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Interim Report **IR-04-031**

On the Stability of Families of Dynamical Systems

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August 2004

Contents

Abstract

The paper proves existence theorems for the common Lyapunov function of a family of asymptotically stable dynamical systems. The theorems generalize and develop the results announced in [1].

Key words: ordinary differential equations, control theory, Lyapunov function, stability
Mathematics Subject Classification (2000): 93D05

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1 Statement of the Problem

In some branches of the control theory, the problem of constructing a common Lyapunov function for a family of dynamical systems turns up. Such a problem occurs in studies of the stability of systems with variable structures, in absolute stability problems, in a number of problems of robust stability, in some problems of fractal compression of information, and so on [2–9].

This paper describes conditions that guarantee the existence of a common Lyapunov function for a family of asymptotically stable dynamical systems governed by autonomous differential equations.

Consider a set of dynamical systems governed by the differential equations:

$$dx/dt = f(x;\mu) \qquad \left(x \in B \subset E^N, \quad \mu \in M\right). \tag{1}$$

Here $x = \{x_1, \dots, x_N\}$ is a point of the N-dimensional Euclidean space E^N,

$$B = \left\{x \in E^N : \ \|x\| \le 1\right\}$$

is the unit ball in E^N, μ is a parameter lying in the set M, and

$$f(x;\mu) = \Big\{f_1\left(x_1, \dots, x_N;\mu\right), \dots, f_N\left(x_1, \dots, x_N;\mu\right)\Big\}$$

is a vector function assumed to be continuous with respect to $x \in B$ for each fixed value of $\mu \in M$.

Suppose that, for each $\mu \in M$, the origin $x = 0$ is a point of equilibrium for system (1), and for each $\mu \in M$, system (1) has a stationary Lyapunov function $V(x;\mu)$ ($x \in B$, $\mu \in M$) defined on B, i.e., a function continuously differentiable with respect to x and such that $V(0;\mu) \equiv 0$ ($\mu \in M$); the point $x = 0$ is a local minimum of the function $V(\cdot\,;\mu)$, and

$$(\nabla V(x;\mu), f(x;\mu)) < 0 \qquad (x \in U, \quad x \ne 0, \quad \mu \in M) \tag{2}$$

in some neighborhood U of the origin (here ∇ denotes the gradient operator with respect to the variable $x\colon\ \nabla = \{\partial/\partial x_1, \dots, \partial/\partial x_N\}$). In that follows, we assume for simplicity that $B \subset U$, i.e.,

$$(\nabla V(x;\mu), f(x;\mu)) < 0 \qquad (x \in B, \quad x \ne 0, \quad \mu \in M), \tag{3}$$

*The work was supported by the Russian Foundation for Basic Research (projects No. 00-01-00641 and Grant of Scientific School NSH-1986.2003.1), the program *Universities of Russia* and *Integration*, and the program *Intellectual Computer Technologies* sponsored by the Presidium of Russian Academy of Sciences.

and $x = 0$ is a point of strict minimum of the function $V(\cdot\,;\mu)$ on B. Therefore, for each $\mu \in M$, there exist continuous functions $\alpha(\cdot\,;\mu), \beta(\cdot\,;\mu) : [0,1] \to \mathbf{R}$ such that $\alpha(0;\mu) = \beta(0;\mu) = 0$, $\alpha(s;\mu) > 0$, $\beta(s;\mu) > 0$ if $0 < s \le 1$, and

$$(\nabla V(x;\mu), f(x;\mu)) \le -\alpha(\|x\|;\mu) \qquad (x \in B, \quad \mu \in M), \tag{4}$$

$$V(x;\mu) \ge \beta(\|x\|;\mu) \qquad (x \in B, \quad \mu \in M). \tag{5}$$

Suppose that there exist continuous functions $\alpha(\cdot), \beta(\cdot) : [0,1] \to \mathbf{R}$ and a constant $C > 0$ such that $\alpha(0) = \beta(0) = 0$ and

$$\alpha(s,\mu) \ge \alpha(s) > 0, \qquad \beta(s,\mu) \ge \beta(s) > 0 \qquad (0 < s \le 1, \quad \mu \in M), \tag{6}$$

$$|\nabla V(x;\mu)| \le C \qquad (x \in B, \quad \mu \in M). \tag{7}$$

By $\mathbf{V}(\mu)$ $(\mu \in M)$ we denote the class of continuous functions $v : B \to \mathbf{R}$, which satisfy the conditions

$$v(0) = 0, \tag{8}$$

$$v(x) \ge \beta(\|x\|) \qquad (x \in B), \tag{9}$$

$$|v(x_1) - v(x_2)| \le C\,\|x_1 - x_2\| \qquad (x_1, x_2 \in B)\,, \tag{10}$$

do not decrease on trajectories of system (1), and whose level surfaces lying in B do not contain entire trajectories of system (1). The class $\mathbf{V}(\mu)$ consists of Lyapunov functions of system with index μ (1) with the Lipschitz property.

It follows from (4)–(10) that $V(\cdot\,;\mu) \in \mathbf{V}(\mu)$ for each $\mu \in M$. For each $\mu \in M$, the class $\mathbf{V}(\mu)$ is a nonempty closed set in the space $C(B)$ of functions $v : B \to \mathbf{R}$ continuous on B. By (10), the class $\mathbf{V}(\mu)$ is a compact set in $C(B)$.

Since elements of the class $\mathbf{V}(\mu)$ are Lyapunov functions of system (1), it follows from the condition

$$\bigcap_{\mu \in M} \mathbf{V}(\mu) \ne \varnothing \tag{11}$$

that the set (1) has a common Lyapunov function.

The forthcoming sections of the paper are devoted to conditions that the meet (11) be nonempty.

2 Main Theorem

Theorem 1 *Suppose that for an arbitrary set of finite parameters $\mu_1, \dots, \mu_k \in M$ and for arbitrary numbers $\lambda_1, \dots, \lambda_k \ge 0$ such that $\lambda_1 + \dots + \lambda_k = 1$, there exists at least one number $\mu_j \in \{\mu_1, \dots, \mu_k\}$ such that*

$$\left(\sum_{i=1}^{k} \lambda_i \nabla V(x;\mu_i)\,, f(x;\mu_j) \right) \le -\alpha(\|x\|) \qquad (x \in B). \tag{12}$$

Then the set of dynamical systems (1) has a common Lyapunov function $V_*(\cdot) : B \to \mathbf{R}$ satisfying the Lipschitz condition and the inequalities

$$|V_*(x)| \ge \beta(\|x\|) \qquad (x \in B), \tag{13}$$

$$|V_*(x_1) - V_*(x_2)| \le C\,\|x_1 - x_2\| \qquad (x_1, x_2 \in B)\,. \tag{14}$$

Proof. Our proof of Theorem 1 is performed by generalizing the well-known Knaster–Kuratowski–Mazurkiewicz lemma [10]. In view of this, let us recall some definitions.

A topological space X is said to be regular if closed neighborhoods of each point $x \in X$ form a basis of neighborhoods for that point.

A topological space X is said to be paracompact if it is regular, and in each of its open coverings, one can inscribe a locally finite open covering, i.e., a covering $\{A_i\}$ such that each point $x \in X$ has a neighborhood U that meets with only a finite number of sets A_i (in general, the number of sets depends on x and U).

Lemma 1 (the generalized Knaster–Kuratowski–Mazurkiewicz lemma). *Let* $K(\alpha)$ $(\alpha \in A)$ *be a set of nonempty compact sets in a paracompact linear topological space* X. *Suppose that there exists a point system* $x_\alpha \in K(\alpha)$ $(\alpha \in A)$ *with the following property*: *for each finite set of parameters* $\alpha_1, \dots, \alpha_k \in A$, *the convex hull* $\operatorname{conv}\{x_{\alpha_1}, \dots, x_{\alpha_k}\}$ *belongs to the join of compact sets* $K(\alpha_1), \dots, K(\alpha_k)$, *i.e.*,

$$\operatorname{conv}\{x_{\alpha_1}, \dots, x_{\alpha_k}\} \subset \bigcup_{i=1}^{k} K(\alpha_i). \tag{15}$$

Then

$$\bigcap_{\alpha \in A} K(\alpha) \neq \varnothing. \tag{16}$$

Proof. Since $K(\alpha)$ $(\alpha \in A)$ are compact sets, it suffices to show that the meet of an arbitrary number of sets of the class $K(\alpha)$ $(\alpha \in A)$ is nonempty. Suppose the contrary. Then

$$\bigcap_{i=1}^{k} K(\alpha_i) = \varnothing \tag{17}$$

for some set $\alpha_1, \dots, \alpha_k$. We set $A_i = X \backslash K(\alpha_i)$ $(i = 1, \dots, k)$. By (17),

$$\bigcup_{i=1}^{k} A_i = X. \tag{18}$$

Since X is paracompact (e.g., see [11]), it follows that the covering (18) corresponds to the partition of the unity coordinated with this covering, i.e., there exist nonnegative continuous functions $\rho_1(x), \dots, \rho_k(x)$ such that

$$\sum_{i=1}^{k} \rho_i(x) \equiv 1, \qquad \operatorname{supp}\{\rho_i(\cdot)\} \subset A_i.$$

Consider the mapping $\phi(x) = \sum_{i=1}^{k} \rho_i(x) x_{\alpha_i}$. It is continuous and maps the convex hull $\operatorname{conv}\big\{x_{\alpha_1}, \dots, x_{\alpha_k}\big\}$ into itself. By the Brauer theorem, it has an immobile point x_*. Let $\rho_i(x_*) > 0$ for i from some index subset $I \subset \{1, \dots, k\}$, and $\rho_i(x_*) = 0$ for $i \notin I$. But then

$$x_* = \sum_{i \in I} \rho_i(x_*) x_{\alpha_i} \in \operatorname{conv}\{x_{\alpha_i} : \ i \in I\} \subset \bigcup_{i \in I} K(\alpha_i).$$

Consequently, by (15), $x_* \in K(\alpha_j)$ for some $j \in I$, i.e., $x_* \notin A_j$, whence $\rho_j(x_*) = 0$. We have a contradiction, whence follows Eq. (16). The proof of the lemma is complete.

Let us proceed to the proof of Theorem 1. In the space $C(B)$, we consider a family of compact sets $\mathbf{V}(\mu)$ $(\mu \in M)$. Let us show that this family satisfies all conditions of Lemma 1. By the Stone theorem [11], $C(B)$ is a paracompact space. Consider an arbitrary set of parameters $\mu_1, \dots, \mu_k \in M$ and numbers $\lambda_1 > 0, \dots, \lambda_k > 0$, $\sum_{i=1}^{k} \lambda_i = 1$.

We set

$$v(x) = \sum_{i=1}^{k} \lambda_i V\left(x; \mu_i\right). \tag{19}$$

Then

$$v(x) \geq \sum_{i=1}^{k} \lambda_i \beta(\|x\|) = \beta(\|x\|) \tag{20}$$

and

$$\left|v\left(x_1\right) - v\left(x_2\right)\right| = \left|\sum_{i=1}^{k} \lambda_i \left(V\left(x_1; \mu_i\right) - V\left(x_2; \mu_i\right)\right)\right| \leq C \left\|x_1 - x_2\right\|. \tag{21}$$

It follows from (20), (21), and (12) that the function $v(\cdot)$ given by (19) belongs to the class $\mathbf{V}\left(\mu_j\right)$ for a positive integer $j \in \{1, \ldots, k\}$, i.e.,

$$\operatorname{conv}\left\{V\left(\cdot\,, \mu_1\right), \ldots, V\left(\cdot\,, \mu_k\right)\right\} \subset \bigcup_{i=1}^{k} \mathbf{V}\left(\mu_i\right).$$

By Lemma 1, condition (11) is satisfied. Therefore, any function $V_*(\cdot) \in \bigcap_{\mu \in M} \mathbf{V}(\mu)$ is a common Lyapunov function for the family of dynamic systems (1) and satisfies conditions (13) and (14). The proof of the theorem is complete.

3 Lyapunov Functions in Parametric Classes

In applications, Lyapunov functions for specific dynamic systems are often constructed so that the resulting functions should belong to definite classes (for example, classes of quadratic or semilinear forms, linear combinations of spherical harmonics, etc.). This section contains assertions concerning the existence of a common Lyapunov function for finite families of dynamic systems lying in a parametric class.

Consider n dynamic systems governed by the equations

$$dx/dt = f_i(x) \qquad \left(x \in B \subset \mathbf{R}^N, \quad i = 1, \ldots, n\right). \tag{22}$$

As in the previous sections, $f_i = \left(f_i^1(x), \ldots, f_i^N(x)\right)$ $(i = 1, \ldots, n)$ is a jointly continuous vector function defined on B. Let each of systems (22) have an asymptotically stable zero equilibrium and a Lyapunov function $V_i : B \to \mathbf{R}$.

Suppose that the Lyapunov functions V_i $(i = 1, \ldots, n)$ of systems (22) lie in a parametric class $\mathbf{V}(c)$ $[c = (c_1, \ldots, c_m)]$ whose elements are Lipschitz functions $V : B \to \mathbf{R}$ such that $V(0) = 0$, and $x = 0$ is an isolated critical point of the function V corresponding to its local minimum. The set $C \subset \mathbf{R}^m$ of values of the parameter c, which defines the class $\mathbf{V}(c)$, is assumed to be convex. We are interested in conditions under which, in the class $\mathbf{V}(c)$, there exists a common Lyapunov function for all systems (22).

By $\mathbf{V}_i$ $(i = 1, \ldots, n)$ we denote the subclass of the parametric class $\mathbf{V}(c)$ $(c \in C \subset \mathbf{R}^m)$ consisting of Lyapunov functions of the ith system of the set (22). The set of values of the parameter $c \in C$ corresponding to functions from $\mathbf{V}_i$ is denoted by C_i. Since we assume that $V_i \in \mathbf{V}_i$, each set C_i $(i = 1, \ldots, n)$ is nonempty. Suppose that C_i $(i = 1, \ldots, n)$ are open sets. Points of these sets corresponding to functions V_i are denoted by c_i $(i = 1, \ldots, n)$.

Theorem 2 *Suppose that, for each parameter subset $I \subset \{1, \ldots, n\}$ and all $\lambda_i \geq 0$ $(i \in I)$ such that*

$$\sum_{i \in I} \lambda_i = 1, \tag{23}$$

the function

$$V\Bigl(\sum_{i\in I}\lambda_i c_i\Bigr) \tag{24}$$

is a Lyapunov function for at least one system of the subset

$$dx/dt = f_i(x) \qquad (i\in I). \tag{25}$$

Then in the parametric class $\mathbf{V}(c)$, *there exists a common Lyapunov function for all systems of the family* (22).

Proof. Theorem 2 is proved following the scheme of the proof of Theorem 1 and is based on the following modification of the Knaster–Kuratowski–Mazurkiewicz lemma.

Lemma 2 *Let* M *be an arbitrary finite set in* $\mathbf{R}^m$, *and* $F : M \to \mathbf{R}^m$ *be a multimapping with bounded open images. Let the convex hull* $\operatorname{conv}\{x_1,\dots,x_k\}$ *lie in the join* $\bigcup_{i=1}^k F(x_i)$ *for any finite set of points* $x_1,\dots,x_k \in M$. *Then* $\bigcap_{x\in M} F(x) \neq \varnothing$.

To prove Theorem 2, we have to set $M = \{c_1,\dots,c_n\}$, $F(c_i) = \mathbf{V}_i$ and use Lemma 2.

In applications, the subclasses $\mathbf{V}_i$ of the parametric class $\mathbf{V}(c)$ $(c\in C\subset \mathbf{R}^m)$ correspond, as a rule, to convex sets C_i in the parametric space $\mathbf{R}^m$. In this case, Theorem 2 can be refined in the following version.

Theorem 3 *Let* $n > m$, *and, for each* $i = 1,\dots,m$, *the subclass* $\mathbf{V}_i$ *correspond to a convex open set* C_i *in the parametric space* $\mathbf{R}^m$. *Suppose that, for each index subset* $I\subset\{1,\dots,n\}$ *such that* $\operatorname{card} I \le m+1$ *and for any* $\lambda_i \ge 0$ $(i\in I)$ *that satisfies condition* (23), *the function* (24) *is a Lyapunov function of at least one of the systems from subset* (25). *Then, in the parametric class* $\mathbf{V}(c)$, *there exists a common Lyapunov function for all systems of family* (22).

The proof of Theorem 3 is based on Theorem 2 and Helly's classical theorem about meets of convex sets (e.g., see [12]).

4 The Lyapunov Function for Families of Linear Systems

Consider a family of n linear stationary asymptotically stable systems governed by the equations

$$dx/dt = A_i x \qquad \bigl(x\in\mathbf{R}^N,\quad i = 1,\dots,n\bigr). \tag{26}$$

Here $A_i = \Bigl[a^i_{j,k}\Bigr]_{j,k=1}^N$ are constant matrices with real entries a^i_{jk} $(i = 1,\dots,n;\ j,k = 1,\dots,N)$. For each system (26), there exists a Lyapunov function V_i that is a quadratic form

$$V_i(x) = (1/2)\,(H_i x, x) \qquad (i = 1,\dots,n).$$

Here $H_i = \Bigl[h^i_{j,k}\Bigr]_{j,k=1}^N$ is a positive definite symmetrical matrix with real entries $h^i_{j,k}$ $(i = 1,\dots,n;\ j,k = 1,\dots,N)$. As was done previously, we determine whether there exists a common quadratic Lyapunov function for linear systems of family (26). Obviously, this problem is equivalent to the solvability of the system of matrix inequalities $HA_i + A_i^* H < 0$ $(i = 1,\dots,n)$ in the class M_N^+ of symmetric positive definite $N\times N$ matrices H. For each $i = 1,\dots,n$, the set

$$\mathcal{H}_i = \bigl\{H\in M_N^+ :\ HA_i + A_i^* H < 0\bigr\}$$

is a nonempty open convex cone in the space M_N^+ of symmetric $N\times N$ matrices. From Theorem 3 derives the following assertion.

Theorem 4 *Let $n > N(N+1)/2$. Suppose that, for each index subset $I \subset \{1, \dots, n\}$ such that* $\operatorname{card} I \le N(N+1)/2$ *and for all $\lambda_i \ge 0$ ($i \in I$, $\sum_{i\in I} \lambda_i = 1$), the quadratic form*

$$V(x) = (1/2) \sum_{i\in I} \lambda_i \, (H_i x, x)$$

is a Lyapunov function of at least one of the systems

$$dx/dt = A_i x \qquad (i \in I).$$

Then for all systems of the set (26)*, there exists a common quadratic Lyapunov function.*

If we have additional information about spectra of the matrices $H_i A_j + A_j^* H_i$ $(i, j = 1, \dots, n)$, the existence conditions for a common Lyapunov function for the set of linear systems (26) can be obtained in a constructive form.

By μ_{ij} $(i, j = 1, \dots, n)$ we denote the maximal eigenvalue of the symmetric matrix $H_i A_j + A_j^* H_i$ $(i, j = 1, \dots, n)$. The following assertion is valid.

Theorem 5 *Let*

$$\min{}_{\lambda_1+\dots+\lambda_n=1} \max{}_{1\le j\le n} \sum_{i=1}^{n} \lambda_i \mu_{ij} < 0 \tag{27}$$

for nonnegative numbers $\lambda_1, \dots, \lambda_n$. Then for the family of systems (26)*, there exists a common quadratic Lyapunov function.*

Proof. We construct a common quadratic Lyapunov function V for the family of linear systems (26) in the form of a convex combination of the quadratic forms $(1/2)\,(H_i x, x)$, i.e.,

$$V(x) = \frac{1}{2} \sum_{i=1}^{n} \lambda_i \, (H_i x, x) \qquad \Big(\lambda_i \ge 0, \quad \sum_{i=1}^{n} \lambda_i = 1\Big). \tag{28}$$

Since H_i are positive definite matrices, the matrix $H_* = \sum_{i=1}^{n} \lambda_i H_i$ is also positive definite for any set $\lambda_1, \dots, \lambda_n \ge 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$.

The quadratic form (28) is a Lyapunov function of the family of systems (26) if and only if

$$(\nabla V(x), A_j x) = \sum_{i=1}^{n} \lambda_i \, (H_i x, A_j x) = \frac{1}{2} \sum_{i=1}^{n} \lambda_i \left(\left(A_j^* H_i + H_i A_j \right) x, x \right) < 0 \quad (x \ne 0) \tag{29}$$

for all $j = 1, \dots, n$. Since

$$\left(\left(A_j^* H_i + H_i A_j \right) x, x \right) < \mu_{ij} (x, x), \tag{30}$$

it follows from (29) that

$$(\nabla V(x), A_j x) \le \left(\frac{1}{2} \sum_{i=1}^{n} \lambda_i \mu_{ij} \right) (x, x). \tag{31}$$

It follows from inequality (27) that for some $\lambda_1 = \lambda_1^*, \dots, \lambda_n = \lambda_n^*$ and for all $j = 1, \dots, n$,

$$\sum_{i=1}^{n} \lambda_i^* \mu_{ij} < 0. \tag{32}$$

Then from (29)–(32), we derive the conclusion that the quadratic form

$$V(x) = \frac{1}{2}\sum_{i=1}^{n} \lambda_i^* \, (H_i x, x)$$

is a common Lyapunov function for the set of systems (26). The proof of the theorem is complete.

Consider the matrix

$$\mathbf{M} = [\mu_{ij}]_{i,j=1}^{n} \, . \tag{33}$$

Since each system (26) is asymptotically stable, we find that the diagonal entries of the matrix $\mathbf{M}$ are negative. Theorem 5 leads to the following assertion.

Corollary. If matrix (33) is a matrix with a dominating leading diagonal, then the family of linear systems (26) has a common quadratic Lyapunov function.

The theorems on the existence of a common Lyapunov function for sets of stable dynamic systems given in this paper are of qualitative nature. Even so, one can easily construct numerical algorithms for calculating common Lyapunov functions in broad parametric classes [13].

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