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Existence of self-sustained oscillations in an ocean circulation box model with turbulent fluxes

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Abstract

Box-models are important for qualitative description of thermohaline circulation. Their bifurcation analysis helps to understand possible mechanisms for the loss of stability. So far, bifurcations in box-models have been studied numerically, except for the saddle-node bifurcation in the Stommel type models. We consider a box-model with turbulent fluxes. We prove that a rapid growth of the transfer function can lead to existence of a limit cycle. This limit cycle collapses to a steady state as the transfer function approaches the step function.

Key words: thermohaline circulation, box-model, transfer function, turbulent flux, limit cycle, flip-flop model, bifurcation, steady state, stability

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1 Introduction

Simple box-models proved to be effective in describing qualitative features of thermohaline circulation (THC), particularly in understanding individual processes and feedbacks. In many cases box models satisfactory reproduce behaviour of general circulation models [1]. In particular, box models have been used recently to mimic THC threshold response to climate change (see, e.g. [2]).

Bifurcations in box models of THC are important since they might reflect real types of instability mechanisms [3, 4]. So far, very few bifurcation types have been completely analytically studied, except for the loss of stability through multiple equilibria in classical two-box Stommel model [5, 6] and its subsequent generalizations (e.g. [7, 8]).

In paper [9] a 2D modification of Stommel model was considered, for which self-sustained oscillations were obtained numerically through a particular choice of the transfer function. We give a complete proof of existence for a limit cycle for that model in a wide class of transfer functions. Choosing one parameter families of functions in that class, one arrives to the so-called soft loss of stability [10] of the steady state in the limit case flip-flop model.

2 The model equations

The model proposed in [9] consists of a single well mixed water layer of fixed depth, overlaying a deep water reservoir of given temperature T_0 and salinity S_0. Turbulent fluxes of heat and salt between the two water bodies are described by a Newtonian law with the transfer function q_0 depending on the density difference: $\Delta\rho = \rho - \rho_0$. The function q_0 is small for negative $\Delta\rho$ but increases rapidly as $\Delta\rho$ becomes positive. The layer is also subjected to external thermohaline forcing from the atmosphere. These fluxes are given by a linear Newtonian law with the adjustment rate for the temperature larger than for salinity: $0 < q_S < q_T$. We also employ the usual assumption (see [3]–[9]) that density depends linearly on temperature and salinity: $\rho = \rho_0 - \alpha T + \beta S$, where α and β are positive coefficients. Thus the temperature and salinity of the layer are governed by the following system of equations:

$$\begin{cases} \dot{T} = q_T(T_A - T) - q_0(\rho)T \\ \dot{S} = q_S(S_A - S) - q_0(\rho)S \end{cases} \tag{1}$$

where T_A and S_A are the temperature and salinity of external forcing, and it is assumed for simplicity that $T_0 = S_0 = \rho_0 = 0$.

3 Existence of an unstable steady state

In non-dimensional coordinates, rewrite system (1) as follows:

$$\begin{cases} \dot{x} = 1 - x - q(z)x \\ \dot{y} = \delta(1-y) - q(z)y \end{cases} \tag{2}$$

Here $0 \le q(z) \le \sigma$ is a function of $z = -x + ry$, and $0 < \delta < 1$. System (2) is considered in the unit square: $0 \le x \le 1$ and $0 \le y \le 1$.

We would like to find conditions when this system has the unique unstable singular point in the vicinity of the line $z = 0$ in this square.

Lemma 1. *Let*

$$1 < r < \frac{\delta + \sigma}{\delta(1+\sigma)}. \tag{3}$$

Let the function $q(z)$ be C^1-smooth, and for sufficiently small negative z the following inequality is satisfied:

$$q'(z)z < -1 - \delta - 2Q(z), \tag{4}$$

where $q = Q(z)$ is the solution to the equation

$$zq^2 + q(1 + z + \delta z - r\delta) - \delta(r - 1 - z) = 0. \tag{5}$$

Then there exits $\varepsilon > 0$ such that system (2) has the unique unstable steady state in the strip $\{0 < x < 1,\ -\varepsilon < z < 0\}$.

PROOF. To simplify calculations let us take z as the new variable instead of the variable y. After this change of coordinates, system (2) takes the form

$$\begin{cases} \dot{x} = 1 - x - q(z)x \\ \dot{z} = -1 + x + q(z)x + r\delta - (\delta + q(z))(z + x) \end{cases} \tag{6}$$

At a singular point of system (6), we have $x = 1/(1+q)$, so that $0 < x < 1$ when q is positive. Substituting for x in the right hand side of the second equation of system (6), we find the equation on z-coordinate of the singular point:

$$(\delta + q)\left(z + \frac{1}{1+q}\right) = r\delta.$$

Multiplying that relation by $1 + q$ and rearranging, we arrive to equation (5).

At $z = 0$ equation (5) has the unique solution, namely, $q_0 = \delta(r-1)/(1-r\delta)$. Moreover, assumption (3) guarantees that q_0 belongs to the interval $(0, \sigma)$. But the derivative of the left hand side of equation (3) with respect to q at the point $q = q_0$ and $z = 0$ is nonzero (it is equal to $1 - r\delta$ as it is easy to see). Hence, due to implicit function theorem, we have the unique smooth solution $q = Q(z)$ for this equation near the origin with $Q(0) = q_0$ and

$$Q'(0) = -\frac{q_0^2 + q_0(\delta+1) + \delta}{1 - r\delta}.$$

The respective x-coordinate of the singular point is $X(z) = 1/(1+Q(z))$ with

$$X(0) = \frac{1}{1+Q(0)} = \frac{1-r\delta}{1-\delta}. \tag{7}$$

The inequality $0 < X(0) < 1$ is secured by assumption (3).

Now calculating the linearization of system (6) at the singular point $(1/(1+Q(z)), z)$, we get the following matrix

$$\begin{pmatrix} -1-Q(z) & -q'(z)/(1+Q(z)) \\ 1-\delta & -\delta - Q(z) - q'(z)z \end{pmatrix} \tag{8}$$

The singular point under the investigation is unstable if both the trace and the determinant of this matrix are positive. The trace is equal to

$$-1-\delta-2Q(z)-q'(z)z.$$

It is positive if $q'(z)z$ has reasonably big negative value for small negative z. More exactly, it is sufficient to satisfy inequality (4). Finally, note that the determinant of matrix (8) is equal to

$$(1+Q(z))(\delta+Q(z)+q'(z)z) + (1-\delta)q'(z)/(1+Q(z))$$

and is always positive for small z.

Remark 1. There are a lot of functions (even smooth) satisfying condition (4). For example, $q(z) = [\arctan(\varepsilon + \sqrt{z}/\varepsilon) + \pi/2]/\pi$ will serve well.

Theorem 1. *Let assumptions of Lemma 1 be fulfilled. Let additionally $q(z) = 0$ for $z \leq -\varepsilon$, and $q(z) = \sigma$ for $z \geq 0$. Then system (2) has a limit cycle inside the unit square around the steady state.*

PROOF. Due to Lemma 1, the strip $\{0 < x < 1, -\varepsilon < z < 0\}$ contains only one steady state, which is unstable. It is easy to see that there are no steady states outside the strip in the unit square. Moreover, the unit square is clearly an attraction domain. Hence [11], there exists a limit cycle around this steady state.

4 Dynamics of the flip-flop model

In this section we show that this limit cycle collapses into a point once $q(z)$ approaches the step function

$$q(z) = \begin{cases} 0, & z \leq 0, \\ \sigma, & z > 0. \end{cases} \tag{9}$$

Solutions to discontinuous (flip-flop) system (2)–(9) are understood in Filippov sense [12].

Theorem 2. *Let inequality (3) holds. Then flip-flop system (2)–(9) has the unique steady state*

$$(\bar{x}, \bar{y}) = \left(\frac{1-\delta r}{1-\delta}, \frac{1-\delta r}{r(1-\delta)}\right). \tag{10}$$

PROOF. Let $z \leq 0$ and $q = 0$. Then the steady state $(x^*, y^*) = (1, 1)$ lies outside the halfplane $x \leq ry$. At the tangency point of the phase curve with the line $x = ry$, the velocity vector $(1 - ry, \delta(1 - y))$ is collinear to $(r, 1)$. This yields point (10). The condition $0 < \bar{y} < 1$ holds if $r < 1/\delta$. The latter is ensured by (3). Note that $\bar{x}$ coincide with expression (7) in Theorem 1.

Let now $z > 0$ and $q = \sigma$. Then the steady state

$$(x^*, y^*) = \left(\frac{1}{1+\sigma}, \frac{\delta}{\delta+\sigma}\right)$$

lies outside the considered halfplane $x > ry$ if (3) is satisfied. It is checked analogously to the previous case that the phase curve is tangent to the line $rx = y$ at the same point (10).

Remark 2. Figure 5b in [9] shows numerical results for the dynamics around the steady state for the flip-flop model with $\sigma = 1$.

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