the consumption index D into the integral (23) one can obtain the following relation for the utility function

$$J = \int_{t_0}^{+\infty} e^{-\lambda(t-t_0)} (\ln y(t) + a_1 \cdot \ln T(t) + a_2 \cdot \ln u(t)) dt +$$

$$+ \int_{t_0}^{+\infty} e^{-\lambda(t-t_0)} (\ln d + h \cdot (\ln c + \chi \cdot t)) dt. \tag{28}$$

Here

$$h = \frac{1-\pi}{\pi} = \frac{1}{\pi} - 1 = \frac{1}{\varepsilon - 1}, \quad a_i = h \cdot \beta_i, \quad i = 1,2. \tag{29}$$

The second term in the utility function (28) does not depend on the basic variables $y(t)$, $T(t)$, $u(t)$. Hence, it does not influence on optimization of R&D investment policy and can be omitted.

The structure of the utility function in equation (28) implies that investors (specifically for the Japan's automotive industry the notion of investors includes auto manufacturers, the government, special corporations, and other non-government institutions) are interested in growth of production, $y(t)$, the accumulative technology stock, $T(t)$, and R&D investment expressed by the technology change $u(t)$.

Inserting expression (16) for the accumulative technology stock, $T(t)$, into functional (28) one can obtain the following relation for the utility function

$$J = \int_{t_0}^{+\infty} e^{-\lambda(t-t_0)} \left(\ln y(t) + a_1 \cdot \ln\left(\mu \cdot T_d(t) + \nu \cdot u(t)\right) + a_2 \cdot \ln u(t)\right) dt. \tag{30}$$

Due to the logarithmic terms in the utility function (30) production, $y(t)$, indigenous technology stock, $T_d(t)$, and change in the indigenous technology stock, $u(t)$, satisfy the following restrictions

$$y = y(t) > 0, \quad T_d = T_d(t) > 0, \quad u = u(t) > 0.$$

Moreover, let us assume that these variables are strictly separated from zero

$$0 < y^l \le y(t), \quad 0 < T_d^l \le T_d(t), \quad 0 < u^l \le u(t). \tag{31}$$

Linearizing in functional (31) the
logarithmic term with respect to variable u

$$a_1 \cdot \ln T = a_1 \cdot \ln\left(\mu \cdot T_d + \nu \cdot u\right) =$$

$$= a_1 \cdot \ln\left(\mu \cdot T_d \left(1 + \frac{\nu}{\mu} \cdot \frac{u}{T_d}\right)\right) = a_1 \cdot \ln \mu + a_1 \cdot \ln T_d + a_1 \cdot \ln\left(1 + \kappa \cdot \frac{u}{T_d}\right) \approx$$

$$\approx a_1 \cdot \ln \mu + a_1 \cdot \ln T_d + a_1 \cdot \kappa \cdot \frac{u}{T_d}, \quad \kappa = \frac{\nu}{\mu}, \tag{32}$$

one can get the following approximation of the utility function

$$I = \int_{t_0}^{+\infty} e^{-\lambda(t-t_0)} \left(\ln y(t) + a_1 \cdot \ln T_d + a_1 \cdot \kappa \cdot \frac{u}{T_d} + a_2 \cdot \ln u(t) \right) dt, \tag{33}$$

which is used for obtaining approximate analytical solutions.

3.2. Identification of Parameters of Utility Function

The elasticity coefficients of equations (28) - (33) are calibrated on the empirical data of the automotive production and its input, and the number of registered patent in Japan's automotive industry over the period 1982-2000 as described in Appendix A.2. and A.3.

The discount factor λ in the utility function (30) is identified at the level 0.105 that similar to the obsolescence rate ρ of technology $T_d(t)$ (Watanabe, 2000 [31]).

Econometric simulations of equations (27), (28) provide the following elasticity coefficients:

(*i*) elasticity for variety of innovative products, π,

$$D(t) = \underset{(4.29)}{0.005} \cdot \left[\int_0^n x^{\underset{(2.01)}{0.09}}(j)dj \right]^{\underset{(2.01)}{11.11}}, \quad adjR^2 = 0.651, \quad DW = 1.38. \tag{34}$$

Therefore, the coefficients h, π, ε have the following values

$$h = \frac{1-\pi}{\pi} = 10.11, \quad \pi = \frac{1}{h+1} = 0.09, \quad \varepsilon = \frac{1}{1-\pi} = 1.0989. \tag{35}$$

Elasticity β_1 of technology, T, and elasticity β_2 of change in the indigenous technology stock, T_d, are identified on the basis of the regression equation with the dummy variable, D_n,

$$\ln n(t) = \underset{(-3.31)}{-8.167} \cdot D_n + \underset{(4.66)}{0.003} \cdot t + \underset{(4.66)}{0.050} D_n \cdot \ln T(t) + \underset{(96.79)}{0.599} \cdot \ln u(t), \tag{36}$$

$adj.R^2 = 0.782$, $DW = 1.22$.

The dummy variable D_n indicates the period before the bubble economy in the Japanese manufacturing industry: $D_n = 1$ in the period 1982-1986, $D_n = 0$ in other years.

The statistical results in equation (34) - (36) demonstrate that all identified coefficients are statistically significant.

Figure 4 illustrates the estimated trends (34), (35) in the consumption index D of Japan's automotive industry over the period 1982-2000.

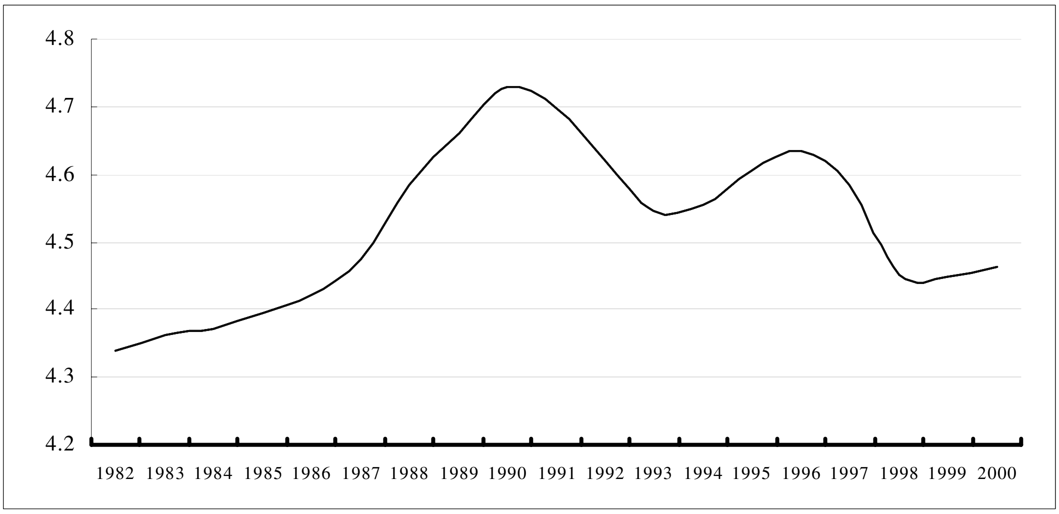

Figure 4. Trends in Consumption Index of Japan's Automotive Industry *(1982-2000*).

Substituting the values of coefficients h, β_1, and β_2 to equation (29) one can obtain coefficients $a_1 = 0.51$ and $a_2 = 6.07$ of the utility function J (30). Figure 5 depicts the values of the utility function for Japan's automotive industry over the last two decades.

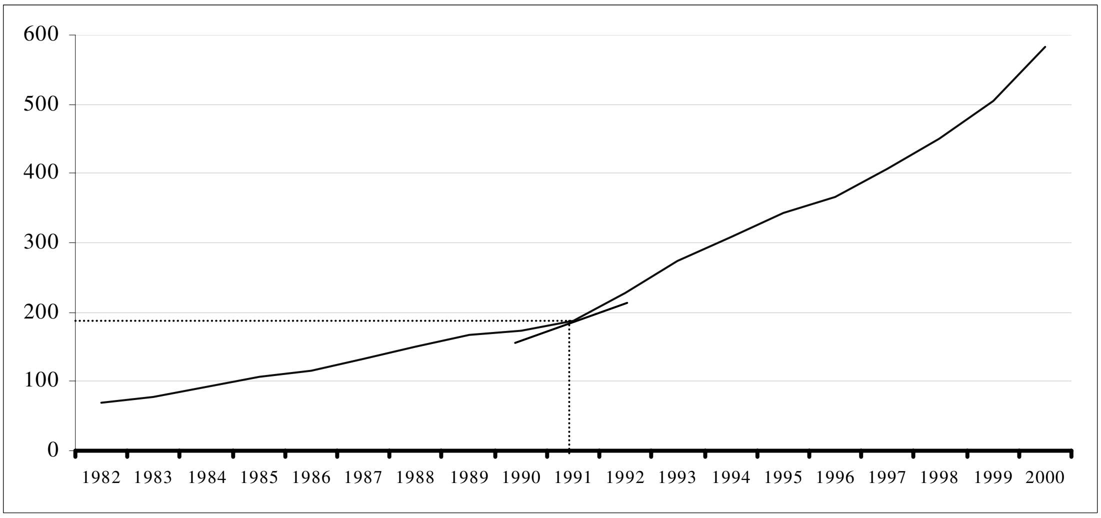

Figure 5. Trends in Utility *(1982-2000)*.

The utility increases significantly in the period of the 1990s after the bursting of the bubble economy. Figure 5 shows also an inflection point in 1991 and indicates the restructuring of the growth slope from the level of 14.42 trillion Yen per year to the level of 41.48 trillion Yen per year in the fixed prices of 1995.

3.3. Model Dynamics

Let us define the dynamics of production by the following differential equation

$$\frac{\dot{y}(t)}{y(t)} = f_1 + f_2 \cdot \left(\frac{T(t)}{y(t)} \right)^{\gamma} - g \cdot \frac{u(t)}{y(t)} . \tag{37}$$

Here parameter f_1 represents the non-R&D contribution into the production growth, $f_1 \geq 0$. Parameter γ is an elasticity of technology to production, $0 \leq \gamma < 1$, and parameter f_2 is a scale coefficient, $f_2 \geq 0$. Parameter g is the discounted marginal productivity of the domestic technology stock T_d. It is assumed that the following inequality is valid

$$g = p - q > 0 . \tag{38}$$

Here parameter p, $p > 0$, demonstrates the decrease in production due to the domestic R&D expenditures, and the marginal productivity of the domestic technology, q, $q > 0$, describes the growth trend. The negative sign in front of the net contribution of R&D investments, $(- g \cdot u(t) / y(t))$, shows that, in the short run, spending into the domestic technology prevails on its rate of return.

Let us introduce the notation

$$r_d(t) = \frac{u(t)}{y(t)} = \frac{\Delta T_d(t)}{y(t)} \tag{39}$$

for R&D intensity. Then dynamics of the domestic technology stock, T_d, is described by the following differential equation

$$\dot{T}_d(t) = r_d(t) \cdot y(t) = u(t) . \tag{40}$$

Let us remind that the accumulative technology stock, T, in equation (37) can be expressed through the domestic technology stock, T_d, and its rate, $u = \Delta T_d$, by relation (16). So, the system of equations (37), (38) forms the closed-loop control system. Production $y(t)$ and the domestic technology stock T_d are the phase variables of this system. The rate $u = \Delta T_d$ of the domestic technology stock, or, equivalently, R&D indensity r_d is the control parameter. The technology spillover pool, T_s, influences on dynamics (37), (38), and utility (30), exogenously through its rate a (8) presented in the model coefficients μ, ν (15).

It is clear that R&D intensity r_d lies in the range between 0 and 100 percent

$$0 \le r_d(t) = \frac{\Delta T_d(t)}{y(t)} \le 1 .$$

Taking into account restrictions (31) it is necessary to separate R&D intensity r_d strictly from zero. Let us assume that there exist lower, r_d^l, and upper, r_d^u, bounds such that the following relations take place

$$0 < r_d^l \le r_d(t) \le r_d^u < 1 .$$

In order to provide the positive trend of the production growth let us assume that parameters in dynamics (37) satisfy the following restriction

$$f_1 - g \cdot r_d(t) > 0 .$$

It means that R&D intensity r_d should satisfy the following inequality

$$r_d < \frac{f_1}{g},$$

and, hence, the upper bound r_d^u should meet the following condition

$$r_d^u < \max\left\{1, \frac{f_1}{g}\right\}.$$

3.4. Optimal Control Problem

The optimal control problem of R&D investment is formulated as follows. It is necessary to find R&D intensity $r_d(t)$ such that maximizes the utility function

$$J = \int_{t_0}^{+\infty} e^{-\lambda(t-t_0)}((1+a_2)\cdot \ln y(t) + a_1 \cdot \ln\big(\mu \cdot T_d(t) + \nu \cdot r_d(t) \cdot y(t)\big) +$$

$$+ a_2 \cdot \ln r_d(t))dt, \tag{41}$$

provided the dynamics is described by differential equations

$$\frac{\dot{y}(t)}{y(t)} = f_1 + f_2 \cdot \left(\frac{\mu \cdot T_d(t) + \nu \cdot r_d(t) \cdot y(t)}{y(t)} \right)^{\gamma} - g \cdot r_d(t), \tag{42}$$

$$\dot{T}_d(t) = r_d(t) \cdot y(t), \tag{43}$$

subject to constrains

$$0 < r_d^l \le r_d(t) \le r_d^u < \max\left\{1, \frac{f_1}{g}\right\}, \tag{44}$$

and initial conditions

$$y(t_0) = y^0, \quad T_d(t_0) = T_d^0. \tag{45}$$

3.5. Approximation of Utility Function

The main difference of the optimal control problem (41)-(45) from the classical problem (see Pontryagin et al., 1962 [19]) consists in the unboundedness of the time interval in the utility function (41). Let us consider an approximation of the utility function (41) restricting the time horizon to a large but a finite interval $[t_0, \vartheta]$, $t_0 \le \vartheta < +\infty$.

The utility function (41) can be presented in the following form

$$J = J_{t_0}^{\vartheta} + J_{\vartheta}. \tag{46}$$

Here the integral $J_{t_0}^{\vartheta}$ is defined on the finite interval of time $[t_0, \vartheta]$

$$J_{t_0}^{\vartheta} = \int_{t_0}^{\vartheta} e^{-\lambda(t-t_0)}((1+a_2)\cdot \ln y(t) + a_1 \cdot \ln\big(\mu \cdot T_d(t) + \nu \cdot r_d(t) \cdot y(t)\big) +$$

$$+ a_2 \cdot \ln r_d(t))dt, \tag{47}$$

and the integral J_{ϑ} is the approximation error.

This error is esimated in the following statement.

Proposition 1. For any initial postion (y^0, T_d^0) and for any realization of control $r_d(t)$ the value of the utility function J is finite. For any initial postion (y^0, T_d^0) and for any parameter $\varepsilon > 0$ there exists a moment of time, $\vartheta = \vartheta(\varepsilon)$, $\vartheta \geq t_0$, such that for any realization of control $r_d(t)$ the value J can be approximated by the value $J_{t_0}^{\vartheta}$ with the given accuracy

$$\left| J - J_{t_0}^{\vartheta} \right| = \left| J_{\vartheta} \right| = J_{\vartheta} < \varepsilon . \tag{48}$$

Proof of Proposition 1. Let us estimate the technology intensity $w = w(t) = T_d(t) / y(t)$. To make this estimation let us prove the following statement.

Lemma 1. There exists an interval $[K_0, K^0]$, $0 < K_0 \leq K^0$, such that it is stronly invariant with respect to the control system (42), (43). It means that if a trajectory $(y(t), T_d(t))$ of the system (42), (43) starts its motion in the interval

$$[K_0, K^0],\ w(t_0) \in [K_0, K^0], \tag{49}$$

then it stays in it forever,

$$w(t) \in [K_0, K^0]. \tag{50}$$

Proof of Lemma 1. To prove this, let us estimate the derivative of $w(t)$ by virtue of the system (42), (43)

$$\dot{w}(t) = \left(\frac{T_d(t)}{y(t)} \right)' = \frac{\dot{T}_d(t)}{y(t)} - w(t) \cdot \frac{\dot{y}(t)}{y(t)} =$$

$$= r_d(t) - w(t) \cdot (f_1 + f_2 \cdot (\mu \cdot w(t) + \nu \cdot r_d(t))^{\gamma} - g \cdot r_d(t)). \tag{51}$$

From (51) one can get

$$\dot{w}(t) \geq r_d^l \cdot (1 + g \cdot w(t)) - f_1 \cdot w(t) - f_2 \cdot w(t) \cdot (\mu \cdot w(t) + \nu \cdot r_d^u)^{\gamma} .$$

To estimate the derivative $\dot{w}(t)$ (51) from below let us choose a number K^0, $K_0 > 0$, such that it satisfies the resolvable inequality

$$0 < \frac{f_1 \cdot K_0 + f_2 \cdot K_0 \cdot (\mu \cdot K_0 + \nu \cdot r_d^u)^\gamma}{1 + g \cdot K_0} < r_d^l \cdot \tag{52}$$

Then at the point $w = K_0$ the derivative $\dot{w}(t)$ (51) is strictly positive for any control r_d (44)

$$\dot{w}(t) \geq r_d^l \cdot (1 + g \cdot K_0) - f_1 \cdot K_0 - f_2 \cdot K_0 \cdot (\mu \cdot K_0 + \nu \cdot r_d^u)^\gamma > 0. \tag{53}$$

On the other hand, from (51) one can obtain

$$\dot{w}(t) \leq r_d^u \cdot (1 + g \cdot w(t)) - f_1 \cdot w(t) - f_2 \cdot w(t) \cdot (\mu \cdot w(t) + \nu \cdot r_d^l)^\gamma .$$

To estimate the derivative $\dot{w}(t)$ (51) from above let us choose a number K^0, $K^0 \geq K_0 > 0$, such that it satisfies the resolvable inequality

$$\frac{f_1}{g} > \frac{f_1 \cdot K^0 + f_2 \cdot K^0 \cdot (\mu \cdot K^0 + \nu \cdot r_d^l)^\gamma}{1 + g \cdot K^0} > r_d^u \cdot \tag{54}$$

Then at the point $w = K^0$ the derivative $\dot{w}(t)$ (51) is strictly negative for any control r_d (44)

$$\dot{w}(t) \leq r_d^u \cdot (1 + g \cdot K^0) - f_1 \cdot K^0 - f_2 \cdot K^0 \cdot (\mu \cdot K^0 + \nu \cdot r_d^l)^\gamma < 0. \tag{55}$$

Inequalities (53), (55) mean that the interval $[K_0, K^0]$, $0 < K_0 \leq K^0$, is strongly invariant with respect to control system (42), (43).

The proof of Lemma 1 is complete.

Using relation (55) let us estimate the production rate (41) from above

$$\frac{\dot{y}(t)}{y(t)} \leq f_1 + f_2 \cdot \left(\mu \cdot K^0 + \nu \cdot r_d^u\right)^\gamma - g \cdot r_d^l = M . \tag{56}$$

Thus,

$$y(t) \leq y^0 \cdot e^{M \cdot (t - t_0)} .$$

Then the integrand in the utility function (47) can be estimated as follows

$$(1+a_2)\cdot\ln y(t)+a_1\cdot\ln\bigl(\mu\cdot T_d(t)+\nu\cdot r_d(t)\cdot y(t)\bigr)+a_2\cdot\ln r_d(t)=$$

$$=(1+a_1+a_2)\cdot\ln y(t)+a_1\cdot\ln\left(\mu\cdot\frac{T_d(t)}{y(t)}+\nu\cdot r_d(t)\right)+a_2\cdot\ln r_d(t)\le$$

$$\le(1+a_1+a_2)\cdot(\ln y^0+M\cdot(t-t_0))+a_1\cdot\ln\bigl(\mu\cdot K^0+\nu\cdot r_d^u\bigr)+a_2\cdot\ln r_d^u\,. \qquad (57)$$

Substituting this inequality to the integral J one can obtain the following estimate

$$J\le\int_{t_0}^{+\infty}e^{-\lambda(t-t_0)}(A+B\cdot t)dt=\frac{(A+B\cdot t_0)}{\lambda}+\frac{B}{\lambda^2}\,. \qquad (58)$$

Here

$$A=(1+a_1+a_2)\cdot(\ln y^0-M\cdot t_0)+a_1\cdot\ln\bigl(\mu\cdot K^0+\nu\cdot r_d^u\bigr)+a_2\cdot\ln r_d^u\,,$$

$$B=(1+a_1+a_2)\cdot M\,.$$

The estimate (58) shows that for any initial position (y^0,T_d^0) and for any realization of control $r_d(t)$ the value of the utility function J is finite.

The analogous substitution of inequality (57) into the integral J_ϑ provides the following estimate

$$J_\vartheta\le\int_{\vartheta}^{+\infty}e^{-\lambda(t-t_0)}(A+B\cdot t)dt=$$

$$=\left(\frac{e^{\lambda t_0}\cdot A}{\lambda}+\frac{e^{\lambda t_0}\cdot B}{\lambda^2}+\frac{e^{\lambda t_0}\cdot B}{\lambda}\cdot\vartheta\right)\cdot e^{-\lambda\vartheta}\,. \qquad (59)$$

Given a positive accuracy parameter $\varepsilon>0$ one can choose a moment of time $\vartheta=\vartheta(\varepsilon)$ in equation (59) in a such way that the necessary estimate (48) is valid.

The proof of Proposition 1 is complete.

Proposition 2. For any initial position (y^0, T_d^0) the upper bound of the utility function J over control realizations $r_d(t)$ is finite. Hence, there exists the finite value of the optimal control problem (41)-(45)

$$V(y^0, T_d^0) = \sup_{r_d} \int_{t_0}^{+\infty} e^{-\lambda(t-t_0)}((1+a_2)\cdot \ln y(t) + a_1 \cdot \ln\big(\mu \cdot T_d(t) + \nu \cdot r_d(t) \cdot y(t)\big) +$$

$$+ a_2 \cdot \ln r_d(t))dt < +\infty \ . \tag{60}$$

The value $V(y^0, T_d^0)$ can be approximated by the values $V_{t_0}^{\vartheta}(y^0, T_d^0)$ of optimal control problems with finite horizon

$$V_{t_0}^{\vartheta}(y^0, T_d^0) = \max_{r_d} \int_{t_0}^{\vartheta} e^{-\lambda(t-t_0)}((1+a_2)\cdot \ln y(t) + a_1 \cdot \ln\big(\mu \cdot T_d(t) + \nu \cdot r_d(t) \cdot y(t)\big) +$$

$$+ a_2 \cdot \ln r_d(t))dt \, . \tag{61}$$

More precisely, for a given accuracy $\varepsilon > 0$ there exists a moment of time $\vartheta = \vartheta(\varepsilon)$, $\vartheta \geq t_0$, such that the following estimate takes place

$$\left| V(y^0, T_d^0) - V_{t_0}^{\vartheta}(y^0, T_d^0) \right| = V(y^0, T_d^0) - V_{t_0}^{\vartheta}(y^0, T_d^0) \leq \varepsilon \, . \tag{62}$$

Proof. The estimate (60) follows immediately from the inequality (58). To prove estimate (62) let us consider inequality

$$J(r_d(t)) < J_{t_0}^{\vartheta}(r_d(t)) + \varepsilon \, ,$$

which follows from relation (59) and is valid for any realization of control parameter $r_d(t)$. From this inequality and definition of the value (61) it follows

$$J(r_d(t)) < V_{t_0}^{\vartheta} + \varepsilon \, .$$

Passing to the upper bound in the last relation over control parameter $r_d(t)$ one can get the necessary estimate (62).

The proof of Proposition 2 is complete.

Remark 1. One can prove the results analogous to the results of Proposition 1 and Proposition 2 for the optimal control problem with dynamics (42)-(45) and utility function (33).

3.6. Hamiltonian System

The Hamiltonian function for the optimal control problem (41)-(45) has the following form

$$H(t, y, T_d, r_d, \psi_1, \psi_2) = e^{-\lambda(t-t_0)} \cdot ((1+a_2) \cdot \ln y + a_1 \cdot \ln(\mu \cdot T_d + \nu \cdot r_d \cdot y) + a_2 \cdot \ln r_d) +$$

$$+\psi_1 \cdot \left(f_1 \cdot y + f_2 \cdot (\mu \cdot T_d + \nu \cdot r_d \cdot y)^{\gamma} \cdot y^{(1-\gamma)} - g \cdot r_d \cdot y\right) + \psi_2 \cdot r_d \cdot y. \qquad (63)$$

One can express the Hamiltonian through the control parameter of the technology change

$$H(t, y, T_d, u, \psi_1, \psi_2) = e^{-\lambda(t-t_0)} \cdot (\ln y + a_1 \cdot \ln(\mu \cdot T_d + \nu \cdot u) + a_2 \cdot \ln u) +$$

$$+\psi_1 \cdot \left(f_1 \cdot y + f_2 \cdot (\mu \cdot T_d + \nu \cdot u)^{\gamma} \cdot y^{(1-\gamma)} - g \cdot u\right) + \psi_2 \cdot u. \qquad (64)$$

Here $\psi_1 = \psi_1(t)$, $\psi_2 = \psi_2(t)$ are the adjoint variables which have the sense of "shadow prices" of production $y = y(t)$, and indigenous technology stock $T_d = T_d(t)$, respectively.

According to the maximum principle of Pontryagin [19] one can introduce the Hamiltonian dynamics for the adjoint variables

$$\dot{\psi}_1(t) = -\frac{\partial H}{\partial y}(t, y(t), T_d(t), r_d(t), \psi_1(t), \psi_2(t)) =$$

$$= -e^{-\lambda(t-t_0)} \cdot \left(\frac{(1+a_2)}{y(t)} + \frac{a_1 \cdot \nu \cdot r_d(t)}{(\mu \cdot T_d(t) + \nu \cdot r_d(t) \cdot y(t))} \right) -$$

$$-\psi_1(t) \cdot \left(f_1 + f_2 \cdot \gamma \cdot \nu \cdot r_d(t) \cdot \left(\frac{\mu \cdot T_d(t) + \nu \cdot r_d(t) \cdot y(t)}{y(t)} \right)^{(\gamma-1)} + \right.$$

$$\left. + f_2 \cdot (1-\gamma) \cdot \left(\frac{\mu \cdot T_d(t) + \nu \cdot r_d(t) \cdot y(t)}{y(t)} \right)^{\gamma} - g \cdot r_d(t) \right) - \psi_2(t) \cdot r_d(t) \quad , \qquad (65)$$

$$\dot{\psi}_2(t) = -\frac{\partial H}{\partial T_d}(t, y(t), T_d(t), r_d(t), \psi_1(t), \psi_2(t)) =$$

$$= -e^{-\lambda(t-t_0)} \cdot \frac{a_1 \cdot \mu}{(\mu \cdot T_d(t) + \nu \cdot r_d(t) \cdot y(t))} -$$

$$-\psi_1(t) \cdot f_2 \cdot \gamma \cdot \mu \cdot \left(\frac{\mu \cdot T_d(t) + \nu \cdot r_d(t) \cdot y(t)}{y(t)} \right)^{(\gamma-1)} . \qquad (66)$$

In the optimal control problem with finite time horizon $[t_0, \vartheta]$, $t_0 \leq \vartheta < +\infty$, the following transversality conditions take place

$$\psi_i(\vartheta) = 0, \qquad i = 1,2 . \qquad (67)$$

Lemma 2. The solutions $\psi_1(t)$, $\psi_2(t)$ of the system (65)-(66) with boundary conditions (67) and variables $y(t)$, $T_d(t)$, $r_d(t)$ subject to dynamics (43)-(45) satisfy inequalities

$$\psi_i(t) > 0, \qquad t \in [t_0, \vartheta), \qquad i = 1,2 . \qquad (68)$$

Proof. The boundary conditions (67) nullify adjoint variables $\psi_i(\vartheta)$, $i = 1,2$, at the right-hand side of the interval $[t_0, \vartheta]$. The right parts of differential equations indicate that velocities $\dot{\psi}_i(t)$, $i = 1,2$, of adjoint variables in the forward time are negative, and, hence in the inverse time they are positive. It means that starting at sero at time ϑ with positive velocities in the inverse time the adjoint variables $\psi_i(t)$, $i = 1,2$, become posive and remain positive on the time interval $[t_0, \vartheta]$.

The proof of Lemma 2 is complete.

Remark 2. The positiveness (68) of the adjoint variables $\psi_i(t)$, $i = 1,2$, agree well with their economic sense of "shadow prices", which, of course, should be positive.

3.7. Concavity Properties of the Hamiltonian

In this section the concavity properties of the Hamiltonian H (64) are analyzed.

Proposition 3. The Hamiltonian $H(t, y, T_d, u, \psi_1, \psi_2)$ (64) is strictly concave with respect to variables y, T_d, u.

Proof. To prove this result let us use the Sylvester's criterion. The first derivatives of the Hamiltonian $H(t, y, T_d, u, \psi_1, \psi_2)$ (64) can be calculated as follows

$$\frac{\partial H}{\partial y} = e^{-\lambda(t-t_0)} \cdot \frac{1}{y} + \psi_1 \cdot f_1 + \psi_1 \cdot f_2 \cdot (1-\gamma) \cdot (\mu \cdot T_d + \nu \cdot u)^{\gamma} \cdot y^{-\gamma}, \tag{69}$$

$$\frac{\partial H}{\partial T_d} = e^{-\lambda(t-t_0)} \cdot \frac{a_1 \cdot \mu}{(\mu \cdot T_d + \nu \cdot u)} + \psi_1 \cdot f_2 \cdot \gamma \cdot \mu \cdot (\mu \cdot T_d + \nu \cdot u)^{(\gamma-1)} \cdot y^{(1-\gamma)}, \tag{70}$$

$$\frac{\partial H}{\partial u} = e^{-\lambda(t-t_0)} \cdot \left(\frac{a_1 \cdot \nu}{(\mu \cdot T_d + \nu \cdot u)} + \frac{a_2}{u} \right) +$$

$$+ \psi_1 \cdot f_2 \cdot \gamma \cdot \nu \cdot (\mu \cdot T_d + \nu \cdot u)^{(\gamma-1)} \cdot y^{(1-\gamma)} - \psi_1 \cdot g + \psi_2 . \tag{71}$$

The second derivatives of the Hamiltonian $H(t, y, T_d, u, \psi_1, \psi_2)$ (64) are defined by relations

$$\frac{\partial^2 H}{\partial y^2} = -e^{-\lambda(t-t_0)} \cdot \frac{1}{y^2} - \psi_1 \cdot f_2 \cdot (1-\gamma) \cdot \gamma \cdot (\mu \cdot T_d + \nu \cdot u)^{\gamma} \cdot y^{-(\gamma+1)}, \tag{72}$$

$$\frac{\partial^2 H}{\partial y \partial T_d} = \frac{\partial^2 H}{\partial T_d \partial y} = \psi_1 \cdot f_2 \cdot (1-\gamma) \cdot \gamma \cdot \mu \cdot (\mu \cdot T_d + \nu \cdot u)^{(\gamma-1)} \cdot y^{-\gamma}, \tag{73}$$

$$\frac{\partial^2 H}{\partial y \partial u} = \frac{\partial^2 H}{\partial u \partial y} = \psi_1 \cdot f_2 \cdot (1-\gamma) \cdot \gamma \cdot \nu \cdot (\mu \cdot T_d + \nu \cdot u)^{(\gamma-1)} \cdot y^{-\gamma}, \tag{74}$$

$$\frac{\partial^2 H}{\partial T_d^2} = -e^{-\lambda(t-t_0)} \cdot \frac{a_1 \cdot \mu^2}{(\mu \cdot T_d + \nu \cdot u)^2} -$$

$$- \psi_1 \cdot f_2 \cdot (1-\gamma) \cdot \gamma \cdot \mu^2 \cdot (\mu \cdot T_d + \nu \cdot u)^{(\gamma-2)} \cdot y^{(1-\gamma)}, \tag{75}$$

$$\frac{\partial^2 H}{\partial T_d \partial u} = \frac{\partial^2 H}{\partial u \partial T_d} = -e^{-\lambda(t-t_0)} \cdot \frac{a_1 \cdot \mu \cdot \nu}{(\mu \cdot T_d + \nu \cdot u)^2} -$$

$$-\psi_1 \cdot f_2 \cdot (1-\gamma) \cdot \gamma \cdot \mu \cdot \nu \cdot (\mu \cdot T_d + \nu \cdot u)^{(\gamma-2)} \cdot y^{(1-\gamma)}, \tag{76}$$

$$\frac{\partial^2 H}{\partial u^2} = -e^{-\lambda(t-t_0)} \cdot \left(\frac{a_1 \cdot \nu^2}{(\mu \cdot T_d + \nu \cdot u)^2} + \frac{a_2}{u^2} \right) -$$

$$-\psi_1 \cdot f_2 \cdot (1-\gamma) \cdot \gamma \cdot \nu^2 \cdot (\mu \cdot T_d + \nu \cdot u)^{(\gamma-2)} \cdot y^{(1-\gamma)}. \tag{77}$$

Let us prove that the matrix of second derivatives (72)-(77) is negative definite. To this end according to the Sylvester's criterion it is necessary to check that the principal minors alternate signs starting from the sign minus

$$\Delta_1 = \frac{\partial^2 H}{\partial y^2} = -e^{-\lambda(t-t_0)} \cdot \frac{1}{y^2} - \psi_1 \cdot f_2 \cdot (1-\gamma) \cdot \gamma \cdot (\mu \cdot T_d + \nu \cdot u)^{\gamma} \cdot y^{-(\gamma+1)} < 0, \tag{78}$$

$$\Delta_2 = \frac{\partial^2 H}{\partial y^2} \cdot \frac{\partial^2 H}{\partial T_d^2} - \left(\frac{\partial^2 H}{\partial y \partial T_d^2} \right)^2 = e^{-2\lambda(t-t_0)} \cdot \frac{a_1 \cdot \mu^2}{y^2 \cdot (\mu \cdot T_d + \nu \cdot u)^2} +$$

$$+ e^{-\lambda(t-t_0)} \cdot (a_1 + 1) \cdot \mu^2 \cdot \psi_1 \cdot f_2 \cdot (1-\gamma) \cdot \gamma \cdot (\mu \cdot T_d + \nu \cdot u)^{(\gamma-2)} \cdot y^{-(\gamma+1)} > 0, \tag{79}$$

$$\Delta_3 = \frac{\partial^2 H}{\partial y^2} \cdot \frac{\partial^2 H}{\partial T_d^2} \cdot \frac{\partial^2 H}{\partial u^2} + 2 \cdot \frac{\partial^2 H}{\partial y \partial T_d} \cdot \frac{\partial^2 H}{\partial y \partial u} \cdot \frac{\partial^2 H}{\partial T_d \partial u} -$$

$$-\frac{\partial^2 H}{\partial y^2} \cdot \left(\frac{\partial^2 H}{\partial y \partial T_d} \right)^2 - \frac{\partial^2 H}{\partial T_d^2} \cdot \left(\frac{\partial^2 H}{\partial y \partial u} \right)^2 - \frac{\partial^2 H}{\partial u^2} \cdot \left(\frac{\partial^2 H}{\partial y \partial T_d} \right)^2 =$$

$$= -e^{-3\lambda(t-t_0)} \cdot \frac{a_1 \cdot a_2 \cdot \mu^2}{y^2 \cdot u^2 \cdot (\mu \cdot T_d + \nu \cdot u)^2} -$$

$$- e^{-2\lambda(t-t_0)} \cdot (a_1 + 1) \cdot a_2 \cdot \mu^2 \cdot \psi_1 \cdot f_2 \cdot (1-\gamma) \cdot \gamma \cdot \frac{(\mu \cdot T_d + \nu \cdot u)^{(\gamma-2)} \cdot y^{-(\gamma+1)}}{u^2} < 0. \tag{80}$$

Proposition 3 is proved.

Let us consider the maximized Hamiltonian

$$\hat{H}(t, y, T_d, \psi_1, \psi_2) = \max_u H(t, y, T_d u, \psi_1, \psi_2), \qquad u \in [r_d^l \cdot y, r_d^u \cdot y]. \tag{81}$$

The maximized Hamiltonian $\hat{H}$ (81) conserves the concavity properties of the Hamiltonian H (64). More precisely, the following proposition is valid.

Proposition 4. The Hamiltonian $\hat{H}(t, y, T_d, \psi_1, \psi_2)$ (81) is a continuously differentiable and strictly concave function with respect to variables y, T_d.

Proof of Proposition 4. Let us indicate the scheme of the proof. The maximum value in the Hamiltonian $\hat{H}$ (81) in variable u, $u \in [r_d^l \cdot y, r_d^u \cdot y]$, can be realized either at boundary points $r_d^l \cdot y$, $r_d^u \cdot y$ of the interval $[r_d^l \cdot y, r_d^u \cdot y]$, or at an internal point $u^0 = u^0(y, T_d) \in (r_d^l \cdot y, r_d^u \cdot y)$. One can prove that all three branches

$$H_r(t, y, T_d, \psi_1, \psi_2) = H(t, y, T_d, r \cdot y, \psi_1, \psi_2), \qquad r = r_d^l, \qquad r = r_d^u, \tag{82}$$

and

$$H_u(t, y, T_d, \psi_1, \psi_2) = H(t, y, T_d, u^0(y, T_d), \psi_1, \psi_2), \tag{83}$$

are continously differentiable and strictly concave in variables y, T_d. Moreover, these Hamiltonians H_r, H_u are smoothly pasted together into the maximized Hamiltonian $\hat{H}$ (81) which is, consequently, continuously differentiable and strictly concave in variables y, T_d..

Lemma 3. The Hamiltonian $H_r(t, y, T_d, \psi_1, \psi_2)$ (82) is a continuously differentiable and strictly concave function in variables y, T_d.

Proof of Lemma 3. Let us prove this result in the general form. Let us denote vector (y, T_d) by the symbol x and emphasize that the Hamiltonian H depends on this phase vector x and control parameter u, $H = H(x, u)$. In the general case one can assume that vector x is n-dimensional and control parameter is m-dimensional. According to Lemma 2 the Hamiltonian $H = H(x, u)$ is strictly concave. By definition it means that the following inequality takes place

$$\lambda_1 \cdot H(x_1, u_1) + \lambda_2 \cdot H(x_2, u_2) < H(\lambda_1 \cdot x_1 + \lambda_2 \cdot x_2, \lambda_1 \cdot u_1 + \lambda_2 \cdot u_2), \tag{84}$$

for all $\quad (x_1, u_1) \neq (x_2, u_2), \quad 0 < \lambda_i < 1, \quad i = 1,2, \quad \lambda_1 + \lambda_2 = 1.$

Assume that control parameter u is lineraly expressed through phase vector x

$$u = u(x) = A \cdot x + b, \tag{85}$$

and define the composition of the Hamiltonian $H = H(x,u)$ and linear function $u = u(x)$ (85)

$$H_r = H_r(x) = H(x,u(x)) = H(x, A \cdot x + b). \tag{86}$$

It is necessary to prove that the Hamiltonian $H_r = H_r(x)$ is strictly concave in x. Really, using relations (84)-(86) one can get the following chain of inequalities

$$\lambda_1 \cdot H_r(x_1) + \lambda_2 \cdot H_r(x_2) = \lambda_1 \cdot H(x_1, A \cdot x_1 + b) + \lambda_2 \cdot H(x_2, A \cdot x_2 + b) <$$

$$< H(\lambda_1 \cdot x_1 + \lambda_2 \cdot x_2, \lambda_1 \cdot (A \cdot x_1 + b) + \lambda_2 \cdot (A \cdot x_2 + b)) =$$

$$= H(\lambda_1 \cdot x_1 + \lambda_2 \cdot x_2, A \cdot (\lambda_1 \cdot x_1 + \lambda_2 \cdot x_2) + b) = H_r(\lambda_1 \cdot x_1 + \lambda_2 \cdot x_2), \tag{87}$$

for all $\quad x_1 \neq x_2, \quad 0 < \lambda_i < 1, \quad i = 1,2, \quad \lambda_1 + \lambda_2 = 1.$

Relation (87) means by definition the property of strict concavity of the Hamiltonian $H_r = H_r(x)$ (86) and proves Lemma 3.

Remark 3. One can express first and second derivatives of the Hamiltonian $H_r = H_r(x)$ (86) by formulas

$$\frac{\partial H_r(x)}{\partial x} = \frac{\partial H}{\partial x} + \frac{\partial H}{\partial u} \cdot \frac{\partial u}{\partial x}, \tag{88}$$

$$\frac{\partial^2 H_r}{\partial x^2} = \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial x \partial u} \cdot \frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^T \cdot \frac{\partial^2 H}{\partial u \partial x} + \left(\frac{\partial u}{\partial x}\right)^T \cdot \frac{\partial^2 H}{\partial u^2} \cdot \frac{\partial u}{\partial x} + \sum_{i=1}^{m} \frac{\partial H}{\partial u_i} \cdot \frac{\partial^2 u_i}{\partial x^2}, \tag{89}$$

$$\frac{\partial u}{\partial x} = A, \qquad \frac{\partial^2 u_i}{\partial x^2} = 0, \qquad i = 1,\ldots,m. \tag{90}$$

In formula (89) the first four terms form a negative definite matrix and the fifth term equals to zero. Hence, the matrix of second derivatives $\frac{\partial^2 H_r}{\partial x^2}$ is negative definite and this implies the property of strict concavity of the Hamiltonian $H_r = H_r(x)$ (86). In the general case, when function $u = u(x)$ is not linear, the fifth term

$$\sum_{i=1}^{m} \frac{\partial H}{\partial u_i} \cdot \frac{\partial^2 u_i}{\partial x^2}$$

may generate the matrix which is not negative definite. One can indicate a case when this matrix is negative definite: the Hamiltonian $H = H(x,u)$ is a monotonically increasing function in u_i, and, hence, first derivative is positive, $\frac{\partial H}{\partial u_i} > 0$; functions $u_i = u_i(x)$ are strictly concave, and, hence, matrix $\frac{\partial^2 u_i}{\partial x^2}$ is negative definite, $i = 1,...,m$.

Lemma 4. The Hamiltonian $H_u(t, y, T_d, \psi_1, \psi_2)$ (83) is a continuously differentiable and strictly concave function in variables y, T_d.

Proof of Lemma 4. The maximum point $u^0 = u^0(x) = u^0(y, T_d)$ in definition of the Hamiltonian $H_u = H_u(x) = H_u(y, T_d)$ (83) should satisfy the necessary and sufficient conditions of maximum of the strictly concave Hamiltonian $H = H(x,u)$ with respect to variable u

$$\frac{\partial H}{\partial u}(x, u^0(x)) = 0. \tag{91}$$

Basing on relation (91) one can calculate first derivatives of the function $u^0 = u^0(x)$

$$\frac{\partial u^0}{\partial x} = -\left(\frac{\partial^2 H}{\partial u^2}\right)^{-1} \cdot \left(\frac{\partial^2 H}{\partial u \partial x}\right)^T. \tag{92}$$

First and second derivatives of the Hamiltonian $H_u = H_u(x) = H_u(y, T_d)$ (83) are expressed by the following relations

$$\frac{\partial H_u}{\partial x} = \frac{\partial H}{\partial x}(x, u^0(x)) + \frac{\partial H}{\partial u}(x, u^0(x)) \cdot \frac{\partial u^0}{\partial x} = \frac{\partial H}{\partial x}, \tag{93}$$

$$\frac{\partial^2 H_u}{\partial x^2} = \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial x \partial u} \cdot \frac{\partial u^0}{\partial x} = \frac{\partial^2 H}{\partial x^2} - \frac{\partial^2 H}{\partial x \partial u} \cdot \left(\frac{\partial^2 H}{\partial u^2}\right)^{-1} \cdot \left(\frac{\partial^2 H}{\partial u \partial x}\right)^T. \tag{94}$$

It is necessary to prove that the matrix of second derivatives $\frac{\partial^2 H_u}{\partial x^2}$ (94) is negative definite, and, hence the Hamiltonian $H_u = H_u(x)$ is strictly concave in x. Really, this fact follows from the property of strict concavity of the Hamiltonian

$H = H(x,u)$. The matrix of second derivatives of the Hamiltonian $H = H(x,u)$ is negative definite and can be presented in the block form

$$\frac{\partial^2 H}{\partial (x,u)^2} = \begin{pmatrix} \dfrac{\partial^2 H}{\partial x^2} & \dfrac{\partial^2 H}{\partial x \partial u} \\ \dfrac{\partial^2 H}{\partial u \partial x} & \dfrac{\partial^2 H}{\partial u^2} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \qquad (95)$$

The inverse matrix is also negative definite and can be presented in the block form

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} A^{11} & -A_{11}^{-1} \cdot A_{12} \cdot A^{22} \\ -A^{22} \cdot A_{21} \cdot A_{11}^{-1} & A^{22} \end{pmatrix},$$

$$A^{11} = (A_{11} - A_{12} \cdot A_{22}^{-1} \cdot A_{21})^{-1}, \qquad A^{22} = (A_{22} - A_{21} \cdot A_{11}^{-1} \cdot A_{12})^{-1}.$$

It means in particular, that matrices A^{11}, A^{22} are negative definite. Hence, matrix $\left(A^{11}\right)^{-1} = \dfrac{\partial^2 H}{\partial x^2} - \dfrac{\partial^2 H}{\partial x \partial u} \cdot \left(\dfrac{\partial^2 H}{\partial u^2}\right)^{-1} \cdot \left(\dfrac{\partial^2 H}{\partial u \partial x}\right)^T = \dfrac{\partial^2 H_u}{\partial x^2}$

is also negative definite and, consequently, the Hamiltonian $H_u = H_u(x)$.

Lemma 4 is proved.

Lemma 5. The Hamiltonians $H_r(t, y, T_d, \psi_1, \psi_2)$ (82), and $H_u(t, y, T_d, \psi_1, \psi_2)$ (83) are smoothly pasted in generating the maximized Hamiltonian $\hat{H}(t, y, T_d, \psi_1, \psi_2)$ (81).

Proof of Lemma 5. To prove this result it is necessary to calculate partial derivatives of the Hamiltonians $H_r(t, y, T_d, \psi_1, \psi_2)$, $H_u(t, y, T_d, \psi_1, \psi_2)$ in variables (y, T_d), and verify that these drivatives coincide with each other at points of sewing of these functions. Points of sewing $x^s = (y^s, T_d^s)$ are defined by relations

$$H(t, y^s, T_d^s, r \cdot y^s, \psi_1, \psi_2) = H_r(x^s) = H_u(x^s) = H(t, y^s, T_d^s, u^0(y^s, T_d^s), \psi_1, \psi_2). \quad (96)$$

Due to strict concavity of the Hamiltomian $H(t, y, T_d, u, \psi_1, \psi_2)$ in variables (y, T_d, u) and uniqueness of the maximum point $u^0(y^s, T_d^s)$ the last relation holds if and only if the following equality is valid

$$r \cdot y^s = u^0(y^s, T_d^s). \tag{97}$$

The last relation implies the following equation

$$\frac{\partial H}{\partial u}(t, y^s, T_d^s, r \cdot y^s, \psi_1, \psi_2) = \frac{\partial H}{\partial u}(t, y^s, T_d^s, u^0(y^s, T_d^s), \psi_1, \psi_2) = 0,$$

which, in turn, provides equality of partial derivatives

$$\frac{\partial H_r}{\partial x}(x^s) = \frac{\partial H_u}{\partial x}(x^s) = \frac{\partial H}{\partial x}(x^s, u^0(x^s)). \tag{98}$$

Lemma 5 is proved.

Summarizing the results of Lemmas 3-5 one can get the statement of Proposition 4 that the maximized Hamiltonian $\hat{H}(t, y, T_d, \psi_1, \psi_2)$ (81) is a continuously differentiable and strictly concave function with respect to variables y, T_d.

Proposition 4 is proved.

Remark 4. The results of Proposition 3 and Proposition 4 are valid for the optimal control problem with dynamics (42)-(45) and utility function (33) if parameter κ is small enough.

3.8. Necessary Conditions of Optimality

One can formulate the Pontryagin maximum principle [19] which provide necessary conditions of optimality for trajectories of the optimal control problem with dynamics (42)-(45) and objective function (47).

Theorem 1. Let $(y^*(t), T_d^*(t), r_d^*(t))$ be an optimal control process in problem with dynamics (42)-(45) and objective function (47) or (33). Then there exists a pair $\psi(t) = (\psi_1(t), \psi_2(t))$ of adjoint variables such that $\psi(t)$ is a solution of the adjont system (65)-(66), taken along the optimal control process $(y^*(t), T_d^*(t), r_d^*(t))$;

the maximum conditions hold

$$H(t, y^*(t), T_d^*(t), r_d^*(t), \psi_1(t), \psi_2(t)) \overset{a.e.}{=} \hat{H}(t, y^*(t), T_d^*(t), \psi_1(t), \psi_2(t)); \tag{99}$$

the transversality condition (67) takes place;

and, moreover, both functions $\psi_1(t)$, $\psi_2(t)$ are strictly positive (68).

3.9. Sufficient Conditions of Optimality

Let us prove the sufficient result for optimality conditions of the Pontryagin maximum principle in the considered optimal control problem.

Theorem 2. Under the conditions of Proposition 4 providing the properties of strcict concavity of the maximized Hamiltonian $\hat{H}(t, y, T_d, \psi_1, \psi_2)$ (81) in variables y, T_d, the Pontryagin maximum principle gives sufficient conditions to find the unique optimal solution in the optimal control problem with dynamics (42)-(45) and objective function (47) or (33).

Proof. Let $(y, T_d, r_d) = (y(t), T_d(t), r_d(t))$ be an arbitrary admissible control process. Denote by symbol x the pair (y, T_d), and by symbol x^* the pair (y^*, T_d^*). Due to the strict concavity of the maximized Hamiltonian $\hat{H}(t, y, T_d, \psi_1, \psi_2)$ (81) in variables $x = (y, T_d)$ the following inequality holds

$$\left\langle \frac{\partial \hat{H}}{\partial x}(t, x^*(t), \psi(t)), x^*(t) - x(t) \right\rangle < \hat{H}(t, x^*(t), \psi(t)) - \hat{H}(t, x(t), \psi(t)), \tag{100}$$

if $x(t) \neq x^*(t)$.

Combining this inequality with adjoint equations (65)-(66) and definition of the Hamiltonians H (63), $\hat{H}$ (81), one can obtain that for $t \in [t_0, \vartheta]$ the following chain of relations takes place

$$\left\langle \dot{\psi}(t), x(t) - x^*(t) \right\rangle = \left\langle \frac{\partial \hat{H}}{\partial x}(x^*(t), \psi(t)), x^*(t) - x(t) \right\rangle < \hat{H}(t, x^*(t), \psi(t)) - \hat{H}(t, x(t), \psi(t)) \leq$$

$$\leq \left\langle \psi(t), \dot{x}^*(t) - x(t) \right\rangle + e^{-\lambda(t-t_0)} \cdot (\ln D(x^*(t), r_d^*(t)) - \ln D(x(t), r_d(t))). \tag{101}$$

Here the symbol $\ln D(x(t), r_d(t))$ denotes the integrand of the objective function (47)

$$\ln D(x(t), r_d(t)) = \ln D(y(t), T_d(t), r_d(t)) =$$

$$= (1 + a_2) \cdot \ln y(t) + a_1 \cdot \ln\big(\mu \cdot T_d(t) + \nu \cdot r_d(t) \cdot y(t)\big) + a_2 \cdot \ln r_d(t)\,. \tag{102}$$

Hence,

$$\frac{d}{dt}\left\langle \psi(t), x(t) - x^*(t) \right\rangle + e^{-\lambda(t-t_0)} \cdot \ln D(x(t), r_d(t)) < e^{-\lambda(t-t_0)} \cdot \ln D(x^*(t), r_d^*(t))\,. \tag{103}$$

Integrating this inequality over $t \in [t_0, \vartheta]$, one can get the following relation

$$\left\langle \psi(\vartheta), x(\vartheta) - x^*(\vartheta) \right\rangle + \left\langle \psi(t_0), x(t_0) - x^*(t_0) \right\rangle + \int_{t_0}^{\vartheta} e^{-\lambda(t-t_0)} \cdot \ln D(x(t), r_d(t)) \cdot dt <$$

$$< \int_{t_0}^{\vartheta} e^{-\lambda(t-t_0)} \cdot \ln D(x^*(t), r_d^*(t)) \cdot dt\,. \tag{104}$$

Taking into account the initial conditions (45) and the transversality conditions (67), one can obtain from relation (104) the necessary inequality

$$\int_{t_0}^{\vartheta} e^{-\lambda(t-t_0)} \cdot \ln D(x(t), r_d(t)) \cdot dt < \int_{t_0}^{\vartheta} e^{-\lambda(t-t_0)} \cdot \ln D(x^*(t), r_d^*(t)) \cdot dt\,. \tag{105}$$

Thus, the process $(x^*(t), r_d^*(t)) = (y^*(t), T_d^*(t), r_d^*(t))$ is the unique optimal solution in the optimal control problem with dynamics (42)-(45) and objective function (47).

Theorem 2 is proved.

Remark 4. The sufficient result of optimality postulated in Theorem 2 can be extended from the optimal control problem with objective function (47) given on the finite interval $[t_0, \vartheta]$ to the optimal control problem with objective function (41) given on the infinite horizon $[t_0, +\infty]$. The transversality conditions (67) due to relation (104) are transferred into the following transversality conditions

$$\lim_{\vartheta \to +\infty} \left\langle \psi(\vartheta), x(\vartheta) \right\rangle = 0\,. \tag{106}$$

4. Basic Solution of Optimal Control Problem

4.1. Optimal R&D Investment Level

In this section we construct a basic analytic solution for the optimal control problem under some simplifications of dynamics (42)-(45) and objective function (33).

Assuming that elasticity γ in dynamics equations (42)-(43) is equal to zero, $\gamma=0$, and introducing notation $f=f_1+f_2$, $T=T_d$, one can get the following dynamics for the system trajectories

$$\frac{\dot{y}(t)}{y(t)}=f-g\cdot\frac{u(t)}{y(t)}, \tag{107}$$

$$\dot{T}(t)=u(t), \tag{108}$$

$$r_d^l\cdot y(t)\leq u(t)\leq r_d^u\cdot y(t), \tag{109}$$

$$y(t_0)=y^0, \qquad T(t_0)=T^0. \tag{110}$$

Let us consider the problem of maximization of the approximate utility function (see (33))

$$I=\int_{t_0}^{+\infty}e^{-\lambda(t-t_0)}\left(\ln y(t)+a_1\cdot\ln T+a_1\cdot\kappa\cdot\frac{u}{T}+a_2\cdot\ln u(t)\right)dt \tag{111}$$

on trajectories of dynamical process (107)-(110).

The value function of the optimal control problem (107)-(109) is defined by the following relation

$$V(y^0,T^0)=\sup_{u(t)}\int_{t_0}^{+\infty}e^{-\lambda(t-t_0)}\cdot\left(\ln y(t)+a_1\cdot\ln T(t)+a_1\cdot\kappa\cdot\frac{u(t)}{T(t)}+a_2\cdot\ln u(t)\right)dt\,, \tag{112}$$

where the process $\left(y(t),T(t),u(t)\right)$ is subject to dynamics (107)-(109) with initial conditions (110).

The Hamiltonian of the optimal control problem (107)-(109) is presented by the following expression

$$H(t,y,T,u,\psi_1,\psi_2)=e^{-\lambda(t-t_0)}\cdot\left(\ln y+a_1\cdot\ln T+a_1\cdot\kappa\cdot\frac{u}{T}+a_2\cdot\ln u\right)+$$

$$+\psi_1\cdot(f\cdot y-g\cdot u)+\psi_2\cdot u, \tag{113}$$

where $\psi_i=\psi_i(t)$, $i=1,2$, are adjoint variables.

Implementing the standard change of adjoint variables

$$\psi_i^s(t)=e^{\lambda(t-t_0)}\cdot\psi_i(t), \qquad i=1,2 \tag{114}$$

one can introduce the stationary Hamiltonian

$$H^s(y,T,u,\psi_1^s,\psi_2^s)=\ln y+a_1\cdot\ln T+a_1\cdot\kappa\cdot\frac{u}{T}+a_2\cdot\ln u+\psi_1^s\cdot(f\cdot y-g\cdot u)+\psi_2^s\cdot u \tag{115}$$

which is connected with the Hamiltonian $H(t,y,T,u,\psi_1,\psi_2)$ by the following relation

$$H(t,y,T,u,\psi_1,\psi_2)=e^{-\lambda(t-t_0)}\cdot H^s(y,T,u,\psi_1^s,\psi_2^s). \tag{116}$$

In what follows let us assume that the optimal control $u^0=u^0(t)$ of the problem (107)-(111) is realized at internal points of boundary condition (109)

$$r_d^l\cdot y(t)<u^0(t)<r_d^u\cdot y(t). \tag{117}$$

It means that the optimal control $u^0=u^0(t)$ should satisfy the following optimality condition

$$\frac{\partial H^s}{\partial u}=\frac{a_1\cdot\kappa}{T}+\frac{a_2}{u^0}-\psi_1^s\cdot g+\psi_2^s=0, \tag{118}$$

and, hence, the value of optimal control can be expressed by the following formula

$$u^0(t)=\frac{a_2}{\left(\psi_1^s(t)\cdot g-\psi_2^s(t)-\dfrac{a_1\cdot\kappa}{T(t)}\right)}. \tag{119}$$

The maximized Hamiltonian $\hat{H}^s=\hat{H}^s(y,T,\psi_1^s,\psi_2^s)$ of the optimal control problem (107)-(111) is defined by the following relation

$$\hat{H}^s(y,T,\psi_1^s,\psi_2^s)=\max_u H(y,T,u,\psi_1^s,\psi_2^s)=$$

$$=\ln y+a_1\cdot\ln T+a_2\cdot(\ln a_2-1)+\psi_1^s\cdot f\cdot y-a_2\cdot\ln\left(\psi_1^s\cdot g-\psi_2^s-\frac{a_1\cdot\kappa}{T}\right). \tag{120}$$

The adjoint variables $\psi_i^s=\psi_i^s(t)$, $i=1,2$, act in equations (115), (119)-(120) as "shadow" prices of production $y=y(t)$ and technology $T=T(t)$, respectively. At points of differentiability of the value function $V=V(y^0,T^0)$ (112) adjoint variables measure the marginal utility

$$\psi_1^s=\frac{\partial V}{\partial y^0}, \tag{121}$$

$$\psi_2^s=\frac{\partial V}{\partial T^0}. \tag{122}$$

The value function $V(y^0,T^0)$ (112) of the optimal control problem (107)-(111) at points of differentiability should satisfy the Hamilton-Jacobi equation

$$-\lambda\cdot V(y^0,T^0)+\max_u H^s\left(y^0,T^0,u,\frac{\partial V(y^0,T^0)}{\partial y^0},\frac{\partial V(y^0,T^0)}{\partial T^0}\right)=0. \tag{123}$$

Taking into account relation (120) for the maximized Hamiltonian $\hat{H}^s$ one can obtain the following form of the Hamilton-Jacobi equation (123)

$$-\lambda\cdot V(y^0,T^0)+\ln y^0+a_1\cdot\ln T^0+a_2\cdot\left(\ln a_2-1\right)+$$

$$+\frac{\partial V(y^0,T^0)}{\partial y^0}\cdot f\cdot y^0-a_2\cdot\ln\left(\frac{\partial V(y^0,T^0)}{\partial y^0}\cdot g-\frac{\partial V(y^0,T^0)}{\partial T^0}-\frac{a_1\cdot\kappa}{T^0}\right)=0. \tag{124}$$

Relations (114) imply the following dynamic for adjoint variables ψ_1^s, ψ_2^s

$$\dot{\psi}_1^s(t)=\lambda\cdot\psi_1^s(t)-\frac{\partial H^s}{\partial y}(y(t),T(t),u(t),\psi_1^s(t),\psi_2^s(t))=$$

$$=\lambda\cdot\psi_1^s(t)-\frac{1}{y(t)}-\psi_1^s(t)\cdot f, \tag{125}$$

$$\dot{\psi}_2^s(t)=\lambda\cdot\psi_2^s(t)-\frac{\partial H^s}{\partial T}(y(t),T(t),u(t),\psi_1^s(t),\psi_2^s(t))=$$

$$= \lambda \cdot \psi_2^s(t) - \frac{a_1}{T(t)} + \frac{a_1 \cdot \kappa \cdot u(t)}{T^2(t)} . \qquad (126)$$

Introducing the "shadow" costs $Z_1 = \psi_1^s \cdot y$, $Z_2 = \psi_2^s \cdot T$ for production y and technology T, respectively, the following equations for their dynamics can be derived

$$\dot{Z}_1(t) = \dot{\psi}_1^s(t) \cdot y(t) + \psi_1^s(t) \cdot \dot{y}(t) = \lambda \cdot \psi_1^s(t) \cdot y(t) - 1 - \psi_1^s(t) \cdot g \cdot u(t) =$$

$$= \lambda \cdot Z_1(t) - 1 - g \cdot Z_1(t) \cdot \frac{u(t)}{y(t)} , \qquad (127)$$

$$\dot{Z}_2(t) = \dot{\psi}_2^s(t) \cdot T(t) + \psi_2^s(t) \cdot \dot{T}(t) = \lambda \cdot \psi_2^s(t) \cdot T(t) - a_1 + \frac{a_1 \cdot \kappa \cdot u(t)}{T(t)} + \psi_2^s(t) \cdot u(t) =$$

$$= \lambda \cdot Z_2(t) - a_1 + \frac{a_1 \cdot \kappa \cdot u(t)}{T(t)} + Z_2(t) \cdot \frac{u(t)}{T(t)} . \qquad (128)$$

Let us introduce the "shadow" total cost of the process (107)-(111)

$$Z = Z(t) = Z_1(t) + Z_2(t) = \psi_1^s(t) \cdot y(t) + \psi_2^s(t) \cdot T(t) . \qquad (129)$$

Summarizing equations (127) and (128) one can obtain the equation for dynamics of the total "shadow" cost $Z(t)$

$$\dot{Z}(t) = \dot{Z}_1(t) + \dot{Z}_2(t) = \lambda \cdot Z(t) - 1 - a_1 - g \cdot Z_1(t) \cdot \frac{u(t)}{y(t)} + Z_2(t) \cdot \frac{u(t)}{T(t)} + \frac{a_1 \cdot \kappa \cdot u(t)}{T(t)} . \qquad (130)$$

Substituting control $u = u(t)$ in dynamics (130) by the optimal control $u^0 = u^0(t)$ and taking into account the maximum condition (118) one can get the optimal dynamics for the $Z(t)$

$$\dot{Z}(t) = \lambda \cdot Z(t) - (1 + a_1 + a_2) . \qquad (131)$$

All solutions of this differential equation are growing exponentially

$$Z = Z(t) = C \cdot e^{\lambda t} + \frac{(1 + a_1 + a_2)}{\lambda} , \qquad (132)$$

except the constant solution

$$Z(t) = \frac{(1 - a_1 - a_2)}{\lambda} = Z^0, \tag{133}$$

which meets the transversality condition (106) of the Pontryagin's maximum principle

$$\lim_{\vartheta\to+\infty}\langle\psi(\vartheta), x(\vartheta)\rangle = \lim_{\vartheta\to+\infty} e^{-\lambda(\vartheta-t_0)} \cdot (\psi_1^s(\vartheta)\cdot y(\vartheta) + \psi_2^s(\vartheta)\cdot T(\vartheta)) = \lim_{\vartheta\to+\infty} e^{-\lambda(\vartheta-t_0)} \cdot Z(\vartheta) =$$

$$= e^{\lambda t_0} \cdot \lim_{\vartheta\to+\infty} e^{-\lambda\vartheta} \cdot Z(\vartheta) = 0. \tag{134}$$

The last relation is equivalent to the transversality condition in the following form

$$\lim_{\vartheta\to+\infty} e^{-\lambda\vartheta} \cdot Z(\vartheta) = 0. \tag{135}$$

Hence, the "shadow" costs $Z_1(t)$, $Z_2(t)$ satisfy the following condition of the constant cost

$$Z = Z(t) = Z_1(t) + Z_2(t) = \frac{(1 - a_1 - a_2)}{\lambda} = Z^0, \qquad 0 \le Z_i(t) \le Z^0, \qquad i = 1,2. \tag{136}$$

Resolving the maximum condition (118) with respect to the R&D intensity $r^0(t) = u^0(t)/y(t)$ one can be obtain the following relation

$$r^0 = r^0(t) = \frac{u^0(t)}{y(t)} = \frac{a_2}{(g \cdot Z_1(t) - Z_2(t)\cdot X(t) - a_1 \cdot \kappa \cdot X(t))}, \tag{137}$$

where $X = X(t) = y(t)/T(t)$ is technology productivity.

The technology productivity $X = X(t)$ is subject to the following differential equation

$$\dot{X}(t) = \frac{d(y(t)/T(t))}{dt} = \frac{\dot{y}(t)\cdot T(t) - y(t)\cdot \dot{T}(t)}{T^2(t)} =$$

$$= \frac{((f \cdot y(t) - g \cdot u(t)\cdot T(t) - y(t)\cdot u(t))}{T^2(t)} = f \cdot \frac{y(t)}{T(t)} - g \cdot \frac{u(t)}{T(t)} - \frac{y(t)}{T(t)} \cdot \frac{u(t)}{T(t)} =$$

$$= f \cdot X(t) - X(t)\cdot (X(t) + g) \cdot \frac{u(t)}{y(t)}. \tag{138}$$

Substituting optimal level of R&D intensity $r^0(t)$ (137) into dynamics of costs $Z_i = Z_i(t)$ (127)-(128), $i = 1,2$, and into dynamics of technology productivity $X = X(t)$ (138), one can get the Hamiltonian system of differential equations

$$\dot{Z}_1(t) = \lambda \cdot Z_1(t) - 1 - \frac{a_2 \cdot g \cdot Z_1(t)}{(g \cdot Z_1(t) - (Z^0 - Z_1(t)) \cdot X(t) - a_1 \cdot \kappa \cdot X(t))} = F_1(Z_1, X), \qquad (139)$$

$$\dot{X}(t) = f \cdot X(t) - \frac{a_2 \cdot X(t) \cdot (X(t) - g)}{(g \cdot Z_1(t) - (Z^0 - Z_1(t)) \cdot X(t) - a_1 \cdot \kappa \cdot X(t))} = F_2(Z_1, X), \qquad (140)$$

$$Z_2(t) = Z^0 - Z_1(t). \qquad (141)$$

Let us denote by symbols Z_1^0, Z_2^0, X^0 the equilibrium point of the Hamiltonian system (139)-(141)

$$\lambda \cdot Z_1^0 - 1 - g \cdot Z_1^0 \cdot r^0 = 0, \qquad (142)$$

$$f \cdot X^0 - X^0 \cdot (X^0 + g) \cdot r^0 = 0, \qquad (143)$$

$$r^0 = \frac{a_2}{(g \cdot Z_1^0 - (Z^0 - Z_1^0) \cdot X^0 - a_1 \cdot \kappa \cdot X^0)}, \qquad (144)$$

$$Z_2^0 = Z^0 - Z_1^0, \qquad 0 \le Z_i^0 \le Z^0, \qquad i = 1,2. \qquad (145)$$

After simplification of these equations the following relations can be derived

$$Z_1^0 = \frac{(X^0 + g)}{(\lambda \cdot (X^0 + g) - g \cdot f)}, \qquad (146)$$

$$Z_1^0 = \frac{a_2}{f} + \frac{(Z^0 + a_1 \cdot \kappa) \cdot X^0}{(X^0 + g)}. \qquad (147)$$

Introducing notations

$$s = X^0 + g, \qquad a = \frac{a_2}{f}, \qquad b = Z^0 + a_1 \cdot \kappa, \qquad (148)$$

and excluding Z_1^0 one can obtained the following relation

$$\frac{s}{(\lambda \cdot s - f \cdot g)} = (a+b) - \frac{b \cdot g}{s}. \tag{149}$$

Resolving this relation with respect to s one can get the quadratic equation

$$((a+b)\cdot \lambda - 1)\cdot s^2 - ((a+b)\cdot f + \lambda \cdot b)\cdot g \cdot s + b \cdot f \cdot g^2 = 0. \tag{150}$$

The unique positive root of this equation is presented by formula

$$s = \frac{g \cdot ((a+b)\cdot f + \lambda \cdot b) + g \cdot (((a+b)\cdot f - \lambda \cdot b)^2 + 4 \cdot b \cdot f)^{1/2}}{2 \cdot ((a+b)\cdot \lambda - 1)}, \tag{151}$$

and, hence, parameter X^0 is defined by relation

$$X^0 = s - g =$$

$$= \frac{g \cdot [((a+b)\cdot f - \lambda \cdot b) + 2 \cdot (1 - a \cdot \lambda) + (((a+b)\cdot f - \lambda \cdot b)^2 + 4 \cdot b \cdot f)^{1/2}]}{2 \cdot ((a+b)\cdot \lambda - 1)}. \tag{152}$$

Parameter Z_1^0 is expressed through s and is determined by formula

$$Z_1^0 = \frac{s}{(\lambda \cdot s - g \cdot f)} =$$

$$= \frac{[((a+b)\cdot f + \lambda \cdot b) + (((a+b)\cdot f - \lambda \cdot b)^2 + 4 \cdot b \cdot f)^{1/2}]}{[2 \cdot f - \lambda \cdot ((a+b)\cdot f - \lambda \cdot b) + \lambda \cdot (((a+b)\cdot f - \lambda \cdot b)^2 + 4 \cdot b \cdot f)^{1/2}]}. \tag{153}$$

The Jacobi matrix of the Hamiltonian system (139)-(140) at the equilibrium point (Z_1^0, X^0) is calculated as follows

$$\frac{\partial F_1}{\partial Z_1} = \lambda + \frac{a_2 \cdot b \cdot g \cdot X^0}{(Z_1^0 \cdot (X^0 + g) - b \cdot X^0)^2} = \lambda + \frac{b \cdot f^2 \cdot g \cdot X^0}{a_2 \cdot (X^0 + g)^2}, \tag{154}$$

$$\frac{\partial F_1}{\partial X} = \frac{a_2 \cdot g \cdot Z_1^0 \cdot (Z_1^0 - b)}{(Z_1^0 \cdot (X^0 + g) - b \cdot X^0)^2} = -\frac{f^2 \cdot g \cdot Z_1^0 \cdot (b - Z_1^0)}{a_2 \cdot (X^0 + g)^2}, \tag{155}$$

$$\frac{\partial F_2}{\partial Z_1} = \frac{a_2 \cdot X^0 \cdot (X^0 + g)^2}{(Z_1^0 \cdot (X^0 + g) - b \cdot X^0)^2} = \frac{f^2 \cdot X^0 \cdot (X^0 + g)^2}{a_2 \cdot (X^0 + g)^2} = \frac{f^2 \cdot X^0}{a_2}, \tag{156}$$

$$\frac{\partial F_2}{\partial X} = f - \frac{a_2 \cdot [Z_1^0 \cdot (X^0 + g)^2 - b \cdot (X^0)^2]}{(Z_1^0 \cdot (X^0 + g) - b \cdot X^0)^2} = -\frac{b \cdot f^2 \cdot g \cdot X^0}{a_2 \cdot (X^0 + g)^2}. \tag{157}$$

The trace of the Jacobi matrix is defined by the following relation

$$TR = \frac{\partial F_1}{\partial Z_1} + \frac{\partial F_2}{\partial X} = \lambda\,. \tag{158}$$

The determinant of the Jacobi matrix is determined by formula

$$DE = \frac{\partial F_1}{\partial Z_1} \cdot \frac{\partial F_2}{\partial X} - \frac{\partial F_1}{\partial X} \cdot \frac{\partial F_2}{\partial Z_1} =$$

$$= -\frac{\lambda \cdot b \cdot f^2 \cdot g \cdot X^0}{a_2 \cdot (X^0 + g)^2} - \frac{b^2 \cdot f^4 \cdot g^2 \cdot (X^0)^2}{a_2^2 \cdot (X^0 + g)^4} + \frac{f^4 \cdot g \cdot Z_1^0 \cdot (b - Z_1^0) \cdot X^0 \cdot (X^0 + g)^2}{a_2^2 \cdot (X^0 + g)^4} =$$

$$= \frac{f^2 \cdot g \cdot X^0}{a_2 \cdot (X^0 + g)^2} \cdot \left[-\lambda \cdot b - \frac{b^2 \cdot f^2 \cdot g \cdot X^0}{a_2 \cdot (X^0 + g)^2} + \frac{f^2 \cdot Z_1^0 \cdot (b - Z_1^0)}{a_2} \right] =$$

$$= \frac{f^2 \cdot g \cdot X^0}{a_2 \cdot (X^0 + g)^2} \cdot \left[-\lambda \cdot b - \frac{f^2}{a_2} \cdot \left(Z_1^0 - \frac{b \cdot X^0}{(X^0 + g)} \right) \cdot \left(Z_1^0 - \frac{b \cdot g}{(X^0 + g)} \right) \right]. \tag{159}$$

Since the following equations take place

$$Z_1^0 - \frac{b \cdot X^0}{(X^0 + g)} = \frac{a_2}{f}, \qquad Z_1^0 - \frac{b \cdot g}{(X^0 + g)} = Z_1^0 - \frac{b \cdot (\lambda \cdot Z_1^0 - 1)}{f \cdot Z_1^0}, \tag{160}$$

then, the determinant DE can be presented by formula

$$DE = -\frac{f^2 \cdot g \cdot X^0}{a_2 \cdot (X^0 + g)^2} \cdot \left(f \cdot Z_1^0 + \frac{b}{Z_1^0} \right) < 0, \tag{161}$$

and is obviously negative.

Let us introduce notations

$$P = \frac{\left(f \cdot Z_1^0 + \frac{b}{Z_1^0} \right)}{b} = f \cdot \frac{Z_1^0}{b} + \frac{1}{b} > 0, \qquad Q = \left| \frac{\partial F_2}{\partial X} \right| = \frac{b \cdot f^2 \cdot g \cdot X^0}{a_2 \cdot (X^0 + g)^2} > 0, \tag{162}$$

$$|DE| = P \cdot Q > 0\,. \tag{163}$$

The characteristic equation for the Jacobian matrix is presented by formula

$$Y^2 - TR \cdot Y + DE = 0, \tag{164}$$

or, equivalently,

$$Y^2 - \lambda \cdot Y - |DE| = 0\,. \tag{165}$$

The roots of this equation (the eigenvalues of the Jacobi matrix) have different signs. One of them

$$Y_1 = \frac{\lambda - (\lambda^2 + 4 \cdot |DE|)^{1/2}}{2} = \frac{\lambda - (\lambda^2 + 4 \cdot P \cdot Q)^{1/2}}{2} < 0 \tag{166}$$

is negative. And another one

$$Y_2 = \frac{\lambda + (\lambda^2 + 4 \cdot |DE|)^{1/2}}{2} = \frac{\lambda + (\lambda^2 + 4 \cdot P \cdot Q)^{1/2}}{2} > 0 \tag{167}$$

is positive.

Hence, the equilibrium point (Z_1^0, X^0) is the saddle point.

The eigenvector $V = (V_1, V_2)$ corresponding to the negative eigenvalue Y_1 of the Jacobi matrix is determined by relation

$$-\frac{\partial F_2}{\partial Z_1} \cdot V_1 + \left(Y_1 - \frac{\partial F_2}{\partial X} \right) \cdot V_2 = 0\,, \tag{168}$$

which can be rewritten as follows

$$- W_1 \cdot V_1 + W_2 \cdot V_2 = 0\,, \tag{169}$$

where

$$W_1 = \frac{\partial F_2}{\partial Z_1} = \frac{f^2 \cdot X^0}{a_2} > 0\,, \tag{170}$$

$$W_2 = \frac{(\lambda + 2 \cdot Q) - ((\lambda + 2 \cdot Q)^2 - 4 \cdot (\lambda + Q - P) \cdot Q)^{1/2}}{2}\,. \tag{171}$$

Let us note that the following inequality takes place

$$\lambda + Q - P = \frac{f}{a_2} \cdot \left(\lambda \cdot Z_1^0 - 1 - a_2 \cdot \frac{Z_1^0}{b} \right) + \frac{1}{Z_1^0} - \frac{1}{b} > 0\,, \tag{172}$$

and, hence,

$$W_2 \geq 0\,. \tag{173}$$

The tangent slope Φ^0 of the optimal trajectory $(Z_1(t), X(t))$ at the equilibrium point (Z_1^0, X^0) coincides with the slope of the eigenvector V

$$\Phi^0 = \frac{W_2}{W_1}. \tag{174}$$

The optimal dynamics of the "shadow" costs Z_1, Z_2 can be approximated by the following linear relations

$$Z_1 = Z_1(X) = Z_1^0 + \Phi^0 \cdot (X - X^0), \tag{175}$$

$$Z_2 = Z_2(X) = Z^0 + Z_1(X). \tag{176}$$

The optimal R&D intensity can be approximated by the following relation constructed on the feedback principle as a function of technology productivity X

$$r = r(X) = \frac{u}{y}(X) = \frac{a_2}{(Z_1(X) \cdot (X + g) - b \cdot X)} =$$

$$= \frac{a_2}{((Z_1^0 + \Phi^0 \cdot (X - X^0)) \cdot (X + g) - b \cdot X)}. \tag{177}$$

Linearization of this formula in X at the equilibrium point X^0 leads to the following relation for a suboptimal feedback of R&D intensity

$$r_L = r_L(X) = \frac{a_2}{(Z_1^0 \cdot (X^0 + g) - b \cdot X^0)} - \frac{a_2 \cdot ((Z_1^0 - b) + \Phi^0 \cdot (X^0 + g))}{(Z_1^0 \cdot (X^0 + g) - b \cdot X^0)^2} \cdot (X - X^0). \tag{178}$$

Let us take into account the obsolescence effect for the indigenous technology T_d. This effect is described by the following relation (see Watanabe, 1992 [30])

$$T_d(t) = u^i + (1 - \sigma^*) \cdot T_d(t-1), \tag{179}$$

or, equivalently,

$$u^i = (1 - \sigma^*) \cdot u + \sigma^* \cdot T_d, \qquad u = T_d(t) - T_d(t-1). \tag{180}$$

Here $u^i = u^i(t - m)$ is the actual level of R&D investment at the initial stage in the investment process with the time lag m; c^* is the obsolescence coefficient, $0 : c^* < 1$.

The actual level of R&D intensity $r^i = u^i / Y$ is expressed through intensity $r = u / Y$ by the following relation

$$r^i = r^i(X) = (1-\sigma^*)\cdot r(X) + \frac{\sigma^*}{X} = \frac{(1-\sigma^*)\cdot a_2}{((Z_1^0 + \Phi^0\cdot(X - X^0))\cdot(X+g) - b\cdot X)} + \frac{\sigma^*}{X}. \quad (181)$$

Linearization of this formula provides a linear approximation for the actual level of R&D intensity

$$r_L^i = r_L^i(X) = \frac{(1-\sigma^*)\cdot a_2}{(Z_1^0\cdot(X^0+g) - b\cdot X^0)} + \frac{\sigma^*}{X^0} -$$
$$-\frac{(1-\sigma^*)\cdot a_2\cdot((Z_1^0 - b) + \Phi^0\cdot(X^0+g))}{(Z_1^0\cdot(X^0+g) - b\cdot X^0)^2}\cdot(X - X^0) - \frac{\sigma^*}{(X^0)^2}\cdot(X - X^0). \quad (182)$$

At the equilibrium point $X = X^0$ the level of R&D intensity r is equal to

$$r = \frac{a_2}{(Z_1^0\cdot(X^0+g) - b\cdot X^0)}, \quad (183)$$

and the actual level of R&D intensity r^i is altered to

$$r^i = \frac{(1-\sigma^*)\cdot a_2}{(Z_1^0\cdot(X^0+g) - b\cdot X^0)} + \frac{\sigma^*}{X^0}. \quad (184)$$

In the transition period, when technology productivity $X(t)$ converges to equilibrium X^0 while time t tends to infinity, intensity r and intensity r^i optimally evolve from the current level (177), and (181), to the equilibrium level (183), and (184), respectively.

Let us analyze the obtained optimal feedbacks for small values of parameter a_1. When values of parameter a_1 are close to zero then the "shadow" costs $Z_1(t)$, $Z_2(t)$ can be approximately given by formulas

$$Z_1 = Z^0, \qquad Z_2 = 0, \quad (185)$$

which correspond to the case

$$Z_1^0 = Z^0, \qquad \Phi^0 = 0. \quad (186)$$

In this case the optimal R&D intensity r is given by the following relation

$$r = r(X) = \frac{u}{y}(X) = \frac{a_2}{(g \cdot Z^0 - a_1 \cdot \kappa \cdot X)}, \tag{187}$$

and the actual level of R&D intensity r^i is determined according to the formula

$$r^i = r^i(X) = \frac{u}{y}(X) = \frac{a_2}{(g \cdot Z^0 - a_1 \cdot \kappa \cdot X)} + \frac{\sigma^*}{X}. \tag{188}$$

Let us examine trends of R&D intensity r and r^i depending on macroeconomic parameters λ, a_1, a_2, g, c^*, κ, and the feedback variable - technology productivity X.

1. From dependence (133) of the cost Z^0 on the discount rate λ and relations (187)-(188) one can easily see that the larger is the discount rate λ the higher should be R&D intensity r and r^i.
2. Analysis of derivatives of optimal feedbacks (187)-(188) and cost Z^0 (133) shows that higher levels of elasticity parameter a_2 for evaluation of technology stock $T(t)$ and R&D investments $\Delta T_d(t)$ stimulate higher levels of R&D intensity r and r^i.
3. The higher level of the discounted marginal productivity g of the domestic technology stock T_d leads to the lesser figures of R&D intensity r and r^i.
4. It is obvious also that the higher level of the obsolescence coefficient c^* implies the higher values for R&D intensity r and r^i.
5. The higher is the coefficient κ (32) characterizing the absorption capacity z (7), the higher are level of R&D intensity r and r^i. Let us remind that the coefficient κ is determined by the formula

$$\kappa = \frac{\nu}{\mu} = \frac{\omega}{((\xi_0 + \omega)^2 + \xi_0^2)}. \tag{189}$$

Here ξ_0 is the mean value of the rate ξ of the domestic technology T_d, and ω is the current rate of the technology spillover pool T_s.

The derivative of the absorption capacity coefficient κ with respect to the rate ω of the technology spillover pool is estimated as follows

$$\kappa'_\omega = \frac{2 \cdot \xi_0^2 - \omega^2}{((\xi_0 + \omega)^2 + \xi_0^2)^2}. \tag{190}$$

It is clear that $\kappa'_\omega < 0$ if the rate ξ_0 of the domestic technology stock T_d is not very high in comparison with the rate a the rate of the technology spillover pool T_s, $\xi_0 < \omega/\sqrt{2}$. In this case the growing trend for the rate a provides the declining trend for the absorption capacity coefficient κ, and, consequently, implies the declining trend for R&D intensity r and r^i. In the opposite case when the rate ξ_0 of the domestic technology stock T_d is rather high $\xi_0 > \omega/\sqrt{2}$, one can observe the inverse relation: the growth of the rate a leads to the growth of the coefficient κ, and, hence, stimulates the growth of R&D intensity r and r^i.

6. Analysis of feedbacks for R&D intensities r and r^i with respect to the technology productivity X demonstrates that R&D intensity r (187) grows with the growth of the technology productivity X. As to behavior of R&D intensity r^i (188), it depends on trends of both terms in the right hand side of formula (188): the first term has the growth trend, while the second term has the decline trend. Depending on ratios of the model macroeconomic parameters the aggregate growth trend of R&D intensity r^i (188) with respect to the technology productivity X can be either positive or negative.

4.2. Econometric Identification of Optimal R&D Investment Level

In this section two scenarios of techno-economic growth are compared within the proposed model based on dynamics (42)-(45) and objective function (47).

In the first scenario it is assumed that the effect of technology assimilation from the technology spillover pool T_s is missing, and the growth of the technology stock T is governed only by the domestic R&D investments ΔT_d. In this scenario the coefficients of absorption capacity are equal to zero, $z = 0$, $\kappa = 0$, and the technology stock T coincides with the domestic technology stock T_d. Macroeconomic parameters a_1, a_2 for this scenario are identified on the basis of relations (29) and the regression model (27) in which $T = T_d$ (see Table 1, row 1).

In the second scenario it is supposed that the spillover technology T_s is assimilated and the assimilation process is modeled by relation (32) for the assimilation coefficient κ and is calculated on the level $\kappa = 1.46$ applying the identified parameters

$a = 0.24$ and $\xi_0 = 0.14$ by means of the regression results for equation (20).

In this scenario macroeconomic parameters a_1, a_2 are defined by relations (29) and the regression model (27) in which $T = \mu \cdot T_d + \nu \cdot \Delta T_d$ (16) (see Table 1, row 2).

Table 1. Parameters of Utility Function in Japan's Automotive Industry (1982-2000).

	π	β_1	β_2	a_1	a_2	r^0
T_d	0.09	0.05	0.60	0.51	6.07	7.42
T	0.09	0.04	0.60	0.40	6.07	7.92

In both scenarios the discounted marginal productivity g is identified according to relation (38) on the level $g = 1.13$ with the following test statistics $t\text{-}value = 9.68$, $adj.R^2 = 0.999$, $DW = 1.87$.

Let us remind that the discount factor λ is identified at the level $\lambda = 0.105$ for both scenarios. This level is used for calculation of the cost Z^0 according to formula (133).

Table 1 compares the level of the optimal R&D intensity r in two scenarios. For calculating the level of the optimal R&D intensity r formula (187) is used. The optimal level for the first scenario is identified on the level $r = 7.42\%$, while the optimal level for the second scenario is calculated on the level $r = 7.92\%$ (see Table 1). This result demonstrates that the optimal R&D intensity r based on the gross technology stock T in the second scenario is higher a little bit than the optimal R&D intensity r based on the domestic technology stock T_d in the first scenario. Let us note that all other economic parameters: production $y(t)$, gross technology stock $T(t)$, and consumption index $D(t)$, are much higher in the second scenario than in the first scenario. One can conclude that comparatively small additional spending $\Delta r / r = (7.92 - 7.42)/7.42 = 6.7\%$ of R&D investment and restructuring of these sources for knowledge absorption could provide a strong leverage for reaching qualitatively higher level of performance of the basic economic parameters, and, consequently, higher levels of the consumption index. In order to reach such high performance of the main economic parameters of the second scenario it is necessary to increase significantly the level of R&D intensity in the first scenario and this increase is quite far from the oprtimal R&D investment policy.

These arguments are justified by Figure 6 which displays dynamics of technology stock T_d in Scenario 1 by the dotted line, and dynamics of technology stock T in Scenario 2 - by the solid line. One can see that in the year 2000 technology stock T is larger than technology stock T_d by $\Delta T / T = (35 - 25)/25 = 40\%$. Calculating the corresponding elasticity $\varepsilon_{T,r} = \dfrac{\Delta T / T}{\Delta r / r} = \dfrac{40}{6.7} = 5.97 > 1$ of technology stock T to R&D intensity r one can conclude that the model demonstrates the effect of increasing returns for the gross technology stock T of the process of R&D investment with restructuring invested sources and directing a part of them to knowledge absorption.

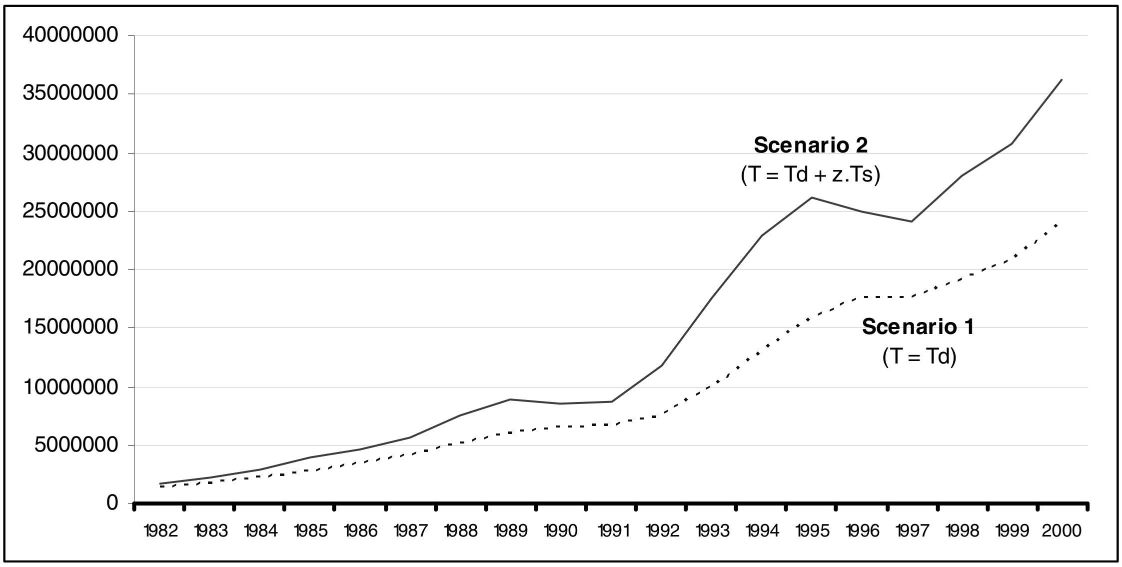

Figure 6. Trends in Technology Stock of Japan's Automotive Industry (1982-2000) in Scenarios 1 and 2 – ***million Yen in 1995 fixed prices.***

The analogous trend is demonstrated by Figure 7 which shows dynamics of production y in Scenario 1 by the dotted line, and in Scenario 2 – by the solid line. The difference between two scenarios in the year 2000 constitutes the value $\Delta y / y = (35 - 34)/34 = 2.9\%$. The corresponding elasticity $\varepsilon_{y,r} = \dfrac{\Delta y / y}{\Delta r / r} = \dfrac{2.9}{6.7} = 0.43$ demonstartes the positive impact of R&D intensity r on production y.

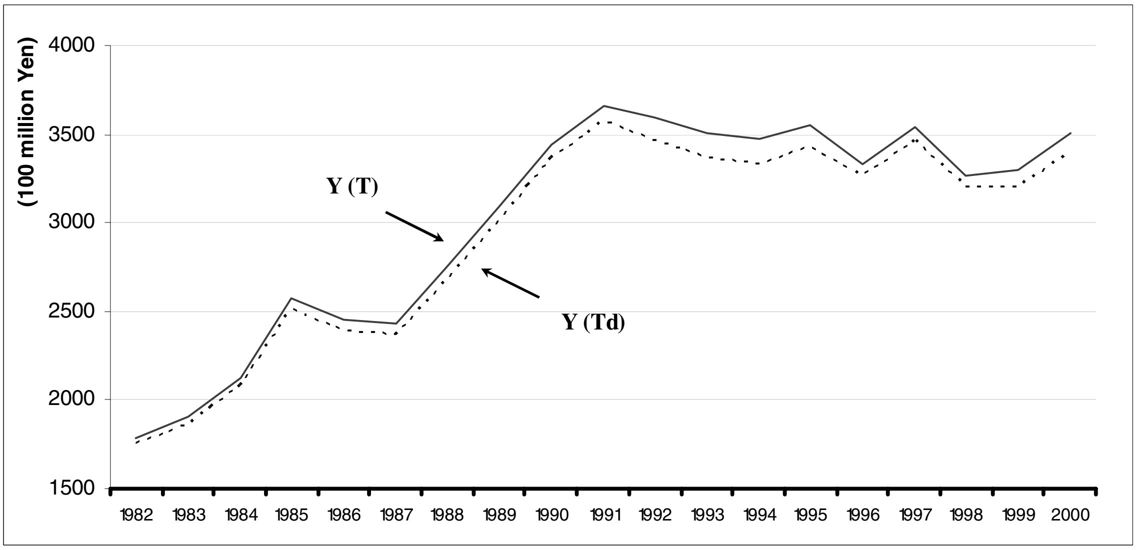

Figure 7. Trends in Production of Japan's Automotive Industry (1982-2000) in Scenarios 1 and 2 – ***million Yen in 1995 fixed prices.***

5. Conclusion

In the paper a dynamic model of optimization of R&D intensity is adjusted to the aggregate data in Japan's automotive industry over the period 1982-2000. The model takes into account that the R&D investment to commercialization leads to redistribution of resources between the technology stock and production factors and provides a risky factor of invention and innovation activity. The model describes dynamic behavior of the technology stock and production factors as a response to the optimal R&D investment policy. The model includes the discounted utility function which correlates the R&D investment and production diversity and reflects simultaneous growth of production, technology stock and rate of technology productivity. The research focuses on the issue of a reasonable balance between the indigenous technology stock and assimilated technology flow.

The Pontryagin's maximum principle is applied to the optimal control design of R&D intensity. The optimality principles are expressed in the nonlinear system of Hamiltonian differential equations. The eigenvalues and eigenvectors of the Jacobi matrix are estimated and on the basis of this analysis the existence and uniqueness result of a saddle-type equilibrium for the Hamiltonian system is proved. It is shown that the optimal solution can be generated from this equilibrium in the direction of the eigenvector corresponding to the negative eigenvalue. For a simplified version of the model the optimal feedback for R&D intensity is constructed analytically and its growth trends are studied. The macroeconomic parameters of the model are calibrated on the aggregate data of Japan's automotive industry over the period 1982-2000. It is shown hat comparatively small additional investments and restructuring of these sources for knowledge absorption could have the effect of increasing returns and provide a strong leverage for reaching qualitatively higher levels of sales, technology development, and consumption index.

The future work will be focused on identification of the optimal trajectory of R&D intensity for auto manufacturers under megacompetition conditions.

Appendix : Data Construction and Sources

Table A.1. Trends in R&D Expenditure and Technology Import in the Japan's Automotive Industry (1982-2000): million Yen at current prices

Year	Intramural expenditure on R&D (disbursement) (r)	R&D funds received (R_r)	R&D funds paid outside (R_p)	Technology import (Im)
1982	529,876	11,865	105,415	16,094
1983	561,024	12,434	127,059	10,644
1984	641,419	12,423	140,275	10,290
1985	727,640	14,625	161,641	11,391
1986	776,815	17,228	180,447	11,289
1987	767,932	20,997	214,798	8,402
1988	885,285	19,057	227,660	6,560
1989	1,028,079	22,548	239,871	7,248
1990	1,223,775	28,564	261,536	7,560
1991	1,231,116	28,825	278,416	8,029
1992	1,218,819	30,209	275,970	17,194
1993	1,040,474	24,467	264,983	8,748
1994	965,095	21,936	276,481	8,700
1995	1,093,416	26,470	300,893	7,511
1996	1,250,391	32,799	341,229	8,556
1997	1,372,413	29,385	392,017	7,536
1998	355,945	24,995	409,159	6,164
1999	1,261,930	19,946	416,376	5,630
2000	1,309,492	21,326	457,771	5,630

Source: Japan Statistics Bureau, Report on the Survey of R&D (annual issues).

Table A.2. Trends in Automotive Production and Its Input (1982-2000): [a]

Year	Production (Y) [a]	Labor (L: man hours)	Capital (K) [a]	Material (M) [a]	Energy (E: cal.)
1982	52,754,838	81,119,165	1,291,304	18,120	1,585
1983	54,622,160	84,740,160	1,382,163	18,607	1,691
1984	56,358,704	89,783,680	1,660,788	20,195	1,793
1985	60,321,660	106,407,536	1,778,290	22,136	1,817
1986	60,266,220	91,900,707	1,585,240	22,691	1,789
1987	60,213,902	86,397,445	1,625,810	22,870	1,792
1988	62,429,102	96,854,957	1,944,562	24,343	1,988
1989	64,031,283	101,140,697	2,164,834	27,646	2,094
1990	66,297,745	106,739,067	2,396,476	31,654	2,340
1991	65,111,259	110,395,696	2,439,122	32,658	2,292
1992	61,443,391	100,623,687	2,129,847	32,940	2,264
1993	55,191,827	93,521,159	2,031,401	31,460	2,195
1994	51,881,432	91,892,163	2,137,216	30,773	2,254

1995	50,118,727	95,976,831	2,325,300	30,034	2,260
1996	50,857,319	90,960,162	2,455,269	30,316	2,527
1997	53,950,806	92,163,409	2,608,546	32,626	2,655
1998	49,402,286	88,947,390	2,469,602	28,938	2,462
1999	48,463,706	87,268,391	1,216,382	29,264	2,503
2000	49,869,552	88,274,205	2,655,882	30,402	2,505

[a] million Yen at 1995 fixed price

Sources:

Y: Japan Automobile Manufacturers Association, Inc., Total Production by Year (2003); Japan Statistics Bureau, Report on the Survey of R&D (annual issues).

L, ***C***, ***M*** and ***E***: Economic and Social Research Institute of Japan, Business and Investment of Incorporated Enterprises (annual issues).

Table A.3. Trends in Number of Registered Patent in Japan's Automotive Industry (1982-2000)

Year	No. of Registered Patent (***P***)	Year	No. of Registered Patent (***P***)
1981	-	1991	661
1982	81	1992	716
1983	154	1993	496
1984	201	1994	450
1985	300	1995	605
1986	292	1996	482
1987	338	1997	358
1988	358	1998	371
1989	345	1999	322
1990	691	2000	304

[b] B60B - B60V International Patent Classification, WIPO (1999).

Source: Japan Patent Office, Industrial Property Digital Library (2003).

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