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THE RECOVERY OF DETAILED MIGRATION PATTERNS FROM AGGREGATE DATA: AN ENTROPY MAXIMIZING APPROACH

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Preface

Interest in human settlement systems and policies has been a critical part of urban-related work at IIASA since its inception. Recently this interest has given rise to a concentrated research effort focusing on migration dynamics and settlement patterns. Four sub-tasks form the core of this research effort:

I. the study of spatial population dynamics;

II. the definition and elaboration of a new research area called demometrics and its application to migration analysis and spatial population forecasting;

III. the analysis and design of migration and settlement policy;

IV. a comparative study of national migration and settlement patterns and policies.

This paper, the eighth in the comparative study series, addresses the problem of estimating, from aggregate data, detailed migration flows for population categories, such as age and nationality. The estimation procedure proposed is the entropy maximization method. This method has been shown to be useful for a number of countries of the Comparative Study, where disaggregate migration data are not available.

Related papers in the comparative study series, and other publications of the migration and settlement study are listed at the end of this report.

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Abstract

The entropy maximization method is used to estimate interregional migration flow matrices for subgroups of the population. The method is presented in lucid terms and a number of practical applications are given, such as the estimation of age-specific migration flows.

Acknowledgments

The practical problem of estimating migration flow matrices by age and by nationality from aggregate data arose in 1974-1975 when I was working with the Brabant Regional Economic Council (GERB-CERB) in Brussels on the development of an urban simulation model for Brussels. I acknowledge with thanks the logistic and financial aid provided at that time by the Council. This paper is a revised version of part of the report on the simulation model (Willekens, 1976).

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The Recovery of Detailed Migration Patterns from Aggregate Data: An Entropy Maximizing Approach

Regional analysis is handicapped by the lack of statistical data. Sophisticated models of regional growth have been developed, but their application is limited since most countries are unable to provide the required input data set. One way to cope with the problem is to estimate or to generate the lacking data. Several estimation methods are known in the literature. One which recently has received considerable attention in urban and regional analysis is entropy maximization.

It is the purpose of this paper to present the method in lucid terms, to allow the practitioner to apply the method in his own situation while realizing the basic assumptions on which the entropy method is based. In the first section, we state the problem for which entropy maximization is relevant. The method is described in the next section. A third section discusses the validity of the method to recover detailed migration patterns from aggregate data. A fourth section applies the method to two practical problems of migration research in Belgium: the estimation of internal migration flows by age, and by nationality.

1. Entropy Maximization

1.1 The Problem

Suppose we have a two-region system. The problem is to find the origins and the destinations of the migrants that moved in a certain time period. Suppose that the only information we have is the departures and the arrivals by region. We assume a closed system, i.e., the total number of arrivals equals the total number of departures.

The multiregional system may be represented by the following origin-destination table $\underset{\sim}{M}$. (Figure 1).

		Origin		Arrivals
		Region 1	Region 2	
Destination	Region 1	m_{11}	m_{21}	$m_{\cdot 1} = 3$
	Region 2	m_{12}	m_{22}	$m_{\cdot 2} = 3$
Departures		$m_{1\cdot} = 4$	$m_{2\cdot} = 2$	$m_{\cdot\cdot} = 6$

Figure 1

The Origin-Destination Table $\underset{\sim}{M}$

where

m_{ij} is the number of migrants going from region i to region j, e.g., m_{12} is the number of people migrating from region 1 to 2 in the unit time interval;

$m_{\cdot 1}$ is the total number of arrivals in region 1

$m_{\cdot 1} = m_{11} + m_{21} = \sum_i m_{i1}$;

$m_{\cdot 2}$ is the total number of arrivals in region 2

$m_{\cdot 2} = m_{12} + m_{22} = \sum_i m_{i2}$;

$m_{1 \cdot}$ is the total number of departures from region 1

$m_{1 \cdot} = m_{11} + m_{12} = \sum_j m_{1j}$;

$m_{2 \cdot}$ is the total number of departures from region 2

$m_{2 \cdot} = m_{21} + m_{22} = \sum_j m_{2j}$;

$m_{\cdot \cdot}$ is the total number of migrants

$m_{\cdot \cdot} = \sum_i m_{i \cdot} = \sum_j m_{\cdot j} = \sum_i \sum_j m_{ij}$.

The row totals and column totals, i.e., the total number of arrivals and departures, are known. The problem is to find the entries m_{11}, m_{12}, m_{21} and m_{22}, such that they add up to the known totals. The constraints imposed on the estimation procedure are therefore:

$$\sum_j m_{ij} = m_{i \cdot} \quad , \qquad \text{for } i = 1,2 \tag{1}$$

$$\sum_i m_{ij} = m_{\cdot j} \quad , \qquad \text{for } j = 1,2 \tag{2}$$

There may be a large number of combinations of entries of $\underset{\sim}{M}$ that satisfy the constraints (1) and (2). In the simple illustration of Figure 1, there are three possible arrangements.

$$\underset{\sim}{M}_a = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \quad \underset{\sim}{M}_b = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \quad \underset{\sim}{M}_c = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \qquad (3)$$

Each arrangement of the entries of $\underset{\sim}{M}$ is called a macrostate of the system.

1.2 The Idea

The true migration flow is represented by one of the three macrostates $\underset{\sim}{M}_a$, $\underset{\sim}{M}_b$ or $\underset{\sim}{M}_c$. Given the limited information we have about the migration behavior, we don't know which macrostate is the true one. Therefore, we must make a guess. It is here that the entropy method comes in. It selects the macrostate which has the highest probability of occurring. A certain macrostate may be generated by various so-called microstates. A microstate is an assignment of individual migrants to the origin-destination table. Consider, for example, the matrix $\underset{\sim}{M}_a$, and denote the individual migrants by m1, m2, m3, m4, m5 and m6. According to $\underset{\sim}{M}_a$, three people migrate from region 1 to 1, i.e., move within the region. Out of the six migrants, we can select the three in 20 different ways, as shown in Figure 2.

	m1	m2	m3	m4	m5	m6
1	x	x	x			
2	x	x		x		
3	x	x			x	
4	x	x				x
5	x		x	x		
6	x		x		x	
7	x		x			x
8	x			x	x	
9	x			x		x
10	x				x	x
11		x	x	x		
12		x	x		x	
13		x	x			x
14		x		x	x	
15		x		x		x
16		x			x	x
17			x	x	x	
18			x	x		x
19			x		x	x
20				x	x	x

Figure 2

Combinations of three out of six

The possible combinations of three people out of six can easily be computed by the familiar combinatorial formula of statistics:

$$\frac{6!}{3!(6-3)!}$$

where ! denotes the factorial operation, e.g.,

$$6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = \prod_{i=1}^{6} i \quad .$$

Once we have made a selection of three people to constitute m_{11}, we must select one person out of the remaining three to constitute m_{12}. There are only three possible ways of selecting one person out of three, or

$$\frac{3!}{1!(3-1)!} = 3 \quad .$$

Finally, the two remaining individuals constitute m_{22}, since $m_{21} = 0$. Therefore, the total number of ways of selecting three out of six, and one out of the remaining three, and two out of the remaining two is

$$\frac{6!}{3!(6-3)!} \times \frac{3!}{1!(3-1)!} \times \frac{2!}{2!} = 20 \times 3 \times 1 = 60 \quad .$$

Each of the 60 ways constitutes a separate microstate, or assignment of individuals. In general, the number of ways in which we can select a <u>particular</u> macrostate from the total number of migrants $m_{..}$ is the combinatorial formula:

$$W = \frac{m_{..}!}{m_{11}!m_{12}!m_{21}!m_{22}!}$$

$$W = \frac{m_{..}!}{\prod\limits_{i,j} m_{ij}!} \quad . \tag{4}$$

Applying (4), we get W = 60 for $\underset{\sim}{M}_a$, W = 180 for $\underset{\sim}{M}_b$ and W = 60 for $\underset{\sim}{M}_c$. The value W is the number of microstates which give rise to a particular macrostate, and is called the <u>entropy</u> of the macrostate.

The question of which macrostate to choose as the best estimate of the true migration flow, may now be answered. We choose the macrostate with the highest entropy-value. The use of this selection criterion relies on two critical assumptions:

- The probability that a macrostate represents the true migration flow matrix, is proportional to the number of microstates of the system which give rise to this macrostate (entropy), and which satisfy the constraints (1) and (2).
- Each microstate is equally probable.

1.3 The Formal Solution

The first assumption, read in a slightly different way, becomes: the true arrangement of a system is one which maximizes the entropy. This is the second law of thermodynamics. This analogy between the behavior of social and physical systems is not accidental. Several authors have attempted to describe social phenomena by laws from physics (e.g., Isard, 1960). This approach is identified as social physics, which is well known in the early regional science literature. It is, however, unfair to evaluate the application of entropy methods in social sciences only by the physical meaning of the entropy concept.

The use of the entropy concept in social sciences may also be justified by means of information theory (Jaynes, 1957), by means of Bayes' theorem for conditional probabilities (Hyman, 1969), or by means of the maximum likelihood estimators (Evans, 1971). See also Wilson (1970, pp. 1-10).

The estimation problem of finding the most probable migration flow matrix which satisfies the constraints (1) and (2), may now be formulated as follows: find the macrostate with maximum entropy, W, subject to constraints (1) and (2). The solution is given by the solution to the mathematical programming problem:

$$\max W = \frac{m_{\bullet\bullet}!}{\prod_{i,j} m_{ij}!} \tag{5}$$

$$\text{subject to } \sum_j m_{ij} = m_{i\cdot} \quad , \qquad \text{for all } i \tag{1}$$

$$\sum_i m_{ij} = m_{\cdot j} \quad , \qquad \text{for all } j \quad . \tag{2}$$

Since the maximum of (5) coincides with the maximum of any monotonic function of W, we may replace W by the Naperian logarithm of W (ℓn W) in the objective function.

$$\ell n \; W = \ell n \; m_{\cdot\cdot}! - \ell n \prod_{i,j} m_{ij}! \tag{6}$$

$$= \ell n \; m_{\cdot\cdot}! - \sum_i \sum_j \ell n \; m_{ij}! \quad . \tag{7}$$

Function (7) is very complex. To make differentiation of (7) easier, we replace $\ell n \; m_{ij}!$ by Stirling's approximation:

$$\ell n \; m_{ij}! = m_{ij} \; \ell n \; m_{ij} - m_{ij} \quad .$$

Since $\ell n \; m_{\cdot\cdot}!$ is a constant, we may write the objective function as

$$\max \; \ell n \; \hat{W} = - \sum_i \sum_j \left[m_{ij} \; \ell n \; m_{ij} - m_{ij} \right] \quad .$$

In most applications of entropy methods, the constraints (1) and (2) are augmented with a budget constraint:

$$\sum_i \sum_j m_{ij} \; c_{ij} = C \tag{8}$$

where c_{ij} is the cost of moving from i to j, and C is the total budget available to all migrants. We introduce this cost function for completeness, although in the applications c_{ij} is set equal to zero for all i and j.

Formally, the entropy problem is:

$$\max \hat{W} = - \sum_i \sum_j \left[m_{ij} \ln m_{ij} - m_{ij} \right] \tag{9}$$

$$\text{subject to } \sum_j m_{ij} = m_{i\cdot} \quad , \quad \text{for all } i \tag{1}$$

$$\sum_i m_{ij} = m_{\cdot j} \quad , \quad \text{for all } j \tag{2}$$

$$\sum_i \sum_j m_{ij} c_{ij} = C \quad . \tag{8}$$

The solution to this problem is found by the method of the Lagrange multipliers. The Lagrangean is

$$L = \ln \hat{W} + \sum_i \lambda_i \left(m_{i\cdot} - \sum_j m_{ij} \right) + \sum_j \mu_j \left(m_{\cdot j} - \sum_i m_{ij} \right)$$

$$+ \nu \left(C - \sum_i \sum_j m_{ij} c_{ij} \right) \quad .$$

The necessary conditions for a maximum are

$$\frac{\delta L}{\delta m_{ij}} = 0 = - \ln m_{ij} - \lambda_i - \mu_j - \nu c_{ij} = 0 \tag{10}$$

$$\frac{\delta L}{\delta \lambda_i} = 0 = m_{i\cdot} - \sum_j m_{ij} = 0 \tag{11}$$

$$\frac{\delta L}{\delta \mu_j} = 0 = m_{\cdot j} - \sum_i m_{ij} = 0 \tag{12}$$

$$\frac{\delta L}{\delta \nu} = 0 = C - \sum_i \sum_j m_{ij} \, c_{ij} = 0 \quad . \tag{13}$$

From (10) we have

$$\ell n \, m_{ij} = - \lambda_i - \mu_j - \nu c_{ij}$$

or

$$m_{ij} = e^{-\lambda_i - \mu_j - \nu c_{ij}} \quad . \tag{14}$$

Writing

$$A_i = e^{-\lambda_i} / m_{i\cdot} \tag{15}$$

$$B_j = e^{-\mu_j} / m_{\cdot j} \tag{16}$$

gives

$$m_{ij} = A_i \, B_j \, m_{i\cdot} \, m_{\cdot j} \, e^{-\nu c_{ij}} \quad . \tag{17}$$

The coefficients A_i and B_j are called <u>balancing factors</u>. They act as repulsion and attraction forces for generating the migration flows. Substituting (17) in (12) yields

$$\sum_j A_i \, B_j \, m_{i\cdot} \, m_{\cdot j} \, e^{-\nu c_{ij}} = m_{i\cdot}$$

$$A_i \, m_{i\cdot} \sum_j B_j \, m_{\cdot j} \, e^{-\nu c_{ij}} = m_{i\cdot}$$

$$A_i = \frac{1}{\sum_j B_j m_{\cdot j} e^{-\nu c_{ij}}} \qquad (18)$$

Substituting (17) in (13) gives

$$\sum_i A_i \; B_j \; m_{i\cdot} \; m_{\cdot j} \; e^{-\nu c_{ij}} = m_{\cdot j}$$

$$B_j = \frac{1}{\sum_i A_i m_{i\cdot} e^{-\nu c_{ij}}} \quad . \qquad (19)$$

The entropy maximization is then given by the solution of the system of nonlinear equations (17), (18) and (19). Since we have no information on the cost of migrating and on the total budget available, we may assume that $C_{ij} = 0$ for all i and j. The system of equations becomes then:

$$m_{ij} = A_i \; B_j \; m_{i\cdot} \; m_{\cdot j} \quad , \qquad \text{for all i and j} \qquad (20)$$

$$A_i = \left[\sum_j B_j \; m_{\cdot j}\right]^{-1} \quad , \qquad \text{for all i} \qquad (21)$$

$$B_j = \left[\sum_i A_i \; m_{i\cdot}\right]^{-1} \quad , \qquad \text{for all j} \quad . \qquad (22)$$

This system has a very simple solution; namely (Raquillet and Willekens, 1977),

$$m_{ij} = \frac{m_{i\cdot} \; m_{\cdot j}}{m_{\cdot\cdot}}$$

Application of this formula to the data in Figure 1 yields:

$$\begin{bmatrix} 3.4/_6 & 3.2/_6 \\ 3.4/_6 & 3.2/_6 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix},$$

which is $\underset{\sim}{M}_b$, the arrangement with the highest entropy.

The first order conditions (10) to (13) only lead to a local extremum of the objective function (9). However, this local extremum coincides with the global extremum since $\ell n\ W$ is a strictly concave function.

The entropy solution represents the most probable or most likely flow between each region of origin i and each region of destination j. This distribution function coincides with the production-attraction constrained gravity model (Wilson, 1970, p.42). Therefore, entropy maximization has been used to provide a theoretical underpinning of the gravity model (Nijkamp, 1975, p.210; Wilson, 1970, pp. 47-49). Nijkamp and Paelinck (1974) have attempted to provide a behavioral interpretation of the entropy and gravity model by formulating the entropy model as a dual geometric programming model. By doing so, the objective function, inherent in the entropy maximizing model, attempts to maximize the imputed net "profits" of a spatial system in terms of the difference between positive and negative interaction stimuli (Nijkamp, 1975, p. 213). We will not elaborate on this theory. The interested reader is referred to the literature.

2. Disaggregation of Migration Flows

2.1 The Problem

Suppose that we know the migration flow matrix of the total population, and that we are interested in the migration patterns of subsets of the population, e.g., sexes, age groups, nationalities, professional categories, etc. Suppose that we also know the number of arrivals and departures of each subset in each region. The system may be represented in a three-dimensional space (Figure 3). We know the outside of the black box, but not its contents. The total migration from i to j is $m_{ij}(\cdot)$, the number of departures from i by category x is $m_{i.}(x)$ and the number of arrivals in j is $m_{.j}(x)$.

2.2 Solution by the Entropy Method

The problem is to estimate $m_{ij}(x)$ given the total flow matrix and the departures and arrivals of each category. Assuming zero cost coefficients, the entropy maximizing model is

$$- \sum_i \sum_j \sum_x [m_{ij}(x) \ \ell n \ m_{ij}(x) - m_{ij}(x)] \qquad (23)$$

$$\text{subject to} \sum_i m_{ij}(x) = m_{.j}(x) \qquad (24)$$

$$\sum_j m_{ij}(x) = m_{i.}(x) \qquad (25)$$

$$\sum_x m_{ij}(x) = m_{ij}(\cdot) \qquad (26)$$

Application of the Lagrangean multiplier technique leads to the following result:

$$m_{ij}(x) = A_i(x) \ B_j(x) \ m_{i.}(x) \ m_{.j}(x) \ T_{ij} \qquad (27)$$

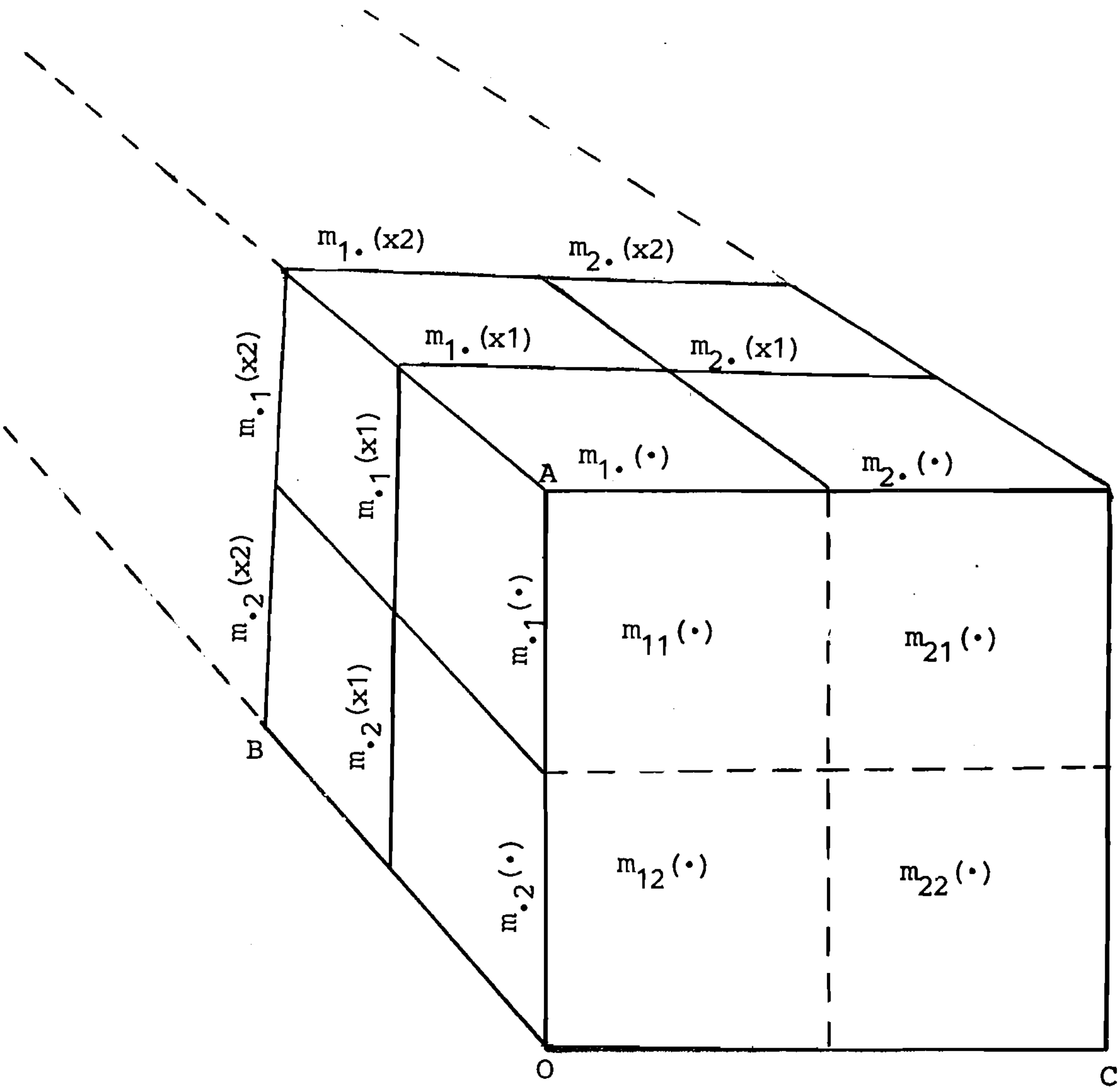

Figure 3

The Origin-Destination Black Box

where $A_i(x)$, $B_j(x)$ and T_{ij} are parameters or balancing factors related to the Lagrange multipliers. To find expressions for the parameters, we substitute (27) in the constraints (24) to (26),

$$(24)\quad \sum_i A_i(x)\ B_j(x)\ m_{i\cdot}(x)\ m_{\cdot j}(x)\ T_{ij} = m_{\cdot j}(x)$$

$$B_j(x)\ m_{\cdot j}(x) \sum_i A_i(x)\ m_{i\cdot}(x)\ T_{ij} = m_{\cdot j}(x)$$

$$B_j(x) = \left[\sum_i A_i(x)\ m_{i\cdot}(x)\ T_{ij}\right]^{-1} \qquad (28)$$

$$(25)\quad \sum_j A_i(x)\ B_j(x)\ m_{i\cdot}(x)\ m_{\cdot j}(x)\ T_{ij} = m_{i\cdot}(x)$$

$$A_i(x)\ m_{i\cdot}(x) \sum_j B_j(x)\ m_{\cdot j}(x)\ T_{ij} = m_{i\cdot}(x)$$

$$A_i(x) = \left[\sum_j B_j(x)\ m_{\cdot j}(x)\ T_{ij}\right]^{-1} \qquad (29)$$

$$(26)\quad \sum_x A_i(x)\ B_j(x)\ m_{i\cdot}(x)\ m_{\cdot j}(x)\ T_{ij} = m_{ij}(\cdot)$$

$$T_{ij} = m_{ij}(\cdot) \left[\sum_x A_i(x)\ B_j(x)\ m_{i\cdot}(x)\ m_{\cdot j}(x)\right]^{-1} . \qquad (30)$$

The system of nonlinear equations (28) to (30) does not have a simple solution as in the two-dimensional case. It must be solved by iteration. The steps involved are:

STEP 1: Give $A_i(x)$ and $B_j(x)$ an arbitrary starting value, say $A_i^{(0)}(x) = B_j^{(0)}(x) = 0.1$

$$T_{ij}^{(1)} = m_{ij}(\cdot) \left[\sum_x A_i^{(0)}(x) \cdot B_j^{(0)}(x) \cdot m_{i\cdot}(x) \; m_{\cdot j}(x) \right]^{-1}$$

STEP 2:

$$B_j^{(1)}(x) = \left[\sum_i A_i^{(0)}(x) \; m_{i\cdot}(x) \; T_{ij}^{(1)} \right]^{-1}$$

STEP 3:

$$A_i^{(1)}(x) = \left[\sum_j B_j^{(1)}(x) \; m_{\cdot j}(x) \; T_{ij}^{(1)} \right]^{-1}$$

STEP 4:

$$T_{ij}^{(2)} = m_{ij}(\cdot) \left[\sum_x A_i^{(1)}(x) \; B_j^{(1)}(x) \; m_{i\cdot}(x) \; m_{\cdot j}(x) \right]^{-1}$$

STEP 5: Go to Step 2.

The stopping criterion of the algorithm may be a combination of the following inequalities.

$$T_{ij}^{(n)} - T_{ij}^{(n-1)} < \varepsilon_1 \tag{31}$$

$$A_i^{(n)} - A_i^{(n-1)} < \varepsilon_2 \tag{32}$$

$$B_j^{(n)} - B_j^{(n-1)} < \varepsilon_3 \quad . \tag{33}$$

The need for iteration in solving the problem described by the expressions (23) to (26) suggests that it is not a generalization of the two-dimensional problem given by (9), (1) and (2). Instead, the extension of the latter problem to three dimensions would be: [1]

$$\max - \sum_i \sum_j \sum_x [m_{ij}(x) \ \ell n \ m_{ij}(x) - m_{ij}(x)] \tag{34}$$

$$\text{subject to } \sum_i \sum_x m_{ij}(x) = m_{.j}(.) \tag{35}$$

$$\sum_j \sum_x m_{ij}(x) = m_{i.}(.) \tag{36}$$

$$\sum_i \sum_j m_{ij}(x) = m_{..}(x) \tag{37}$$

It can be shown that the entropy solution to this problem is:

$$m_{ij}(x) = \frac{m_{i.}(.) \ m_{.j}(.) \ m_{..}(x)}{[m_{..}(.)]^2} \tag{38}$$

The problems (23) - (26) and (34) - (37) differ in the constraints, i.e. in the known data. The first problem assumes that all sides of the box Figure 3 are known [matrices $m_{i.}(x)$, $m_{.j}(x)$ and $m_{.j}(x)$]. In the latter only the axes OA, OB and OC are given [vectors $m_{i.}(.)$, $m_{.j}(.)$ and $m_{..}(x)$]. If no other information on migration would be available than the total number of arrivals and departures by region, and the composition of migrant categories for all migrants, then (38) would give the most probable origin-destination flows by category.

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[1] I am grateful to Andras Por for pointing out this extension and its solution.

2.3 Validity of the Estimation Procedure

To assess the validity of the entropy maximizing method, we disaggregate the 1970 migration flow matrix of the total population of Belgium into male and female migration flows, and compare the estimates with the true values published by the National Institute for Statistics. We consider five regions: Brussels, Flemish Brabant, Walloons Brabant, rest of Flanders, and rest of Wallonia.

The input data are given in Table 1.

Table 1

Internal Migration in Belgium, 1970

a. Migration flow matrix of total population.

from / to	Brussels	Fl. Brab.	W. Brab.	R. Flanders	R. Wallonia	Total
Brussels	67,600	8,376	3,091	5,709	10,230	95,006
Fl. Brabant	13,753	23,459	927	7,986	1,765	47,890
W. Brabant	5,551	1,469	6,673	448	3,463	17,604
R. Flanders	5,407	6,870	342	180,346	4,488	197,453
R. Wallonia	8,383	1,675	3,047	4,136	163,733	180,974
Total	100,694	41,849	14,080	198,625	183,679	538,927

b. Regional distribution of departures and arrivals by sex.

Region	Departures male	Departures female	Arrivals male	Arrivals female
Brussels	47,395	47,611	50,548	50,146
Fl. Brabant	23,968	23,922	20,902	20,947
W. Brabant	8,795	8,809	7,022	7,058
R. Flanders	98,602	98,851	98,753	99,872
R. Wallonia	90,130	90,844	91,665	92,014
Total	268,890	270,037	268,890	270,037

Applying the algorithm described by Step 1 to Step 5, we obtain the following values for the balancing factors:

$$\left[T_{ij}\right] = \begin{bmatrix} 0.001413 & 0.000570 & 0.000626 & 0.000054 & 0.000092 \\ 0.000421 & 0.002341 & 0.000399 & 0.000166 & 0.000044 \\ 0.000462 & 0.000275 & 0.005384 & 0.000025 & 0.000239 \\ 0.000060 & 0.000168 & 0.000026 & 0.000920 & 0.000023 \\ 0.000117 & 0.000040 & 0.000214 & 0.000025 & 0.000985 \end{bmatrix}$$

$$\{A_i(1)\} = \begin{bmatrix} 0.100707 \\ 0.099893 \\ 0.100822 \\ 0.098492 \\ 0.101220 \end{bmatrix} \qquad \{A_i(2)\} = \begin{bmatrix} 0.099293 \\ 0.100099 \\ 0.099185 \\ 0.101520 \\ 0.098800 \end{bmatrix}$$

$$\{B_j(1)\} = \begin{bmatrix} 0.099230 \\ 0.100083 \\ 0.099310 \\ 0.101872 \\ 0.099068 \end{bmatrix} . \qquad \{B_j(2)\} = \begin{bmatrix} 0.100773 \\ 0.099917 \\ 0.100694 \\ 0.098172 \\ 0.100935 \end{bmatrix} .$$

Entering the balancing factors in equation (27), yields the solution to the entropy maximization problem. The estimates of the migration flows by sex are given in Table 2. The numbers in parentheses are the observed values of the migrations.

Table 2

Internal Migration by Sex in Belgium, 1970

a. Males

from / to	Brussels	Fl. Brab.	W. Brab.	R. Flanders	R. Wallonia	Total
Brussels	33836 (33970)	4137 (4059)	1539 (1489)	2767 (2833)	5116 (5035)	47395
Fl. Brabant	6965 (6936)	11726 (11798)	467 (451)	3917 (3871)	893 (912)	23968
W. Brabant	2785 (2728)	727 (729)	3330 (3364)	218 (213)	1736 (1761)	8795
R. Flanders	2780 (2776)	3487 (3462)	175 (172)	89854 (89746)	2306 (2446)	98602
R. Wallonia	4182 (4129)	825 (854)	1512 (1546)	1998 (2090)	81614 (81511)	90130
Total	50548	20902	7022	98753	91665	268890

b. Females

from / to	Brussels	Fl. Brab.	W. Brab.	R. Flanders	R. Wallonia	Total
Brussels	33764 (33621)	4239 (4317)	1552 (1602)	2942 (2876)	5116 (5195)	47611
Fl. Brabant	6788 (6817)	11733 (11661)	460 (476)	4069 (4115)	872 (853)	23922
W. Brabant	2766 (2823)	742 (740)	3343 (3309)	230 (235)	1727 (1702)	8809
R. Flanders	2627 (2631)	3383 (3408)	167 (170)	90492 (90600)	2182 (2042)	98851
R. Wallonia	4201 (4254)	850 (821)	1535 (1501)	2138 (2046)	82119 (82222)	90844
Total	50146	20947	7058	99872	92014	270037

The first observation is that the method tends to underestimate high values of $m_{ij}(x)$, and to overestimate low values. A similar observation has been made by Chilton and Poet (1973, p. 140) and Nijkamp (1975, p. 221). This feature may in part be related to the use of Stirling's approximation. Recall that we have set

$$\ell n\ m_{ij}! = m_{ij}\ \ell n\ m_{ij} - m_{ij} \quad .$$

Therefore, the derivative is

$$\frac{d\ \ell n\ m_{ij}!}{d\ m_{ij}} = \ell n\ m_{ij} \quad . \tag{39}$$

Chilton and Poet (1973, p. 137) show that the maximum error, associated with (39) is

$$\ell n(m_{ij} + 1) - \ell n\, m_{ij} = \ell n\left(1 + \frac{1}{m_{ij}}\right) \quad ,$$

which is small providing m_{ij} is large. However, the error becomes greater as m_{ij} becomes small. If we get errors associated with small values, then, because of the constraints imposed by the totals, we will get equal and compensating errors in high values.

Another more serious cause of the deviation between the estimated and the observed values is the fact that people do not behave according to the entropy maximizing principle. Random elements cause the behavior to deviate from the most likely one. However, given the limited information we have, it is our best available means for recovering hidden migration patterns. The estimated flow matrix has the highest probability to be the true one.

2.4 Application of the Estimation Procedure

The purpose of presenting the entropy method was to come up with a technique to recover detailed migration patterns from aggregate data. The detailed information we are interested in is the migration behavior of aliens living in Belgium, and the age distribution of the migrants. The estimation of this unknown information is the subject of this section.

A. Estimation of migration pattern by nationality.

Each year, the National Institute for Statistics (N.I.S.) publishes the origin-destination table of migrations by sex between and within the 43 arrondissements[1,2]. No information is available on the breakdown by nationality or by age. The published data on the internal migration by nationality are limited to the annual number of arrivals and departures by sex and arrondissement for the aliens as a whole[3,4]. The arrivals and departures of the Belgians may then be found by subtracting the aliens from the total population. For the system of five regions, the totals of arrivals and departures by nationality are given in Table 3. The number of departures of the aliens have been adjusted such that their total is equal to the total number of arrivals. The data for Belgians

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[1]See for example, N.I.S., (1975) pp. 148-153.

[2]The intra-arrondissement migrations are between municipalities.

[3]See for example, N.I.S., (1975) pp. 132-133.

[4]Each year there is a fictitious positive net migration for the country as a whole. The reason is that some people leave their municipality without informing the local authorities, but register in the municipality of destination. They are added to the total number of arrivals, but not to the number of departures. For our analysis, we have adjusted the number of departures such that the total net internal migration is zero.

are residuals between total population (Table 1b) and aliens.

Table 3

Regional Distribution of Departures and Arrivals
by Nationality. Belgium, 1970.

Region	Departures			Arrivals	
	Aliens		Belgians	Aliens	Belgians
	Published	Adjusted			
Brussels	20,402	20,495	80,199	20,538	74,468
Fl. Brabant	2,454	2,466	39,383	3,056	44,834
W. Brabant	1,475	1,482	12,598	1,623	15,981
R. Flanders	11,133	11,184	187,441	11,139	186,314
R. Wallonia	27,242	27,366	156,313	26,637	154,337
Total	62,706	62,993	469,502	62,993	469,502

The migration flows of the total population have been given in Table 1a.

Given the information in Tables 1 and 3, we may apply the entropy method to recover the origin-destination table for aliens and Belgians. The result is given in Table 4.

Table 4

Internal Migration in Belgium by Nationality, 1970

a. Aliens

from / to	Brussels	Fl. Brab.	W. Brab.	R. Flanders	R. Wallonia	Total
Brussels	16,300	919	513	695	2,111	20,538
Fl. Brabant	1,437	1,015	63	387	154	3,056
W. Brabant	660	73	520	25	344	1,623
R. Flanders	624	331	26	9,725	433	11,139
R. Wallonia	1,474	128	359	352	24,324	26,639
Total	20,495	2,466	1,482	11,184	27,366	62,993

b. Belgians

from / to	Brussels	Fl. Brab.	W. Brab.	R. Flanders	R. Wallonia	Total
Brussels	51,300	7,457	2,578	5,014	8,119	74,468
Fl. Brabant	12,316	22,444	864	7,599	1,611	44,834
W. Brabant	4,891	1,396	6,153	423	3,119	15,981
R. Flanders	4,783	6,539	316	170,621	4,055	186,314
R. Wallonia	6,909	1,547	2,688	3,784	139,409	154,337
Total	80,199	39,383	12,598	187,441	156,313	475,934

B. Estimation of migration pattern by age.

The second application of the entropy maximization is the disaggregation of the total migration flow of Table 1 into age-specific flows. To perform this task, we need to know the age structure of outmigrants and inmigrants in each region, in addition to Table 1. The N.I.S. does not publish the age structure of the migrants. The required input information, however, is provided by Delanghe (1974). He publishes, in percentages, the age structure of the emigrants and the inmigrants, by sex, for the nine Belgian provinces and for Brussels, Flemish Brabant and Walloons Brabant. The statistics are based on a sample of 100 municipalities. In each municipality of the sample, the characteristics of the people who located in the municipality in 1968 were collected from the Family Registration Forms (Delanghe, 1974, p. 6). The total number of inmigrants in the sample was 51,784, or 9.76% of the total number of internal migrants in 1968. Table 5 gives the age structures of the out-migrants and the inmigrants for the five regions under consideration. The percentage distributions are weighted averages of the male and female age structures published by Delanghe (1974, pp. 25-26 BIS). The number of migrants in each age group has been found by applying the percentage distribution to the total number of outmigrants and inmigrants of each region in 1970. These totals are the row and column totals in Table 1 minus the diagonal elements.

Table 5

Age Structure of Total Population, Outmigrants and Inmigrants, 1970

brussels

age	population absolute	population percent	departures absolute	departures percent	arrivals absolute	arrivals percent
0	68417.	6.364	3786.	11.441	2764.	10.087
5	70242.	6.533	2232.	6.745	1722.	6.282
10	68582.	6.379	1638.	4.951	1436.	5.241
15	66443.	6.180	1638.	4.950	1393.	5.082
20	78894.	7.338	5156.	15.579	4881.	17.809
25	69793.	6.492	4515.	13.644	3716.	13.559
30	66528.	6.188	2702.	8.164	2293.	8.365
35	69149.	6.432	2408.	7.276	2073.	7.564
40	73328.	6.820	1669.	5.044	1391.	5.076
45	79091.	7.356	1483.	4.480	1239.	4.522
50	57350.	5.334	803.	2.427	790.	2.883
55	67656.	6.293	1239.	3.745	878.	3.203
60	66088.	6.147	1064.	3.214	855.	3.120
65	60717.	5.647	985.	2.977	702.	2.561
70	48965.	4.554	745.	2.252	537.	1.960
75	33571.	3.122	496.	1.499	373.	1.361
80	19550.	1.818	353.	1.066	155.	0.564
85	10772.	1.002	181.	0.546	209.	0.761
total	1075136.	100.000	33094.	100.000	27406.	100.000

fl.brab

age	population absolute	population percent	departures absolute	departures percent	arrivals absolute	arrivals percent
0	63028.	7.276	2403.	13.069	3013.	12.333
5	71958.	8.307	1210.	6.581	1671.	6.839
10	71366.	8.239	746.	4.057	1022.	4.185
15	65435.	7.554	1091.	5.931	1295.	5.300
20	62816.	7.252	4467.	24.288	5405.	22.125
25	54335.	6.273	2759.	15.004	3812.	15.602
30	55555.	6.413	1429.	7.770	1911.	7.820
35	60489.	6.983	919.	4.997	1300.	5.322
40	61417.	7.090	654.	3.555	1011.	4.139
45	59658.	6.887	494.	2.685	805.	3.293
50	40308.	4.653	342.	1.861	489.	2.000
55	47531.	5.487	454.	2.469	651.	2.664
60	45895.	5.298	345.	1.874	502.	2.054
65	40000.	4.618	381.	2.074	532.	2.177
70	30647.	3.538	219.	1.189	349.	1.428
75	19464.	2.247	239.	1.299	318.	1.303
80	10554.	1.218	156.	0.848	228.	0.934
85	5785.	0.668	83.	0.449	118.	0.482
total	866241.	100.000	18390.	100.000	24431.	100.000

Table 5: Continued

w.brab

age	population absolute	population percent	departures absolute	departures percent	arrivals absolute	arrivals percent
0	15949.	6.787	765.	10.327	1310.	11.987
5	19316.	8.220	805.	10.866	992.	9.076
10	19039.	8.102	514.	6.937	599.	5.478
15	17123.	7.287	554.	7.481	730.	6.676
20	16110.	6.855	1412.	19.058	1422.	13.006
25	12737.	5.420	632.	8.536	1236.	11.306
30	14159.	6.025	553.	7.471	786.	7.193
35	15880.	6.758	475.	6.407	674.	6.164
40	16763.	7.133	329.	4.448	505.	4.623
45	17041.	7.252	238.	3.208	730.	6.679
50	11234.	4.781	250.	3.378	393.	3.596
55	13360.	5.685	314.	4.237	281.	2.567
60	13345.	5.679	172.	2.318	412.	3.768
65	12034.	5.121	184.	2.490	262.	2.397
70	9468.	4.029	92.	1.243	243.	2.224
75	6013.	2.559	66.	0.885	150.	1.370
80	3509.	1.493	53.	0.710	113.	1.031
85	1916.	0.815	0.	0.000	94.	0.859
total	234996.	100.000	7407.	100.000	10931.	100.000

r.fland

age	population absolute	population percent	departures absolute	departures percent	arrivals absolute	arrivals percent
0	355725.	7.818	2302.	12.593	2182.	12.756
5	399619.	8.782	1295.	7.083	1215.	7.102
10	386067.	8.484	849.	4.647	803.	4.696
15	358562.	7.880	1101.	6.022	1050.	6.136
20	341483.	7.505	4154.	22.726	3829.	22.384
25	285688.	6.278	2674.	14.629	2461.	14.388
30	293193.	6.443	1419.	7.763	1350.	7.889
35	303889.	6.678	959.	5.249	909.	5.311
40	296436.	6.515	697.	3.814	647.	3.782
45	288742.	6.346	543.	2.968	513.	2.998
50	199772.	4.390	340.	1.860	314.	1.833
55	245508.	5.395	396.	2.166	382.	2.231
60	240077.	5.276	414.	2.266	405.	2.370
65	212537.	4.671	384.	2.102	371.	2.170
70	158802.	3.490	295.	1.616	274.	1.604
75	100465.	2.208	199.	1.089	189.	1.103
80	54578.	1.199	168.	0.921	135.	0.791
85	29199.	0.642	89.	0.486	78.	0.456
total	4550342.	100.000	18279.	100.000	17107.	100.000

Table 5: Continued

r.wall.

age	population absolute	population percent	departures absolute	departures percent	arrivals absolute	arrivals percent
0	208119.	7.117	2350.	11.780	2037.	11.817
5	227685.	7.786	1665.	8.349	1433.	8.312
10	227685.	7.786	1112.	5.573	965.	5.597
15	220137.	7.528	1377.	6.902	1192.	6.916
20	223076.	7.629	3567.	17.884	3097.	17.961
25	153241.	5.240	2339.	11.726	1985.	11.513
30	161738.	5.531	1494.	7.491	1283.	7.441
35	180068.	6.158	1222.	6.125	1068.	6.192
40	201052.	6.875	949.	4.756	820.	4.759
45	210554.	7.200	820.	4.111	676.	3.919
50	143208.	4.897	514.	2.579	442.	2.565
55	168968.	5.778	585.	2.931	527.	3.054
60	170330.	5.825	505.	2.532	433.	2.511
65	157146.	5.374	467.	2.341	414.	2.403
70	122313.	4.183	353.	1.769	319.	1.851
75	80672.	2.759	292.	1.462	251.	1.458
80	44189.	1.511	191.	0.959	177.	1.025
85	24048.	0.822	146.	0.730	122.	0.706
total	2924229.	100.000	19946.	100.000	17241.	100.000

Given Table 5 together with Table 1 we may apply the entropy method to decompose the total migration flow into age-specific flows. Note that we have ignored intra-regional movements, i.e., the diagonal elements of Table 1 are set equal to zero. Therefore, the diagonal elements of the $\left[T_{ij}\right]$ matrix are zero in each iteration. The solution of the entropy maximization is given in Table 6. Rounding errors cause the number of arrivals and the total number of migrants (column totals) to deviate slightly from the data in Tables 1 and 5. The departures are recovered exactly.

3. Conclusion

The objective of this paper was to demonstrate the use of entropy maximization in the estimation of detailed migration patterns. The entropy method has received considerable attention in the regional science literature. This paper reviews the important features of the method and applies it to a practical problem.

The basic idea of the entropy method is that the best estimate is given by the most probable migration pattern. This is the pattern with maximum entropy.

The method is particularly useful to derive origin-destination migration flows for several population categories from data consisting of

i. the flow matrix of the total population (all categories) and

ii. the number of arrivals and departures by region and category (i.e., age, nationality). The entropy method is applied to estimate the origin-destination migration flows by age and by nationality for a five-region system of Belgium.

The entropy method belongs to a broader class of sum-constrained optimization problems. A comparison with other methods, in particular the RAS method, is given in a forthcoming paper. (Raquillet and Willekens, 1977).

Table 6: Internal Migration by Age, Belgium, 1970

age group	brussels departures	brussels arrivals	migration from brussels to brussels	fl.brab	w.brab	r.fland	r.wall.
0	3786.	2839.	0.	1637.	602.	608.	939.
5	2232.	1765.	0.	871.	434.	334.	593.
10	1638.	1447.	0.	588.	307.	280.	463.
15	1638.	1418.	0.	630.	286.	250.	472.
20	5156.	4915.	0.	2562.	563.	848.	1182.
25	4515.	3636.	0.	2140.	621.	772.	981.
30	2702.	2286.	0.	1138.	424.	479.	660.
35	2408.	2060.	0.	894.	431.	428.	655.
40	1669.	1367.	0.	651.	288.	262.	469.
45	1483.	1119.	0.	493.	404.	209.	376.
50	803.	733.	0.	282.	201.	110.	210.
55	1239.	966.	0.	508.	195.	189.	347.
60	1064.	820.	0.	334.	266.	200.	263.
65	985.	739.	0.	379.	172.	174.	260.
70	745.	532.	0.	239.	160.	140.	206.
75	496.	376.	0.	207.	78.	64.	147.
80	353.	176.	0.	148.	59.	45.	101.
85	181.	168.	0.	59.	37.	22.	63.
total	33094.	27362.	0.	13763.	5528.	5413.	8389.

age group	fl.brab departures	fl.brab arrivals	migration from fl.brab to brussels	fl.brab	w.brab	r.fland	r.wall.
0	2403.	3091.	950.	0.	221.	988.	244.
5	1210.	1712.	466.	0.	138.	472.	134.
10	746.	1029.	350.	0.	65.	262.	69.
15	1091.	1317.	404.	0.	113.	440.	133.
20	4467.	5438.	1797.	0.	291.	1945.	433.
25	2759.	3726.	1302.	0.	191.	1053.	214.
30	1429.	1904.	706.	0.	102.	509.	112.
35	919.	1291.	547.	0.	57.	253.	62.
40	654.	993.	360.	0.	48.	192.	55.
45	494.	726.	278.	0.	55.	125.	36.
50	342.	453.	185.	0.	38.	92.	28.
55	454.	715.	266.	0.	29.	123.	36.
60	345.	481.	199.	0.	29.	97.	20.
65	381.	560.	207.	0.	27.	119.	29.
70	219.	345.	123.	0.	16.	64.	15.
75	239.	321.	122.	0.	19.	71.	26.
80	156.	260.	55.	0.	18.	61.	22.
85	83.	94.	46.	0.	8.	20.	9.
total	18390.	24457.	8362.	0.	1465.	6885.	1678.

Table 6: Continued

age group	w.brab departures	w.brab arrivals	migration from w.brab to brussels	fl.brab	w.brab	r.fland	r.wall.
0	765.	1345.	268.	111.	0.	40.	347.
5	805.	1017.	269.	106.	0.	39.	391.
10	514.	603.	220.	51.	0.	23.	220.
15	554.	743.	172.	70.	0.	27.	285.
20	1412.	1431.	495.	239.	0.	76.	602.
25	632.	1209.	278.	92.	0.	32.	230.
30	553.	784.	256.	65.	0.	26.	206.
35	475.	669.	267.	38.	0.	18.	152.
40	329.	496.	161.	32.	0.	12.	124.
45	238.	659.	127.	20.	0.	8.	83.
50	250.	364.	124.	23.	0.	9.	95.
55	314.	308.	162.	30.	0.	11.	111.
60	172.	395.	101.	12.	0.	7.	52.
65	184.	276.	94.	17.	0.	8.	65.
70	92.	240.	51.	7.	0.	4.	31.
75	66.	151.	27.	7.	0.	2.	29.
80	53.	129.	14.	8.	0.	2.	29.
85	0.	75.	0.	0.	0.	0.	0.
total	7407.	10895.	3085.	928.	0.	343.	3051.

age group	r.fland departures	r.fland arrivals	migration from r.fland to brussels	fl.brab	w.brab	r.fland	r.wall.
0	2302.	2240.	581.	1099.	61.	0.	560.
5	1295.	1245.	324.	578.	44.	0.	350.
10	849.	809.	294.	312.	25.	0.	219.
15	1101.	1069.	261.	484.	33.	0.	322.
20	4154.	3855.	993.	2190.	73.	0.	898.
25	2674.	2408.	838.	1265.	56.	0.	516.
30	1419.	1345.	504.	582.	33.	0.	300.
35	959.	902.	452.	294.	21.	0.	191.
40	697.	636.	276.	247.	17.	0.	158.
45	543.	463.	237.	170.	21.	0.	115.
50	340.	291.	135.	116.	12.	0.	77.
55	396.	420.	165.	139.	8.	0.	84.
60	414.	389.	208.	113.	14.	0.	79.
65	384.	391.	159.	134.	9.	0.	82.
70	295.	271.	140.	83.	8.	0.	64.
75	199.	190.	63.	80.	5.	0.	51.
80	168.	154.	33.	81.	5.	0.	49.
85	89.	63.	34.	27.	3.	0.	26.
total	18279.	17142.	5696.	7996.	446.	0.	4141.

Table 6: Continued

age group	r.wall. departures	arrivals	migration from r.wall. to brussels	fl.brab	w.brab	r.fland	r.wall.
0	2350.	2090.	1040.	244.	461.	605.	0.
5	1665.	1468.	707.	157.	401.	401.	0.
10	1112.	971.	584.	77.	206.	244.	0.
15	1377.	1213.	580.	133.	310.	353.	0.
20	3567.	3116.	1630.	447.	504.	985.	0.
25	2339.	1941.	1219.	229.	341.	551.	0.
30	1494.	1278.	820.	118.	225.	331.	0.
35	1222.	1060.	794.	64.	159.	205.	0.
40	949.	806.	571.	63.	144.	170.	0.
45	820.	610.	478.	43.	179.	120.	0.
50	514.	410.	290.	31.	113.	80.	0.
55	585.	579.	373.	39.	77.	97.	0.
60	505.	415.	313.	21.	86.	85.	0.
65	467.	436.	279.	29.	68.	90.	0.
70	353.	316.	218.	16.	55.	63.	0.
75	292.	253.	163.	26.	49.	53.	0.
80	191.	201.	75.	23.	47.	46.	0.
85	146.	98.	88.	9.	28.	21.	0.
total	19946.	17260.	10220.	1770.	3455.	4501.	0.

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