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Sensitivity and Cost-benefit Analyses of Emission-constrained Technological Growth Under Uncertainty in Natural Emission

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Abstract

The paper addresses the issue of control of world technological development under prescribed constraints on the emission of greenhouse gases. We use a stylized mathematical model of the world GDP whose growth leads to the increase of industrial emission provided investment in "cleaning" technology is insufficient. The proportion between investment in industrial technology and investment in "cleaning" technology acts as a control parameter in the model. The optimal control maximizing a standard economic utility index is described. Two components in total emission are distinguished: industrial emission and natural emission. It is assumed that natural emission is uncertain. We use IPCC scenarios for the world GDP and fossil fuel emission to calibrate the model. For a given range of uncertain values of natural emission, we construct a bundle of optimal trajectories of the GDP, industrial emission, "production" technology stock and "cleaning" technology stock. We analyze the sensitivity of the optimal trajectories and optimal utility to variations in the values of natural emission. Finally, we introduce a modified control policy assuming reduction of natural emission prior to intensive development of "cleaning" technology, and carry out its cost-benefit analysis.

About the Author

Elena Rovenskaya is a graduate student at the Department of Computational Mathematics and Cybernetics, Moscow State University. The work presented in this paper was performed as a DYN/YSSP project in summer 2005.

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Sensitivity and Cost-benefit Analyses of Emission-constrained Technological Growth Under Uncertainty in Natural Emission

Elena Rovenskaya[*]

1 Introduction

Climate change is an important global issue widely discussed nowadays (see, e.g., [2], [3], [6], [8], [12]). The fact that climate is changing is neither new nor unexpected – the Earth's evolution has been escorted by climate change since the birth of our planet. Events related to climate change have led to creating and destroying civilizations. However, today we see an extremely high speed of climate change. Such a high sensitivity of the economy to climate change has never been observed in the history. Even little change in climate is believed to influence much on humans life. This concern is especially actual for developed European countries whose economy is based on a unique combination of a great amount of factors, in which the climate factor plays an important role.

The greenhouse effect is one of the most serious climate factors on the Earth. Due to the greenhouse gases the average temperature on the Earth's surface is kept within a narrow corridor allowing various forms of life, including human life, to exist. The greenhouse effect comprises two components: the natural greenhouse effect created via natural processes on the Earth, and the anthropogenic greenhouse effect resulting from human activities.

It is hard to estimate, in percent, the contributions of different gases to the total greenhouse effect (a serious difficulty on this way is that the respective infra-red spectra of different gases overlap). Water vapor, H_2O, is considered to cause about 60% of the Earth's natural greenhouse effect. Other gases contributing to the greenhouse effect include carbon dioxide, CO_2 (about 26%), ozone, O_3 (8%), methane, CH_4, and nitrous oxide, N_2O (totaly about 6%), etc. [8].

The human activity resulting in emission of greenhouse gases to the atmosphere is viewed as the most important factor in the global greenhouse effect. Coal-burning power plants, automobile exhausts, factory smokestacks, and other waste vents of the human environment contribute about 22 million tons of carbon dioxide and other greenhouse gases into the atmosphere each year [6]. About half of human emissions has remained in the atmosphere. This is considerably higher than at any time during the last 420,000 years, the period for which reliable data has been extracted from ice cores [1]. About three-quarters of the anthropogenic emissions of CO_2 to the atmosphere during the past 20 years is due to fossil fuel burning. The rest is predominantly due to land-use change, especially deforestation.

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The Kyoto Protocol assumes that carbon dioxide contributes the greatest part to the total anthropogenic greenhouse effect [11]. Since the start of the Industrial Revolution, the atmospheric CO_2 concentration has increased by approximately $110\mu L/L$ or about 40%; most of it has been released since 1945. [3]. The Global Warming Theory (GWT) predicts that increased amounts of CO_2 in the atmosphere tend to enhance the greenhouse effect and thus contribute to global warming [6].

The reduction of industrial GHG emission requires investment in developing "cleaning" technology components. However, too intensive investment in "cleaning" technology in a starting period may lead to a recession in industrial technological growth, which, in turn, may create a shortage of resources for developing "cleaning" technology in the future; as a result, the concentration of the GHGs in the atmosphere may continue to grow with an unacceptable rate. An optimal investment policy aims at maximizing the effectiveness of the economy, together with keeping a "safe" level of the anthropogenic impact on the environment.

It should be taken into account that the estimated average lifetime of CO_2 in the atmosphere is 10 years, and the greenhouse effect cannot be reduced immediately even if the humans stop their industrial activity. Therefore, preventive measures should be based on reliable quantitative estimates. Modelling economic growth scenarios and their impact on climate change is a practical basis for producing and comparing such quantitative estimates. The underlying process is highly complex, it includes a number of feedbacks, uncertain components and random factors. This complexity gives rise to a variety of modelling approaches emphasizing different aspects of the process, which leads to a diversification in resulting estimates.

Nordhaus [13] suggests a relatively simple modelling framework, implemented in the well-known DICE model. It captures principal features of climate-economy interaction and produces aggregated scenarios of global development. We follow this conceptual modelling framework in our study, considering a simple model of global economic growth [4] and introducing a constraint on annual GHG emission. We calibrate the model using data from various sources (including IPCC scenarios for the world GDP, fossil fuel emission and land use emission), simulate the optimal technological growth trajectory, analyze its sensitivity to the uncertain levels of natural emission, and carry out a cost-benefit analysis of preventive investment in reducing natural emission.

2 Model

2.1 Economic component

In our model, the global economy is characterized by three parameters:

$Y(t)$,	the annual production output (the GDP),
$T(t)$,	the "production" technology stock used for production,
$C(t)$,	the stock of "cleaning" technology used to reduce GHG emissions.

Here and in what follows, t is time varying from 0 to ϑ where $\vartheta > 0$ is a fixed (medium-size) time horizon. Using the simplest Cobb-Douglas production function we assume

$$Y(t) = aT(t), \tag{2.1}$$

where $a > 0$ is a constant parameter.

Let u^* be the fraction of the production output, allocated annually for developing both "production" and "cleaning" technologies (u^* is located between 0 and 1). Thus, it is assumed that yearly a financial resource of size $u^*Y(t)$ is allocated for developing the entire technology stock. This amount is split in two parts: resource $u(t)Y(t)$ where $u(t)$ is chosen between 0 and u^*, is allocated for developing the "production" technology stock, $T(t)$, and resource $(u^* - u(t))Y(t)$ is allocated for developing the "cleaning" technology stock, $C(t)$.

Based on this and using (2.1), we set

$$\begin{aligned} \dot{T}(t) &= u(t)aT(t), \\ \dot{C}(t) &= (u^* - u(t))aT(t). \end{aligned} \tag{2.2}$$

We also denote

$$T(0) = T_0, \quad C(0) = C_0. \tag{2.3}$$

The time-varying fraction of the GDP, allocated for "production" technology, $u(t)$, is viewed as a control variable. Applying different controls one can produce different scenarios of economic growth and use them to analyze the impact of various factors on global economic development.

For system (2.2) we introduce the utility index

$$J = \int_0^{\vartheta} e^{-\rho t} \ln Y(t) dt \tag{2.4}$$

commonly used to evaluate economic growth. Here $\rho > 0$ is the discount rate.

2.2 Environmental component

The global GHG emission process is driven by a variety of factors including climate change, agriculture activities, industrial growth, development of "cleaning" technology, etc. We distinguish two global sources of GHG emission: industrial emission, $E(t)$, and natural emission, $L(t)$.

Industrial emission, $E(t)$, is positively related to the size of the total production output, $Y(t)$, and negatively related to the size of the "cleaning" technology stock, $C(t)$. We use a simplest model for such relations and set $E(t)$ to be proportional to $Y(t)$ and inverse proportional to $C(t)$:

$$E(t) = \alpha_0 \frac{Y(t)}{C(t)} = \alpha_0 a \frac{T(t)}{C(t)} = \alpha \frac{T(t)}{C(t)}; \tag{2.5}$$

here $\alpha_0 > 0$ is a fixed parameter and $\alpha = \alpha_0 a$.

We represent natural emission, $L(t)$, as a deterministic scenario. For simplicity, we consider constant scenarios only:

$$L(t) = L^* \quad (t \in [0, \vartheta]). \tag{2.6}$$

Due to uncertainty in quantifying the impact of GHG emission on climate, various approaches to setting upper limits for total annual GHG emission have been proposed. Here we restrict ourselves to the simplest approach prescribing to keep total annual GHG emission below a fixed level over the considered medium-horizon time period. Thus, we impose the constraint

$$E(t) + L^* \leq E^* \tag{2.7}$$

where $E^* > 0$ is the maximum admissible level of emission.

We assume that at the initial time, $t = 0$, constraint (2.7) is satisfied:

$$E_0 + L^* < E^*, \quad E_0 = E(0) = \alpha \frac{T_0}{C_0}, \ L_0 = L(0). \tag{2.8}$$

Our modelling approach views E^* as a threshold between a relatively slow accumulation of negative changes in the environment while $E(t) + L^* \leq E^*$, and an abrupt strong negative impact of the accumulated changes, caused by $E(t) + L^* > E^*$. In line with this viewing, we assume that the production dynamics "does not feel" slow changes in the environment, i.e., the model dynamics (2.2) does not change while $E(t) + L^* \leq E^*$, and it changes significantly as soon as we get into $E(t) + L^* > E^*$.

3 Calibration of the model

In this section we define the values of the model's parameters (see (2.1), (2.2) – (2.3), (2.4), (2.5)).

We set the year 2000 to be the initial time, 0, and the year 2050 to be the final time, ϑ.

The IPCC special report [6] gives us the initial values for industrial emission and for the GDP:

$$E_0 = 6.97 \text{ GtC/year}, \quad Y_0 = 26.7 \text{ trillion USD} \tag{3.1}$$

(we use prices of 1990 everywhere).

From the parameter values used for the DICE model [13] we easily identify the value of the proportionality coefficient in the production function (2.1):

$$a = 4. \tag{3.2}$$

We assume that in the period preceding the year 2000 the extreme "production"-oriented control, $u(t) = u^*$, has been applied. Then using (2.1), (2.2), we find that the GDP growth trajectory in that period is the exponential function with the parameter au^*. Applying the method of least squares to the historical data on the GDP growth (see [6]) we estimate the parameter of the exponent:

$$au^* = 0.024. \tag{3.3}$$

From (3.3) and (3.2) we find the fraction of the GDP, allocated for technology:

$$u^* = 0.006. \tag{3.4}$$

From (2.1) and (3.2) we derive the initial value for the "production" technology stock:

$$T_0 = 6.6 \text{ trillion USD}. \tag{3.5}$$

As estimated in [7], approximately 10% of the entire technology stock can be attributed to "cleaning" technology. This estimate and (3.5) give us the initial value for the "cleaning" technology stock:

$$C_0 = 0.73 \text{ trillion USD}. \tag{3.6}$$

Substituting (3.6), (3.5) and (3.1) in (2.5) we identify the proportionality coefficient in (2.5):

$$\alpha = 0.7 \text{ GtC/year}.$$

We take the discount rate value from [13]:

$$\rho = 0.03 \text{ year}^{-1}.$$

The issue of defining the maximum admissible level of CO_2 emission (represented as a constant E^* in our model) is widely discussed in the literature today. We take one of the IPCC scenarios for the basis: the A1B scenario – "rapid" economic growth with a balance across fossil and non-fossil energy sources [6]). Averaging the A1B scenario over the period between 2000 and 2050 gives us the upper bound for annual emission:

$$E^* = 13 \text{ GtC/year}.$$

Table 1 summarizes the above model calibration results.

Table 1: The model's parameters.

parameter	**notation**	**value**	**unit**	**source**
initial time	0	2000	year	
final time	ϑ	2050	year	
"production" technology stock in 2000	T_0	6.6	trillion USD	derived
"cleaning" technology stock in 2000	C_0	0.73	trillion USD	[7]
GDP in 2000	Y_0	26.7	trillion USD	[6]
industrial emission in 2000	E_0	6.97	GtC/year	[6]
coefficient in the production function (2.1)	a	4.0		[13]
fraction of the GDP allocated for technology	u^*	0.006		[6, 13]
coefficient in the model of industrial emission, (2.5)	α	0.7	GtC/year	derived
upper bound for annual emission	E^*	13.0	GtC/year	[6]
discount rate	ρ	0.03	year^{-1}	[13]

In order to estimate the size of the total annual natural emission (see (2.6)) we represent it as

$$L^* = l_1 + l_2 \quad (t \in [0, \vartheta]).$$

Here l_1 is the average annual land use emission and l_2 the average annual emission due to the forest fires. To estimate the average annual land use emission we use the A1B IPCC scenario [6]. We approximate it by a constant trend and find:

$$l_1 = 1.0 \text{ GtC/year with about 30\% uncertainty.}$$

Having poor access to data on global forest burning emission, we refer to the European Space Agency's Global Burnt Scar Satellite Product (GLOBSCAR), which estimates CO_2 emission totals as 5716 Tg (or about 1.5 GtC) in 2000 [5]. Based on that, we set

$$l_2 = 1.5 \text{ GtC/year with about 30\% uncertainty}$$

(we take the same level of uncertainty as for the average annual land use emission).

Table 2 summarizes the above estimates (for simplicity, we fix the same size of uncertainty for l_1 and l_2, 0.5 GtG/year, which is 30% of the maximum of the estimated average values of l_1 and l_2).

Table 2: Estimated average size of annual natural emission

parameter	**notation**	**value**	**unit**	**source**
average annual land use emission	l_1	1.0 ± 0.5	GtC/year	[6]
average emission due to the forest fires	l_2	1.5 ± 0.5	GtC/year	[5]
average natural emission	L^*	2.5 ± 1.0	GtC/year	[6, 5]

4 Optimal control

Recall that model (2.2), (2.3) describes the process of global economic growth, and $u(t)$ acts as a control determining the growth trajectory. We define a control $u(t)$ to be *optimal* if $u(t)$ maximizes the utility index J (2.4) under the constraint on the total annual emission, given by (2.7). An optimal control produces the "best" economic growth trajectory, given the environmental constraint (2.7).

More formally, using the terminology of theory of optimal control [16] we define an optimal control $u(t)$ to be a solution to the following dynamic optimization problem (the problem of optimal control):

$$\begin{aligned}
\text{maximize} \quad J_0 &= \int_0^{\vartheta} e^{-\rho t} \ln Y(t) dt, \\
Y(t) &= aT(t), \\
\dot{T}(t) &= u(t) aT(t), \\
\dot{C}(t) &= (u^* - u(t)) aT(t), \\
T(0) &= T_0, \quad C(0) = C_0, \\
u(t) &\in [0, u^*], \\
\alpha \frac{T(t)}{C(t)} &\leq E^* - L^*, \\
t &\in [0, \vartheta];
\end{aligned} \tag{4.1}$$

according to the standards of theory of optimal control, the admissible controls $u(t)$ are supposed to be measurable functions of time; however, in our study it is sufficient to view those as piece-wise continuous functions of time.

An exact solution to problem (4.1), obtained in [9][1], states that the following control is optimal:

$$u(t) = \begin{cases} u^* & \text{if } t \leq \xi, \\ bu^*/(b+1) & \text{if } t > \xi; \end{cases} \tag{4.2}$$

here

$$b = (E^* - L^*)/\alpha \tag{4.3}$$

and ξ is the minimum time, $t \geq 0$, at which industrial emission $E(t) = \alpha T(t)/C(t)$ (see (2.5)) corresponding to the extreme control $u(t) = u^*$ hits the upper bound $E^* - L^*$.

In our further analysis we will use the following equivalent formula for the optimal control (4.2), (4.3):

$$u(t) = \begin{cases} u^* & \text{if } t \leq \xi, \\ hu^* & \text{if } t > \xi \end{cases} \tag{4.4}$$

where

$$h = 1 - \frac{\alpha}{E^* - L^* + \alpha}. \tag{4.5}$$

The optimal control strategy is therefore as follows. For small t one sets $u(t) = u^*$ providing maximum investment in "production" technology and zero investment in "cleaning" technology. Following this extreme "production"-oriented mode, one observes the current value of industrial emission, $E(t) = \alpha T(t)/C(t)$. If $E(t)$ lies below the upper bound $E^* - L^*$ until the final time $t = \vartheta$, one never changes the extreme control mode. If $E(t)$ hits the the upper bound $E^* - L^*$ at a $t = \xi < \vartheta$, one immediately switches to

[1] A more general result is presented in [15]

$u(t) = hu^*$ and keeps this control value till the final time $t = \vartheta$. Following this mode, one keeps emission at the critical level, $E(t) = E^* - L^*$, and annually invests resources $hu^*Y(t)$ and $(1-h)u^*Y(t)$ in "production" technology and in "cleaning" technology, respectively.

In conclusion let us note that zero investment in "cleaning" technology in the period preceding the critical time ξ (as suggested by the optimal control), does not mean that environment protection technology is not developed in this period at all. A proper interpretation is that in this period the "business-as-usual" R&D investment strategy (extending the one that has been implemented before 2000) is kept. In this period environment protection technology is part of "production" technology, and its proportion in the entire technology stock is determined by the "business-as-usual" investment policy. In this interpretation, the stock of environment protection technology developed before 2000 is estimated as C_0 and identified with the initial "cleaning" technology stock for the period subsequent to 2000. The critical time, ξ, is interpreted then as a time, at which the "business-as-usual" R&D investment policy is reconsidered and new serious efforts in developing "cleaning" technology are undertaken.

5 Optimal trajectories and optimal utility

Let us find the trajectories of the "production" technology stock, $T(t)$, "cleaning" technology stock, $C(t)$, industrial emission, $E(t)$, and GDP, $Y(t)$, corresponding to the optimal control $u(t)$ given by (4.4), (4.5), i.e., the *optimal trajectories* in problem (4.1).

From (2.2) and (2.3) we easily get:

$$T(t) = T_0 e^{ap(t)} \tag{5.1}$$

where

$$p(t) = \int_0^t u(s)ds,$$

and

$$\begin{aligned} C(t) &= C_0 + \int_0^t (au^*T(s) - \dot{T}(s))ds \\ &= C_0 + T_0 - T(t) + au^*T_0 \int_0^t e^{ap(s)}ds. \end{aligned}$$

Substituting $u(t)$ (4.4), (4.5), and using (2.5) and (2.1), we find:

$$p(t) = \begin{cases} u^*t & \text{if} \quad t \le \xi, \\ u^*\xi + hu^*(t-\xi) & \text{if} \quad t > \xi, \end{cases} \tag{5.2}$$

$$T(t) = \begin{cases} T_0 e^{au^*t} & \text{if} \quad t \le \xi, \\ T_0 e^{au^*(\xi + h(t-\xi))} & \text{if} \quad t > \xi, \end{cases} \tag{5.3}$$

$$C(t) = \begin{cases} C_0 & \text{if} \quad t \le \xi, \\ C_0 + T_0 \frac{1-h}{h} e^{au^*\xi}\left(e^{au^*h(t-\xi)} - 1\right) & \text{if} \quad t > \xi, \end{cases} \tag{5.4}$$

$$E(t) = \alpha \frac{T(t)}{C(t)} = \begin{cases} E_0 e^{au^*t} & \text{if} \quad t \le \xi, \\ E^* - L^* & \text{if} \quad t > \xi, \end{cases} \tag{5.5}$$

$$Y(t) = aT(t). \tag{5.6}$$

The critical time ξ, at which the optimal control $u(t)$ switches from u^* to hu^* is found from the equation

$$E(\xi) = E^* - L^*,$$

or (see (5.5))

$$E_0 e^{au^*\xi} = E^* - L^*.$$

Thus,

$$\xi = \frac{1}{au^*} \ln \frac{E^* - L^*}{E_0}. \tag{5.7}$$

Consider the utility index J (2.4). Taking into account (2.1) and (5.1), we get

$$\begin{aligned} J &= \int_0^\vartheta e^{-\rho t}(\ln a + \ln T(t))dt \\ &= \ln a \frac{1 - e^{-\rho\vartheta}}{\rho} + \int_0^\vartheta e^{-\rho t} \ln T(t) dt \\ &= \ln a \frac{1 - e^{-\rho\vartheta}}{\rho} + \int_0^\vartheta e^{-\rho t}(\ln T_0 + ap(t))dt \\ &= c_0 + a \int_0^\vartheta e^{-\rho t} p(t) dt \end{aligned}$$

where

$$c_0 = \ln(aT_0) \frac{1 - e^{-\rho\vartheta}}{\rho}.$$

Substituting $p(t)$ given by (5.2), we find the *optimal value* for J in problem (4.1):

$$J^* = c_0 + a \int_0^\xi e^{-\rho t} u^* t dt + \int_\xi^\vartheta e^{-\rho t}(u^*\xi + hu^*(t - \xi))dt.$$

A direct calculation gives

$$J^* = c_1 + au^* I^*, \tag{5.8}$$

where

$$I^* = \frac{h-1}{\rho} \left(\frac{e^{-\rho\xi}}{\rho} + e^{-\rho\vartheta}\xi \right) - \frac{h}{\rho} \left(\frac{e^{-\rho\vartheta}}{\rho} + e^{-\rho\vartheta}\vartheta \right), \tag{5.9}$$

$$c_1 = c_0 + \frac{au^*}{\rho^2}.$$

6 Impact of uncertainty in natural emission

As noted in Section 3, there is considerable uncertainty in the estimated value of natural emission, L^* (see Table 2). Figure 1 presents uncertainty in L^* graphically.

Recall that the obtained solution to problem (4.1) depends on L^*. Different values for L^* lead to different optimal controls $u(t)$ (4.4), different optimal trajectories $T(t)$, $C(t)$, $E(t)$, $Y(t)$ (5.3) – (5.6), and different optimal values of the utility, J^* (5.8). In this situation where parameter L^* is uncertain, the following decision making scheme can be suggested.

Prior to choosing a control optimizing economic growth under the emission constraint (2.7), one fixes a particular value for L^* within the given uncertainty interval. One should view L^* as a "guaranteed" upper bound for the actual value of natural emission, L, which can be unknown. For the upper bound L^* one finds the optimal control in problem (4.1), $u(t)$, and implements it giving rise to the corresponding optimal trajectories $T(t)$,

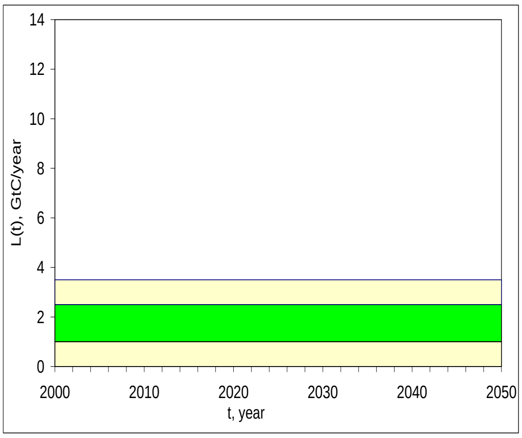

Figure 1: Deterministic natural emission scenarios, L^*. The dark green strip shows the "most probable" "central" 50% interval of the uncertain values of L^*, and two light green strips show two 25% "extreme" intervals.

$C(t)$, $E(t)$, $Y(t)$. The latter trajectories being optimal for the upper bound L^* are no longer optimal for the actual value of natural emission, L, if $L < L^*$. However, the loss in optimality, which is unavoidable since any L not exceeding L^* is admissible, is compensated by the fact that the critical constraint on total emission, $E(t) + L \leq E^*$ (see (2.7)), is satisfied "with the guarantee".

The described decision making pattern is off-line; it does not take into account that in the source of the control process, the initially set upper bound L^* can be lowered based on current observations, and the control values $u(t)$ can be updated using the feedback principle. However, the design and analysis of feedback controllers coupled with appropriate on-line observers lie beyond the frame of the present study (in this context we refer to [10] where an observer-controller pattern is developed for a simplified model of the global carbon cycle).

Thus, we adopt the above described decision making scheme. Note that different considerations may lead to different choices of L^*. For example, one can employ the worst-case approach and set L^* to be the maximum value given in Table 2: $L^* = 3.5$ GyC/year. However, the worst-case choice may lead to economically unacceptable trajectories, which may make one lower the upper bound L^*, with the acceptance of a certain measure of risk. To arrive at a reasonable value for L^* risk assessment and a detailed analysis of data on natural emission should be combined with the analysis of economic outcomes corresponding to different values of L^*. In the latter analysis, the first step is to understand how the optimal control, optimal trajectories and optimal utility value in problem (4.1) depend on L^*. Our closest goal is to study these dependencies.

Figures 2, 3 and 4 show the ranges of the optimal trajectories of industrial emission, $E(t)$, the GDP, $Y(t)$, and the "cleaning" technology stock, $C(t)$, corresponding to the given range of uncertain values of natural emission L^* (see Figure 1). Note that the simulated trajectories show a good fit with some predictions obtained using other methodologies (see [14], [6]); this could be an evidence of the applicability of the proposed model-based approach to analysis of global economic growth under emission constraints.

We also introduce carbon intensity, $D(t)$, usually defined as industrial emission per

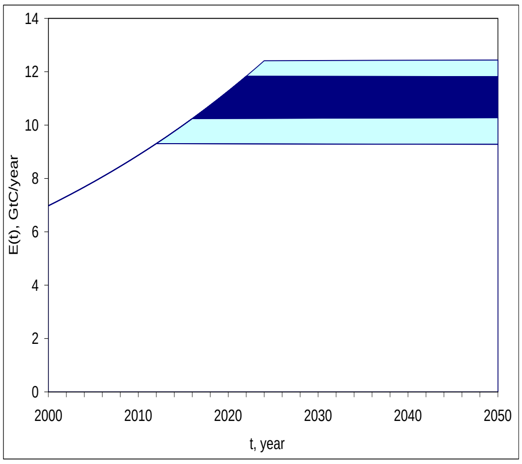

Figure 2: The range of the optimal trajectories of industrial emission, $E(t)$, corresponding to the given range of natural emission, L^*. The dark blue corridor shows the "most probable" trajectories corresponding to the "central" 50% interval of the uncertain values of L^*, and two light blue corridors correspond to two 25% "extreme" intervals of the values of L^*.

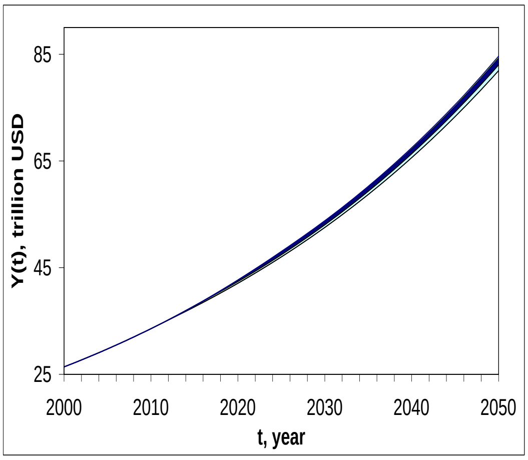

Figure 3: The range of the optimal trajectories of the GDP, $Y(t)$, corresponding to the given range of natural emission, L^*. The dark blue corridor shows the "most probable" trajectories corresponding to the "central" 50% interval of the uncertain values of L^*, and two light blue corridors correspond to two 25% "extreme" intervals of the values of L^*.

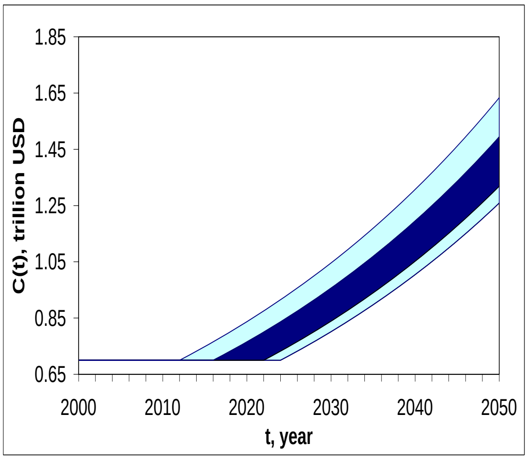

Figure 4: The range of the optimal trajectories of the "cleaning" technology stock, $C(t)$, corresponding to the given range of natural emission, L^*. The dark blue corridor shows the "most probable" trajectories corresponding to the "central" 50% interval of the uncertain values of L^*, and two light blue corridors correspond to two 25% "extreme" intervals of the values of L^*.

unit of the GDP (see, e.g., [2]). For our model of industrial emission (see (2.5)), we have

$$D(t) = \frac{E(t)}{Y(t)} = \frac{\alpha_0}{C(t)}.$$

Figure 5 shows the optimal trajectories of carbon intensity.

Table 3 shows the maximum variations in the values of industrial emission $E(t)$, the GDP, $Y(t)$, the "cleaning" technology stock, $C(t)$, and carbon intensity, $D(t)$, along their optimal trajectories corresponding to different values of L^*. We see that among these four indicators the "cleaning" technology stock, $C(t)$, is most sensitive to the variations in L^*, whereas the GDP, $Y(t)$, and carbon intensity, $D(t)$, are rather insensitive to these variations.

Table 3: Maximum variations in the optimal trajectories in 2000 – 2050.

parameter	**notation**	**maximum variation**	**year**
industrial emission	$E(t)$	25 %	2025 – 2050
GDP	$Y(t)$	4 %	2050
"cleaning" technology stock	$C(t)$	30 %	2050
carbon intensity	$D(t)$	6 %	2025

Let us consider how variations in L^* change two parameters of the optimal control $u(t)$ (4.4): ξ, the critical time at which investment in "cleaning" technology starts, and h, the coefficient determining the size of annual investment in "cleaning" technology, $(1-h)u^*Y(t)$, for $t \geq \xi$. We have (see (5.7) and (4.5)):

$$\xi = \xi(L^*) = \frac{1}{au^*} \ln \frac{E^* - L^*}{E_0},$$

$$h = h(L^*) = 1 - \frac{\alpha}{E^* - L^* + \alpha}.$$

These formulas show that both $\xi(L^*)$ and $h(L^*)$ decrease as L^* grows. Thus, higher values of L^* bring the critical time ξ closer to the current year and moreover require

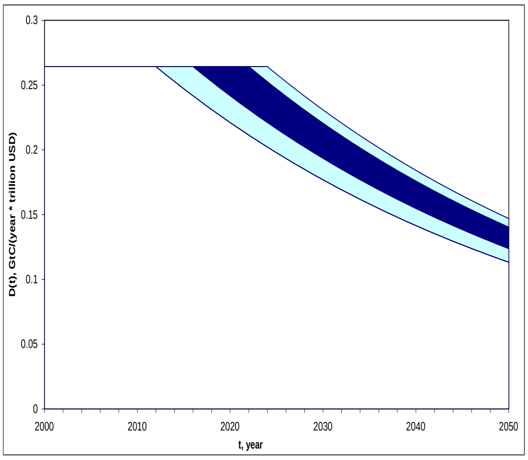

Figure 5: The range of the optimal trajectories of carbon intensity, $D(t)$, corresponding to the given range of natural emission, L^*. The dark blue corridor shows the "most probable" trajectories corresponding to the "central" 50% interval of the uncertain values of L^*, and two light blue corridors correspond to two 25% "extreme" intervals of the values of L^*.

that in the subsequent period higher fractions of the GDP ($(1-h)u^*$) are invested in "cleaning" technology and smaller ones (hu^*) in "production" technology. Consequently, if one decides to increase L^* one should realize that the GDP will switch to a lower growth rate at an earlier time ξ and in the subsequent period the GDP growth rate will be lower.

The (negative) derivatives $\xi'(L^*)$ and $h'(L^*)$ decrease, implying that the rates of decline of $\xi(L^*)$ and $h(L^*)$ grow as L^* grows; in other words, the functions $\xi(L^*)$ and $h(L^*)$ are concave. Table 4 gives the values of ξ and h corresponding to four selected values of L^*. Figures 6 and 7 show the graphs of the critical time, $\xi(L^*)$, and the fraction of the GDP, invested in "cleaning" technology in the subsequent period, $u^*(1-h(L^*))$. In particular, Figure 7 shows that while natural emission grows from 0 to 3.5 GtC/year, the fraction of the GDP invested in "cleaning" technology, increases for approximately 30%, which is a significant change in the R&D investment policy. In this context we may conclude that the question of reducing natural emission and estimation of the associated costs and benefits, is highly important economically.

From Table 4 we see that ξ varies between 2012 and 2024 as L^* covers the entire interval of its admissible values; investment in "cleaning" technology starts in 2024 if one assumes $L^*=0$ and in 2012 if one assumes the worst-case scenario, $L^*=3.5$ GtC/year.

Table 4: Optimal values of the critical time ξ and investment coefficient h.

natural emission, L^* (GtC/year)	critical time, ξ (year)	investment coefficient, h
0	2024	0.944
1	2022	0.941
2.5	2017	0.931
3.5	2012	0.926

Note that the size of the uncertainty interval covering all admissible values of natural emission, L^*, is relatively small compared to the size of annual industrial emission, whereas the distance between the maximum and minimum critical times, ξ, is relatively large, 12 years. Noticing that the size of the uncertain interval for L^* (3.5 GtC/year) is

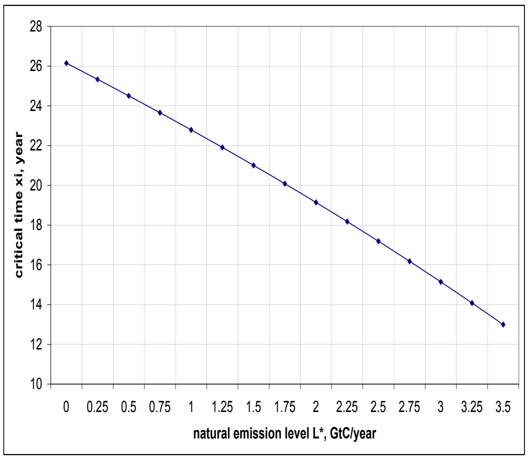

Figure 6: The graph of the critical time, $\xi(L^*)$.

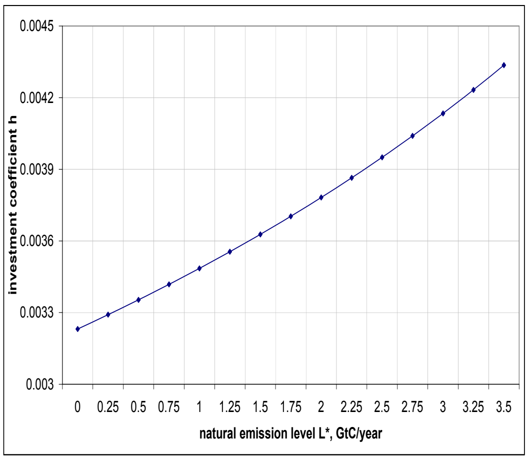

Figure 7: The graph of the fraction of the GDP, invested in "cleaning" technology, $u^*(1 - h(L^*))$.

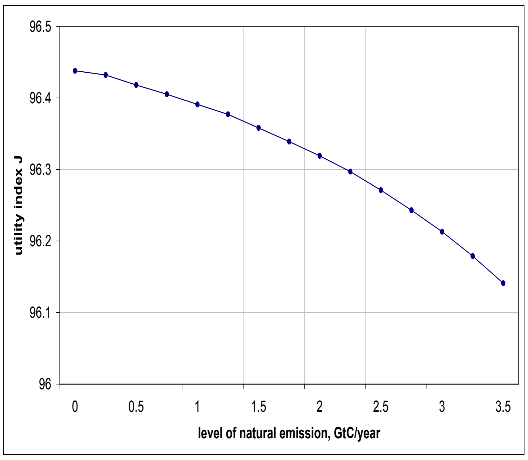

Figure 8: The graph of the optimal utility value, $J^*(L^*)$.

approximately two times less than the size of industrial emission in 2000, E_0 (see Table 1), and treating E_0 as a unit for measuring annual industrial emission (for a moment, let us call this unit "AIE"), we find that the rate of the shift of the critical time is about 24 years/"AIE", which seems quite high if we recall that one "AIE" is emitted in one year.

Let us consider the optimal utility value, J^*, as a function of L^*. An elementary analysis of formulas (5.8) and (5.9) defining $J^* = J^*(L^*)$, shows that $J^*(L^*)$ declines as L^* grows; moreover the derivative $J^{*'}(L^*)$ decreases in L^* as well, implying that the rate of loss in utility grows as L^* grows (in other words, the function $J^*(L^*)$ is concave). Figure 8 shows the graph of $J^*(L^*)$.

7 Reduction of natural emission: a cost-benefit analysis

The analysis presented in the previous Section shows that the values of key economic indicators such as the GDP growth rate and the utility index fall down at increasingly growing rates as one increases the upper bound for uncertain natural emission, L^*, determining the optimal R&D investment policy (the optimal control in problem (4.1)). The reason for that is that as L^* increases, the starting time for forcing investment in "cleaning" technology, ξ, decreases rapidly, and, moreover, in the subsequent period the fraction of the GDP, invested in "production" technology, hu^*, decreases in favor of the growing fraction of the GDP, invested in "cleaning" technology. Therefore, there is a strong economic need to lower the value of L^*.

Lowering the initially chosen (nonimprovable) value of L^* for ΔL^* implies that the size of uncertain natural emission, L, originally estimated from above by L^*, is reduced for ΔL^* due to some economic effort (controlling the deforestation process is one of the efforts of the kind). Let us discuss how to define a cost for that effort.

In a sense, reduction of natural emission is similar to development of "cleaning" technology: both measures aim at satisfying the upper constraint on total emission (see (2.7)). Therefore, we assume that investment in the reduction of natural emission comes from the total R&D investment budget whose annual size is $u^*Y(t)$.

More specifically, we assume that if it is decided to reduce natural emission for ΔL^*, then the following *modified control* is implemented:

(i) in the period between

$$\xi = \xi(L^*) \tag{7.1}$$

(the planned starting time for investment in "cleaning" technology) and some

$$\eta = \eta(L^*, \Delta L^*) \geq \xi(L^*) \tag{7.2}$$

"cleaning" technology is not developed, the "cleaning" technology budget, $(1-h)u^*Y(t)$, is invested in reducing natural emission, and the planned resource, $hu^*Y(t)$, is invested in "production" technology;

(ii) the final time in the process of the reduction of natural emission, η, is defined to be the minimum time $t \geq \xi$, at which total emission with the reduced natural emission component reaches the upper emission constraint:

$$E(t) + L^* - \Delta L^* = E^*; \tag{7.3}$$

(iii) in the period subsequent to η control $v(t)$ keeping total emission at the critical level E^*, i.e., ensuring (7.3), is used.

Now let us assume that a nonimprovable upper bound for natural emission, L^*, is chosen and the modified control is implemented.

Looking at (i), we immediately come to the following definition: the *cost* for reducing natural emission for ΔL^* equals the total investment in "cleaning" technology over the time interval $[\xi, \eta]$. Our model (see (2.2) and (2.1)) suggests that the latter cost is given by

$$\Delta P(L^*, \Delta L^*) = C(\eta | L^*) - C_0; \tag{7.4}$$

here and in what follows $C(t|L^*)$ is the optimal trajectory of the "cleaning" technology stock without reducing natural emission. The value

$$p(L^*) = \lim_{\Delta L^* \to 0} \frac{\Delta P(L^*, \Delta L^*)}{\Delta L^*} \tag{7.5}$$

gives us is investment per unit of reduced natural emission; we call $p(L^*)$ the *investment price.*

Let $Y(t|L^*)$ be the optimal GDP trajectory without reducing natural emission and $Y(t|L^*, \Delta L^*)$ be the GDP trajectory defined by the modified control. For $t \geq \eta$

$$\Delta Y(t|L^*, \Delta L^*) = Y(t|L^*, \Delta L^*) - Y(t|L^*) \tag{7.6}$$

and

$$\Delta B(t|L^*, \Delta L^*) = \int_{\eta(L^*, \Delta L^*)}^{t} \Delta Y(s|L^*, \Delta L^*) ds \tag{7.7}$$

represent, respectively, the increment in the GDP and accumulated increment in the GDP at time t; we treat the latter as the *benefit* gained at time t due to the implementation of the modified control – instead of the optimal control without reducing natural emission. The value

$$\nu(t, L^*) = \lim_{\Delta L^* \to 0} \frac{\Delta Y(t|L^*, \Delta L^*)}{\Delta L^*} \tag{7.8}$$

gives us the GDP increment per unit of reduced natural emission, achieved at time t and

$$b(t, L^*) = \lim_{\Delta L^* \to 0} \frac{\Delta B(t|L^*, \Delta L^*)}{\Delta L^*} \tag{7.9}$$

gives us the benefit per unit of reduced natural emission, achieved at time t; we call $b(t, L^*)$ the *benefit price*.

In the rest of this Section we carry out a cost-benefit analysis of the modified control (including reduction of natural emission via measures (i) – (iii)). In Appendix, the following formulas for the final time of investment in reducing natural emission, $\eta(L^*, \Delta L^*)$ (7.2), the cost, $\Delta P(L^*, \delta L^*)$ (7.4), the investment price, $p(L^*)$ (7.5), the benefit, $\Delta B(t|L^*, \Delta L^*)$ (7.7), and the benefit price, $b(t, L^*)$ (7.9) are derived:

$$\eta(L^*, \Delta L^*) = \frac{1}{au^*h(L^*)} \ln \frac{E^* - L^* + \Delta L^*}{E^* - L^*} + \xi(L^*); \tag{7.10}$$

$$\Delta P(L^*, \Delta L^*) = \frac{C_0}{E^* - L^*} \Delta L^*; \tag{7.11}$$

$$p(L^*) = \frac{C_0}{E^* - L^*}; \tag{7.12}$$

$$\begin{aligned} \Delta B(t|L^*, \Delta L^*) \quad = \quad & \frac{(E^* - L^* + \Delta L^*)C_0(e^{au^*h(L^* - \Delta L^*)(t - \eta(L^*, \Delta L^*))} - 1)}{\alpha u^* h(L^* - \Delta L^*)} - \\ & \frac{(E^* - L^* + \Delta L^*)C_0(e^{au^*h(L^*)(t - \eta(L^*, \Delta L^*))} - 1)}{\alpha u^* h(L^*)}; \end{aligned} \tag{7.13}$$

$$b(t, L^*) = \frac{(E^* - L^*)C_0}{(E^* - L^* + \alpha)^2} \int_{\xi(L^*)}^{t} e^{au^*h(L^*)(s - \xi(L^*))} a^2 u^*(s - \xi(L^*)) ds. \tag{7.14}$$

An explicit form of the latter formula is as follows:

$$b(t, L^*) = \frac{C_0}{(E^* - L^*)u^*} \left[e^{au^*h(L^*)(t - \xi(L^*))} (au^*h(L^*)(t - \xi(L^*)) - 1) + 1 \right]. \tag{7.15}$$

Let us compare the investment price, $p(L^*)$, and benefit price, $b(t, L^*)$, as functions of the upper bound for natural emission, L^*. Recall that $b(t, L^*)$ (7.15) is defined for $t \geq \xi(L^*)$ only. Note that $\xi(L^*)$ is the limit of the times $\eta(L^*, \Delta L^*)$, at which the process of reducing natural emission is terminated (the equality $\lim_{\Delta L^* \to 0} \eta(L^*, \Delta L^*) = \xi(L^*)$ follows easily from (7.10)). Therefore, in the representation of $t \geq \xi(L^*)$ as $t = \xi(L^*) + \delta$ with $\delta \geq 0$, parameter δ acts as a *delay* following the limit final time of reducing natural emission.

The inequality

$$b(\xi(L^*) + \delta, L^*) > p(L^*)$$

means that in δ time units the economic benefit due to the switch to the modified control (including reduction of natural emission for a small value ΔL^*) exceeds the cost of reduction of natural emission; in other words, with delay δ the modified control becomes more profitable than the optimal control without reducing natural emission. The inequality

$$b(\xi(L^*) + \delta, L^*) < p(L^*)$$

means that delay δ is yet not enough for making the economic benefit higher than the cost, or, equivalently, for making the modified control more profitable than the optimal control without reducing natural emission. The value

$$m(\delta, L^*) = b(\xi(L^*) + \delta, L^*) - p(L^*)$$

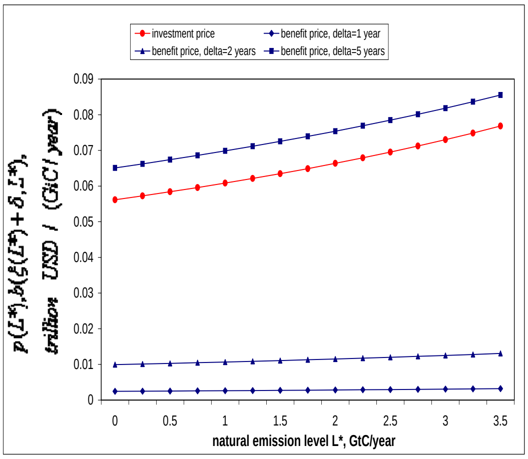

Figure 9: The graph of the investment price, $p(L^*)$, and the graphs of the benefit price, $b(\xi(L^*) + \delta, L^*)$, for several values of the delay parameter, δ.

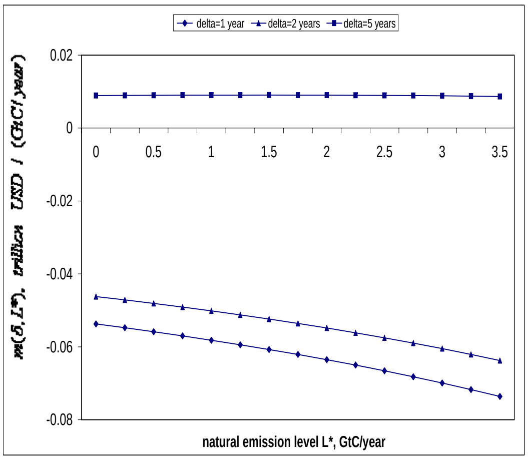

Figure 10: The graphs of the marginal benefit price, $m(\delta, L^*)$ for several values of the delay parameter, δ.

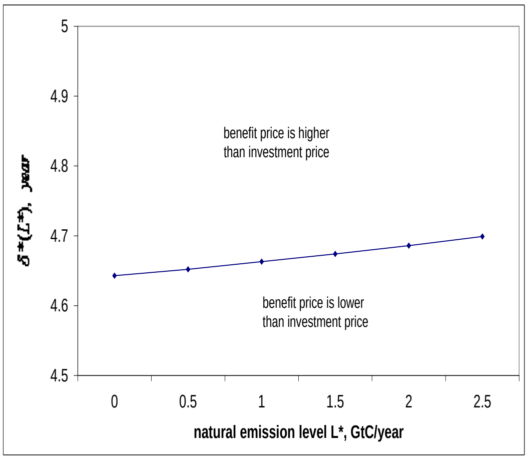

Figure 11: The graph of the investment return delay, $\delta^*(L^*)$.

is the *marginal benefit price* of a unit of reduced natural emission, established with the time delay δ.

Figure 9 depicts the graphs of the investment price, $p(L^*)$, and benefit price, $b(\xi(L^*)+\delta, L^*)$, for several values of delay δ, and Figure 10 shows the graph of the marginal benefit price, $m(\delta, L^*)$, for several values of δ.

The value $\delta^*(L^*) > 0$ such that $m(\delta, L^*) = 0$ is the *investment return delay*; with delay $\delta^*(L^*)$ investment in reduction of natural emission (for a small ΔL^*) is compensated by the economic benefit gained due to a higher rate of economic growth (compared with that without reduction of natural emission) in the post-reduction period. Figure 11 shows the graph of the investment return delay, $\delta^*(L^*)$.

Thus the investment return delay weakly depends on the natural emission level $L*$, taking its values between 4.6 and 4.7 years on considered values of $L*$.

Similarly, we compare the cost, $\Delta P(L^*, \Delta L^*)$ (7.11), and benefit, $\Delta B(t|L^*, \Delta L^*)$ (7.13), for a relatively large value of reduced natural emission, ΔL^*. Recall that $\Delta B(t|L^*, \Delta L^*)$ (7.7) is defined for $t \geq \eta(L^*, \Delta L^*)$ only, where $\eta(L^*, \Delta L^*)$ is the time, at which the process of reducing natural emission is terminated. We represent $t \geq \eta(L^*, \Delta L^*)$ as $t = \eta(L^*, \Delta L^*) + \delta$ where $\delta \geq 0$ is a *delay* parameter. The inequality

$$\Delta B(\eta(L^*, \Delta L^*) + \delta | L^*, \Delta L^*) > \Delta P(L^*, \Delta L^*)$$

means that with delay δ the modified control becomes more profitable than the optimal control without reducing natural emission, and the inequality

$$\Delta B(\eta(L^*, \Delta L^*) + \delta | L^*, \Delta L^*) < \Delta P(L^*, \Delta L^*)$$

means that delay δ is yet not enough for making the modified control more profitable than the optimal control without reducing natural emission. The value

$$M(\delta, L^* \Delta L^*) = \Delta B(\eta(L^*, \Delta L^*) + \delta | L^*, \Delta L^*) - \Delta P(L^*, \Delta L^*)$$

is the *marginal benefit* of reducing natural emission for ΔL^*, gained with the time delay δ.

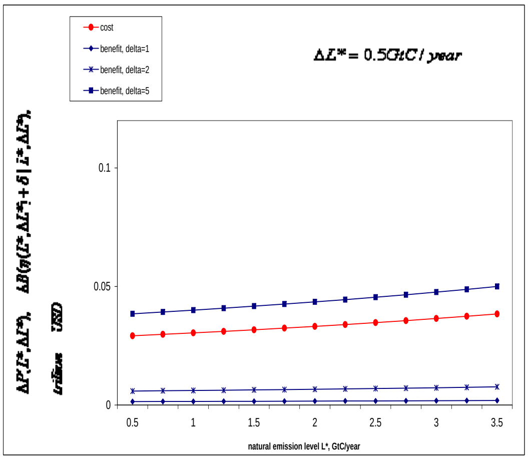

Figure 12:
The graphs of the cost, $\Delta P(L^*, \Delta L^*)$,
and the graphs of the benefit,
$\Delta B(\eta(L^*, \Delta L^*) + \delta | L^*, \Delta L^*)$,
for the emission reduction, $\Delta L^* = 0.5 GtC$,
and several values of the delay parameter, δ.

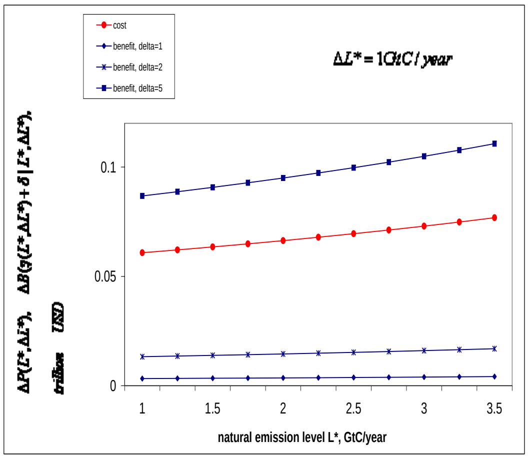

Figure 13:
The graphs of the cost, $\Delta P(L^*, \Delta L^*)$,
and the graphs of the benefit,
$\Delta B(\eta(L^*, \Delta L^*) + \delta | L^*, \Delta L^*)$,
for the emission reduction, $\Delta L^* = 1 GtC$,
and several values of the delay parameter, δ.

Figure 12, 13 depicts the graphs of the cost, $\Delta P(L^*, \Delta L^*)$ and benefit, $\Delta B(\eta(L^*, \Delta L^*) + \delta | L^*, \Delta L^*)$, for two values of ΔL^* and several values of δ, and Figure 14, 15 shows the graph of the marginal benefit, $M(\delta, L^* \Delta L^*)$ for the same values of ΔL^* and δ.

The value $\delta^*(L^*, \Delta L^*) > 0$ such that $M(\delta, L^* \Delta L^*) = 0$, is the *investment return delay* in reduction of natural emission for ΔL^*. Figure 16 shows the graph of the investment return delay, $\delta^*(L^*, \Delta L^*)$, for several values of ΔL^*.

Thus in case considered values ΔL^* equal to 0.5 GtC/year and 1 GtC/year the investment return delay lies approximately in the same interval $(4.6 - 4.7 GtC/year)$ as it does in case of small ΔL^* (i.e. $\Delta L^* \to 0$) and has the same tendency of growing.

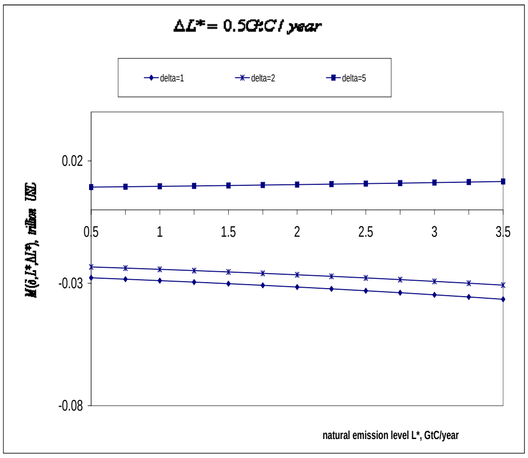

Figure 14:
The graphs of the marginal benefit, $M(\delta, L^*, \Delta L^*)$,
for the emission reduction, $\Delta L^* = 0.5GtC$,
and several values of the delay parameter, δ.

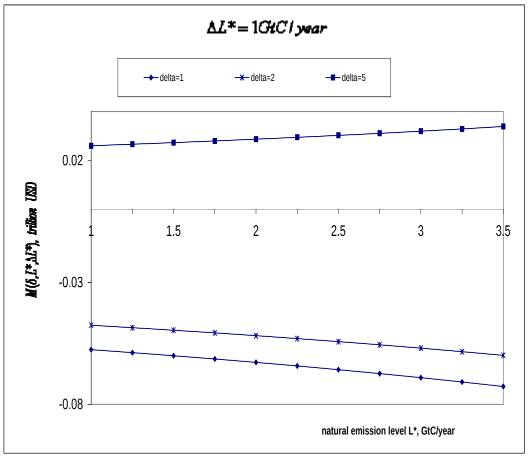

Figure 15:
The graphs of the marginal benefit, $M(\delta, L^* \Delta L^*)$,
for the emission reduction, $\Delta L^* = 1GtC$,
and several values of the delay parameter, δ.

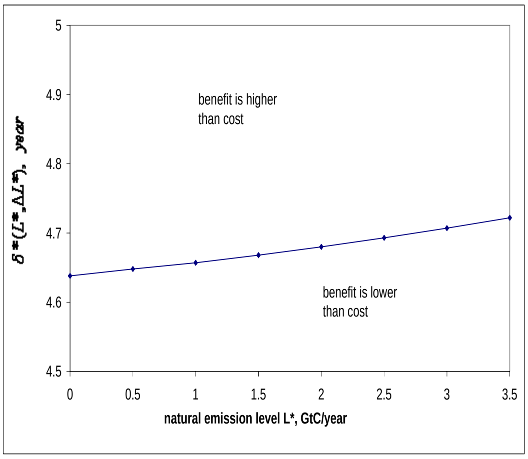

Figure 16: The graph of the investment return delay, $\delta^*(L^*, \Delta L^*)$ for several values of ΔL^*.

8 Conclusion and discussion

We presented an approach to quantifying optimal investment in "production" and "cleaning" technologies under constraints on emission of greenhouse gases. Our methodology is based on application of mathematical control theory to a stylized model of global economic growth. The optimal investment policy (optimal control) maximizing an integral utility index over a given time period and corresponding optimal trajectories of the world GDP, industrial emission, "production" technology stock and "cleaning" technology stock are described analytically. We calibrated the model using IPCC scenarios for the world GDP and fossil fuel emission. The simulated optimal trajectories showed a good fit with some predictions presented in the literature (see [14] and [6]), which could be viewed as an evidence of the applicability of the proposed approach to analysis of global economic growth under emission constraints.

In our setting, the optimal trajectories depend on the choice of the value L^* estimating the size of uncertain natural emission, L. We treat L^* as a "guaranteed" upper bound for L. Although the trajectories optimal for L^* are not optimal for $L < L^*$, the unavoidable loss in optimality is compensated by the fact that the prescribed constraint on total emission is satisfied "with the guarantee". To find a reasonable value for L^* risk assessment and a detailed analysis of data on natural emission should be combined with the analysis of economic outcomes corresponding to different values of L^*.

This kind of argument motivates our first study: the analysis of the sensitivity of the optimal investment policy and optimal utility to variations in L^*. We show that as L^* increases, both parameters of the optimal investment policy, ξ (the starting time for intensive investment in "cleaning" technology) and h (the coefficient characterizing the fraction of total R&D investment directed to "production" technology) decrease at increasing rates. The same holds for the optimal value of the utility index. This qualitative observation shows the importance of reducing natural emission (through, for example, an appropriate control of the deforestation process).

To estimate costs of and benefits from reducing natural emission, we introduce a modified investment policy (a modified control) that assumes that at the start of the planned "cleaning" technology investment period the "cleaning" technology budget is invested, instead, in reducing natural emission; the total investment in measures to reducing natural emission defines the cost for emission reduction. The modified investment policy allows one to reach a higher rate of the GDP growth, compared to that defined by the optimal investment policy without reducing natural emission. However, the resulting benefit compensates the cost for emission reduction with a certain delay. That is providing the reduction of natural emission long enough one could get more profitable optimal strategy in comparison with nonreducing policy in spite of necessary of investment in the reduction. We derived explicit formulas for the cost, the benefit, the investment price and the benefit price and, based on these, provided expressions for the marginal benefit and marginal benefit price.

We used the obtained explicit formulas to carry out a numerical cost/benefit analysis whose principal results are as follows:

(a) the investment return delay is highly robust to the choice of the "guaranteed" upper bound for uncertain natural emission, L^*: as L^* runs through the entire uncertainty interval, the investment return delay keeps staying close to 5 years;

(b) with a 5-years delay, the benefit price exceeds the investment price, practically irrespective of the size of L^*;

(c) with a 5-years delay, the benefit exceeds the investment cost, practically irrespective of the size of L^* and also of the size of the reduced natural emission, ΔL^*;

(d) the marginal benefit grows at an increasing rate as either ΔL^* or the delay grows.

These observations show that an early effort on reducing natural emission for any ΔL^* is economically profitable with an approximately 5-years delay; moreover, any increase of ΔL^* (implying a prolongation of the period of reducing natural emission and also an increase of total investment in the natural emission reduction effort) yields a considerable increment in the marginal benefit, and the latter increment grows over time at an increasing rate.

The approach presented here can be extended in several aspects. For example:

1. The introduction of technology obsolescence coefficients would make the economic growth model more realistic.

2. To better represent the IPCC scenarios, one could assume that the upper constraint on annual emission is time-dependent.

3. More complex models of natural emission, particularly those involving strong random releases of greenhouse gases, could be considered.

4. One could extend the cost-benefit analysis by considering a more general problem of optimal control, in which R&D investment is dynamically redistributed between three factors: "production" technology, "cleaning" technology and technology of reducing natural emission.

5. Instead of the suggested off-line control pattern, one could introduce feedback ovserver-controller (learning) patterns that would allow one to reduce uncertainty in natural emission on-line and thus raise the effectiveness of the investment policy.

It is expected that extended models will be constructed, calibrated and analyzed at next steps of the study presented in this paper.

9 Appendix

Here we derive formulas (7.10) – (7.15) for the final time of investment in reducing natural emission, $\eta(L^*, \Delta L^*)$ (7.2), the cost, $\Delta P(L^*, \delta L^*)$ (7.4), the investment price $p(L^*)$ (7.5), the benefit, $\Delta B(t|L^*, \Delta L^*)$ (7.7), and the benefit price, $b(t, L^*)$ (7.9).

1. Let us derive formula (7.10) for $\eta(L^*, \Delta L^*)$.

Point (i) in the definition of the modified control (Section 7) implies that in the period preceding η investment in "production" technology is the same as that defined by the optimal control without reducing natural emission (see (4.4)). Hence, on interval $[0, \eta]$ the trajectory of the "production" technology stock under the modified control coincides with the optimal trajectory of the "production" technology stock without reducing natural emission. Denoting the former and latter trajectories by $T(t|L^*, \Delta L^*)$ and by $T(t|L^*)$, respectively, we can write:

$$T(t|L^*, \Delta L^*) = T(t|L^*) \quad (t \in [0, \eta]).$$

Using (5.3), we get

$$T(t|L^*, \Delta L^*) = T_0 e^{au^*(\xi + h(t-\xi))} \quad (t \in [\xi, \eta]) \tag{9.1}$$

where

$$h = h(L^*).$$

Point (i) implies no investment in "cleaning" technology on interval $[0, \eta]$; therefore, on this interval the trajectory of the "cleaning" technology stock under the modified control – denote it $C(t|L^*, \Delta L^*)$ – goes along C_0:

$$C(t|L^*, \Delta L^*) = C_0 \quad (t \in [0, \eta]) \tag{9.2}$$

Industrial emission under the modified control is given by

$$E(t) = E(t|L^*, \Delta L^*) = \alpha \frac{T(t|L^*, \Delta L^*)}{C(t|L^*, \Delta L^*)} \tag{9.3}$$

(see (2.5)). Then by (9.1) and (9.2)

$$E(t) = \alpha \frac{T_0}{C_0} e^{au^*(\xi + h(t-\xi))} = E_0 e^{au^*(\xi+h(t-\xi))} \quad (t \in [0, \eta]) \tag{9.4}$$

(see (2.8)).

By definition (see (ii)) η is the minimum time $t \geq \xi$, at which $E(t) + L^* - \Delta L^* = E^*$. Then

$$E(\xi) = E^* - L^* + \Delta L^* \tag{9.5}$$

and in view of (9.4)

$$\begin{aligned} \eta = \eta(L^*, \Delta L^*) &= \frac{1}{h(L^*)} \left[\frac{1}{au^*} \ln \frac{E^* - L^* + \Delta L^*}{E_0} - (1 - h(L^*))\xi(L^*) \right] \\ &= \frac{1}{h(L^*)} \left[\frac{1}{au^*} \ln \frac{E^* - L^* + \Delta L^*}{E_0} - \xi(L^*) \right] + \xi(L^*). \end{aligned} \tag{9.6}$$

Substituting the expression for $\xi(L^*)$ (see (5.7)), we get

$$\begin{aligned} \eta(L^*, \Delta L^*) &= \frac{1}{h(L^*)} \left[\frac{1}{au^*} \ln \frac{E^* - L^* + \Delta L^*}{E_0} - \frac{1}{au^*} \ln \frac{E^* - L^*}{E_0} \right] + \xi(L^*) \\ &= \frac{1}{au^* h(L^*)} \ln \frac{E^* - L^* + \Delta L^*}{E^* - L^*} + \xi(L^*). \end{aligned} \tag{9.7}$$

Formula (7.10) is derived.

2. Let us derive formula (7.11) for the cost $\Delta P(L^*, \Delta L^*)$ and formula (7.12) for the investment price $p(L^*)$.

Using (7.4), (5.4), we find:

$$\Delta P(L^*, \Delta L^*) = T_0 \frac{1 - h(L^*)}{h(L^*)} e^{au^*\xi(L^*)} [e^{au^* h(L^*)(\eta(L^*, \Delta L^*) - \xi(L^*))} - 1]. \tag{9.8}$$

By (9.7)

$$au^* h(L^*)(\eta(L^*, \Delta L^*) - \xi(L^*)) = \ln \frac{E^* - L^* + \Delta L^*}{E^* - L^*}.$$

Substituting in (9.8) and using (5.7) again, we get

$$\begin{aligned} \Delta P(L^*, \Delta L^*) &= T_0 \frac{1 - h(L^*)}{h(L^*)} e^{au^*\xi(L^*)} \left[\frac{E^* - L^* + \Delta L^*}{E^* - L^*} - 1 \right] \\ &= T_0 \frac{1 - h(L^*)}{h(L^*)(E^* - L^*)} e^{au^*\xi(L^*)} \Delta L^* \\ &= T_0 \frac{1 - h(L^*)}{h(L^*)(E^* - L^*)} \frac{E^* - L^*}{E_0} \Delta L^* \\ &= \frac{T_0}{E_0} \frac{1 - h(L^*)}{h(L^*)} \Delta L^* \end{aligned}$$

The substitution of the formula for $h(L^*)$ (see (4.5)) yields

$$\Delta P(L^*, \Delta L^*) = \frac{T_0}{E_0} \frac{\alpha}{E^* - L^*} \Delta L^* = \frac{C_0}{E^* - L^*} \Delta L^* \tag{9.9}$$

(in the last equality we use $E_0 = \alpha T_0/C_0$, see (2.7)). Formula (7.11) is derived. For the investment price (see (9.10)) we have

$$p(L^*) = \lim_{\Delta L^* \to 0} \frac{\Delta P(L^*, \Delta L^*)}{\Delta L^*} = \frac{C_0}{E^* - L^*} \tag{9.10}$$

Formula (7.12) is derived.

3. Let us derive formula (7.13) for the benefit, $\Delta B(t|L^*, \Delta L^*)$.

Let $T(t|L^*)$ be the optimal trajectory of the "production" technology stock without reducing natural emission and $T(t|L^*, \Delta L^*)$ be the trajectory of the "production" technology stock, defined by the modified control. Measure (i) implies that on interval $[\xi, \eta]$ these trajectories coincide:

$$T(t|L^*, \Delta L^*) = T(t|L^*) \quad (t \in [\xi, \eta]). \tag{9.11}$$

Hence, the corresponding GDP trajectories coincide on $[\xi, \eta]$ too; therefore (see (7.6))

$$\Delta Y(t) = \Delta Y(t|L^*, \Delta L^*) = 0 \quad (t \in [\xi, \eta]). \tag{9.12}$$

According to (iii) on interval $[\eta, \vartheta]$ control $v(t)$ is chosen so that industrial emission $E(t)$ (2.2) equals $E^* - L^* + \Delta L^*$. Using model (2.1), we easily find that necessarily

$$v(t) = h(L^* - \Delta L^*)u^* \tag{9.13}$$

From (9.5), (9.3) and (9.11) we get

$$T(\eta|L^*, \Delta L^*) = T(\eta|L^*) = \frac{(E^* - L^* + \Delta L^*)C_0}{\alpha}. \tag{9.14}$$

Referring to model (2.2), we conclude that on interval $[\eta, \vartheta]$ trajectories $T(t|L^*)$ and $T(t|L^*, \Delta L^*)$ solve the equations

$$\dot{T}(t) = au(t)T(t)$$

and

$$\dot{T}(t) = av(t)T(t)$$

where

$$u(t) = h(L^*)u^*$$

(see (4.4)) and $v(t)$ is given by (9.13). Then in view of (9.14)

$$T(t|L^*) = \frac{(E^* - L^* + \Delta L^*)C_0}{\alpha} e^{au^* h(L^*)(t-\eta)},$$

$$T(t|L^*, \Delta L^*) = \frac{(E^* - L^* + \Delta L^*)C_0}{\alpha} e^{au^* h(L^* - \Delta L^*)(t-\eta)}$$

for $t \in [\eta, \vartheta]$. Hence, given a $t \in [\eta, \vartheta]$, for

$$\Delta T(t|L^*, \Delta L^*) = T(t|L^*, \Delta L^*) - T(t|L^*) \tag{9.15}$$

we have

$$\Delta T(t|L^*, \Delta L^*) = \frac{(E^* - L^* + \Delta L^*)C_0}{\alpha} (e^{au^* h(L^* - \Delta L^*)(t-\eta)} - e^{au^* h(L^*)(t-\eta)}). \tag{9.16}$$

Using the production function (2.1), we find that

$$\Delta Y(t|L^*, \Delta L^*) a \Delta T(t|L^*, \Delta L^*),$$

hence, the increment in the GDP (7.6) is given by

$$\begin{aligned}\Delta Y(t|L^*, \Delta L^*) &= a\frac{(E^* - L^* + \Delta L^*)C_0}{\alpha} e^{au^* h(L^* - \Delta L^*)(t - \eta(L^*, \Delta L^*))} - \\ & a\frac{(E^* - L^* + \Delta L^*)C_0}{\alpha} e^{au^* h(L^*)(t - \eta(L^*, \Delta L^*))}). \end{aligned} \quad (9.17)$$

Therefore, for the benefit (7.7) we get:

$$\begin{aligned}\Delta B(t|L^*, \Delta L^*) &= a\frac{(E^* - L^* + \Delta L^*)C_0}{\alpha} \int_{\eta(L^*, \Delta L^*)}^{t} e^{au^* h(L^* - \Delta L^*)(s - \eta(L^*, \Delta L^*))} ds - \\ & a\frac{(E^* - L^* + \Delta L^*)C_0}{\alpha} \int_{\eta(L^*, \Delta L^*)}^{t} e^{au^* h(L^*)(s - \eta(L^*, \Delta L^*))}) ds \\ &= \frac{(E^* - L^* + \Delta L^*)C_0(e^{au^* h(L^* - \Delta L^*)(t - \eta(L^*, \Delta L^*))} - 1)}{\alpha u^* h(L^* - \Delta L^*)} - \\ & \frac{(E^* - L^* + \Delta L^*)C_0(e^{au^* h(L^*)(t - \eta(L^*, \Delta L^*))} - 1)}{\alpha u^* h(L^*)}. \end{aligned} \quad (9.18)$$

Formula (7.13) is derived.

4. Let us derive formulas (7.14) and (7.15) for the benefit price, $b(t, L^*)$.
By (7.7) for the benefit price (7.9) we have

$$b(t, L^*) = \lim_{\Delta L^* \to 0} \frac{1}{\Delta L^*} \int_{\eta(L^*, \Delta L^*)}^{t} \Delta Y(s|L^*, \Delta L^*) ds = b_1(t, L^*) - b_2(t, L^*) \quad (9.19)$$

where

$$\begin{aligned} b_1(t, L^*) &= \lim_{\Delta L^* \to 0} \int_{\xi(L^*)}^{t} \frac{\Delta Y(s|L^*, \Delta L^*)}{\Delta L^*} ds, \\ b_2(t, L^*) &= \lim_{\Delta L^* \to 0} \int_{\xi(L^*)}^{\eta(L^*, \Delta L^*)} \frac{\Delta Y(s|L^*, \Delta L^*)}{\Delta L^*} ds. \end{aligned} \quad (9.20)$$

Using the definition of the GDP increment per unit of reduced natural emission, $\nu(s|L^*)$ (see (7.8)), we find that

$$b_1(t, L^*) = \int_{\xi(L^*)}^{t} \nu(s|L^*) ds. \quad (9.21)$$

Consider $b_2(t, L^*)$ (9.20). As shows (9.7),

$$|\eta(L^*, \Delta L^*) - \xi(L^*)| \leq K_1 \Delta L^*$$

with some constant K_1. From (9.17) we get

$$\max_{s \in [\xi, t]} |\Delta Y(s|L^*, \Delta L^*)| \leq K_2 \Delta L^*$$

with some constant K_2. Then

$$|b_2(t, L^*)| \leq \lim_{\Delta L^* \to 0} \frac{1}{\Delta L^*} K_1 K_2 \Delta L^{*2} = 0;$$

consequently,

$$b_2(t, L^*) = 0.$$

Combining with (9.21), we find the formula for the benefit price (9.19):

$$b(t, L^*) = \int_{\xi(L^*)}^{t} \nu(s|L^*)ds. \tag{9.22}$$

Let us specify (9.22) by computing $\nu(s|L^*)$. Let us come back to (9.16) and continue it as follows:

$$\begin{aligned}\Delta T(t|L^*, \Delta L^*) &= \frac{(E^* - L^* + \Delta L^*)C_0}{\alpha}(e^{au^*h(L^*-\Delta L^*)(t-\eta)} - e^{au^*h(L^*)(t-\eta)}) \\ &= -\frac{(E^* - L^* + \Delta L^*)C_0}{\alpha}\frac{d}{dL^*}e^{au^*h(\bar{L}^*)(t-\eta(L^*,\Delta L^*))}\Delta L^* \\ &= -\frac{(E^* - L^* + \Delta L^*)C_0}{\alpha}e^{au^*h(\bar{L}^*)(t-\eta(L^*,\Delta L^*))}au^*h'(\bar{L}^*)(t-\eta(L^*, \Delta L^*))\Delta L^*\end{aligned}$$

with some $\bar{L}^* \in [L^* - \Delta L^*, L^*]$. By (9.6)

$$\lim_{\Delta L^* \to 0} \eta(L^*, \Delta L^*) = \xi(L^*).$$

Hence,

$$\begin{aligned}\tau(t, L^*) &= \lim_{\Delta L^* \to 0} \frac{\Delta T(t|L^*, \Delta L^*)}{\Delta L^*} \\ &= -\frac{(E^* - L^*)C_0}{\alpha}e^{au^*h(\bar{L}^*)(t-\xi(L^*))}au^*h'(L^*)(t - \xi(L^*)).\end{aligned}$$

The latter value gives us the increment of the "production" technology stock per unit of reduced natural emission, achieved at time $t > \xi(L^*)$. Substituting

$$h'(L^*) = -\frac{\alpha}{(E^* - L^* + \alpha)^2}$$

(here we use the formula for $h(L^*)$, (4.5)), we get

$$\tau(t, L^*) = \frac{(E^* - L^*)C_0}{(E^* - L^* + \alpha)^2}e^{au^*h(\bar{L}^*)(t-\xi(L^*))}au^*(t - \xi(L^*)). \tag{9.23}$$

Referring to the production function (2.1) and using (7.6), we express the GDP increment per unit of reduced natural emission, $\nu(t, L^*)$ (7.8), through the increment of the "production" technology stock per unit of reduced natural emission, $\tau(t, L^*)$ (9.23):

$$\begin{aligned}\nu(t, L^*) &= a\tau(t, L^*) \\ &= \frac{(E^* - L^*)C_0}{(E^* - L^* + \alpha)^2}e^{au^*h(L^*)(t-\xi(L^*))}a^2u^*(t - \xi(L^*)).\end{aligned} \tag{9.24}$$

The substitution of (9.24) in formula (9.19) for the benefit price implies

$$b(t, L^*) = \frac{(E^* - L^*)C_0}{(E^* - L^* + \alpha)^2}\int_{\xi(L^*)}^{t} e^{au^*h(L^*)(s-\xi(L^*))}a^2u^*(s - \xi(L^*))ds. \tag{9.25}$$

The integration by parts yields

$$\begin{aligned}
b(t,L^*) &= \frac{(E^*-L^*)C_0}{(E^*-L^*+\alpha)^2}\frac{a}{h(L^*)}e^{au^*h(L^*)(t-\xi(L^*))}(t-\xi(L^*)) - \\
&\quad \frac{(E^*-L^*)C_0}{(E^*-L^*+\alpha)^2}\frac{1}{h^2(L^*)}(e^{au^*h(L^*)(t-\xi(L^*))}-1) \\
&= \frac{(E^*-L^*)C_0}{(E^*-L^*+\alpha)^2}\left[e^{au^*h(L^*)(t-\xi(L^*))}\left(\frac{a(t-\xi(L^*))}{h(L^*)}-\frac{1}{u^*h^2(L^*)}\right)+\frac{1}{u^*h^2(L^*)}\right]. \\
&= \frac{C_0}{(E^*-L^*)u^*}\left[e^{au^*h(L^*)(t-\xi(L^*))}\left(au^*h(L^*)(t-\xi(L^*))-1\right)+1\right].
\end{aligned} \tag{9.26}$$

Formulas (7.14) and (7.15) are derived.

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