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A Dynamic Model of Stochastic Innovation Race: Leader-Followers Case

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Contents

Abstract

We provide steps towards analysis of rational bahaviors of innovators acting on a market of a technological product. The situation when a technological leader competes with a large number of identical followers is in the focus. The process of development of new generations of the product is treated as a Poisson-type cyclic stochastic process. The technology spillovers effect acts as a driving force of the technological progress. We obtain an analytic characterization of optimal leader's R&D and manufacturing investment policies. Numerical simulations and economic interpretations are presented as well.

Key words: R&D, stochastic innovation race, technology spillovers, imitation, optimal allocation of resources, dynamic optimization
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"Almost by definition, it is hopeless to develop a model which will genuinely predict innovations: an innovation is something new, and if you know what it will be in the future, you know it now. There are, it must be admitted, some loopholes in this argument. It can be held that we have probabilistic beliefs about future innovations. Experts will frequently claim that such-and-such lines of development are promising. I believe that analysis of technological forecast are subject to a great deal of error. However, I do not conclude from this that dynamic models which incorporate technical change are useless. What they give you is not any prediction of specific innovations, but an idea of the statistical properties of technological progress. We may have some useful idea of the average rate of technological change, of the degree of fluctuations and the kinds of surprises that may find in the future. We cannot, of cause, predict a surprise, that is a contradiction in terms. But we can predict the kinds of surprises that might occur. From the point of view of public policy, this knowledge may be very useful. It indicates the information we will need to react to, the range of possibilities we may encounter. It gives an idea of the policies of protection and precaution that would be useful to invest in."

Arrow, K.J., 1991.

1 Introduction

It is beyond argument that R&D is one of the most important determinants of firms' competitiveness especially in high-tech fields. However, there are different types of R&D and their effects on firms' performances are also different.

Innovations based on certain technological systems or dominant designs give clear patterns of continuous or cumulative technological improvements. However, when technological development reaches a certain turning point, a discontinuous innovation often occurs. Many authors define such type of technological development as the technological trajectory: a technology is developing along a certain trajectory, and when it saturates a new trajectory occurres together with a shift in the corresponding scientific paradigm (see, for example, Kuhn, 1970; and Dosi, 1982).

Along with this argument, innovations can be classified into two types. Innovations of the first type create new technological trajectories (radical or discontinuous innovations); innovations of the second type develop a product improvements along a certain technological trajectory (incremental or continuous innovations).

Invention of the transistor, which provided the basis of the semiconductor industry, gives an example of a radical technological innovation which created a new technological trajectory. It needs no argument that innovations of this type create the strongests aftereffects on the markets.

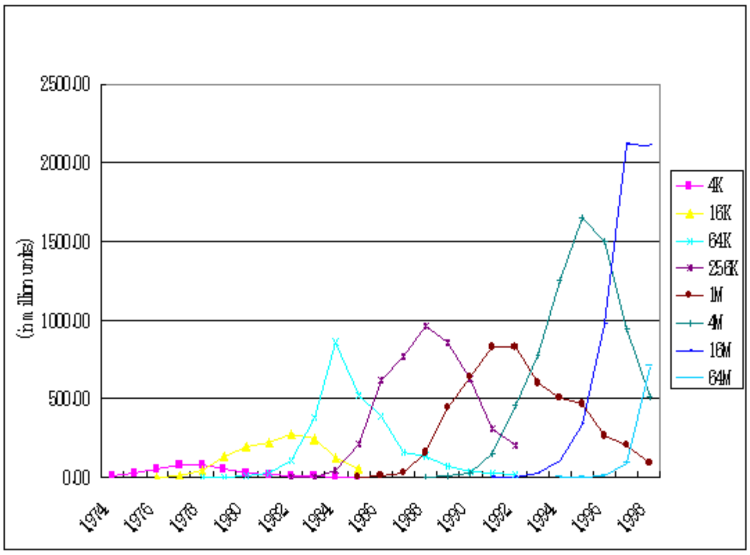

Figure 1: Global DRAM shipment by IC density. Data Source: Victor, 2002.

Nevertheless, it is important to recognize that outcomes of continuous innovations can be not less than outcomes of discontinuous ones. Of course, each incremental innovation can produce less outcome than a radical one, but often their cumulative outcome is bigger. For example, the process of developing high density integrated circuites is a sequence of typical continuous innovations which create a large market by interacting with computer and other information technology products and services.

It is interesting from the point of view of industrial dynamics that firms succeeding in the radical innovations often lose their leading positions during the following phase of technological development at which incremental innovations are dominant.

Technological developments along certain technological trajectories are mainly performed as innovation races among private companies. Regular emergences of new generations of products are observed in the industries where innovations along technological trajectories dominate. For example, the transistors appeared on the market and started a new technological trajectory in the middle of the 1950s. The transistors quickly swept away vacuum tubes which had in 100 times larger market share at that time. However, the transistors themselves were swept away by the integrated circuits (IC) in the 1960s. During this process, the firms severely fought with each other and this severe competition led to the withdrawal of the top producers of vacuum tubes from the semiconductor market. After the 1970s, Japanese firms appeared as leading producers of large scaled integrated circuits (LSI). After the 1990s, the US firms revived, and Korean and Taiwanese firms appeared as the major players on this market. This innovation race is now continuing. Figure 1 demonstrates the technological dynamics in the semiconductor industry in the period 1974 - 1998. In this process the autonomous evolutions of new generations of the product are driven by innovation race among companies. Furthermore, one can observe a regularity in this technological dynamics. In particular, Figure 1 demonstrates the famous Moore's law in the semiconductor industry. It asserts that the complexity for minimum component costs increases at a rate of roughly a factor of two per year (Moore, 1965).

Similar industrial dynamics can be observed also in other areas. For example, the leading producer of hard disk drives is changing with each major innovation (Christensen, 1997).

But, however, there are many examples of different types of industrial dynamics. For example, in the same semiconductor industry the market of microprocessors for personal computers is dominated by Intel which innovated the microprocessor firstly in the world.

From the point of view of the innovation race, technology spillovers[1] (including imitation) provide an important influence on the industrial dynamics through each firm's decision making. We should recognise that imitation is an important type of firms' behavior even in developed countries although usually it is related mainly to the North-South problems (Mukoyama, 2003).

There are various opportunities and routes for technology spillovers (Porter, 1985):

i) Reverse-engineering of products and observation of operations;

ii) Spillovers from vendors of equipments or components;

iii) Spillovers from consultants or specialists;

iv) Spillovers from purchasers which desire varieties of suppliers;

v) Movement of engineers to rivalry firms or spin-out;

vi) Analysis of patents or presentations in academic societies.

Many authors assert that imitations have negative impacts on the innovation process. Imitations erode profits of innovators, and this erosion causes shrinking the efforts of the innovators. Usually, "appropriability of innovations," which means the possibility for an innovator to ensure the profit from innovations, is determined by the extent of legal protection (intellectual property rights), imitation facilities, and accessability of complimentary assets (see Levin, Klevorick, Nelson and Winter, 1988; and Teece, 1986). Goto and Nagata, 1997 assert that early introduction of new products on the market is effective for appropriability of innovations in high-tech fields in Japan. However, Mansfield, Schwartz and Wagner, 1981 found that 34 new products out of 48 samples had been imitated during the sample periods. They report that the average time until the sample products were imitated is about 70% of the time it took to bring the innovation to the market.

The problem of choosing the most rational type of the business strategy is an important point for firms' managers. From the point of view of the innovation race, business strategies can be classified into two types. Strategies of the first type are the leader's startegies, which are to firstly develop new generations of the product. Strategies of the second type are the follower's strategies, which are to penetrate on the market exploited by the leader.

The earliest release of a new generation of the product on the market is directly connected with the profitability of R&D. Particularly, in the fields where learning effects are large, the first penetrator can get the largest share of the market. However, a larger R&D investment may result in a shorter product's lifecycle. It means that the return of R&D might be smaller. A smaller R&D investment allows a follower exploiting technology spillovers to get a larger share of the market. In this respect the problem how much of resources should be allocated to R&D is also an important point for firms' managers because now the required R&D investments are increasing to the level which governs the existence of firms in many fields especially in high-tech fields.

Of course, not only the effect of a certain targeted environment should be taken into account in the decision making process but also the effect of other larger environments. For example, the Japanese firms in the electric home appliances market reached highest competitiveness in the world in the 1980s due to a strategy of introducing new generations

[1] In this paper we identify technology spillovers mainly with imitation.

of certain products on the market in short cycles in order to win the innovation race on the domestic market (Hamel and Prahalad, 1992).

According to Lieberman and Montgomery, 1988 merits of leader's and follower's strategies can be particularized as follows:

Merits of a leader's strategy:

i) First-mover's advantage. The innovator or first-mover can monopolize the market by using a physical lead time. The first mover can improve the products by responding to its customers and often make a dominant design of the products. It might lead to establishing a superior brand image among the customers. The first mover can occupy rare resources;

ii) Learning effect. The first mover can get cost competitiveness by using a learning or experience effect. Experience of production of a certain product can increase efficiency of workers and improve the manufacturing process. If the first mover constructs such kind of cost competitiveness, a follower can not enter the market. That is why the semiconductor makers challenged to be the first movers for new DRAM generations;

iii) Legal protection. The first mover can get legal protection by intellectual property rights such as patents and copyrights. Xerox for copy machines and GE for electric bulbs dominated the market for a long time by exploiting their patents;

iv) Transfer cost of customers. In the case when customers should pay some cost to change their suppliers, the first mover can take an advantage (the mileage system is an example);

v) Network externality. If the new products or services have network externalities, the fist mover can get an advantage to the followers. Network externality has been defined as a change in the benefit, or surplus, that an agent derives from a good when the number of other agents consuming the same kind of good changes. As fax machines increase in popularity, for example, your fax machine becomes increasingly valuable since you have greater use of it.

Merits of a follower's strategy:

i) Free-ride of leader's effort. The followers can freely use many things which the first-mover built at its cost. It is difficult to exclude the use of knowledge, the outcomes of innovations, by others. In many case, imitation is cheaper than the original innovation although it costs larger than generally expected. For example, investment in educating customers how to use certain innovative products, and infrastructure development for new products (for example, repair services) can be freely used by the followers;

ii) Low market risk for introducing products. The first mover should take risk of uncertain innovative products and bear investments which can eventually be found unnecessary as well as the cost of try and errors. Followers can avoid unnecessary investments or failures by exploiting the experiences of the first mover. Actually, Japanese car makers learn many lessons from the experience of the German car maker Volks Wagen which precedently penetrated into the US market and failed when Japanese car makers managed to produce their cars in the US;

iii) Conservation of initial cost by the spillovers effect, etc.. In the case of environmental changes, the first mover often faces difficulties. There are some cases where the environments on the market or technology change after the first mover commits to certain assets or processes and embarks full-scale investments in them. In such cases, the followers can get an advantage over the first mover. Because of the institutional inertia it is difficult for the first mover to respond to environmental change, diminishing the merits of existing assets or processes and to sunk the cost of them. Ford succeeded in the passenger car market by focusing on the production of the T-type Ford. However, Ford could not respond to the preference of the customers in the differentiation of products and gave up the top position to General Motors.

It is a difficult problem for the firms' managers to choose the rational type of business strategy (the leader's strategy or the follower's one). Usually the innovators' decision making processess are affected by properties they already occupied, institutional environments and strategies of the competitors. For example, Nintendo had a dominant power on the 8-bit video game console's market at the 1980s. For Nintendo, the change of the video game market from 8-bit consoles to 16-bit ones meant the loss of its monopoly profit. Sega had already introduced a 16-bit video game console on the market at that time but, however, its diffusion was rather slow because Sega was satisfied with the high price and monopoly profit from the 16-bit console's market. In such situation, Nintendo delayed the release of the new generation of the 16-bit video game console "Super Famicom" (Brandenburger and Nalebuff, 1997) earning a relatively large revenue from the 8-bit consoles. On the contrary, Nintendo's early introduction of the 16-bit console could lead to a price competition between Sega and Nintendo, and, as a consequence, could induce an early diffusion of 16-bit consoles and Nintendo's loss of its monopoly profit from the 8-bit consoles. Nintendo faced this dilemma, and it took the strategy to delay the introduction of "Super Famicom" on the market.

In the present paper we provide steps towards analysis of rational behaviors of innovators acting on a market of a technological product. We develop a dynamic model of optimal investment in R&D and manufacturing. The situation when a technological leader competes with a large number of identical technological followers is in the focus.

The model is developed according to the templates of the economic growth theory (see Arrow and Kurz, 1970; Grosmann and Helpman, 1991; Baro and Sala-i-Martin, 1995). The essential feature of the model is that the innovation process performed by the innovators (the technological leader and the followers) is treated as a Poisson-type cyclic stochastic process on the infinite time interval. The main assumption is that the probability of the development of a new generation of the product on a small time interval is proportional to the length of this interval and to the innovator's knowledge capital which is associated with accumulated innovator's investments of some resource in R&D.

In our model we consider the case of the development along a certain technological tragectory with a vertical products differentiation (see Grossman and Helpman, 1991). This means that every innovation gives a significant improvement in the product's quality and (or) in the level of services that the product provides.

Further, we assume that the innovators acting on the market have two sectors: an R&D sector and a manufacturing one. All innovators make costly investments to both R&D and manufacturing. Consequently the leader's R&D sector developes new generations of the product and the leader's manufacturing sector performs their production. Selling the products on the market, the technological leader gets a revenue for its R&D and manufacturing efforts. The goal of the leader is to maximize the aggregated discounted profit by optimizing both R&D and manufacturing investment policies. Concerning the technological followers we assume that they act as imitators. When the latest generation of the product developed by the leader appears on the market the product's attributes become available for the followers. This provides the followers with the ability to improve their economic performance by developing corresponding imitations (their own versions of the newest product). In this way a technology spillovers effect is taken into account. When the followers bring their versions of the latest generation of the product on the market, the product's offer increases. As a consequence the product's price decreases together with the leader's profit. This stimulates the technological leader to develop the next generation of the product. Thus, in our model, the technology spillovers effect plays a role of a driving force in the technological progress. Choosing appropriate R&D and manufacturing investment policies, the technological leader maximizes the expectation of the value of its aggregated discounted profit.

The paper is organized as follows. In Section 2 we consider a process of new product's generations development performed by the technological leader and the followers. In Section 3 we describe an accepted market price formation mechanism and design a technological leader's goal functional. In Section 4 the dynamic optimization problem the technological leader is faced at is considered. We obtain an analytic expression for the optimal value of the goal functional. For the case where the technological leader competes with a large number of followers numerical simulations and economic interpretations are presented in Section 5. Section 6 presents concluding remarks.

2 Innovation process

First, we consider the process of the development of new generations of the product, performed by the technological leader.

At the initial instant of time $T_0 = 0$ the leader makes a decision on the amount $u(t) \equiv u_1 \geq 0$ of a resource which will be allocated to R&D. It can be leader's labor, capital, energy or another resource which is recognized as the determinant of the leader's R&D activity. We assume that this amount $u(t)$ of the resource is fixed till the instant of time $T_1 > T_0$ when the next generation of the product will be developed by the leader's R&D sector. Therefore, the leader's R&D investment policy $u(t)$ is assumed to be fixed on the time interval $[T_0, T_1]$: $u(t) \equiv u_1$ for all $t \in [T_0, T_1]$.

Starting from the instant of time T_1 the developed new technology is implemented in manufacturing and the leader's research sector starts development of the next generation of the product. At the time T_1 the technological leader makes a decision on the amount $u(t) \equiv u_2 \geq 0$ of the resource which will be allocated to R&D. We assume that the leader's R&D investment policy $u(t)$ is fixed till the next instant of time $T_2 > T_1$ when the next generation of the product will be available, i.e., $u(t) \equiv u_2$ for all $t \in [T_1, T_2]$.

This process is repeated infinitely many times.

Thus, we have an infinite sequence of instants of time $T_0 = 0$, $T_{n-1} < T_n$, $n = 1, 2, \ldots$, at which the technological leader starts development of the new generations of the product. On each n-th time interval $[T_{n-1}, T_n]$ the leader's R&D investment policy (an instantaneous amount $u(t)$ of the resource allocated to R&D) is fixed: $u(t) \equiv u_n \geq 0$ for all $t \in [T_{n-1}, T_n]$, $n = 1, 2, \ldots$.

We consider the length $l_n = T_n - T_{n-1}$ of each time interval $[T_{n-1}, T_n]$, $n = 1, 2, \ldots$, as a random variable of the Poisson type.

Let $K_n(t)$ be the leader's knowledge capital accumulated in R&D at the instant of time $t \in [T_{n-1}, T_n]$. We assume that for any $n = 1, 2, \ldots$ we have $K_n(T_{n-1}) = 0$. In our model the current leader's knowledge capital $K_n(t)$, $t \in [T_{n-1}, T_n]$, is associated with the accumulated investment of the resource in R&D, i.e.,

$$\dot{K}_n(t) = u(t) \equiv u_n, \qquad t \in [T_{n-1}, T_n];$$

$$K_n(T_{n-1}) = 0.$$

Thus, we have

$$K_n(t) = u_n \cdot (t - T_{n-1}) \qquad \text{for all} \quad t \in [T_{n-1}, T_n], \quad n = 1, 2, \ldots. \tag{2.1}$$

Further, T_n is the instant of time when the n-th generation of the product becomes available. We assume that for any $n = 1, 2, \ldots$ the length $l_n = T_n - T_{n-1}$ of the n-th research interval is a random variable with a smooth distribution $P_n(\tau) = P(l_n < \tau)$, $\tau > 0$, satisfying the equality

$$P(l_n < \tau + \Delta\tau | l_n \geq \tau) = \rho K_n(T_{n-1} + \tau)\Delta\tau + o(\Delta\tau), \quad \rho > 0.$$

Here $\lim_{\Delta\tau \to 0} \frac{o(\Delta\tau)}{\Delta\tau} = 0$, all lengths $l_n = T_n - T_{n-1}$, $n = 1, 2, \ldots$, are considered as independent random variables and $\rho > 0$ is a constant parameter characterizing the efficiency of the leader's R&D sector.

These assumptions provide a complete characterization of the random variables $l_n = T_{n-1} - T_n$, $n = 1, 2, \ldots$, as the Poisson-type random variables (see, for example, Gnedenko, 1962). Indeed:

$$P(l_n < \tau + \Delta\tau) = P(l_n < \tau) + P(l_n < \tau + \Delta\tau | l_n \geq \tau) P(l_n \geq \tau)$$

$$= P(l_n < \tau) + (\rho K_n(T_{n-1} + \tau)\Delta\tau + o(\Delta\tau))(1 - P(l_n < \tau)).$$

Hence, we have

$$\frac{P(l_n < \tau + \Delta\tau) - P(l_n < \tau)}{\Delta\tau} = (1 - P(l_n < \tau))\Big(\rho K_n(T_{n-1} + \tau) + \frac{o(\Delta\tau)}{\Delta\tau}\Big).$$

The last equality implies that the smooth distribution $P_n(\tau) = P(l_n < \tau)$ is a solution to the ordinary differential equation

$$\frac{d}{dt} P(\tau) = \rho K_n(T_{n-1} + \tau)(1 - P(\tau))$$

satisfying the initial condition $P(0) = 0$. Hence, the random variable $l_n = T_{n-1} - T_n$ has the following Poisson-type distribution and density:

$$P_n(\tau) = 1 - e^{-\rho \int_0^\tau K_n(T_{n-1} + s) ds};$$

$$p_n(\tau) = \frac{d}{dt} P_n(\tau) = \rho K_n(T_{n-1} + \tau) e^{-\rho \int_0^t K_n(T_{n-1} + s) ds}.$$

In particular due to (2.1) we have

$$P_n(\tau) = 1 - e^{-\rho u_n \frac{\tau^2}{2}}, \qquad p_n(\tau) = \rho u_n \tau e^{-\rho u_n \frac{\tau^2}{2}}. \tag{2.2}$$

Thus, for any $n = 1, 2, \ldots$, the density $p_n(\tau)$ of the random variable l_n is defined completely by the fixed leader's R&D policy u_n accepted on the n-th time interval $[T_{n-1}, T_n]$.

Further, (2.2) implies that the mean value $E(l_n)$ of the random variable l_n, $n = 1, 2, \ldots$, is the following:

$$E(l_n) = \sqrt{2}\rho u_n \int_0^\infty \sqrt{s} e^{-\rho u_n s} ds.$$

Consider now the process of the development of new generations of the product performed by N_f identical technological followers. In this paper we assume that all followers are independent and act as imitators. This means that when the latest generation of the product developed by the leader appears on the market at the instant of time T_{n-1}, $n = 1, 2, \ldots$, the followers observe its attributes and start development of its imitations (their own versions of the newest product).

We assume that all followers allocate equal (and fixed) amounts u_f of the same resource as the technological leader to R&D on every time interval $[T_{n-1}, T_n]$, $n = 1, 2, \ldots$. Then, analogously to the case of the technological leader (see (2.1)) the value of each i-th follower's knowledge capital $K_f^i(t)$, $i = 1, 2, \ldots N_f$, at the instant of time $t \in [T_{n-1}, T_n]$ can be represented by the following formula:

$$K_f^i(t) = u_f \cdot (t - T_{n-1}). \tag{2.3}$$

Further, similarly to the case of the technological leader we assume that the length l_n^i, $n = 1, 2, \ldots$, $i = 1, 2, \ldots, N_f$, of the time interval needed for the i-th follower to develop its own version of the newest product is the Poisson-type random variable characterized by an efficiency parameter $\beta > 0$ (which assumed to be the same for all followers) and the corresponding amount of the accumulated knowledge capital $K_f^i(t)$ (see 2.3), i.e.,

$$P(l_n^i < \tau + \Delta\tau | l_n^i \geq \tau) = \beta K_f^i(T_{n-1} + \tau)\Delta\tau + o(\Delta\tau).$$

In this case for all $i = 1, 2, \ldots, N_f$ and all $n = 1, 2, \ldots$ the distribution $P^f(\tau) = P(l_n^i < \tau)$ and the corresponding density $p^f(\tau) = \frac{d}{dt}P^f(\tau)$ of the random variable l_n^i are independent of i and n and can be represented by the following formulas (see (2.2)):

$$P^f(\tau) = 1 - e^{-\beta u_f \frac{\tau^2}{2}}, \qquad p^f(\tau) = \beta u_f \tau e^{-\beta u_f \frac{\tau^2}{2}}. \tag{2.4}$$

Let $0 \leq N(t) \leq N_f$ be the number of followers operating on the market at the instant of time $t \in [T_{n-1}, T_n]$, $n = 1, 2, \ldots$, with their versions of the latest generation of the product. As far as all followers are assumed to be independent, and the probability of the event that the i-th follower has developed its version of the newest product before time $t \in [T_{n-1}, T_n]$ is equal to $P^f(t - T_{n-1}) = 1 - e^{-\beta u_f \frac{(t-T_{n-1})^2}{2}}$ (see (2.4)) for any $M = 0, 1, \ldots, N_f$, we have

$$P(N(t) = M) = \frac{N_f!}{M!(N_f - M)!}\left(1 - e^{-\beta u_f \frac{(t-T_{n-1})^2}{2}}\right)^M e^{-(N_f - M)\beta u_f \frac{(t-T_{n-1})^2}{2}}$$

$$= C_{N_f}^M \left(1 - e^{-\beta u_f \frac{(t-T_{n-1})^2}{2}}\right)^M e^{-(N_f - M)\beta u_f \frac{(t-T_{n-1})^2}{2}}. \tag{2.5}$$

Hence the mean value $E(N(t))$ of the random variable $N(t)$ at the instant of time

$t \in [T_{n-1}, T_n]$, $n = 1, 2, \ldots$, can be calculated as follows:

$$E(N(t)) = \sum_{M=0}^{N_f} M \frac{N_f!}{M!(N_f - M)!} \Big(1 - e^{-\beta u_f \frac{(t-T_{n-1})^2}{2}}\Big)^M e^{-(N_f - M)\beta u_f \frac{(t-T_{n-1})^2}{2}}$$

$$= N_f \Big(1 - e^{-\beta u_f \frac{(t-T_{n-1})^2}{2}}\Big) \sum_{M=1}^{N_f} \frac{(N_f - 1)!}{(M-1)!(N_f - M)!} \Big(1 - e^{-\beta u_f \frac{(t-T_{n-1})^2}{2}}\Big)^{(M-1)}$$

$$\times e^{-(N_f - M)\beta u_f \frac{(t-T_{n-1})}{2}} = N_f \Big(1 - e^{-\beta u_f \frac{(t-T_{n-1})^2}{2}}\Big) \sum_{K=0}^{N_f - 1} C_{N_f - 1}^K \Big(1 - e^{-\beta u_f \frac{(t-T_{n-1})^2}{2}}\Big)^K$$

$$\times e^{-(N_f - 1 - K)\beta u_f \frac{(t-T_{n-1})^2}{2}} = N_f \Big(1 - e^{-\beta u_f \frac{(t-T_{n-1})^2}{2}}\Big). \tag{2.6}$$

3 Manufacturing and profit maximization

In this section we describe the accepted market price formation mechanism and design the technological leader's goal functional.

Usually in economic literature, the firm's instantaneous production (production rate) $Y(t)$ at the instant of time $t \geq 0$ is viewed as a function of quantities of resources accumulated in manufacturing such as labor, $L(t)$, capital, $C(t)$, materials, $M(t)$, and energy, $E(t)$ (see, e.g., Arrow and Kurz, 1970; Intriligator, 1971):

$$Y(t) = F(L(t), C(t), M(t), E(t)).$$

In this paper we assume that the leader's production rate at every instant of time $t \geq 0$ is determined by the corresponding instantaneous investment $v(t) \geq 0$ of some resource. We do not concretize the nature of this resource. It can be capital, labor, energy or any other particular resource which is invested in manufacturing. This resource can be different from the resource which the leader allocates to R&D. In particular, we represent the leader's production rate $Y_l(t)$ (a number of units of the product produced in the unit of time) as a function of the current investment $v(t)$ of the chosen resource as follows:

$$Y_l(t) = \sigma v^{\gamma}(t). \tag{3.7}$$

Here $\sigma > 0$ and $\gamma > 0$ are the model parameters[2]: parameter σ characterizes the productivity level of the leader's manufacturing sector; parameter γ represents the leader's marginal productivity. In that follows we assume that the leader's instantaneous production rate is strictly positive (i.e., $v(t) > 0$ for all $t \geq 0$).

Such form of the production function can be interpreted by different ways.

In particular, if the current leader's production rate $Y_l(t)$ can be represented by means of the Cobb-Duglas formula

$$Y_l(t) = \sigma_0 L^{\gamma_1}(t) C^{\gamma_2}(t) M^{\gamma_3}(t) E^{\gamma_4}(t) \tag{3.8}$$

where $\sigma_0 > 0$ and $\gamma_i > 0$, $i = 1, 2, 3, 4$, $\sum_{i=1}^4 \gamma_i = 1$; and the instantaneous amounts of the leader's labor, capital, materials and energy can be represented as functions of this resource:

$$L(t) = \kappa_1 v^{\alpha_1}(t), \quad C(t) = \kappa_2 v^{\alpha_2}(t), \quad M(t) = \kappa_3 v^{\alpha_3}(t), \quad E(t) = \kappa_4 v^{\alpha_4}(t),$$

[2] In this formulation, we allow that the economics of scale exists ($\gamma > 1$).

where $\kappa_i > 0$; $\alpha_i > 0$, $i = 1,2,3,4$; then substituting in (3.8) we represent the leader's instantaneous production $Y_l(t)$ in the form (3.7).

In our model the technological leader's manufacturing sector produces the latest version of the product developed by its research lab. We have a sequence $T_0 = 0 < T_1 < \cdots < T_n < \ldots$ of random instants of time T_n, $n = 1,2,\ldots$, at which the leader's R&D sector develops new generations of the product. We assume that at each instant of time T_{n-1}, $n = 1,2,\ldots$, the leader makes a decision about amount $v(t) \equiv v_n > 0$ of the resource which will be allocated to manufacturing on the current time interval $[T_{n-1}, T_n]$[3].

So, we assume that the leader's manufacturing investment policy $v(t)$ is fixed on each time interval $[T_{n-1}, T_n]$: $v(t) \equiv v_n > 0$ for all $t \in [T_{n-1}, T_n]$, $n = 1,2,\ldots$. According to (3.7) in this case on every time interval $[T_{n-1}, T_n]$, $n = 1,2,\ldots$, the leader has fixed manufacturing sector's production rate

$$Y_l(t) \equiv Y_n = \sigma v_n^{\gamma}, \qquad t \in [T_{n-1}, T_n].$$

Now we describe the accepted market price formation mechanism.

At the instant of time $t \in [T_{n-1}, T_n]$, $n = 1,2,\ldots$, a random number $0 \le N(t) \le N_f$ of followers produce their versions of the newest generation of the product. As far as all followers are supposed to be identical, each of them produces the same number

$$y_f = \sigma_f v_f^{\gamma_f}$$

of units of the product in the unit of time. Here $\sigma_f > 0$ and $\gamma_f > 0$ are parameters analogous to whose of the technological leader (followers' level of productivity and marginal productivity respectively), and v_f is an amount of the resource allocated by each follower to manufacturing. In this case the current aggregated rate of total followers' production at the instant of time $t \in [T_{n-1}, T_n]$ is $Y_f(t) = N(t)y_f$. Thus, (see (3.7)) the total supply rate of the product to the market at the instant of time $t \in [T_{n-1}, T_n]$, $n = 1,2,\ldots$, is

$$Y(t) = Y_l(t) + Y_f(t) = \sigma v_n^{\gamma} + N(t)y_f. \tag{3.9}$$

Let $d(t)$ be the current market price of the unit of the product at the instant of time $t \in [T_{n-1}, T_n]$, $n = 1,2,\ldots$.

Assume that the market's demand is equal to a constant $d_0 > 0$ at any instant of time $t \ge 0$. Then taking into account the equilibrium condition $d(t)Y(t) = d_0$, due to (3.9) we get the following formula for the current price $d(t)$ at the instant of time $t \in [T_{n-1}, T_n]$:

$$d(t) = \frac{d_0}{Y(t)} = \frac{d_0}{\sigma v_n^{\gamma} + N(t)y_f}.$$

Then due to (2.5) the mean value $E(d(t))$ of the price $d(t)$ at the instant of time $t \in [T_{n-1}, T_n]$, $n = 1,2,\ldots$, is the following:

$$E(d(t)) = \sum_{M=0}^{N_f} \frac{d_0}{\sigma v_n^{\gamma} + M y_f} C_{N_f}^{M} \left(1 - e^{-\beta u_f \frac{(t-T_{n-1})^2}{2}}\right)^M e^{-\beta u_f (N_f - M)\frac{(t-T_{n-1})^2}{2}}. \tag{3.10}$$

Now, suppose that the leader should pay prices $p_1 > 0$ and $p_2 > 0$ for units of resources $u(t)$ and $v(t)$ allocated to R&D and manufacturing respectively at any instant of time $t \ge 0$.

[3]The technological leader terminates the production of the old product as soon as the new one is developed. We don't consider the market of the old products after their withdrawal explicitly. In this model, the new entrants to the market of current leading-edge products are not limited by old products which were on the market of the previous product's generations (see paper by Winter et al., 2000 focused on the modelling of new entrants).

As far as the leader's instantaneous revenue at the instant of time $t \in [T_{n-1}, T_n]$ is equal to $d(t)Y_n$, the instantaneous leader's profit $D(t)$ is represented by the formula

$$D(t) = d(t)Y_n - (p_1 u_n + p_2 v_n).$$

Let $\alpha > 0$ be a subjective discount parameter. Then the discounted aggregated leaders profit J_n on the n-th random time interval $[T_{n-1}, T_n]$, $n = 1, 2, \ldots$, can be defined as follows:

$$J_n = \int_{T_{n-1}}^{T_n} D(t)e^{-\alpha t}dt = \int_{T_{n-1}}^{T_n} [d(t)Y_n - (p_1 u_n + p_2 v_n)]e^{-\alpha t}dt.$$

Introducing the new integration variable $\tau = t - T_{n-1}$ we get

$$J_n = \prod_{i=1}^{n-1} e^{-\alpha(T_i - T_{i-1})} \cdot \int_0^{T_n - T_{n-1}} [d(\tau)Y_n - (p_1 u_n + p_2 v_n)]e^{-\alpha\tau}d\tau$$

$$= \prod_{i=1}^{n-1} e^{-\alpha l_i} \cdot \int_0^{l_n} [d(\tau)Y_n - (p_1 u_n + p_2 v_n)]e^{-\alpha\tau}d\tau.$$

As far as all random variables l_1, l_2, $\ldots$, l_n are independent we have

$$E(J_n) = \prod_{i=1}^{n-1} E(e^{-\alpha l_i}) \cdot E\Big(\int_0^{l_n} [d(\tau)Y_n - (p_1 u_n + p_2 v_n)]e^{-\alpha\tau}d\tau\Big). \tag{3.11}$$

Due to the equality $p_i(\tau) = \rho u_i \tau e^{-\rho u_i \frac{\tau^2}{2}}$, $i = 1, 2, \ldots$, (see (2.4)) the following equality takes place

$$E(e^{-\alpha l_i}) = \int_0^\infty e^{-\alpha\tau - \rho u_i \frac{\tau^2}{2}} \rho u_i \tau d\tau, \quad i = 1, 2, \ldots.$$

Integrating the last equality by parts we get

$$E(e^{-\alpha l_i}) = 1 - \alpha \int_0^\infty e^{-\alpha\tau - \rho u_i \frac{\tau^2}{2}} d\tau, \quad i = 1, 2, \ldots. \tag{3.12}$$

Our next result gives a tool for calculation of the mean value $E(J_n)$, $n = 1, 2, \ldots$ (see (3.11)):

Proposition 1. *For any $n = 1, 2, \ldots$, the following equality takes place:*

$$E\Big(\int_0^{l_n} d(\tau)e^{-\alpha\tau}d\tau\Big) = \int_0^\infty E(d(\tau))e^{-\alpha\tau - \rho u_n \frac{\tau^2}{2}} d\tau. \tag{3.13}$$

Proof. For arbitrary $T > 0$ and any $N = 1, 2, \ldots$, put $\Delta_N^T = \frac{T}{N}$, $t_j = j\Delta_N^T$, $j = 1, \ldots, N$, and consider the random process

$$\xi_{l_n}(\tau) = 1 \quad \text{if} \quad l_n \geq \tau, \qquad \xi_{l_n}(\tau) = 0 \quad \text{if} \quad l_n < \tau$$

and the random variable

$$J_N^T = \sum_{j=1}^N d(t_j)e^{-\alpha t_j}\xi_{l_n}(t_j)\Delta_N^T.$$

The random variable J_N^T and the random process $d(t)$ are completely characterized on the time interval $[T_{n-1}, T_n]$ by a finite number of random parameters $\tau_1 \geq 0 \ldots, \tau_{N_f} \geq 0$ which are the instances of time when the corresponding followers appear on the market.

Further, for any realization of $\tau_1, \ldots, \tau_{N_f}$ the corresponding realization of

$$d(t) = d_{\tau_1,\ldots,\tau_{N_f}}(t) = \frac{d_0}{\sigma v_n^\gamma + N(t) y_f}.$$

is a piece-wise constant function having not more than N_f points of discontinuity. Hence, due to the presence of the discounting factor $e^{-\alpha t}$ in the integral $\int_0^{l_n} d(\tau) e^{-\alpha\tau} d\tau$ for arbitrary $\epsilon > 0$ there is a $T_\epsilon > 0$ such that for any realization of random variables $\tau_1, \ldots, \tau_{N_f}$, any $T > T_\epsilon$ and any realization of the random variable l_n we have

$$\Big| \int_0^{l_n} d(\tau) e^{-\alpha\tau} d\tau - J_N^T \Big| \leq \epsilon. \tag{3.14}$$

Consider the random variable J_N^T (depending on the random variables $\tau_1, \ldots, \tau_{N_f}$ and l_n). We have

$$E(J_N^T) = E\Big(\sum_{j=1}^{N} d(t_j) e^{-\alpha t_j} \Delta_N^T \xi(l_n) \Big) = \sum_{j=1}^{N} E(d(t_j)) e^{-\alpha t_j} E(\xi_{l_n}(t_j)) \Delta_N^T.$$

As far as

$$E(\xi_{l_n}(t_j)) = P(l_n \geq t_j) = 1 - P_n(t_j) = e^{-\rho u_n \frac{t_j^2}{2}}$$

(see (2.2)) we get

$$E(J_N^T) = \sum_{j=1}^{N} E(d(t_j)) e^{-\alpha t_j - \rho u_n \frac{t_j^2}{2}} \Delta_N^T.$$

Due to (3.10) the mean value $E(d(t))$ is a continuous bounded function. Hence, due to (3.14) for arbitrary $\epsilon > 0$ there is $T_\epsilon > 0$ such that for any $T > T_\epsilon$ passing to a limit in the above equality as $N \to \infty$ we get

$$\Big| E\Big(\int_0^{l_n} d(\tau) e^{-\alpha\tau} d\tau \Big) - \int_0^T E(d(\tau)) e^{-\alpha\tau - \rho u_n \frac{\tau^2}{2}} dt \Big| \leq \epsilon.$$

Passing to a limit as $T \to \infty$ in the last inequality we get (3.13). Proposition 1 is proved.

Due to Proposition 1 for arbitrary $n = 1, 2, \ldots,$ we get the following formula for the mean value $E(J_n)$ (see (3.11) and (3.12)):

$$E(J_n) = \prod_{i=1}^{n-1} \Big(1 - \alpha \int_0^\infty e^{-\alpha\tau - \rho u_i \frac{\tau^2}{2}} d\tau \Big) \cdot \int_0^\infty [E(d(t)) Y_n - (p_1 u_n + p_2 v_n)] e^{-\alpha\tau - \rho u_n \frac{\tau^2}{2}} d\tau. \tag{3.15}$$

Further, on each time interval $[T_{n-1}, T_n]$, $n = 1, 2, \ldots,$ we posit the following constraint on the leader's investments policies[4] $u_n \geq 0$ and $v_n > 0$

$$E(J_n) \geq 0. \tag{3.16}$$

[4]Conditions (3.16) implicitly assumes that the financial source exists outside the firm; $E(J_n) \geq 0$ means that the leader should not run faster its cash flow.

Here $E(J_n)$ is the mean value (expectation) of the aggregated discounted random leader's profit J_n on the n-th time interval $[T_{n-1}, T_n]$, $n = 1, 2, \ldots$. Note that (3.16) and obvious inequality $E(d(t)) \le d_0$ imply inequalities $u_n \le \frac{d_0}{p_1}$ and $v_n \le \frac{d_0}{p_2}$, $n = 1, 2, \ldots$.

Summarizing (3.15) we get the following total expectation of the aggregated discounted leader's profit on the infinite time interval $[0, \infty)$:

$$J = \sum_{n=1}^{\infty} E(J_n) = \sum_{n=1}^{\infty} \int_{T_{n-1}}^{T_n} [E(d(t))Y_n - (p_1 u_n + p_2 v_n)] e^{-\alpha t} dt$$

$$= \prod_{i=1}^{n-1} \Big(1 - \alpha \int_0^{\infty} e^{-\alpha\tau - \rho u_i \frac{\tau^2}{2}} d\tau\Big) \cdot \int_0^{\infty} \Bigg[E\Big(\frac{d_0 \sigma v_n^{\gamma}}{\sigma v_n^{\gamma} + N(t) y_f}\Big) - (p_1 u_n + p_2 v_n) \Bigg] e^{-\alpha\tau - \rho u_n \frac{\tau^2}{2}} d\tau.$$

We assume that the technological leader's goal is to maximize the value of this functional J.

4 Dynamic optimization problem

The technological leader faced the following dynamic optimization problem:

Problem (P):

$$J = \sum_{n=1}^{\infty} E(J_n) = \prod_{i=1}^{n-1} \Big(1 - \alpha \int_0^{\infty} e^{-\alpha\tau - \rho u_i \frac{\tau^2}{2}} d\tau\Big)$$

$$\times \int_0^{\infty} \Bigg[E\Big(\frac{d_0 \sigma v_n^{\gamma}}{\sigma v_n^{\gamma} + N(t) y_f}\Big) - (p_1 u_n + p_2 v_n) \Bigg] e^{-\alpha\tau - \rho u_n \frac{\tau^2}{2}} d\tau \to \max. \tag{4.17}$$

Here quantities $u_n \ge 0$, $v_n > 0$, $n = 1, 2, \ldots$, are control parameters satisfying the constraint

$$E(J_n) \ge 0 \qquad n = 1, 2, \ldots \tag{4.18}$$

(see (3.16)); $d_0 > 0$, $\alpha > 0$, $\beta > 0$, $\sigma > 0$, $\gamma > 0$, $u_f \ge 0$, $y_f = \sigma_f v^{\gamma_f} > 0$; quantities $T_0 = 0$, $T_{n-1} \le T_n$, $n = 1, 2, \ldots$, are random instants of time when technological leader develops n-th generations of the product; random variables $l_n = T_n - T_{n-1} > 0$, $n = 1, 2, \ldots$, are independent and their distributions are given by the formula (see (2.2))

$$p_n(\tau) = \rho u_n \tau e^{-\rho u_n \frac{\tau^2}{2}}.$$

Further, $0 \le N(t) \le N_f$ is a random number of identical followers operating on the market at the instant of time $t \ge 0$; N_f is a total number of followers and the distribution of $N(t)$ is given by the formula (see (2.5))

$$P(N(t) = M) = C_{N_f}^M \Big(1 - e^{-\beta u_f \frac{(t - T_{n-1})^2}{2}}\Big)^M e^{-(N_f - M)\beta u_f \frac{(t - T_{n-1})^2}{2}}.$$

Let us introduce auxiliary functions $\mu(u)$ on $[0, \infty)$ and $\eta(u, v)$ on $[0, \infty) \times [0, \infty)$ as follows:

$$\mu(u) = 1 - \alpha \int_0^{\infty} e^{-\alpha\tau - \rho u \frac{\tau^2}{2}} d\tau \quad \text{for all} \quad u \ge 0;$$

$$\eta(u, v) = \int_0^{\infty} \Bigg[E\Big(\frac{d_0 \sigma v^{\gamma}}{\sigma v^{\gamma} + N(t) y_f}\Big) - (p_1 u + p_2 v) \Bigg] e^{-\alpha\tau - \rho u \frac{\tau^2}{2}} d\tau \quad \text{for all} \quad v > 0, u \ge 0;$$

$$\eta(u, 0) = -p_1 u \int_0^{\infty} e^{-\alpha\tau - \rho u \frac{\tau^2}{2}} d\tau \quad \text{for all} \quad u \ge 0.$$

Obviously the function $\eta(u,v)$ is bounded on $[0,\infty)\times[0,\infty)$ and for all $u\geq 0$ the function $\mu(u)$ satisfies the inequality

$$0\leq \mu(u)<1.$$

Further, due to the constraint $E(J_n)\geq 0$, $n=1,2,\ldots$, (see (4.18)) all quantities u_n and v_n satisfy the inequalities $u_n\leq \frac{d_0}{p_1}$ and $v_n\leq\frac{d_0}{p_2}$ respectively. Hence, as far as

$$\frac{d\mu(u)}{du}=\frac{\alpha\rho}{2}\int_0^\infty e^{-\alpha\tau-\rho u\frac{\tau^2}{2}}\tau^2 d\tau>0 \qquad \text{for all} \quad u\geq 0$$

we have

$$\max_{u\in[0,\frac{d_0}{p_1}]}\mu(u)=\mu\Big(\frac{d_0}{p_1}\Big)=1-\alpha\int_0^\infty e^{-\alpha\tau-\frac{\rho d_0}{p_1}\frac{\tau^2}{2}}d\tau<1.$$

Thus, the row representing functional J see (4.17) converges absolutely and we can group its summands by an arbitrary way:

$$J=\sum_{n=1}^\infty E(J_n)=\sum_{n=1}^\infty\Big(\eta(u_n,v_n)\prod_{i=1}^{n-1}\mu(u_i)\Big)$$

$$=\eta(u_1,v_1)+\eta(u_2,v_2)\mu(u_1)+\eta(u_3,v_3)\mu(u_1)\mu(u_2)+\cdots+\eta(u_n,v_n)\mu(u_1)\cdot\cdots\cdot\mu(u_{n-1})+\ldots$$

$$=\eta(u_1,v_1)+\mu(u_1)\Big(\eta(u_2,v_2)+\mu(u_2)\Big(\eta(u_3,v_3)+\mu(u_3)\Big(\ldots\Big)\Big)\Big).$$

The last equality implies the following representation for the maximal value J_* of the functional J:

$$J_*=\sup_{u_i\geq0,v_i\geq0;i=1,2,\ldots}\Big(\eta(u_1,v_1)+\mu(u_1)\Big(\eta(u_2,v_2)+\mu(u_2)\Big(\eta(u_3,v_3)+\mu(u_3)\Big(\ldots\Big)\Big)\Big)\Big)$$

$$=\sup_{u_1\geq0,v_1\geq0}\Big(\eta(u_1,v_1)+\mu(u_1)\sup_{u_i\geq0,v_i\geq0;i=2,3,\ldots}\Big(\eta(u_2,v_2)+\mu(u_2)\Big(\ldots\Big)\Big)\Big)$$

$$=\sup_{u\geq0,v\geq0}(\eta(u,v)+\mu(u)J_*)=\inf\{r:r\geq\eta(u,v)+\mu(u)r, \quad \text{for all} \quad u\geq0,v\geq0\}$$

$$=\sup\{r:r\geq\frac{\eta(u,v)}{1-\mu(u)}, \quad \text{for all} \quad u\geq0,v\geq0\}=\sup_{u\geq0,v\geq0}\frac{\eta(u,v)}{1-\mu(u)}.$$

As far as functions $\eta(u,v)$ and $\mu(u)$ are continuous, quantities u and v are bounded ($0\leq u\leq\frac{d_0}{p_1}$, $0\leq v\leq\frac{d_0}{p_2}$) and $\mu(u)<1$ for all $u\in\Big[0,\frac{d_0}{p_2}\Big]$, the ratio $\frac{\eta(u,v)}{1-\mu(u)}$ reaches its maximal value. Hence

$$J_*=\max_{u\geq0,v\geq0}\frac{\eta(u,v)}{1-\mu(u)}.$$

It is easy to see that any pair (u_*,v_*) which provides the maximum in the above equality is an optimal time-invariant leaders strategy, i.e. the strategies $u_n=u_*$ and $v_n=v_*$, $n=1,2,\ldots$, give the maximum to the functional J.

Summarizing we get the following result concerning the optimal value J_* of the goal functional J in the dynamic optimization problem (P):

Proposition 2. *An optimal R&D and manufacturing leaders strategy exists. The optimal value J_* of the functional J in problem (P) is represented by the formula*

$$J_* = \max_{u\geq 0, v>0} \frac{\int_0^\infty \left[E\left(\frac{d_0\sigma v^\gamma}{\sigma v^\gamma + N(\tau)y_f}\right) - (p_1 u + p_2 v)\right] e^{-\alpha\tau - \rho u\frac{\tau^2}{2}}\, d\tau}{\alpha \int_0^\infty e^{-\alpha\tau - \rho u\frac{\tau^2}{2}}\, d\tau}.$$

Any pair (u_*, v_*) which provides the maximum in the above equality is an optimal time-invariant leaders strategy.

Consider now the situation when the number N_f of identical followers is large (i.e. $N_f \to \infty$). Passing N_f to infinity we assume that the maximal total followers' production rate $Y_f = N_f y_f = N_f \sigma_f v_f^{\gamma_f}$ is a constant.

Proposition 3. *Assume that there is a constant $Y_f > 0$ such that for all sufficiently large numbers N_f we have $N_f y_f = Y_f$. Then for all $\tau > 0$ the following equality takes place:*

$$\lim_{N_f\to\infty} E\Big(\frac{d_0\sigma v^\gamma}{\sigma v^\gamma + N(\tau)y_f}\Big) = \frac{d_0\sigma v^\gamma}{\sigma v^\gamma + \Big(1 - e^{-\beta u_f \frac{\tau^2}{2}}\Big)Y_f}. \tag{4.19}$$

Proof. Due to the definition of the mean value of the random variable $N(\tau)$ at the instant of time $\tau > 0$ we have

$$\begin{aligned} E\Big(\frac{d_0\sigma v^\gamma}{\sigma v^\gamma + N(\tau)y_f}\Big) &= \sum_{M=0}^{N_f} \frac{d_0\sigma v^\gamma}{\sigma v^\gamma + M y_f} P(N(\tau) = M) \\ &= \sum_{M=0}^{N_f} \frac{d_0\sigma v^\gamma}{\sigma v^\gamma + \frac{M}{N_f} Y_f} P\Big(\frac{N(\tau)}{N_f} = \frac{M}{N_f}\Big). \end{aligned} \tag{4.20}$$

Further, due to the Bernoulli theorem (see Gnedenko, 1962) for arbitrary $\epsilon > 0$ and $\delta > 0$ there is an integer $\bar{N}$ such that for all $N_f \geq \bar{N}$ the following inequality takes place:

$$P\left(\Big|\frac{N(\tau)}{N_f} - P^f(\tau)\Big| \leq \epsilon\right) \geq 1 - \delta. \tag{4.21}$$

Here $P^f(\tau) = 1 - e^{-\beta u_f \frac{\tau^2}{2}} > 0$ (see (2.4)) is the probability of the development of the newest product by a single follower before the time $\tau \geq 0$.

Due to (4.20) we have

$$\begin{aligned} E\Big(\frac{d_0\sigma v^\gamma}{\sigma v^\gamma + N(\tau)y_f}\Big) = &\sum_{M:\left|\frac{M}{N_f} - P^f(\tau)\right| \leq \epsilon} \frac{d_0\sigma v^\gamma}{\sigma v^\gamma + \frac{M}{N_f}Y_f} P\Big(\frac{N(\tau)}{N_f} = \frac{M}{N_f}\Big) \\ &+ \sum_{M:\left|\frac{M}{N_f} - P^f(\tau)\right| > \epsilon} \frac{d_0\sigma v^\gamma}{\sigma v^\gamma + \frac{M}{N_f}Y_f} P\Big(\frac{N(\tau)}{N_f} = \frac{M}{N_f}\Big). \end{aligned}$$

As far as

$$\Bigg| \sum_{M:\left|\frac{M}{N_f}-P^f(\tau)\right|\le\epsilon} \frac{d_0\sigma v^\gamma}{\sigma v^\gamma + \frac{M}{N_f}Y_f} P\Big(\frac{N(\tau)}{N_f} = \frac{M}{N_f}\Big)$$

$$- \sum_{M:\left|\frac{M}{N_f}-P^f(\tau)\right|\le\epsilon} \frac{d_0\sigma v^\gamma}{\sigma v^\gamma + P^f(\tau)Y_f} P\Big(\frac{N(\tau)}{N_f} = \frac{M}{N_f}\Big)\Bigg| \le \epsilon \frac{d_0}{P^f(\tau)}$$

and

$$\sum_{M:\left|\frac{M}{N_f}-P^f(\tau)\right|>\epsilon} \frac{d_0\sigma v^\gamma}{\sigma v^\gamma + \frac{M}{N_f}Y_f} P\Big(\frac{N(\tau)}{N_f} = \frac{M}{N_f}\Big) \le \delta d_0$$

for all sufficiently large $N_f \ge \bar{N}$ we get

$$\Bigg| E\Big(\frac{d_0\sigma v^\gamma}{\sigma v^\gamma + N(\tau)Y_f}\Big) - \frac{d_0\sigma v^\gamma}{\sigma v^\gamma + P^f(\tau)Y_f}\Bigg| \le \epsilon \frac{d_0}{P^f(\tau)} + \delta d_0.$$

Hence, (4.19) holds true. Proposition 3 is proved.

Combing Propositions 2 and 3 we arrive to the following concluding result:

Proposition 4. *An optimal R&D and manufacturing leader's policy in problem* (P) *exists. If there is a constant* Y_f *such that for all sufficiently large* N_f *the equality* $N_f y_f = Y_f$ *takes place then the following asymptotical formula for the maximal value* J_* *of the functional* J *holds true:*

$$\lim_{N_f\to\infty} J_* = \max_{u\ge0, v>0} \frac{\int_0^\infty \left[\frac{d_0\sigma v^\gamma}{\sigma v^\gamma + \left(1-e^{-\beta u_f \frac{\tau^2}{2}}\right)Y_f} - (p_1 u + p_2 v)\right] e^{-\alpha\tau - \rho u \frac{\tau^2}{2}} d\tau}{\alpha \int_0^\infty e^{-\alpha\tau-\rho u\frac{\tau^2}{2}} d\tau}. \tag{4.22}$$

Formula (4.22) gives a tool for analytical and numerical analysis of optimal leader's strategies in the case of large number of identical followers.

5 Numerical simulations and discussion

In this section, using numerical simulations, we analyze some features of the developed model in the case when the technological leader competes with a large number of identical followers. In this situation the asymptotics for the optimal value J_* of the leader's goal functional is given by formula (4.22).

In the reference case (see the table below), the leader invests 10 units of corresponding resources in both R&D and manufacturing. Further, the leader's production rate is equal to 50. Hence, at the initial time $T_0 = 0$, when there are no followers operating on the market yet, the price of the unit of the product is 20 because the market size is assumed to be 1,000. Thus, the leader's profit is 800 at the starting instant of time $T_0 = 0$. This profit is very high in respect to the level of total investment (200). However, at some instant of time $t_* > 0$ ($E(t_*) \simeq 1.542$) the total followers' instantaneous production level will be 300. It is in 6 times larger than the instantaneous leader's production. From the dynamic point of view, these competition conditions are rather severe.

u	Input volume of the leader's investment in R&D	10
v	Input volume of the leader's investment in manufacturing	10
p_1	Price of the unit of the resource invested in R&D	10
p_2	Price of the unit of the resource invested in manufacturing	10
α	Discount rate	0.1
ρ	Leader's efficiency of R&D	0.1
σ	Leader's level of productivity	5
γ	Leader's marginal productivity	1
d_0	Market size	1000
β	Followers' efficiency of R&D	0.1
u_f	Input volume of a typical follower's investment in R&D	3
Y_f	Potential followers' penetration size	1000

Table: Model Parameters in Reference Case.

Figure 2 demonstrates simulation results in the reference case. The vertical axis indicates the mean value of accumulated discounted leader's profit, and two horizontal axes indicate the leader's instantaneous R&D and manufacturing investment levels respectively. The surface represents the shape of the function

$$\chi(u,v) = \frac{\int_0^\infty \left[\frac{d_0\sigma v^\gamma}{\sigma v^\gamma + \left(1-e^{-\beta u_f \frac{\tau^2}{2}}\right)Y_f} - (p_1 u + p_2 v)\right] e^{-\alpha\tau - \rho u\frac{\tau^2}{2}}\, d\tau}{\alpha \int_0^\infty e^{-\alpha\tau-\rho u \frac{\tau^2}{2}}\, d\tau}.$$

Due to Proposition 4 the maximal value of accumulated discounted leader's profit corresponds to the maximum value of function $\chi(u,v)$; and values $u_* \geq 0$ and $v_* > 0$, which maximize $\chi(u,v)$, are the leader's optimal time-invariant R&D and manufacturing investment strategies respectively.

We perform sensitivity analysis of the model in respect to the following parameters:

i) Market size (demand) d_0;

ii) R&D price p_1;

iii) Discount rate α;

iv) Productivity of the leader γ;

v) R&D efficiency of the leader ρ;

vi) Level of a typical follower's investment in R&D u_f (level of competition).

The effects of parameters are easily predictable in some areas and vague in the others. One can expect that higher demand will cause higher levels of the leader's both optimal R&D and manufacturing investment levels, and also a higher value of the accumulated discounted profit. Also, one can expect that higher R&D price will cause lower level of the leader's optimal R&D investments and lower accumulated profit. Higher discount rate means lower present value of future profit. So, it will cause lower optimal R&D investment and lower accumulated profit. It is obvious that higher productivity of the leader yields higher accumulated profit. However, it is vague whether higher productivity causes higher

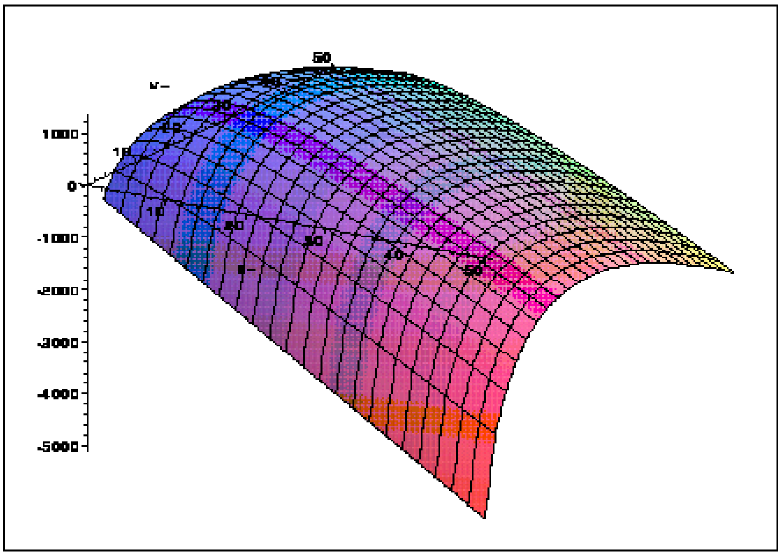

Figure 2: Shape of Function $\chi(u, v)$ in Reference Case. Maple Simulation.

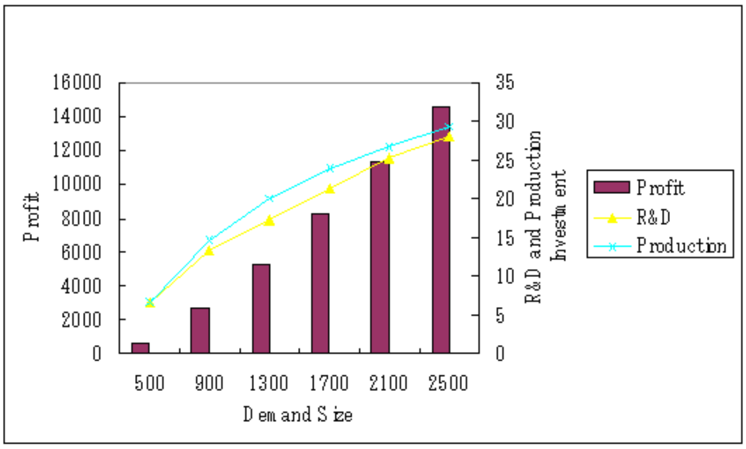

Figure 3: Sensitivity Analysis in Market Size.

optimal R&D and manufacturing investments. The expectation for the effect of the leaders productivity is similar to the expectations for the effects of the leader's R&D efficiency and the level of competition. Higher R&D leader's efficiency yields higher accumulated profit and higher level of competition yields lower accumulated profit. However, effects of both the leader's R&D efficiency and the level of competition to the leader's optimal R&D and manufacturing investments are vague.

1) Sensitivity in the market size.

Figure 3 demonstrates the results of sensitivity analysis in the value of market size. Higher market size, it means relatively strong demand, yields higher leader's accumulated profit, and higher optimal R&D and manufacturing investments. It is basically consistent with the expectations. This analysis shows that higher market size stimulates rapid new products development through the higher R&D investments. It suggests that the product life cycle would become shorter and shorter in the industry where the demand rapidly increases. However, despite that, the demand growth is very fast in the semiconductor industry but the product's life cycle is relatively stable. This difference is due to the fact

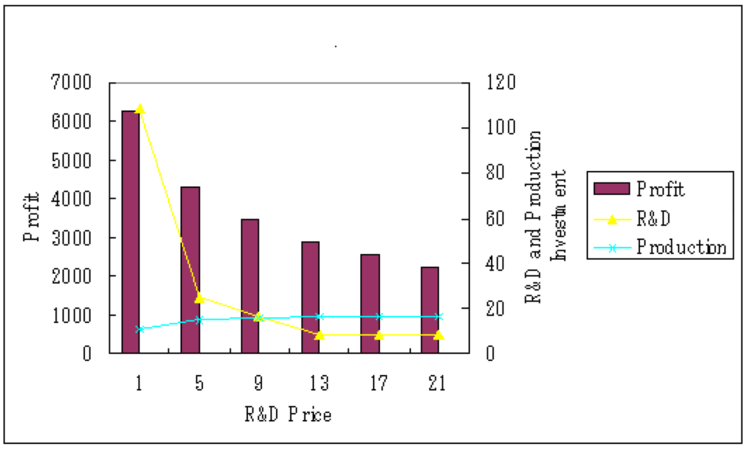

Figure 4: Sensitivity Analysis in R&D Price.

that the leader's decision makings are different in the case of the increasing market and in the case of the stable one. It means that the model should be extended to the case of increasing markets.

2) Sensitivity in R&D price.

Figure 4 demonstrates the result of sensitivity analysis in R&D price. Higher R&D price yields lower accumulated profit of the leader and causes lower optimal R&D investments. It is basically consistent with the expectations. Optimal manufacturing investments become larger as optimal R&D investments become smaller. However, from the quantitative point of view, the increase of optimal manufacturing investments is not equivalent to the decrease of optimal R&D investments. Optimal level of manufacturing investments is not so sensitive to the change of R&D price. This is because the market size and productivity of the leader are fixed in the model. The relatively large sensitivity of the level of optimal R&D investments in R&D price suggests that the decreasing of R&D cost would be very effective political instrument for the fostering new products development.

3) Sensitivity in discount rate.

Figure 5 demonstrates the result of sensitivity analysis in discount rate. Higher discount rate yields lower accumulated profit, and also causes lower optimal R&D investment. It is basically consistent with the expectations. The effect of discount rate to the optimal manufacturing investment level is very small. If the financial market is perfect competitive, then discount rate is equal to the interest rate. In this case, the rise of the interest rate decelerates the product development through the decreasing of the level of R&D investment.

4) Sensitivity in the leader's productivity.

Figure 6 demonstrates the result of sensitivity analysis in the leader's productivity. The result is basically consistent with the expectations. Higher productivity yields higher accumulated profit. Below a certain productivity level (in this case, 0.9 for R&D investments and 1.3 for manufacturing investments), higher productivity causes higher optimal R&D and manufacturing investments. While, above the threshold, higher productivity causes lower optimal R&D and production investments. High productivity brings the strong dominant power to the leader. In such situation, the leader can earn sufficient profit even in the case of relatively low R&D and manufacturing investments. Thus, higher productivity of the leader provides a negative impact on the new products development through the leaders dominant power. It is interesting that this relatively simple model can handle the

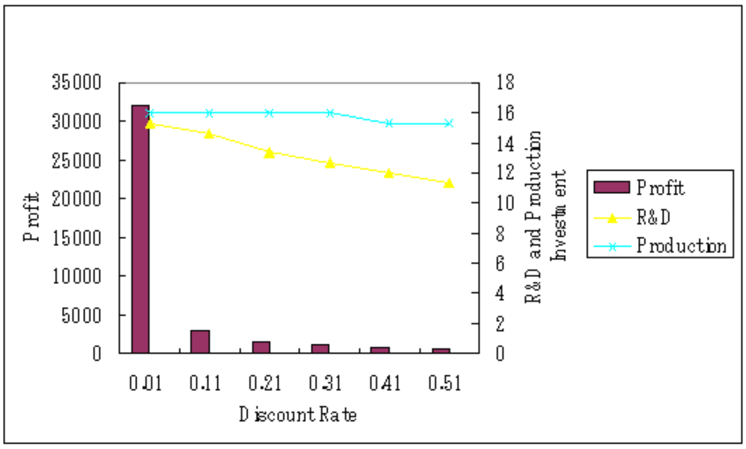

Figure 5: Sensitivity Analysis in Discount Rate.

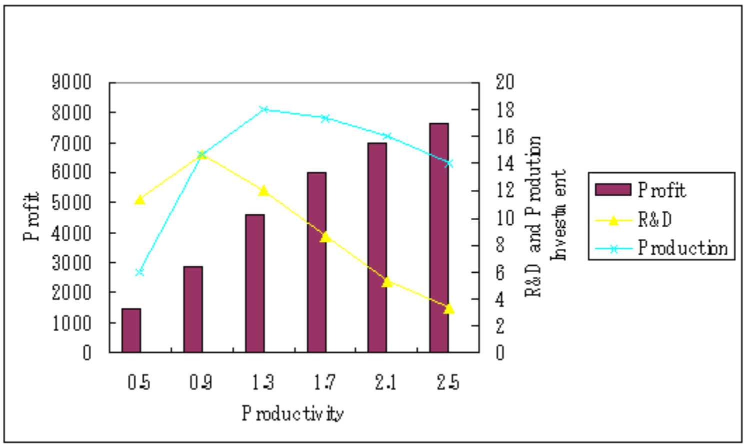

Figure 6: Sensitivity Analysis in Productivity of Leader.

monopolistic situation. Note, that the leader's productivity is assumed to be fixed in this model. The result of this simulations suggests that it should be interesting to analyse how the productivity affects industrial dynamics by learning effects or R&D investments. It produces a new dilemma which the leader should consider: what is the an optimal balance between R&D and manufacturing investment levels?

5) Sensitivity in the leader's R&D efficiency.

Figure 7 demonstrates the result of sensitivity analysis in the leader's R&D efficiency. It is basically consistent with the expectations. Higher leader's R&D efficiency yields higher accumulated profit. Above a certain R&D efficiency level, higher R&D efficiency causes lower optimal R&D and manufacturing investments. While, below the threshold, higher R&D efficiency causes higher optimal R&D and manufacturing investments. It is the similar result to the one of sensitivity analysis in the productivity of the leader. However, the effect of the leaders's productivity to the accumulated profit is larger than the effect of the leader's R&D efficiency. This is because the leader's productivity increases its accumulated profit directly through the decreasing of the production cost, while the R&D efficiency defends only from the penetration of the followers.

6) Sensitivity in the level of competition .

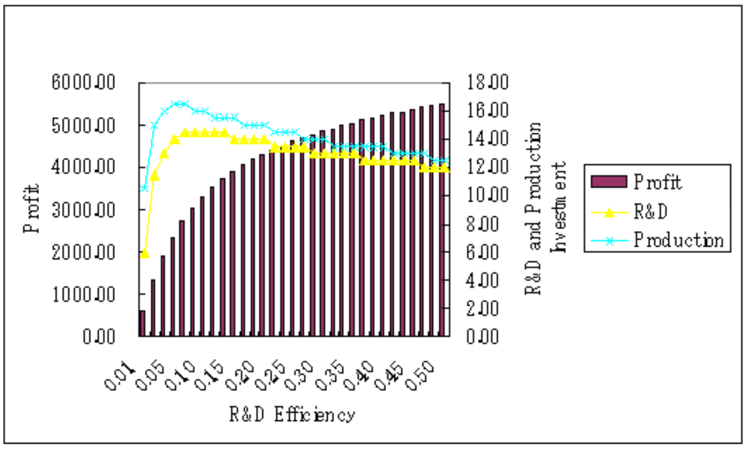

Figure 7: Sensitivity Analysis in R&D Efficiency of Leader.

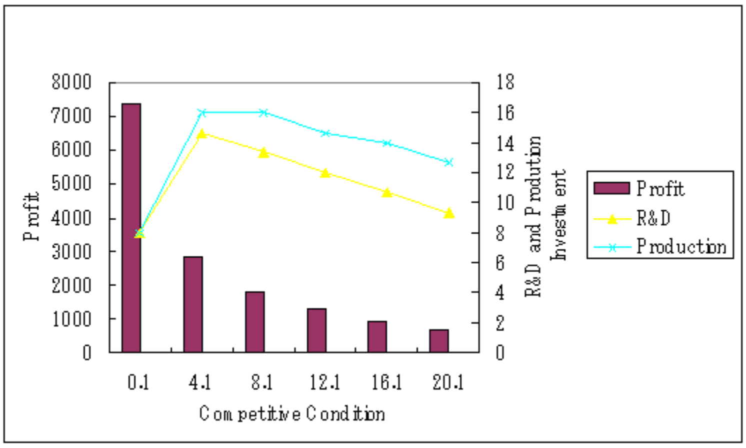

Figure 8: Sensitivity Analysis in Competition Level.

Figure 8 demonstrates the result of sensitivity analysis in the level of competition. It is consistent with the expectation that the harder competition conditions yield less leaders accumulated profit. For optimal R&D and manufacturing investment policies, we see the concave curves for both optimal levels of R&D and manufacturing investments respectively. Below a certain level of competition (in this case, 4.1), higher competition, which means higher penetration rate of the followers, causes higher optimal R&D and manufacturing investments. While, above the threshold, higher competition causes lower levels of both optimal R&D and manufacturing investments. This result means that it is very important for the leader to know what kind of investment strategies the followers take. And of course, it is also very important for the followers to know what kind of investment strategy the leader takes when they decide their strategies. At this moment, the model does not take not into account the followers decision-making process. So, it should be exptended in this direction.

6 Concluding remarks

Innovation race is one of the most important forms of the technological dynamics. However, due to its complicated nature, this process is not still deeply understood.

In present paper we consider the innovation race from the point of view of a technological leader. We develop a dynamic model of optimal investment in R&D and manufacturing. The developed model is based on natural assumptions concerning probabilistic features of innovation processes. Nevertheless, our main result (Proposition 4) is completely deterministic. It provides a useful tool for analytic analysis and numerical simulations.

In the case when the technological leader competes with a large number of identical followers, the following effects of the model parameters are obtained due to the numerical simulations:

i) Higher demand yields higher accumulated profit of the leader and higher optimal R&D and manufacturing investments;

ii) Higher R&D price yields lower accumulated profit of the leader and lower optimal R&D investments. It causes relatively stable optimal manufacturing investments;

iii) Higher discount rate yields lower accumulated profit and causes lower optimal R&D investments level. It has a little effect to the manufacturing investments level;

iv) Higher productivity of the leader yields higher accumulated profit. Below a certain productivity level, higher productivity causes higher optimal R&D and manufacturing investments. While, above the threshold, higher productivity causes lower optimal R&D and manufacturing investments;

v) Higher R&D efficiency of the leader yields higher accumulated profit. Above a certain R&D efficiency level, higher R&D efficiency causes lower optimal R&D and manufacturing investments. While, below the threshold, higher R&D efficiency causes higher optimal R&D and manufacturing investments;

vi) Higher level of competition yields smaller leader's accumulated profit. Below a certain competition level, higher competition, which means higher penetration rate of the followers, causes higher optimal R&D and manufacturing investments. While, above the threshold, higher competition causes lower optimal R&D and manufacturing investments.

For decision making analysis, the concave character of dependances of the optimal levels of both leader's R&D and manufacturing investments from the competition level is especially interesting. If the same character of dependance of optimal follower's R&D and manufacturing investment startegies from the leader's R&D investment policy will appear also in a complementing model, treating the innovation race from the follower's point of view, then multiple equilibria can happen. If so, we can expect an explanation of the difference of average firms' investment levels between countries in the same industry along with the comparative institutional analysis.

We believe that the developed approach could be extended to the following situations:

i) Increasing demand size;

ii) Innovation processes with learning effects;

iii) Follower's decision making case;

iv) Leader-leader and leader-challenger cases.

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