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AN INCENTIVE-TAX MODEL FOR OPTIMIZATION OF AN INSPECTION PLAN FOR NUCLEAR MATERIALS SAFEGUARDS

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I. Introduction

The goal of nuclear materials safeguards is to guard against diversion of the nuclear materials which in themselves must be used for peaceful purposes. In nuclear fuel processing facilities the materials are controlled by accounting systems of various measurements, but the systems are always accompanied by some measurement errors or losses, and further, by some other normal losses inherent in the operation of each facility. All these losses are the factors yielding what we call the Material Unaccounted For, MUF. In principle the losses may be reduced by improving the accounting system and the operational manner. From both the technical and the economic standpoint, it is almost impossible to reduce the MUF to nothing.

As far as nuclear fuel processing facilities are concerned, the purpose of safeguards inspection consists mainly in discriminating between the MUF due to diversion and the MUF due to normal losses as mentioned above. It is impossible to perfectly discriminate between the two in so far as the generation of normal MUF is inevitable. Therefore, for an inspection authority, it is of great importance to determine the detection limit concerning the MUF due to diversion; for each facility, it is an important problem to set the allowable limit of technical and economic efforts for reducing the normal MUF. It is to be noted here that both of these limits should be fixed, not independently, but interdependently.

Though both the limits are determined from reason, it is still necessary for an inspection authority to draw up an inspection plan to verify the normality of MUF within the limits. It is intuitive to say the more rigorous the limits, the stricter and more frequent the inspections must be. Hence, in determining the limits, the feasibility of the corresponding inspection plan must be taken into account.

The aim of this paper is to construct the mathematical model which describes the comprehensive relationship between the limits mentioned above and the corresponding inspection effort. Using the model it is possible to optimize inspection effort in accordance with these limits.

MUF is defined by the difference between the book inventory (BI), the amount of material which is supposed to be present in the inventory, and the physical inventory (PI), the amount of material which is estimated to be in the inventory by direct inventory measurement techniques. Figure 1 is a schematic illustration of the material balance area where MUF for a campaign interval T is described by the relation:

$$MUF = BI - PI \quad , \tag{1}$$

where

$$BI = \sum_{i=1}^{I} Z_i \quad ; \qquad Z_i = A(t_i) - B(t_i) \quad , \tag{2}$$

and

$$PI = \sum_{j=1}^{J} (w_j(O) - w_j(T)) \quad . \tag{3}$$

One question is how an inspection authority should verify the normality of the MUF of Eq. (1) reported from each facility. It follows from Eq. (1) that this question should be divided into two questions: one for verification of BI and the other for verification of PI. In either of the two, however, it requires a great deal of inspection effort to verify all of the individual data Z_i, or $w_j(O)$ and $w_j(T)$. Thus it is natural that we should consider the sampling plan of drawing samples from the population of $(Z_1,\dots,Z_I)$ or the populations of $(w_1(O),\dots,w_J(O))$ and $(w_1(T),\dots,w_J(T))$ by making use of the statistical behavior of these data.

A number of works have been written concerning this kind of sampling plan, and reference [8] gives us the critical reviews of some of them. The direct sampling method, based on the paired comparison test, was developed by Gmelin [5]. The feasibility of the mixed variable/attribute sampling plan for the safeguarding of the Pu-fuel store of the Zebra zero-energy reactor was shown by Brown et al. [3]. Stewart [16] proposed the cost/effectiveness model where the variance on MUF was minimized subject to a cost constraint, and the sample size which gave adequate protection was determined.

Avenhaus et al. [1] treated the problem of optimizing inspection plans by a game theoretical method, selecting the probability of detection as a criterion for mini-max optimization. Jaech [9] constructed the statistical model for inventory verification on measured data showing the numerical examples for fuel fabrication facilities. And Constanzi et al. [4] made a model to optimize the overall inspection costs subject to a given constraint on the vulnerability index, an index of the relative likelihood of attempted diversion assigned to the various forms of nuclear materials.

In addition to these works, some further papers have been reported. Among them, Bouchey, Koen and Beightler [2] improved Stewart's model by using the dynamic programming algorithm; Servais and Goldschmidt [15] represented another stochastic model which enabled us to quantitatively assess the efficiency of detecting diversion.

Almost all these models were based upon the Neyman-Pearson theory of testing hypotheses and upon the supposition of diversion strategies. According to the theory of statistical testing, a choice is made between acceptance and rejection of the null hypothesis H_n against the alternate hypothesis H_a. It is well known that there are two kinds of error: an error of type I, rejection of H_n when it is actually true, and an error of type II, acceptance of H_n when it is actually false. It is the usual way to determine a sampling plan that first the maximum tolerable probability α of an error of type I is fixed customarily as .1 or .05, and then the probability β of an error of type II is minimized subject to the constraints on costs for the sampling plan.

In the models briefly reviewed above, the null hypothesis,

H_n: there is no degree of diversion in the MUF, is tested against the alternate hypothesis,

H_a: there is some degree of diversion in the MUF.

Generally speaking the probability β of an error of type II depends upon not only the accuracy of sampling plan, but also the value of the statistic in question which is selected in H_a. Therefore in the problem of statistical testing for nuclear materials safeguards, β is dependent on the degree of diversion in H_a as well as on the accuracy of inspection procedures [7]. From this consideration there are two different methods to determine a sampling plan for inspection. One is the method of minimizing β or maximizing the accuracy (the inverse of the variance) for a specified degree of diversion, and the other is the method of minimizing the degree of diversion for a fixed β.

It is to be noted here that the following two inevitable difficulties are involved in the orthodox approach stated above:

(1) It is difficult to explain the reasonability for selecting the value of α which is strongly related to the limit of detecting the MUF due to diversion.

(2) It is difficult to take into account all the possibilities of diversion strategies so as to evaluate the degree of diversion in H_a.

As mentioned earlier the determination of the detection limit is a critical problem to be resolved between an inspection authority and each facility. Hence it may be a crucial assumption to fix the value of α not reasonably but customarily. The idea of deciding an inspection plan by selecting the degree of diversion in H_a is a straightforward approach as the goal of safeguards consists in the prevention of diversion. However, it is impossible to comprehend all the possibilities of diversion strategies. Even if possible, it is difficult to formulate the possibility as a probability because the possibility of diversion is a hypothetical danger and the frequency distribution of diversion is not given.

The first difficulty is solved by using Bayesian decision theory [14]. The theory requires not the preassignment of the maximum tolerable probability α, but its derivation from the consideration of the risk and benefit associated with decision making under uncertainty. The uncertainty is due to the fact that it is impossible for a decision maker to know the true value of a basic variable for making a decision. The basic variable of the problem of a sampling plan for inspection is the variable to be verified by an inspection authority, the true value of which is unknown. Therefore some uncertainty is involved in the problem treated here, and uncertainty is always associated with risk and benefit in decision making. It is obvious that such risk and benefit is closely related to the detection limit concerning the MUF due to diversion, and the allowable limit of technical and economic efforts for reducing the normal MUF. This implies that Bayesian decision theory is useful for formulating the problem.

The second difficulty makes it necessary to introduce a measure to describe the hypotheticality of diversion [6]. In this study a kind of incentive-tax system is supposed for the purpose of evaluating such a measure. Here too it is apparent that MUF is to be as low as possible and yet it is also apparent that it is impossible to reduce MUF to nothing. In the case of a non-nuclear materials accounting system, the question whether any amount of MUF should be traced or not is settled by comparing the cost required for tracing it with the cost required for re-purchasing the same amount. In other words MUF should not be traced if the tracing cost is more than the re-purchasing cost; otherwise MUF should be traced. The amount of MUF which makes both costs equal is a threshold value in this case.

In the case of nuclear materials accounting systems, however, the threshold amount of MUF is to be fixed not from such an economic consideration but from the standpoint of nuclear materials safeguards. That is to say the threshold amount of MUF is defined as the minimum value of the amount required for producing a nuclear weapon.[1] The threshold amount fixed in such a manner is usually far smaller than the threshold amount which would be fixed from economic considerations. This necessitates additional technical or economic efforts for each nuclear fuel processing facility to reduce MUF.

To reduce the hypothetical danger by making these additional technical or economic efforts easier, the following incentive-tax system is useful. Each facility can get an incentive when the amount of the basic variable is less than a fixed value on the one hand, and on the other hand a facility must pay a tax when the amount of the basic variable exceeds the fixed value. This sort of incentive-tax system is founded upon utility theory [13]. A linear utility function is used in the model presented in this paper.

This paper shows a new model to optimize a sampling plan for inspection by using Bayesian decision theory under supposition of an incentive-tax system. The mathematical framework of the model is delineated in section II and the numerical examples for a fuel fabrication plant, a fuel reprocessing plant and a fuel enrichment plant are illustrated in section III.

II. Mathematical Model

A. Basic Random Variable

Now we consider the problem of how to determine an optimal sampling plan for verification of the normality of the BI or the PI in Eq. (1). First of all the following

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[1]See IAEA, 1972, p. 3.2.

three assumptions are made for the purpose of simplifying the discussion:

(a) Concerning the problem of how to verify the BI, a plant is in stationary state and therefore the true value of individual data $Z_i (i=1,...,I)$ is kept constant during campaign interval.

Under this assumption the true value of BI is I times the true value of individual data. Next, the problem is how to estimate the true value of individual data by drawing I' samples ($I' \leq I$) from the population of I data.

(b) Concerning the problem of how to verify the PI, the true values of individual data $w_j(O)$ and $w_j(T) (j=1,...,J)$ are constant at the beginning and the end of the campaign respectively.

Under this assumption the true value of PI is J times the difference between the two values. Next, the problem is how to estimate each true value of individual data by drawing J' samples ($J' \leq J$) from the population of J data.

(c) The measuring process for individual samples is fixed and therefore the parameter to characterize the sampling plan is only sample size.

Under this assumption the problem of optimizing the sampling plan is equivalent to the problem of optimizing sample size.

Owing to assumptions (a) and (b), both of the problems for verification of the BI and PI are described in the following fundamental form: an operator states that N data $(x_1,...,x_N)$ have been measured by a prescribed measuring process and that as a result of the measurement the true value of individual data has been estimated as a certain value. In order to verify this statement it is necessary for an inspector to draw n samples $(x_1',...,x_n')$ from N data $(x_1,...,x_N)$, estimate the true value of individual data independently, and compare his estimated value with the value estimated by an operator.

It is necessary to examine the difference between the two estimated values for the sake of this comparison. Thereupon let ξ_{ope} and ξ_{ins} denote the true values of individual data estimated by an operator and an inspector respectively.

Both values should be equal in an ideal case. In practice, however, they are not necessarily equal because of the statistical behavior of individual data. Hence the difference δ between ξ_{ins} and ξ_{ope} becomes a key measure for the verification, and we consequently choose δ as a basic random variable. The definition is

$$\delta = \xi_{ins} - \xi_{ope} \quad . \tag{4}$$

B. Two-Action Problem

Collecting the information on the value ξ_{ins} from an inspector and on the value ξ_{ope} from an operator, and comparing these two values, an inspection authority needs to decide whether any further action should be conducted or not. If there is no difference between the values ξ_{ins} and ξ_{ope}, or if the value δ is nil, there will be no reason for an authority to bring any further action against a facility. On the contrary, if the value δ is by far larger, and especially if it exceeds the threshold amount fixed earlier, then some further action will have to be applied.

So as to formulate this decision-making problem we now consider the following two-action problem: an authority selects either

action 1: "Accept the operator's data"

or

action 2: "Reject the operator's data"

after observing n samples drawn from the population of N data. A criterion for the selection is supposed to be given from the costs of each action.

Action 1 means no further action, while action 2 means some further action depending on the value of δ. Therefore action 1 costs nothing, while the cost of action 2 is dependent on the basic random variable. Let C_A and C_R denote the cost of action 1 (act of acceptance) and the cost of action 2 (act of rejection) respectively, and we can describe the equations:

$$C_A = 0 \quad , \tag{5}$$

and

$$C_R = C_R(\delta) \quad . \tag{6}$$

It is supposed here that the problem to choose either the act of acceptance or the act of rejection is equivalent to the one to compare these costs C_A and C_R. In other words it is supposed that an optimal action for an inspection authority is the act of acceptance if $C_A < C_R$, or the act of rejection if $C_A > C_R$.

If the value of the basic random variable could be known exactly, an optimal act could be chosen deterministically since the cost C_R of Eq. (6) could be computed exactly. As mentioned earlier, however, an inspector can only estimate the value δ statistically, and it is inevitable for an inspection authority to decide an optimal action under uncertainty. Hence using $P(\delta)$ to denote the probability distribution of a basic random variable, we can describe the discriminative condition for optimality as below:

1) if $EVC_R \geq EVC_A = 0$, then act of acceptance is optimal

or

2) if $EVC_R < EVC_A = 0$, then act of rejection is optimal

where

$$EVC_A = \int_{-\infty}^{\infty} C_A(\delta) \cdot P(\delta)d\delta = 0 \quad , \tag{7}$$

and

$$EVC_R = \int_{-\infty}^{\infty} C_R(\delta) \cdot P(\delta)d\delta \quad . \tag{8}$$

EVC_A and EVC_R mean the expected costs of acts of acceptance and rejection respectively under the probability distribution $P(\delta)$ of a basic random variable. The distribution $P(\delta)$ is estimated with the aid of the historical data of δ, and by making use of the observed information on n samples $(x'_1,...,x'_n)$ drawn from N data $(x_1,...,x_N)$. If the distribution $P(\delta)$ is preassigned prior to the observation of samples, then an optimal act under the distribution can be determined by computing the corresponding EVC_R. After observing samples, however, the newly estimated distribution becomes different from the preassigned one, and it may bring about the revision of decision making on an optimal act. Therefore it is of great importance to assess the probability distribution $P(\delta)$ of a basic random variable in order to determine an optimal act.

In the model described here the effectiveness of the assessment on $P(\delta)$ is defined by the value of the observed information required to select either of the two acts. On the other hand it costs some amount to obtain the observed information. This susggests that it is necessary to consider cost/effectiveness analysis for the purpose of optimization on the observed information. The observed information has a close relation to sample size, and an optimal sample size is to be determined from the optimization analysis.

C. Incentive-Tax System

In order to solve such a two-action problem as mentioned above it is necessary to assess the costs of actions as well as the distribution of a basic random variable. As a result, the following incentive-tax system is thought out hypothetically.

Suppose first that the threshold amount δ_{TA} of a basic random variable is preassigned from the standpoint of nuclear materials safeguards. Then suppose that if it is proved that the value of δ exceeds δ_{TA}, a facility must pay the tax $C_T(\delta)$, which is proportional to the exceeded value $\delta - \delta_{TA}$, or an inspection authority should levy the tax $C_T(\delta) \propto \delta - \delta_{TA}$. On the other hand, it is supposed that if it is proved that the value of δ is smaller than the threshold amount δ_{TA}, a facility can receive the incentive $C_I(\delta)$, which is proportional to the difference $\delta_{TA} - \delta$, or an authority should pay the incentive $C_I(\delta) \propto \delta_{TA} - \delta$.

In the event that the threshold amount δ is set to the utmost limit of permissibility, a kind of safety factor should be taken into account. In such a case, using the value δ_{BE}, which is smaller than δ_{TA}, we can formulate the incentive-tax system:

> if $\delta \leq \delta_{BE}$, then the incentive $C_I(\delta) = k_I(\delta_{BE} - \delta)$, will be paid to a facility by an inspection authority, while on the contrary
>
> if $\delta > \delta_{BE}$, then the tax $C_T(\delta) = k_T(\delta - \delta_{BE})$, will be paid to an inspection authority by a facility.

Now suppose that the act of rejection is synonymous with taking part in the hypothetical game where an inspector takes some further action and then according to the above incentive-tax system, pays the incentive or levys the tax. It is supposed additionally that the cost of the further action is negligibly small compared with the incentive or the tax. The additional supposition implies that an authority has no need to take the cost of the further action into account in choosing the decision.

Upon these suppositions Eq. (6) is to be written as below:

$$C_R(\delta) = \begin{cases} C_I(\delta) = k_I(\delta_{BE} - \delta) \quad , & \text{if } \delta \leq \delta_{BE} \\ \\ -C_T(\delta) = k_T(\delta_{BE} - \delta) \quad , & \text{if } \delta > \delta_{BE} \end{cases} \quad . \tag{9}$$

Substituting the $C_R(\delta)$ of Eq. (9) for the $C_R(\delta)$ of Eq. (8) the expected cost of act of rejection is

$$EVC_R = \int_{-\infty}^{\delta_{BE}} C_I(\delta)P(\delta)d\delta - \int_{\delta_{BE}}^{\infty} C_T(\delta)P(\delta)d\delta \quad , \tag{10}$$

and therefore the discriminative condition for optimality on actions is rewritten as in the following:

$$1) \quad \text{if} \int_{-\infty}^{\delta_{BE}} C_I(\delta)P(\delta)d\delta \geq \int_{\delta_{BE}}^{\infty} C_T(\delta)P(\delta)d\delta \quad ,$$

then act of acceptance is optimal;

or

$$2) \quad \text{if} \int_{-\infty}^{\delta_{BE}} C_I(\delta)P(\delta)d\delta < \int_{\delta_{BE}}^{\infty} C_T(\delta)P(\delta)d\delta \quad ,$$

then act of rejection is optimal.

The first term of the right hand side of Eq. (10) implies the authority's risk associated with the act of rejection, or the expected utility associated with the facility's effort to realize $\delta \leq \delta_{BE}$. The second term implies the facility's risk associated with the act of rejection, or the expected utility associated with the authority's effort to prove $\delta > \delta_{BE}$. Hence it is to be understood that Eq. (9) represents a utility function in terms of δ.

For the purpose of simplicity we assume here that $k_I = k_T = k_R$. Then Eq. (9) is reduced to a linear utility function, i.e.,

$$C_R = k_R(\delta_{BE} - \delta) \qquad \text{for every } \delta. \tag{11}$$

Assume that the incentive $C_I(0)$ for $\delta = 0$, and the tax $C_T(\delta_{TA})$ for $\delta = \delta_{TA}$, are given the values k_R and δ_{BE} are obtained from:

$$k_R = (C_I(0) + C_T(\delta_{TA}))/\delta_{TA} \quad , \tag{12}$$

and

$$\delta_{BE} = \Pi \cdot \delta_{TA} \quad ; \quad \Pi = C_I(0)/(C_I(0) + C_T(\delta_{TA})) \tag{13}$$

respectively. The ratio Π is a sort of safety factor.

Provided that these assumptions for simplification are made, Eq. (10) is written in a most simple form:

$$EVC_R = k_R(\delta_{BE} - E(\delta)) \quad , \tag{14}$$

where

$$E(\delta) = \int_{-\infty}^{\infty} \delta \cdot P(\delta) d\delta \tag{15}$$

represents the expected value of δ.

Consequently according to the incentive-tax system an optimal act in the two-action problem is to be decided from the following:

1) if $E(\delta) \leq \delta_{BE}$, then act of acceptance is optimal,

or

2) if $E(\delta) > \delta_{BE}$, then act of rejection is optimal.

Thus an essential parameter of our problem is the expected value $E(\delta)$ of a basic random variable. If the expectation $E(\delta)$ is larger than the break-even value δ_{BE} as a result of observing n data from N data and estimating $E(\delta)$, then an inspection authority should bring some further action against a facility. And furthermore if, after the further action it is still true that $E(\delta) > \delta_{BE}$, then a facility should pay the corresponding tax. On the contrary if it turns out that $E(\delta) \leq \delta_{BE}$, then an inspection authority should give the corresponding incentive to a facility. Figure 2 shows the procedure of inspection for nuclear materials safeguards according to the incentive-tax system.

D. Nuisance Parameters

Generally speaking the errors associated with observation of samples are divided into two component parts: random error and systematic error. The definitions are as follows and are valid for both operator's data and inspector's data.

Let μ denote the mean of an infinite number of measurements. Random error γ is defined as the difference between individual measured data x and the mean μ. In symbols,

$$\gamma = x - \mu \quad . \tag{16}$$

Then in many if not most situations μ would not be exactly equal to the true value ξ of the quantity being measured. Systematic error θ is defined as the difference between these two values. In symbolic forms,

$$\theta = \mu - \xi \quad . \tag{17}$$

From Eqs. (16) and (17) the formula which shows the three component parts of any individual measured data is obtained:

$$x = \xi + \theta + \gamma \quad . \tag{18}$$

If the systematic error θ is kept constant during the campaign interval, then Eq. (18) leads to the relation for any sample mean

$$\bar{x} = \xi + \theta + \bar{\gamma} \quad . \tag{19}$$

In words rather than symbols, any sample mean can be regarded as the sum of the true value of the quantity measured, the fixed systematic error of the measuring process and the mean of the random error of the individual observations in the sample. This implies that the distribution of the mean $\bar{\gamma}$ of random error and the distribution of systematic error θ are to be assessed so as to yield the distribution of the sample mean $\bar{x}$.

The distribution of the mean $\bar{\gamma}$ is assessed in a common manner. It follows from the definition of random error that the expected value $E(\bar{\gamma})$ of the mean $\bar{\gamma}$ is zero:

$$E(\bar{\gamma}) = 0 \quad . \tag{20}$$

The variance $\sigma^2(\bar{\gamma})$ of the mean $\bar{\gamma}$ is estimated by using the equation:

$$\sigma^2(\bar{\gamma}) = \frac{1}{n}\,\sigma^2(\gamma)\,\frac{N-n}{N-1} \quad , \tag{21}$$

where

$$\sigma^2(\gamma) \simeq \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \quad . \tag{22}$$

Equation (21) is the formula to be used when a sample of γ's is drawn without replacement from a finite population the size of which is N. Equation (22) is the formula to be used when the number of measurements under the same condition is not large enough to justify us in treating $\bar{x}$ as certainly equal to μ. It should be noted here that if the conditions under which the measuring process operated were not constant or if the systematic error varied even though the true value ξ remained constant, then the value $\sigma^2(\gamma)$ estimated by Eq. (22) would tend to overstate the true random variance $\sigma^2(\gamma)$ because the x's in Eq. (22) would contain variance due to the random error. As concerns the shape of the distribution of $\bar{\gamma}$, the central limit theorem insists that the distribution of $\bar{\gamma}$ in large samples will be often exactly Normal even though the distributions of the individual γ's are quite far from Normal.

On the other hand the distribution of systematic error θ cannot be assessed in such a common manner because systematic error often results from many sorts of factors inherent in individual facilities. As far as a nuclear materials processing facility is concerned, however, the factors are to be decomposed into the following two. Factor one is related to a material flow pattern of individual material balance areas. We cannot assess this factor without repeating the integral experiments for various patterns of material flow. Hence it is assumed in the model that a facility operates under the stationary condition such that the pattern of material flow is kept unchanged, and it is also assumed that the component of systematic error due to this factor is known in advance with aid of the historical data. Factor two concerns the measuring process, with a systematic error component that is always involved in any measured data. To assess the error component, observed data must be calibrated at appropriate time intervals and therefore it is assumed here that the error component can be estimated on occasion by the calibration.

Consequently, provided that the assumptions: (a) the pattern of material flow is kept stationary throughout the campaign interval, (b) observed data are calibrated at appropriate intervals and (c) the shape of the distribution of systematic error is Normal are made, then the distribution of systematic error is to be specified by the two parameters: the expected value $E(\theta)$ and the variance $\sigma^2(\theta)$. And then it is supposed in the model that these parameters are given beforehand:

$$E(\theta) = E_0(\theta) \quad , \tag{23}$$

$$\sigma^2(\theta) = \sigma_0^2(\theta) \quad . \tag{24}$$

Using Eqs. (20), (21), (23) and (24) the expected value $E(\bar{x})$ and the variance $\sigma^2(\bar{x})$ of sample mean in Eq. (19) are described in the formulae:

$$E(\bar{x}) = \xi + E(\theta) \tag{25}$$

and

$$\sigma^2(\bar{x}) = \sigma^2(\theta) + \sigma^2(\bar{\gamma}) \quad , \tag{26}$$

respectively.

Now in connection with basic random variable δ let $\Delta\bar{x}$ denote the difference between the mean $\bar{x}_{ins}$ of inspector's measurements and the mean $\bar{x}_{ope}$ of operator's measurements, and then $\Delta\bar{x}$ is written:

$$\Delta\bar{x} = \delta + \theta_{ins} - \theta_{ope} + \bar{\gamma}_{ins} - \bar{\gamma}_{ope} \quad , \tag{27}$$

where

θ_{ins}: systematic error of inspector's measurements

θ_{ope}: systematic error of operator's measurements

$\bar{\gamma}_{ins}$: the mean of random error of inspector's measurements

$\bar{\gamma}_{ope}$: the mean of random error of operator's measurements.

The expected value $E(\Delta\bar{x})$ and the variance $\sigma^2(\Delta\bar{x})$ are given by the definitions,

$$E(\Delta\bar{x}) = \delta + E(\Delta\theta) \tag{28}$$

and

$$\sigma^2(\Delta\bar{x}) = \sigma^2(\Delta\theta) + \sigma^2(\Delta\bar{\gamma}) \quad , \tag{29}$$

where

$$\Delta\bar{x} = \bar{x}_{ins} - \bar{x}_{ope} \quad , \tag{30}$$

$$\Delta\theta = \theta_{ins} - \theta_{ope} \tag{31}$$

and

$$\Delta\bar{\gamma} = \bar{\gamma}_{ins} - \bar{\gamma}_{ope} \quad . \tag{32}$$

In case that both θ_{ins} and θ_{ope} are assessed by the same method, and that an operator measures all of N data, Eq. (29) is rewritten as

$$\sigma^2(\Delta\bar{x}) = 2\sigma^2(\theta_{ins}) + \sigma^2(\bar{\gamma}_{ins}) \quad . \tag{33}$$

E. Bayesian Decision Making

The value of basic random variable δ is calculated from Eq. (28) by observing the value of $E(\Delta\bar{x})$. Prior to the observation, however, it is possible to a certain degree to guess the value of δ with aid of the design information and the historical data.

Suppose now that $P_O(\delta)$ denotes the prior distribution of δ which is estimated roughly prior to observation and and assume that the prior distribution $P_O(\delta)$ is a Normal distribution with the expectation:

$$E(\delta) = E_O(\delta) \quad , \tag{34}$$

and the variance:

$$\sigma^2(\delta) = \sigma_O^2(\delta) \quad . \tag{35}$$

An optimal act under $P_O(\delta)$ is decided from the conditions:

1) if $E_O(\delta) \leq \delta_{BE}$, then act of acceptance is optimal;

or

2) if $E_O(\delta) > \delta_{BE}$, then act of rejection is optimal.

Hence an optimal decision under $P_O(\delta)$ is dependent only on $E_O(\delta)$ and regardless of $\sigma_O^2(\delta)$.

The variance $\sigma_O^2(\delta)$, however, implies a sort of the unreliability of the value of δ being estimated, or the inverse of $\sigma_O^2(\delta)$ implies a sort of the accuracy of the value of δ. Therefore it is risky to select an optimal action under the prior distribution with large variance. Usually the expected value estimated without any observation is not so reliable and then it becomes valuable to observe samples. It is to be noted here that if the relation

$$E_O(\delta) = \delta_{BE} \tag{36}$$

is made, there is no distinction concerning the optimality between the acts of acceptance and rejection; either of the two is optimal regardless of the value of $\sigma_O^2(\delta)$.

Suppose then that as a result of observing n samples from the population of N data a sample mean $\Delta\bar{x}$ is obtained and that $P_1(\delta)$ denotes the posterior distribution of δ which is estimated after observation. The posterior distribution $P_1(\delta)$ is given by Bayes' theorem:

$$P_1(\delta) = P_O(\delta)P(\Delta\bar{x}|\delta)d\delta \Big/ \int_{-\infty}^{\infty} P_O(\delta)P(\Delta\bar{x}|\delta)d\delta \quad , \qquad (37)$$

where

$$P(\Delta\bar{x}|\delta) = P_N'(u)/\sigma(\Delta\bar{x}) \quad ;$$
$$u = (\Delta\bar{x} - E(\Delta\bar{x}))/\sigma(\Delta\bar{x}) \quad . \qquad (38)$$

The probability $P(\Delta\bar{x}|\delta)$, the conditional probability of the event $\Delta\bar{x}$ given the event δ, implies a sort of likelihood.

Given that both the prior distribution of δ and the sampling distribution of $\Delta\bar{x}$ are Normal, the posterior distribution also is Normal, and the formula Eq. (37) is able to be solved analytically. The mean $E_1(\delta)$ and the variance $\sigma_1^2(v)$ of the posterior distribution are in the following:[2]

$$E_1(\delta) = \frac{I_O E_O(\delta) + I_{\Delta\bar{x}}(\Delta\bar{x} - E(\Delta\theta))}{I_O + I_{\Delta\bar{x}}} \qquad (39)$$

and

$$\sigma_1^2(\delta) = 1/I_1 = 1/(I_O + I_{\Delta\bar{x}}) \quad , \qquad (40)$$

where

$$I_O = 1/\sigma_O^2(\delta) \qquad (41)$$

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[2]See R. Schlaifer, 1959, p. 441.

and

$$I_{\Delta\bar{x}} = 1 / \sigma^2(\Delta\bar{x}) \quad . \tag{42}$$

The meaning of Eq. (39) is clear. The inverse of variance (I_0, $I_{\Delta\bar{x}}$ or I_1) represents the accuracy of the corresponding statistic and may be called the quantity of information on the statistic. Thus the mean of the posterior distribution of δ is a weighted average of the prior mean and the sample mean, the weight of each estimate being the quantity of information.

Once the mean $E_1(\delta)$ and $\sigma_1^2(\delta)$ are assessed from the observation of samples, an optimal act can be decided by comparing the value of $E_1(\delta)$ and the break-even value δ_{BE}. Unless the quantity of information I_1 is sufficient to make a decision, it is necessary to reobserve more samples and re-estimate the posterior distribution $P_2(\delta)$ by regarding $P_1(\delta)$ as a renovated prior distribution. Taking this procedure iteratively, we have a sequential decision making problem in the sampling plan.

Now there is a question of how to determine the optimal quantity of information to make a decision. A solution here is obtained from the counterbalance between the effectiveness and the cost of observed information. First, let us define effectiveness of observed information.

An optimal act under the assumption that the value of δ is known deterministically is chosen easily by inspecting Fig. 3 which shows the cost of each act as a function of δ. If $\delta = \delta'$, for example, then the act of acceptance is optimal since $C_A(\delta') < C_R(\delta')$. Hence the cost $C_*(\delta)$ of the optimal act under the assumption of deterministicality is illustrated as the bold line $\overline{OPQ}$ in Fig. 3.

Now let us define the opportunity loss of each act, L_A or L_R, by the difference between the cost C_* and the cost of each act. In symbolic form

$$L_A = C_A - C_* = \begin{cases} 0 & ; \quad \delta \leq \delta_{BE} \\ k_R(\delta - \delta_{BE}) & ; \quad \delta > \delta_{BE} \end{cases} \tag{43}$$

and

$$L_R = C_R - C_* = \begin{cases} k_R(\delta_{BE} - \delta) & ; \quad \delta \leq \delta_{BE} \\ 0 & ; \quad \delta > \delta_{BE} \end{cases} \quad . \tag{44}$$

Furthermore let EVL_A and EVL_R denote the expected values of opportunity losses of acts of acceptance and rejection respectively, i.e.,

$$EVL_A = \int_{-\infty}^{\infty} L_A(\delta)P(\delta)d\delta \tag{45}$$

and

$$EVL_R = \int_{-\infty}^{\infty} L_R(\delta)P(\delta)d\delta \quad . \tag{46}$$

It is evident that the expected value of opportunity loss of the optimal act under uncertainty, EVL_* is subject to

$$EVL_* = \min_{A \text{ or } R} \{EVL_A, EVL_R\} \quad . \tag{47}$$

The value EVL_* of Eq. (47) is the expected value of opportunity loss which is by all means inevitable as far as there is any uncertainty on a basic random variable. Hence it may be called the cost of uncertainty or the expected value of the perfect information which if it were available would enable us to make a decision perfectly. So let $EVPI|_{P(\delta)}$ denote the expected value of perfect information under the distribution $P(\delta)$. The definition is

$$EVPI|_{P(\delta)} = EVL_* \tag{48}$$

The observation of samples changes the distribution of δ from $P_0(\delta)$ to $P_1(\delta)$, and therefore it changes the expected value of perfect information from $EVPI|_{P_0}$ to $EVPI|_{P_1}$. The definition of Eq. (48) apparently insists that the inequality

$$EVPI|_{P_1} \leq EVPI|_{P_0} \tag{49}$$

should be satisfied for the observation to be of worth. Then we define the value of the observed information, VOI_1 which changes the distribution $P_0(\delta)$ into the distribution $P_1(\delta)$ by the difference between $EVPI|_{P_0}$ and $EVPI|_{P_1}$. In symbols,

$$VOI_1 = EVPI|_{P_0} - EVPI|_{P_1} \quad . \tag{50}$$

The value of observed information defined by the above equation is known after observing samples and getting the distribution $P_1(\delta)$. However, the effectiveness of observed information needs to be assessed before observing samples. To do so it is necessary first to estimate the distribution of posterior mean $E_1(\delta)$ of a basic random variable, and then to assess the expected value of VOI_1 under the distribution of $E_1(\delta)$. That is to say, using $P(E_1)$ and $EVOI_1$ to denote the distribution of $E_1(\delta)$ and the expected value of observed information respectively, we should define the effectiveness of observed information by the equation:

$$EVOI_1 = (EVPI|_{P_0} - EVPI|_{P_1})|_{P(E_1)} \quad . \tag{51}$$

Provided that the prior distribution of δ and the distributions of θ and γ are Normal, the distribution of $E_1(\delta)$ is also Normal and it is given in the following forms:[3] the expectation,

$$E(E_1(\delta)) = E_0(\delta) \quad , \tag{52}$$

[3] See R. Schlaifer, 1959, pp. 525-530.

and the variance,

$$\sigma^2(E_1(\delta)) = \frac{\sigma_{00}^2(\delta)\cdot\varepsilon_\gamma}{\varepsilon'_\gamma} \cdot \left(1 + \varepsilon_{\Delta\theta} + \frac{1}{n}\;\varepsilon'_\gamma\,\frac{N-n}{N-1}\right) \quad , \tag{53}$$

where,

$$\varepsilon_\gamma = \sigma^2(\gamma)/\sigma_{00}^2(\delta) \quad , \tag{54}$$

$$\varepsilon_{\Delta\theta} = \sigma^2(\Delta\delta)/\sigma_{00}^2(\delta) \tag{55}$$

and

$$\varepsilon'_\gamma = (1 + \varepsilon_{\Delta\theta})n + \varepsilon_\gamma \quad . \tag{56}$$

There are two sorts of usage of Eq. (53), depending on the variance $\sigma_{00}^2(\delta)$ of the original distribution of δ:

(1) In the case that we have no useful knowledge for assessing the prior distribution of δ, it is necessary first of all to take n_0 pilot samples ($n_0 << n$) from N data for the sake of the provisional assessment of the prior distribution. In this case the variance $\sigma_{00}^2(\delta)$ guessed prior to taking pilot samples is equal to infinity. Therefore Eq. (53) is reduced to

$$\sigma^2(E_1(\delta)) = \frac{\sigma^2(\gamma)}{n_0}\Big/\left(1 + \frac{n_0}{n}\,\frac{N-n}{N-1}\right) \quad , \tag{57}$$

since $\varepsilon_\gamma = \varepsilon_{\Delta\theta} = 0$ and $\varepsilon'_\gamma = n_0$.

(2) In the case that we have any aid from design information, historical data and so forth to assess the prior distribution of δ, it is possible to give the prior distribution a certain finite value without taking any pilot sample. Therefore Eq. (53) is rewritten as

$$\sigma^2(E_1(\delta)) = \sigma_0^2(\delta)/(1 + \varepsilon_{\Delta\theta} + \frac{1}{n}\varepsilon_\gamma \frac{N-n}{N-1}) \ , \qquad (58)$$

since $n_0 = 0$, $\sigma_{00}^2(\delta) = \sigma_0^2(\delta)$ and $\varepsilon_\gamma' = \varepsilon_\gamma$.

Given the distribution of the posterior mean in Eqs. (52) and (53), the formula below[4] is used to compute the expected value of observed information defined by Eq. (51):

$$EVOI_1 = k_R \sigma(E_1(\delta)) G(D_{E_1}) \ , \qquad (59)$$

where

$$D_{E_1} = |\delta_{BE} - E_0(\delta)|/\sigma(E_1(\delta)) \qquad (60)$$

and

$$G(D_{E_1}) = P_N'(D_{E_1}) - D_{E_1} \cdot P_N(D_{E_1}' > D_{E_1}) \ . \qquad (61)$$

Equation (59) is the concrete expression of the effectiveness of observed information to be derived in the model.

It follows from inspection of Eq. (58) that the variance $\sigma^2(E_1(\delta))$ increases monotonously in terms of sample size n. Since the variable D_{E_1} is inversely proportional to $\sigma^2(E_1(\delta))$, the variable D_{E_1} decreases monotonously in terms of sample size n. Furthermore Fig. 4 shows that the loss integral function $G(D_{E_1})$ is a monotonously decreasing function of the argument

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[4]See R. Schlaifer, 1959, p. 532.

D_{E_1}, and that the value of the function $G(D_{E_1})$ increases monotonously in terms of sample size n. Therefore the expected value of observed information, $EVOI_1$ is a monotonously increasing function of sample size n. This is due to the fact that the expected value $EVOI_1$ corresponds to the gross gain of observed information.

In order to determine an optimal sample size it is necessary to take into consideration the cost $C_S(n)$ for observing the information on n samples, and then to define the net gain $NGO_1(n)$ of observed information by the equation

$$NGO_1(n) = EVOI_1(n) - C_S(n) \quad . \tag{62}$$

It is natural that the cost $C_S(n)$ should increase monotonously in terms of sample size n. The simplest expression of the cost $C_S(n)$ is a linear equation in terms of n:

$$C_S(n) = K_S + k_S n \quad . \tag{63}$$

Since both $EVOI_1(n)$ and $C_S(n)$ are monotonously increasing functions of n, the net gain of observed information, $NGO_1(n)$, possibly has a maximum where the values of the individual differential coefficients are equal.

Thus the ultimate description of the problem of optimizing sample size is as follows: find an n subject to the constraints

$$\frac{d}{dn} NGO_1(n) = 0 \quad , \quad \frac{d^2}{dn^2} NGO_1(n) < 0 \quad , \tag{64}$$

$$NGO_1(n) > 0 \tag{65}$$

and

$$1 \leq n \leq N \quad . \tag{66}$$

III. Calculation Results

A. Illustrated Examples

In order to demonstrate the calculating procedures according to the mathematical model presented here, and also to show the optimal solutions of sample size, the following three nuclear fuel processing facilities are taken as examples:

1) 100 tons of UO_2/year fuel fabrication facility (FFF),

2) 200 tons of U/year fuel reprocessing facility (FRF), and

3) 8,750 tons SWU/year fuel enrichment facility (FEF).

The relevant specifications of the individual facilities are given in Table 1. The specifications for the FFF are taken from the results of the simulation study using the historical data which have been obtained actually. As for the FRF, the data in Table 1 are made artificially with the aid of the design information [10]. It is supposed that the reprocessed fuel is the discharged fuel of a light water moderated reactor which includes 0.75% plutonium. The FEF is characterized by analogy from the data published in [11] and [12]. Here, the top product is 4% enriched uranium, and for the purpose of simplicity it is supposed that an inspector observes only samples of the product.

It is to be noted here, however, that there are the remarkable differences between the values of systematic error given for the individual facilities. For the FFF both the expectation and the standard deviation are negligibly small; for the FRF, both the expectation and the standard deviation are approximately 1% of the population mean; and for the FEF the expectation is negligible, and yet the standard deviation is comparable to the standard deviation of random error. These differences will have a significant effect on the calculation results.

Table 1 also shows the inspection parameters assigned for the individual facilities. The threshold amount (T.A.) is fixed in accordance with footnote one. It is supposed here that the incentive $C_I(0)$ for $\delta = 0$ should be equal to the tax $C_T(\delta_{TA})$ for $\delta = \delta_{TA}$, and that the value of $C_I(0)$ should be assigned impartially for each facility. Concerning the cost of observing samples, however, the costs for the FRF and the FEF are fixed at ten times the cost for the FFF because of the complicated measuring process for the FRF and the FEF.

The calculation results for the FFF, the FRF and the FEF, obtained by using the basic input data in Table 1, are represented in Tables 2, 3 and 4 respectively. In each of these the following four cases are taken in connection with the assignment of the prior distribution $P_O(\delta)$:

Case 1: the expected value of $P_O(\delta)$ is zero, $E_O(\delta) = 0$;

Case 2: the expected value of $P_O(\delta)$ is comparable to the break-even value δ_{BE} but a little smaller, $E_O(\delta) < \delta_{BE}$;

Case 3: the expected value of $P_O(\delta)$ is identical to δ_{BE}, $E_O(\delta) = \delta_{BE}$; and

Case 4: the expected value of $P_O(\delta)$ is comparable to δ_{BE} but a little larger, $E_O(\delta) > \delta_{BE}$.

Case 3 implies that the corresponding optimal act under the distribution $P_O(\delta)$ is undecided since the costs of acceptance and rejection are equivalent. Cases 1 and 2 make the act of acceptance optimal under $P_O(\delta)$, while on the contrary, Case 4 makes the act of rejection optimal under $P_O(\delta)$.

In addition it is presumed that the variance of $P_O(\delta)$ for the FFF is assessed by taking n_O pilot samples for the sake of verifying the variance $\sigma^2(\gamma)$ of individual measurements. On the other hand, the variances of $P_O(\delta)$ for the FRF and the FEF are assessed by making use of the design information and the experimental data.

Displaying the net gain of observation NGO_1 for the FFF as a function of sample size n, Fig. 5 shows that:

(a) there is no feasible solution in Case 1 (FFF 1),

(b) the net gains for Cases 2 and 4 (FFF 2,4) are exactly identical,

(c) the net gain for Case 3 (FFF 3) is the highest for any n, and

(d) the optimal sample sizes for FFF 2, 3 and 4 are almost the same ($n^* \simeq 205$).

It follows from the definition of NGO_1 (Eq. (62)) that the differences between the values of $E_O(\delta)$ affect only the values of the loss integral $G(D_{E_1})$. Hence we should give a reason for these four facts by considering differences between the values of $G(D_{E_1})$. That is to say, fact (a) is due to the value of D_{E_1} for FFF 1 being too large and therefore the value of $G(D_{E_1})$ being extremely small (< \$$10^{-7}$) as compared with the cost (> \$4). This means that it is of little value to observe samples if the value of $E_0(\delta)$ preassigned by a decision maker is so different from the break-even value, and if at the same time the preassignment is done with such accuracy as $\sigma(E_1(\delta)) < 23\%$ of δ_{BE}. Fact (b) is a self-evident truth because the absolute values of the difference $E_0(\delta)$ and δ_{BE} for FFF 2 and 4 are given as identical. Fact (c) results from the value of $G(D_{E_1})$ which has the maximum value for $D_{E_1} = 0$, as shown in Fig. 4. And finally, fact (d) originates in the value of $G(D_{E_1})$ being almost constant in the neighborhood of the optimal solution n*.

Figures 6 and 7 are placed to display the net gains of observation for FRF and FEF respectively as a function of sample size. According to these figures, we can see facts, except for (a), similar to the ones for FFF. In the cases of FRF and FEF the values of $\sigma_0(\gamma)$, which from Eq. (58) are proportional to $\sigma(E_1(\delta))$, are assigned to be more than ten times the break-even value δ_{BE}. From this the value of D_{E_1} is close to zero ($\lesssim 0.1$) regardless of the value of $E_0(\delta)$ assigned in the examples. Therefore the value of $G(D_{E_1})$ is roughly constant for any of the assigned values of $E_0(\delta)$. This is why the optimal sample size exists, and takes the same value for any cases of FRF and FEF.

It is also worthy of notice that the net gain of observation does not significantly vary with sample size n in the cases of FRF and FEF. For instance, even the values of NGO_1 for the minimum sample size in FRF and FEF amount to approximately 95% and 85% of the values of NGO_1 for the optimal sample size respectively. This is caused by the following facts:

(a) $EVOI_1(n) >> C_S(n)$ for any n,

(b) the change of $EVOI_1$ associated with n is nearly proportional to the change of $\sigma(E_1(\delta))$ associated with n, and

(c) the value of $\sigma(E_1(\delta))$ does not remarkably vary with sample size n because of $\varepsilon_{\Delta\theta} \approx \varepsilon_\gamma$.

Generally speaking, the increase of sample size yields the reduction of the ambiguity associated with random error, and yet it has no relation to the decrease of the ambiguity associated with systematic error. Therefore the increasing rate of $EVOI_1(n)$ is by far lower for $\varepsilon_{\Delta\theta} \approx \varepsilon_\gamma$ than for $\varepsilon_{\Delta\theta} = 0$.

With the view of comparing the optimal act and the cost of uncertainty, EVPI under the posterior distribution $P_1(\delta)$ with the optimal act and the cost EVPI under the prior distribution $P_0(\delta)$, Tables 2, 3 and 4 show the examples assuming certain inspection results concerning the values of $\Delta\bar{x}$ and $\sigma(\gamma)$. For every facility it is assumed that an inspector obtained the value of $\Delta\bar{x}$ or $\Delta\bar{x} - E(\Delta\theta)$ which was close to and a little smaller than the break-even value, and also that he obtained the value of $\sigma(\gamma)$ which was equal to or a little larger than the value given from the specification. Since the value of $E_1(\delta)$, computed in accordance with the imaginary results of inspection, is smaller for every case than the break-even value δ_{BE}, the optimal act under $P_1(\delta)$ for every case is the act of acceptance. With regard to the cost of uncertainty, the values of $EVPI|_{P_1}$ under the posterior distribution $P_1(\delta)$ for FFF 3, FRF 3 and FEF 3, for example, are reduced to $\$2.77 \times 10^3$ (7.7% of $EVPI|_{P_0}$), $\$7.53 \times 10^5$ (37.8% of $EVPI|_{P_0}$) and $\$9.94 \times 10^5$ (58.4% of $EVPI|_{P_0}$) respectively. While the reduction rates for FRF and FEF may be still unsatisfactory, the reduced amounts are enormous in comparison with FFF. Furthermore, taking into consideration the ratio of the cost of uncertainty for the maximum sample size (n=N) to the value $EVPI|_{P_0}$ (see Table 5), the optimal sample size obtained here can be considered as reasonable.

B. Asymptotic Solution

The calculation results mentioned above suggest to us that an optimal sample size can be found regardless of the expectation $E(E_1(\delta))$ of posterior mean $(E(E_1(\delta)) = E_0(\delta)$, from Eq. (52) so long as the prior distribution has been assigned so that samples from measured data may be worthy of observing. Additionally it is apparent from the relation between $E_0(\delta)$ and $G(D_{E_1})$, Eqs. (60) and (61), that given all the parameters other than $E_0(\delta)$ the optimal sample size has the maximum value for $E_0(\delta) = \delta_{BE}$. Owing to these mathematical properties of the optimal sample size n* it is worthwhile to consider the relationship between the optimal solution n* for $E_0(\delta) = \delta_{BE}$ and the other relevant parameters.

The net gain of observation for $E_0(\delta) = \delta_{BE}$ is obtained from Eqs. (52) to (63) as below:

$$NGO_1(n) = \frac{k_R \cdot \sigma_{00}(\delta) \cdot (\varepsilon_\gamma / \varepsilon'_\gamma)^{\frac{1}{2}}}{\sqrt{1 + \varepsilon_{\Delta\theta} + \frac{1}{n} \varepsilon'_\gamma \frac{N-n}{N-1}}} \cdot G(0) - (K_S + k_S \cdot n) \quad , \tag{67}$$

where

$$G(0) = 0.3989 \quad . \tag{68}$$

Now setting aside the constraints of Eqs. (65) and (66), we suppose that an optimal solution is subject only to Eq. (64). In considering the differential equation, Eq. (64) it is convenient to rewrite Eq. (67) in the following form:

$$\frac{NGO_1(n) + K_S}{k_S} = G(0) \cdot \frac{\lambda}{\sqrt{1 + \omega/n}} - n \quad , \tag{69}$$

where

$$\lambda = \frac{k_R \cdot \sigma_{00}(\delta) \cdot (\varepsilon_\gamma)^{\frac{1}{2}}}{k_S \cdot (\varepsilon'_\gamma)^{\frac{1}{2}}} \sqrt{1 + \varepsilon_{\Delta\theta} - \frac{\varepsilon'_\gamma}{N - 1}} \tag{70}$$

and

$$\omega = \varepsilon'_\gamma \frac{N}{N - 1} (1 + \varepsilon_{\Delta\theta} - \frac{\varepsilon'_\gamma}{N - 1}) \quad . \tag{71}$$

Equation (69) implies that the parameters, which have a substantial effect on an optimal solution, are λ of Eq. (70) and ω of Eq. (71). And it is obviously understood that as far as such a function of n of Eq. (69) is concerned, the optimal solution satisfying the condition, Eq. (64), exists for any $\lambda > 0$ and $\omega > 0$. Figure 8 illustrates the contour of the optimal solution n* in the λ-to-ω chart where the units are in logarithmic scale.

From Fig. 8 we find that every one of the contours is divided into two parts: (a) for $\lambda/\omega \gtrsim 10$, the optimal solution n* increases with ω and (b) for $\lambda/\omega \lesssim 10$, the optimal solution n* decreases with ω. Now for the purpose of explaining this property, we can consider the simplest case assuming $N \rightarrow \infty$ (infinite population), $\varepsilon_{\Delta\theta} = 0$ (no systematic error) and $\varepsilon'_\gamma = \varepsilon_\gamma$ or $\sigma_{00}(\delta) = \sigma_0(\delta)$ (no pilot sample). In this simplest case, since

$$\lambda = \sigma_0(\delta) \cdot k_R/k_S \tag{72}$$

and

$$\omega = \sigma^2(\gamma)/\sigma_0^2(\delta) \quad , \tag{73}$$

the value of λ is proportional to k_R/k_S and the value of ω is proportional to $\sigma^2(\gamma)$, for constant $\sigma_0^2(\delta)$. Hence the increase of ω corresponds to the increases of $\sigma^2(\gamma)$, and then it results in the increase of n* so as to improve the accuracy of observed value. This is the reason for property (a) mentioned above. However, if the value of ω increases too far

($\omega \gtrsim 1$ for $n^* \simeq 1$), then it becomes less valuable to observe samples, since prior to observation we had the value of $\sigma_O(\delta)$ which is less than $\sigma(\gamma)$. In other words, Fig. 8 suggests that there is a limit concerning the accuracy of sampling measurement which is available for observation of samples. The limit is written by the relation:

$$\lambda/\omega \gtrsim 10 \quad . \tag{74}$$

Nevertheless it is to be noticed that every one of the contours is represented asymptotically by a straight line in the the λ-to-ω chart as the value of λ increases and the value of ω decreases. This implies that it is possible to express the optimal sample size for a domain of λ and ω by an asymptotic solution. According to Fig. 8 the asymptotic solution is approximated by the formula:

$$n^*_{(a)} = \sqrt{\lambda \cdot \omega/5} \quad ; \quad \lambda/\omega \gtrsim 10^3 \quad . \tag{75}$$

The comparison between the asymptotic solution and the numerical solution for the examples, FFF, FRF and FEF is given by Table 6 and it shows that the formula, Eq. (75), gives almost the exact solution, since each of the examples is subject to the condition $\lambda/\omega \gtrsim 10^3$. Even for $10^2 \leq \lambda/\omega \leq 10^3$ it follows from Fig. 8 that the formula, Eq. (75), overstates the optimal sample size, and therefore it is useful for practical purposes; the difference due to the overstatement is less than 25% of the exact optimal sample size.

From Eqs. (70), (71) and (75) we obtain the relationship:

$$n^*_{(a)} \propto \left(\frac{k_R}{k_S}\right)^{\frac{1}{2}} \cdot (\sigma_{OO}(\delta))^{\frac{1}{2}} \cdot (\varepsilon_\gamma)^{\frac{1}{4}} \cdot (\varepsilon'_\gamma)^{\frac{1}{4}} \cdot \left(\frac{N}{N-1}\right)^{\frac{1}{2}} \cdot \left(1 + \varepsilon_{\Delta\theta} - \frac{\varepsilon'_\gamma}{N-1}\right)^{-\frac{3}{4}} \quad . \tag{76}$$

Furthermore assuming $N\to\infty$ and $\varepsilon'_\gamma = \varepsilon_\gamma$ and using Eqs. (12), (13), (54) and (55), Eq. 76 is reformed into

$$n^*_{(a)} \propto \left(\frac{C_I(O)}{\pi \cdot \delta_{TA} \cdot k_S}\right)^{\frac{1}{2}} \cdot \sigma(\gamma) \cdot \sigma_O(\delta) \cdot \left(\sigma_O^2(\delta) + \sigma^2(\Delta\theta)\right)^{-\frac{3}{4}} \quad . \tag{76}$$

The safety factor π is related to the detection limit concerning the MUF due to diversion, and the lower value of π raises the limit. Therefore Eq. (76) implies that if the detection limit rises, then the corresponding sample size increases. With regard to the variances $\sigma^2(\gamma)$ and $\sigma^2(\Delta\theta)$ it is natural that

$$\sigma^2(\gamma_{ins}) \geq \sigma^2(\gamma_{ope}) \quad , \quad \sigma^2(\theta_{ins}) \geq \sigma^2(\theta_{ope}) \quad , \tag{77}$$

because of the difference between the quantities of the source data available for assessing $\sigma^2(\gamma)$ and $\sigma^2(\theta)$. Hence it is to be considered that the variances $\sigma^2(\gamma)$ and $\sigma^2(\Delta\theta)$ in Eq. (76) should be improved by individual facilities. From this consideration the variances $\sigma^2(\gamma)$ and $\sigma^2(\Delta\theta)$ are related to the technical and economic efforts of individual facilities. According to Eq. (76) the effort to improve the accuracy associated with random error brings about the decrease of sample size in proportion to $\sigma(\gamma)$. On the contrary the effort to reduce the ambiguity associated with systematic error gives rise to the increase of sample size. This is due to the fact that the reduction of $\sigma(\Delta\theta)$ promotes the expected value of observed information.

Consequently Eq. (76) should be regarded as the equation representing the relationship between the limits concerning the inspection plan mentioned earlier and the corresponding optimal sample size. Equation (76) has been derived without regard to the constraints, Eqs. (65) and (66). In applying Eq. (76) for a practical problem, the limits concerning the inspection plan should be selected so that the optimal solution obtained from Eq. (76) may exist within the feasible domain given by the constraints, Eqs. (65) and (66).

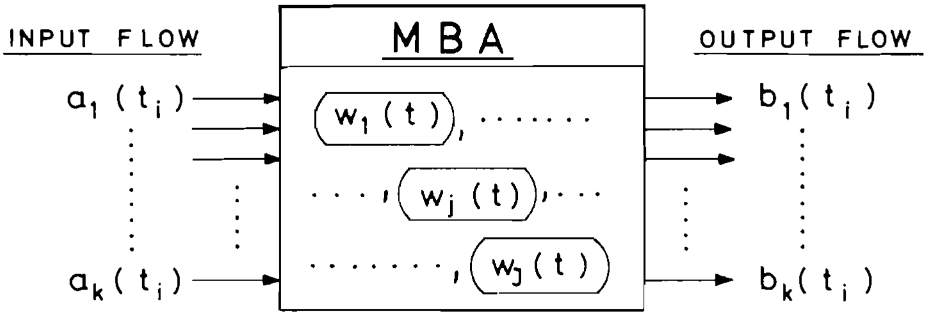
IV. Concluding Remarks

So as to determine the optimal sample size for nuclear materials safeguards inspection and to show the explicit expression of the relationship between the optimal sample size and the parameters which represent the limits concerning the inspection plan, the problem of optimizing the observation plan was formulated by describing the hypotheticality associated with the safeguards problem as an incentive-tax system. Bayesian decision theory was applied to solve the problem: it was found that the formulated model gives the optimal sample size, which is of practical use, and that the asymptotic solution derived from the model is applicable for the sake of understanding the relationship between the optimal sample size and the limits concerning the inspection plan.

It is prescribed in the model that the purpose of the inspection plan is to collect the observed information which is required for making the decision on the two-action problem, whether the operator's data should be accepted or rejected, while evaluating the costs for acts of acceptance and rejection in an incentive-tax system.

Some uncertainties are inevitably involved in the decision making because of the statistical behavior of the variable δ in question, and therefore the effectiveness of the observed information is defined by the expected value of observed information which is expressed by the difference of the costs of uncertainty prior to the observation and posterior to the observation. Thus by subtracting the cost for observing samples from the expected value of observed information, the net gain of observation is defined and the optimal sample size is obtained by maximizing the net gain of observation.

In conclusion, the calculation results based on the formulation suggest to us that (a) while the optimal sample size depends on the break-even value δ_{BE}, which corresponds to a sort of the detection limit of MUF due to diversion, and on the variances of random error and systematic error, $\sigma^2(\gamma)$ and $\sigma^2(\Delta\theta)$, which are related to the technical and economic effort of individual facilities, it is possible to select these parameters by taking the feasibility of the corresponding optimal sample size into consideration, and (b) the ambiguity associated with systematic error is not reduced by taking samples from measured data, but rather systematic error reduces the expected value of observed information, and therefore it is of great importance to research and develop the method to evaluate systematic error.



TOTAL INPUT FLOW AT $t = t_i$: $A(t_i) = \sum_{k=1}^{k} a_k(t_i)$

TOTAL OUTPUT FLOW AT $t = t_i$: $B(t_i) = \sum_{k=1}^{k} b_k(t_i)$

THE DIFFERENCE AT $t = t_i$: $z_i = A(t_i) - B(t_i)$

BEGINNING INVENTORY($t = 0$): $\sum_{j=1}^{J} w_j(0)$

ENDING INVENTORY ($t = T$): $\sum_{j=1}^{J} w_j(T)$

Figure 1: A scheme of material balance area.

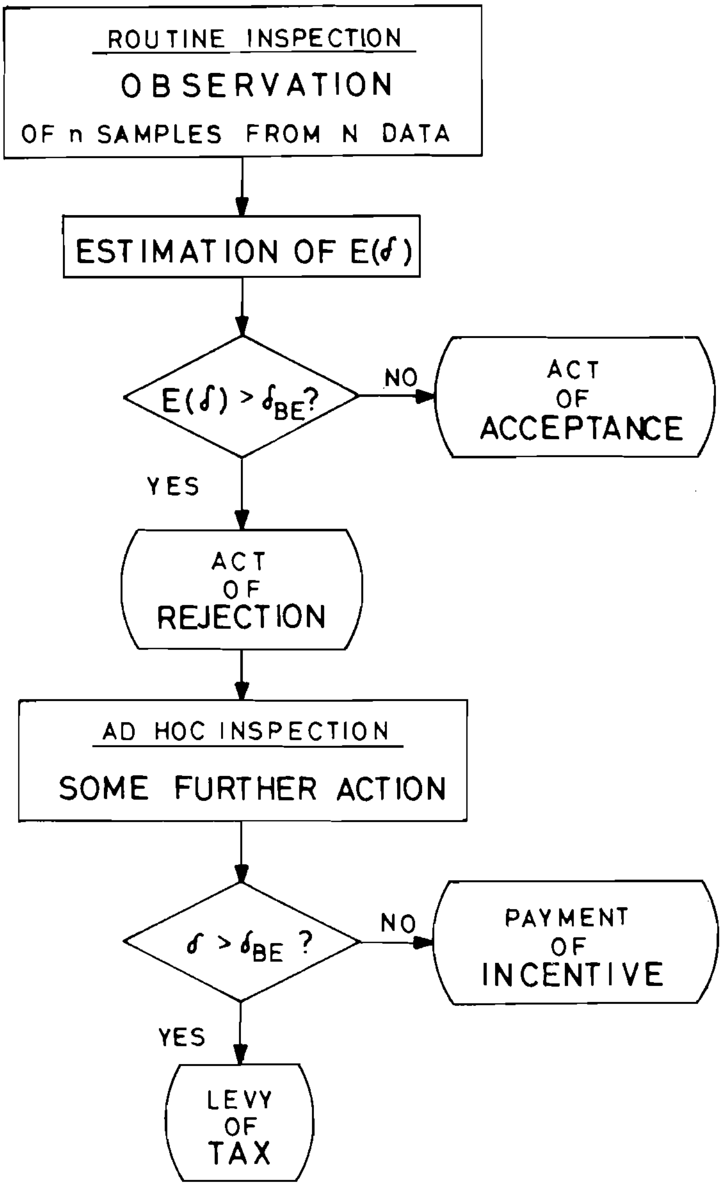
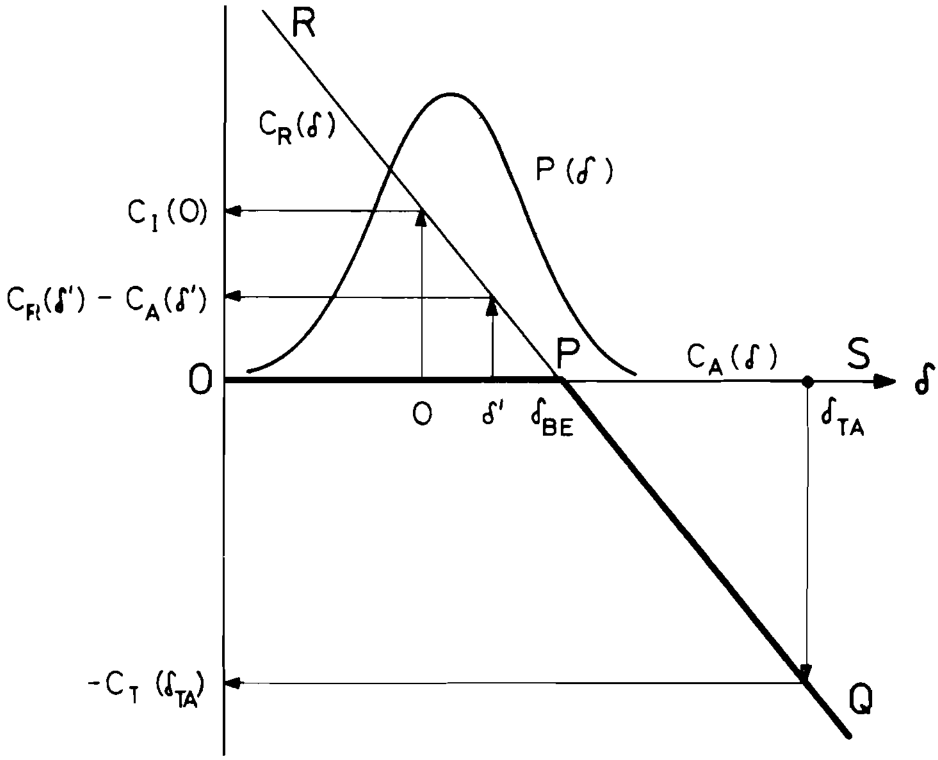


Figure 2: Inspection procedure based on the two-action problem in an incentive-tax system.



STRAIGHT LINE $\overline{OPS}$: $C_A(\delta)$
STRAIGHT LINE $\overline{RPQ}$: $C_R(\delta)$
BENT LINE $\overline{OPQ}$: $C_*(\delta)$

Figure 3: Costs of acts, $C_A(\delta)$ and $C_R(\delta)$ and cost of optimal act for given δ, $C_*(\delta)$.

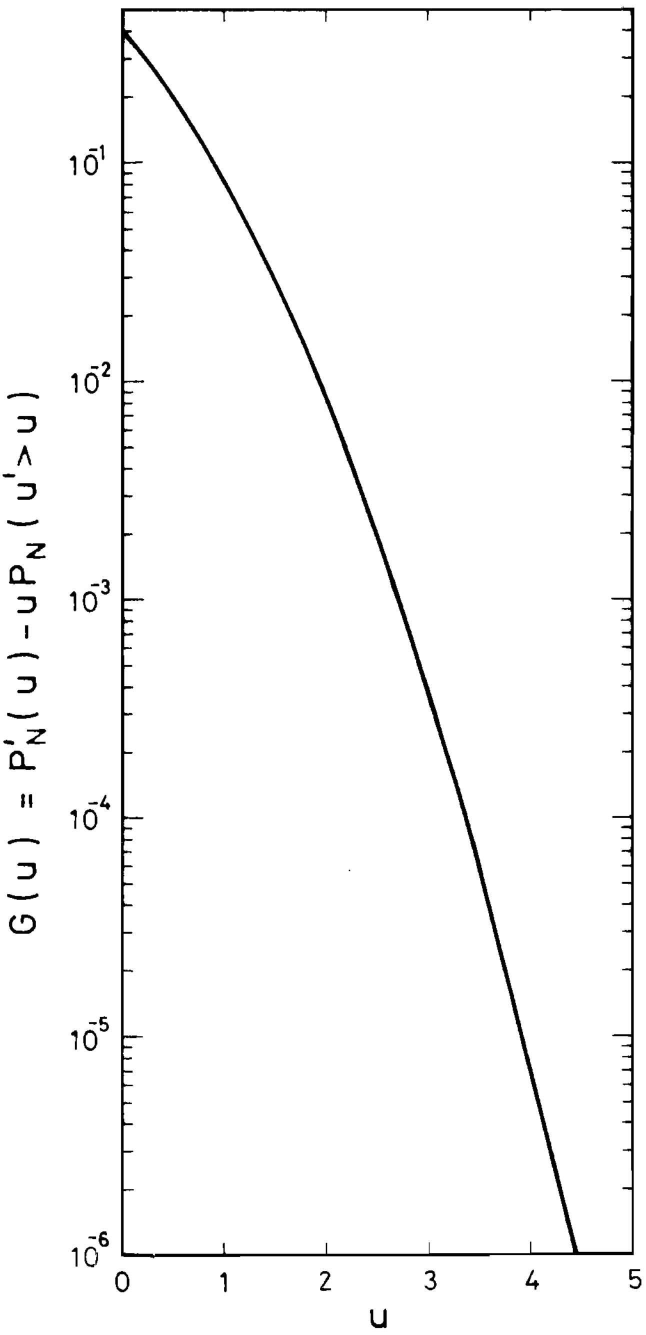


Figure 4: Unit Normal Loss Integral G(u).

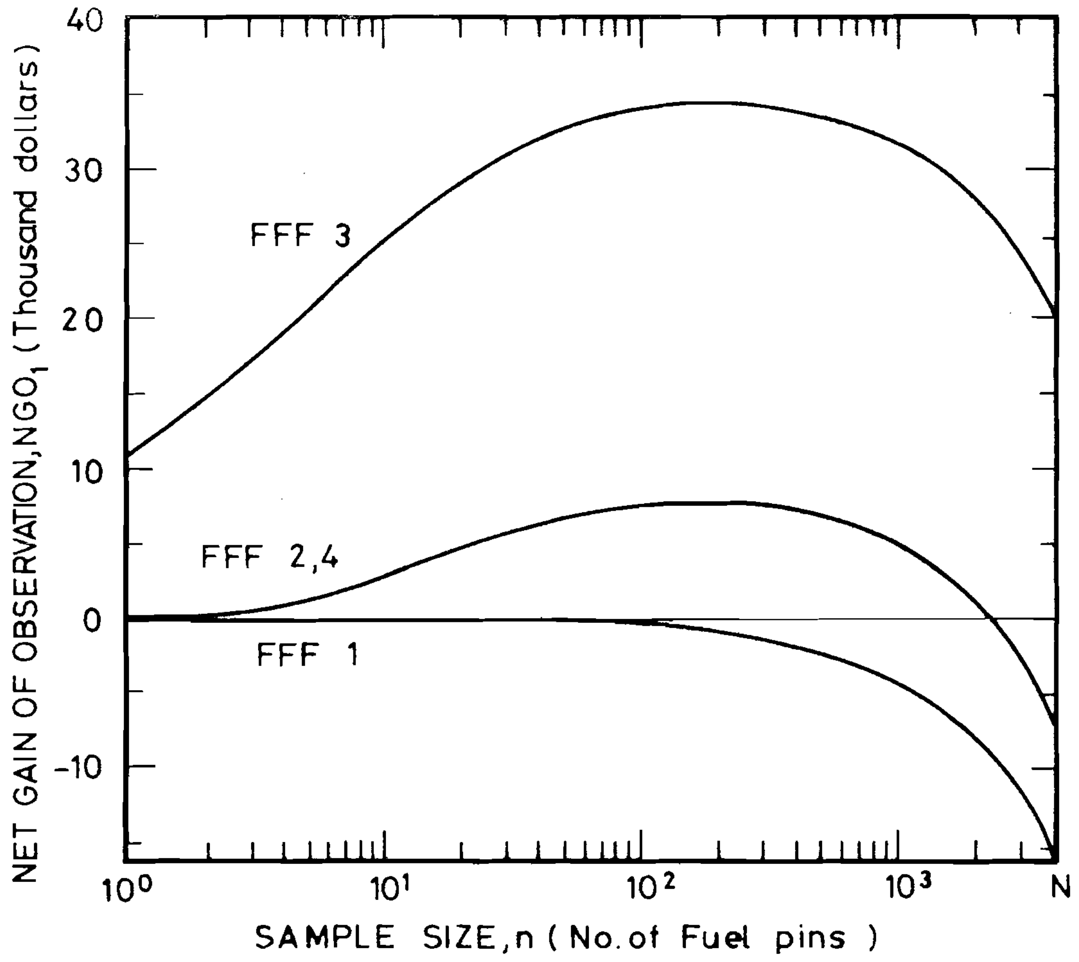


Figure 5: Net gain of observation vs. sample size (for fuel fabrication facility).

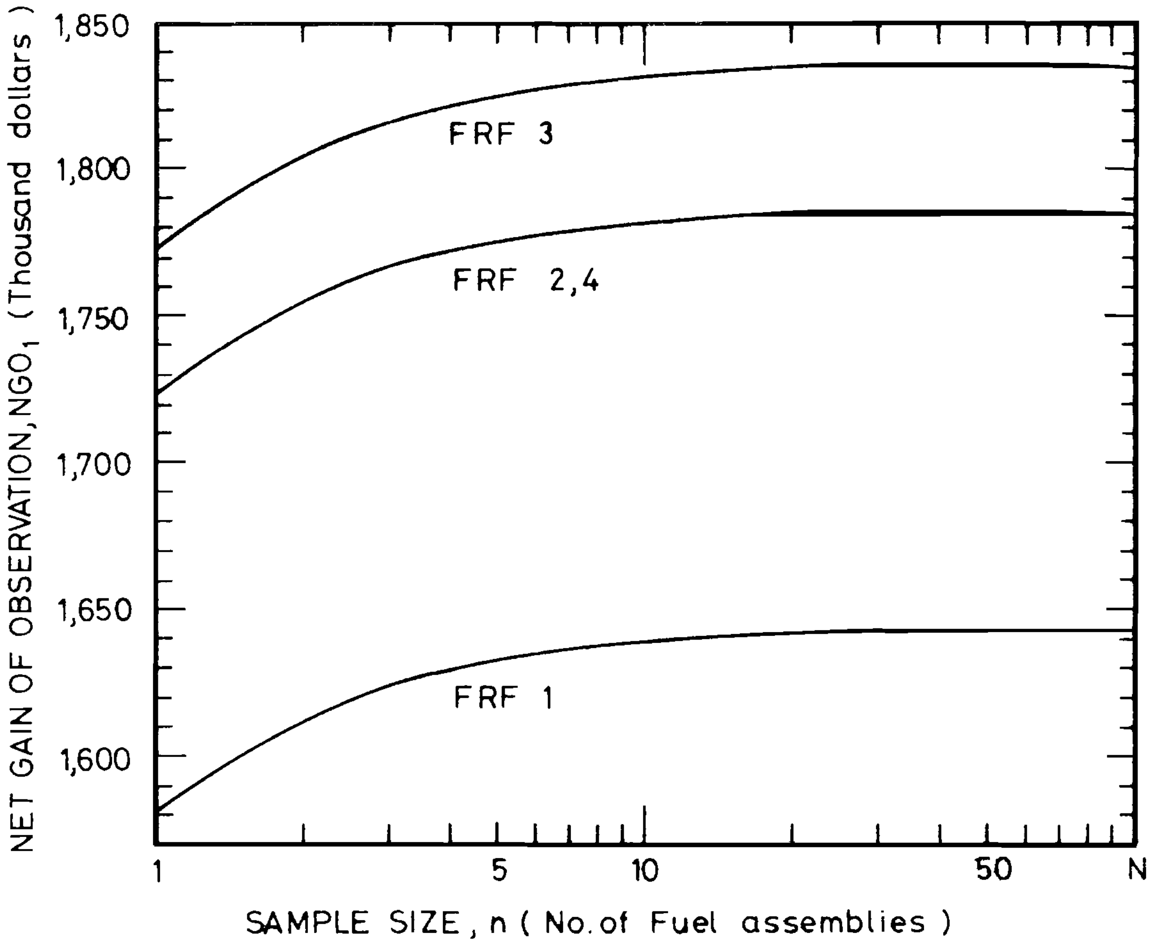


Figure 6: Net gain of observation vs. sample size (for fuel reprocessing facility.

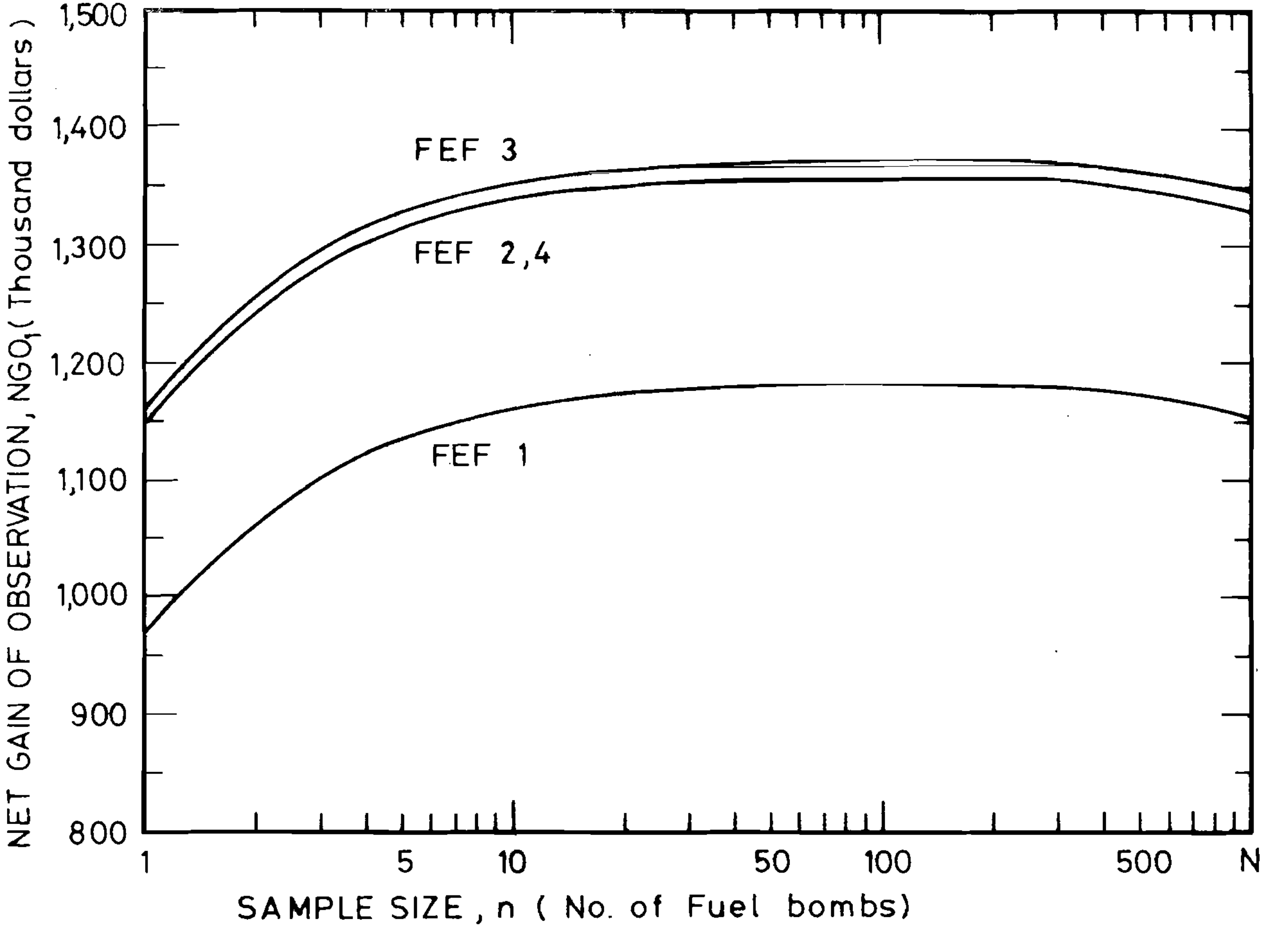


Figure 7: Net gain of observation vs. sample size (for fuel enrichment facility).

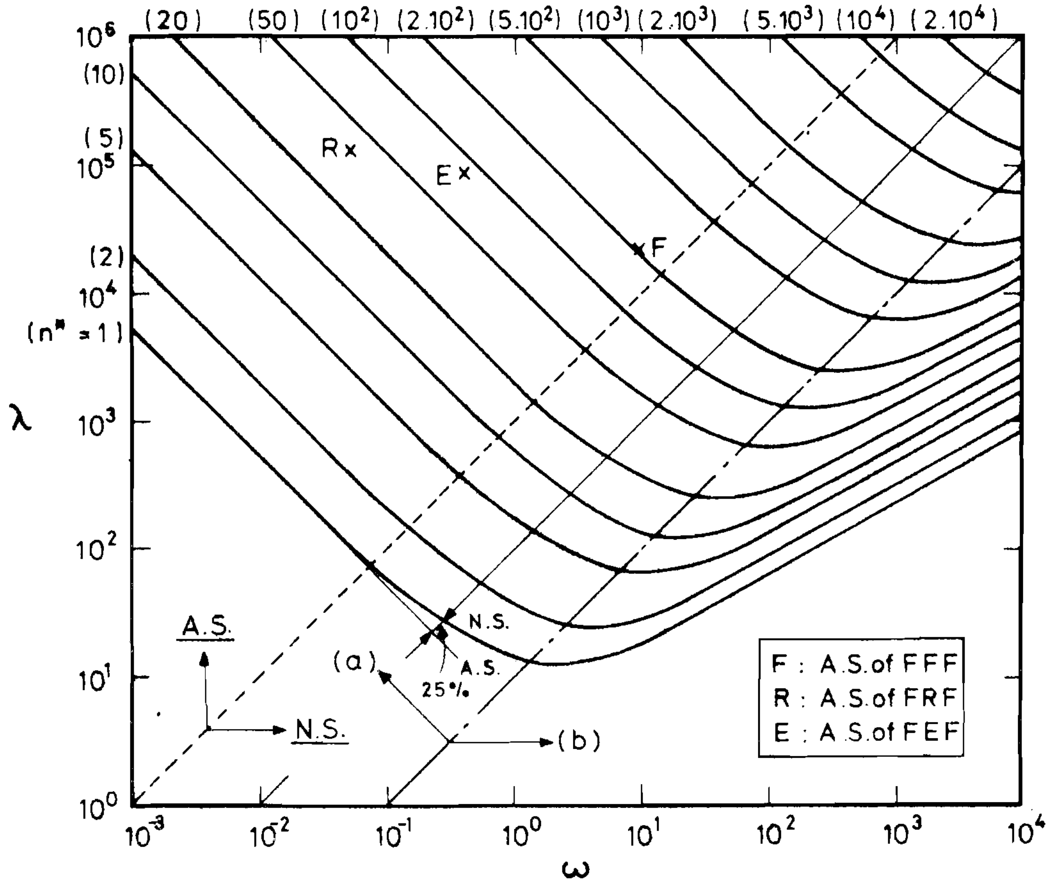


Figure 8: Contour of optimal sample size,
N.S.: Numerical solution n^*
A.S.: Asymptotic solution $n^*_{(a)}$
(a) : the domain where $\partial n^*/\partial\omega > 0$
(b) : the domain where $\partial n^*/\partial\omega < 0$.

Table 1: Relevant data on the specifications and the inspection parameters of the reference facilities.

		Uranium-Fuel Fabr. Facility (FFF)	Fuel Reprocess. Facility (FRF)	Fuel Enrichm. Facility (FEF)
A) Specifications:				
Capacity/year		100 tons of UO_2	200 tons of U	1972 tns. UF_6
No. of Campaigns		5/year	5/year	1/year
Campaign interval		60 days	60 days	300 days
Total pieces/Camp.	N	4165 fuel pins	100 fuel assm.s	986 bombs
Population mean	μ	4482.9 g of 2.5w/O EU	3000 g of Pu	2tns.of UF_6(4%)
E. of r. error	$E(\gamma)$	0 g "	0 g "	0 kg "
S.D. of r. error	$\sigma(\gamma)$	17.8 g "	30 g "	4.0 kg "
E. of s. error	$E(\Delta\theta)$	0 g "	30 g "	0 kg "
S.D. of s. error	$\sigma(\Delta\theta)$	0 g "	42 g "	3.6 kg "
B) Inspection Parameters:				
Threshold Amount/year		25 kg of ^{235}U	8 kg of Pu	25kg of ^{235}U
$C_I(0)$	in Eq. (12)	$ $4x10^5$	$ $4x10^5$	$ $4x10^5$
$C_T(\delta_{TA})$	in Eq. (12)	$ $4x10^5$	$ $4x10^5$	$ $4x10^5$
K_S	in Eq. (63)	$ 0	$ 0	$ 0
k_S	in Eq. (63)	$ 4	$ 40	$ 40
δ_{TA}, T.A./piece		50 g of 2.5 w/O EU	16 g of Pu	938g UF_6(4%)
π	from Eq. (13)	0.5	0.5	0.5
δ_{BE}	from Eq. (13)	25 g of 2.5 w/O EU	8 g of Pu	469g UF_6(4%)
k_R	from Eq. (12)	$ $1.6x10^4$/g	$ $5x10^4$/g	$ $8.53x10^2$/g

Table 2: Calculation results for the fuel fabrication facility.

		FFF 1	FFF 2	FFF 3	FFF 4
1) Prior Distribution $P_0(\delta)$					
$E_0(\delta)$	g of 2.5 w/o EU	0	20.0	25.0	30.0
$\sigma_0(\delta)$	g of 2.5 w/o EU	5.63	5.63	5.63	5.63
$(\sigma_{00} \rightarrow \infty;\ N_0 = 10)$					
Optimal Act under $P_0(\delta)$		Accept	Accept	Either	Reject
$EVPI\|_{P_0}$ a)		7.52×10^{-2}	9.04×10^3	3.58×10^4	9.04×10^3
2) Optimal Sample Size					
n*	from Eqs.(64-66)	----	205	205	205
$EVOI_1$	from Eq. (59)	----	8.68×10^3	3.51×10^4	8.68×10^3
C_S	from Eq. (63)	----	8.20×10^2	8.20×10^2	8.20×10^2
NGO_1	from Eq. (62)	----	7.46×10^3	3.43×10^4	7.46×10^3
3) Inspection Results					
$\overline{\Delta x}$	g of 2.5 w/o EU	----	24.0	24.0	24.0
$\overline{\Delta x} - E(\Delta\theta)$	g of 2.5 w/o EU	----	24.0	24.0	24.0
$\sigma(\gamma)$	from Eq. (22)	----	20.0	20.0	20.0
$\sigma(\overline{\gamma})$	from Eq. (21)	----	1.36	1.36	1.36
$\sigma(\overline{\Delta x})$	from Eq. (33)	----	1.36	1.36	1.36
4) Posterior Distribution $P_1(\delta)$					
I_0	from Eq. (41)	----	0.032	0.032	0.032
$I_{\overline{\Delta x}}$	from Eq. (42)	----	0.540	0.540	0.540
I_1	from Eq. (40)	----	0.572	0.572	0.572
$v_1(\delta)$	from Eq. (40)	----	1.32	1.32	1.32
$E_1(\delta)$	from Eq. (39)	----	23.7	24.1	24.3
Optimal Act under $P_1(\delta)$		----	Accept	Accept	Accept
$EVPI\|_{P_1}$ b)		----	1.89×10^3	2.77×10^3	3.93×10^3

a) $EVPI|_{P_0} = k_R \sigma_0(\delta) G(D_0)$
where,
$D_0 = |\delta_{BE} - E_0(\delta)| / \sigma_0(\delta)$

b) $EVPI|_{P_1} = k_R \sigma_1(\delta) G(D_1)$,
where
$D_1 = |\delta_{BE} - E_1(\delta)| / \sigma_1(\delta)$

Table 3: Calculation results for the fuel reprocessing facility.

	FRF 1	FRF 2	FRF 3	FRF 4
1) Prior Distribution, $P_0(\delta)$				
$E_0(\delta)$ g of Pu	0	6.0	8.0	10.0
$\sigma_0(\delta)$ g of Pu	100	100	100	100
($\sigma_{00}=\sigma_0$, $n_0=0$)				
Optimal Act under $P_0(\delta)$	Accept	Accept	Either	Reject
$EVPI\|_{P_0}$	1.80×10^6	1.94×10^6	1.99×10^6	1.94×10^6
2) Optimal Sample Size				
n^* from Eqs. (64-66)	42	42	42	42
$EVOI_1$ from Eq. (59)	1.64×10^6	1.79×10^6	1.84×10^6	1.79×10^6
C_S from Eq. (63)	1.68×10^3	1.68×10^3	1.68×10^3	1.68×10^3
NGO_1 from Eq. (62)	1.64×10^6	1.79×10^6	1.84×10^6	1.79×10^6
3) Inspection Results				
$\Delta\bar{x}$ g of Pu	37.0	37.0	37.0	37.0
$\Delta\bar{x} - E(\Delta\theta)$ g of Pu	7.0	7.0	7.0	7.0
$\sigma(\gamma)$ from Eq. (22)	35.0	35.0	35.0	35.0
$\sigma(\bar{\gamma})$ from Eq. (21)	4.15	4.15	4.15	4.15
$\sigma(\Delta\bar{x})$ from Eq. (33)	42.2	42.2	42.2	42.2
4) Posterior Distribution, $P_1(\delta)$				
I_0 from Eq. (41)	1.00×10^{-4}	1.00×10^{-4}	1.00×10^{-4}	1.00×10^{-4}
$I_{\Delta\bar{x}}$ from Eq. (42)	5.64×10^{-4}	5.64×10^{-4}	5.64×10^{-4}	5.64×10^{-4}
I_1 from Eq. (40)	6.64×10^{-4}	6.64×10^{-4}	6.64×10^{-4}	6.64×10^{-4}
$\sigma_1(\delta)$ from Eq. (40)	38.8	38.8	38.8	38.8
$E_1(\delta)$ from Eq. (39)	5.95	6.85	7.15	7.45
Optimal Act under $P_1(\delta)$	Accept	Accept	Accept	Accept
$EVPI\|_{P_1}$	7.24×10^5	7.45×10^5	7.53×10^5	7.60×10^5

Table 4: Calculation results for the fuel enrichment facility.

	FEF 1	FEF 2	FEF 3	FEF 4
1) Prior Distribution $P_0(\delta)$				
$E_0(\delta)$ g of UF_6 (4%)	0	438	469	500
$\sigma_0(\delta)$ g of UF_6 (4%) ($\sigma_{00}=\sigma_0$, $n_0=0$)	5000	5000	5000	5000
Optimal Act under $P_0(\delta)$	Accept	Accept	Either	Reject
$EVPI\|_{P_0}$	1.51×10^6	1.69×10^6	1.70×10^6	1.69×10^6
2) Optimal Sample Size				
n* from Eqs. (64-66)	85	85	85	85
$EVOI_1$ from Eq. (59)	1.19×10^6	1.36×10^6	1.38×10^6	1.36×10^6
C_S from Eq. (63)	3.40×10^3	3.40×10^3	3.40×10^3	3.40×10^3
NGO_1 from Eq. (62)	1.18×10^6	1.36×10^6	1.37×10^6	1.36×10^6
3) Inspection Results				
$\Delta\bar{x}$ g of UF_6 (4%)	450	450	450	450
$\Delta\bar{x} - E(\Delta\theta)$ g of UF_6 (4%)	450	450	450	450
$\sigma(\gamma)$ from Eq. (22)	4000	4000	4000	4000
$\sigma(\bar{\gamma})$ from Eq. (21)	415	415	415	415
$\sigma(\Delta\bar{x})$ from Eq. (33)	3624	3624	3624	3624
4) Posterior Distribution, $P_1(\delta)$				
I_0 from Eq. (41)	4.0×10^{-8}	4.0×10^{-8}	4.0×10^{-8}	4.0×10^{-8}
$I_{\Delta\bar{x}}$ from Eq. (42)	7.6×10^{-8}	7.6×10^{-8}	7.6×10^{-8}	7.6×10^{-8}
I_1 from Eq. (40)	11.6×10^{-8}	11.6×10^{-8}	11.6×10^{-8}	11.6×10^{-8}
$\sigma_1(\delta)$ from Eq. (40)	2936	2936	2936	2.936
$E_1(\delta)$ from Eq. (39)	295	442	457	467
Optimal Act under $P_1(\delta)$	Accept	Accept	Accept	Accept
$EVPI\|_{P_1}$	9.27×10^5	9.88×10^5	9.94×10^5	9.99×10^5

Table 5: Comparison of the cost of uncertainty for various sample size.

	Under Prior Distribution $P_0(\delta)$	Under Posterior Distribution, $P_1(\delta)$		
		n=1	n=n*	n=N
Fuel Fabrication Facility (FFF 3) n*=205, N=4165	$\$3.58\times10^4$	$\$3.38\times10^4$ (94.3%)	$\$2.77\times10^3$ (7.7%)	\$ 0 (0%)
Fuel Reprocessing Facility (FRF 3) n*=42, N=100	$\$1.99\times10^6$	$\$9.38\times10^5$ (47.1%)	$\$7.53\times10^5$ (37.8%)	$\$7.51\times10^5$ (37.7%)
Fuel Enrichment Facility (FEF 3) n*=85, N=986	$\$1.70\times10^6$	$\$1.24\times10^6$ (73.1%)	$\$9.94\times10^5$ (58.4%)	$\$9.89\times10^5$ (58.2%)

( ): the ratio to the cost of uncertainty under $P_0(\delta)$

Table 6: Comparison between asymptotic solutions and numerical solutions.

			FFF	FRF	FEF
Population Size		N	4,165	100	986
S.D. of original distribution		$\sigma_{00}(\delta)$	∞	100	5,000
Pilot Sample Size		n_0	10	0	0
ε_γ	from Eq. (54)		0	0.09	0.64
$\varepsilon_{\Delta\theta}$	from Eq. (55)		0	0.18	0.52
ε'_γ	from Eq. (56)		10	0.09	0.64
λ	from Eq. (70)		$2.25\text{x}10^4$	$1.15\text{x}10^5$	$8.65\text{x}10^4$
ω	from Eq. (71)		$1.00\text{x}10^1$	$7.73\text{x}10^{-2}$	$4.22\text{x}10^{-1}$
<u>Asymptotic Solution</u>		$n^*_{(a)}$	<u>212</u>	<u>42</u>	<u>85</u>
<u>Numerical Solution</u>		n^*	<u>205</u>	<u>42</u>	<u>85</u>

References

[1] Avenhaus, R., "Application of Theory of Games to Safeguards Problems," Proceedings of the Symposium on Implementing Nuclear Safeguards (Oct. 1971).

[2] Bouchey, G.D., Koen, B.V., and Beightler, C.S., "Optimization of Nuclear Materials Safeguards Sampling Systems by Dynamic Programming," Nucl. Tech., 12, 1 (Sept. 1971), 18-25.

[3] Brown, F. et al., "Progress in the Development of a System for the Safeguards Inspection of the Fuel Store of a Zero-Energy Fast Reactor," UKAEA Rep. COS 20 (Oct. 1970).

[4] Constanzi, F. et al., "Optimization of Inspections, KSU/DS-25 Final Report, Political and Scientific Effectiveness in Nuclear Materials Control," Kansas State University, U.S.A. (June 1972).

[5] Gmelin, W., "An Approximation for the Determination of the Sample Size for Paired Comparisons," IAEA/STR/36.

[6] Haefele, W., "Hypotheticality and the New Challenges, the Pathfinder Role of Nuclear Energy," Minerva, 10 (July 1974), 303-322.

[7] Haefele, W., "Systems Analysis in Safeguards of Nuclear Material," Proceedings of the Fourth International Conference on Peaceful Uses of Atomic Energy, Vol. 9 (Sept. 1971), 303-322.

[8] International Atomic Energy Agency, "Safeguards Technical Working Group on Accuracy of Nuclear Material Accountancy and Technical Effectiveness of Safeguards," IAEA PL-488 (Aug. 1972), 4.1-4.13.

[9] Jaech, J., "Inventory Verification Based on Measured Data," Jersey Nuclear Co., U.S.A. (April 1972).

[10] Nakajima, K., Koizumi, T., Suyama, N., and Kishimoto, Y., "Analysis of Accountability System in a Reprocessing Plant," IAEA/SM-133/12 (July 1970).

[11] Oak Ridge Operation Office, United States Atomic Energy Commission, "Data on New Gaseons Diffusion Plants," ORO-685 (April 1972).

[12] Otto, R., Wenk, E., Kraska, E., Avenhaus, R., and Gupta, D., "Assessment of MUF and Inspection Efforts in a Centrifuge Plant," paper to be presented at the IAEA Working Group Meeting on Safeguards Procedures for Isotopic Enrichment Facilities (June 1972).

[13] Raiffa, H. Decision Analysis: Introductory Lectures on Choices under Uncertainty, Reading, Mass. Addison-Wesley, 1968.

[14] Schlaifer, R., Probability and Statistics for Business Decisions, Kogakusha, Tokyo, McGraw-Hill Co., 1959.

[15] Servais, F., and Goldschmidt, P., "Safeguards Inspection--An Analysis of Detection Methods and Diversion Strategies," Nucl. Tech., 12, 3 (Nov. 1971), 290-297.

[16] Stewart, K., "A Cost Effectiveness Approach to Inventory Verification," IAEA/SM-133/59 (July 1970).