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THE SPATIAL REPRODUCTIVE VALUE: THEORY AND APPLICATIONS

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Preface

Interest in human settlement systems and policies has been a critical part of urban-related work at IIASA since its inception. Recently this interest has given rise to a concentrated research effort focusing on migration dynamics and settlement patterns. Four sub-tasks form the core of this research effort:

I. the study of spatial population dynamics;

II. the definition and elaboration of a new research area called demometrics and its application to migration analysis and spatial population forecasting;

III. the analysis and design of migration and settlement policy;

IV. a comparative study of national migration and settlement patterns and policies.

This paper, the tenth in the spatial population dynamics series, deals with a concept which is currently receiving great interest in the demographic literature: the reproductive value. It reformulates the notion of reproductive value and generalizes it to multiregional demographic systems. The usefulness of this concept for demographic analysis is demonstrated in the applications.

Related papers and other publications of the migration and settlement study are given on the back page of this report.

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Abstract

What is important for population growth is not the number of people, but the biological potential. This observation led Fisher in 1929 to the development of the concept of reproductive value as part of a theory of natural selection. Recently, mathematical demographers have explored this concept and shown how it provides solutions to problems of population dynamics that are governed by fertility and mortality. This paper reformulates the theory of reproductive value, and generalizes it to multiregional population systems, the dynamics of which are determined by fertility, mortality, and _migration_. Births are considered as investments in _lives or_ individuals by the society. The growth of the population depends on the _number_ of investments (births), and on _when_ and _where_ _they_ take place. A number of applications of the spatial reproductive value concept are indicated. The computations of the stable birth trajectory and population growth path are discussed in detail. Numerical illustrations are used throughout the paper.

Acknowledgements

The idea of generalizing the reproductive value concept to multiregional systems came from Professor A. Rogers. I am most grateful for his invaluable comments and suggestions during the development of this paper.

I am also indebted to Professor N. Keyfitz for several discussions on the topic while he was at IIASA in September 1976. Linda Samide has typed the manuscript with great skill and a surprising amount of patience.

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The Spatial Reproductive Value: Theory and Applications

It is not uncommon in science that an old and almost forgotten concept is picked up decades later and is then received with interest. The concept of reproductive value was developed in 1929 by R.A. Fisher. Recently Goodman (1967, 1971) and Keyfitz (1975) have studied the concept and have indicated a wide field of application.

It is the purpose of this paper to generalize the idea of reproductive value to a multiregional population system which includes internal migration. The introduction of the spatial reproductive value in the analysis enables us to solve in an elegant way problems of multiregional stable and stationary populations, problems which previously could not be solved at all or only in a complicated manner.

1. THE THEORY OF SPATIAL REPRODUCTIVE VALUE

Fisher (1929, p. 27) regarded birth as a loan of life to a child. In the course of this life, the individual must pay back this debt with interest by producing new life, i.e., offspring. The interest rate to be paid has a simple and particularly useful demographic interpretation. It is the intrinsic growth rate of the population, i.e., the growth rate at stability or steady state equilibrium.

In this section we propose a slightly different conceptual approach to the reproductive value. By doing so the concept of spatial reproductive value becomes more meaningful, and the demographic characteristics underlying the notion itself as considered by Rogers (1975), are given a broader interpretation (see also Rogers and Willekens, 1976b).

1.1 The Multiregional Characteristic Matrix

Consider a multiregional female population. Suppose that the probability that a girl, born in region i, will survive to age a and be in region j at that time is ${}_i\hat{\ell}_j(a)$. The chance of her having a child in region j between ages a and a + da is $m_j(a)$ da. The expected number of births in region j during this small interval, to a woman born in region i a years ago is

$$m_j(a)\ {}_i\hat{\ell}_j(a)\ da \quad . \tag{1.1}$$

For the multiregional system, we may express (1.1) in matrix notation as

$$\underset{\sim}{m}(a)\ \underset{\sim}{\hat{\ell}}(a)\ da \tag{1.2}$$

where $\underset{\sim}{m}(a)$ is a diagonal matrix of age-specific regional fertility rates, and

$$\underset{\sim}{\hat{\ell}}(a) = \begin{bmatrix} {}_1\hat{\ell}_1(a) & {}_2\hat{\ell}_1(a) & \cdots \\ {}_1\hat{\ell}_2(a) & {}_2\hat{\ell}_2(a) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} .$$

This quantity summed over all reproductive ages is the net reproduction matrix $\underset{\sim}{R}(0)$ (Rogers, 1975, p. 106), defined as

$$\underset{\sim}{R}(0) = \int_{\alpha}^{\beta} \underset{\sim}{m}(a)\ \underset{\sim}{\hat{\ell}}(a)\ da \quad , \tag{1.3}$$

where α is the youngest age of childbearing and β the oldest. The matrix $\underset{\sim}{R}(0)$ represents the expected number and spatial distribution of children, by which a girl child in each region will be replaced in the stationary population. It is the multiregional analogue of the Net Reproduction Rate. An element ${}_iR_j(0)$ denotes the expected number of offspring born in region j of a woman who is herself born in region i. The value of this element is determined not only by the fertility and mortality behavior of the woman born in i, but also by her migration behavior.

By way of illustration, consider the two-region system of Slovenia and the rest of Yugoslavia.[1] The matrix $\underset{\sim}{R}(0)$ is numerically evaluated by the formula

$$\underset{\sim}{R}(0) = \sum_{a=\alpha-5}^{\beta-5} \underset{\sim}{M}(a)\ \underset{\sim}{L}(a) \quad ,$$

where $\underset{\sim}{M}(a)$ is the diagonal matrix of average annual fertility rates of the age group a to a + 4, and $\underset{\sim}{L}(a)$ is the matrix of the person-years lived between ages a and a + 5 by the life table population. For the year 1961, $\underset{\sim}{R}(0)$ was

$$\underset{\sim}{R}(0) = \begin{bmatrix} 0.961876 & 0.010687 \\ & \\ 0.122364 & 1.174812 \end{bmatrix} \quad .$$
$$\begin{matrix} \overline{1.084240} & \quad \overline{1.185499} \end{matrix}$$

The total number of female offspring of a woman born in Slovenia is 1.08 on the average. A part of them, namely 0.96, will be born in Slovenia. The rest, 0.12, will be born in the rest of Yugoslavia. On the other hand, a girl born in the rest of Yugoslavia is expected to be replaced by 1.19 daughters, 0.01 of them born in Slovenia.

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[1]The numerical results in this paper deviate from the illustrations in Rogers and Willekens (1976b) due to a slightly different approach to the computation of the multiregional life table. The life table probabilities have been computed using the Rogers-Ledent method (Rogers and Ledent, 1976).

To enable the comparison of the demographic benefits and costs of life, the expected number of offspring of a girl child must be discounted back to the time of her birth. The value at birth of an offspring born to a woman of age a is e^{-ra}, r being the rate of discount. For positive r, an offspring is worth more if it comes at a young age a. Introducing the discounting therefore adds a time preference to the fact of having children. The exact demographic interpretation of r will be explored in the next section.

The expected number of offspring, discounted at the time of birth of the mother at a rate r, is given by the multiregional characteristic matrix $\underset{\sim}{\Psi}(r)$ (Rogers, 1975, p. 93), defined as

$$\underset{\sim}{\Psi}(r) = \int_{\alpha}^{\beta} e^{-ra} \, \underset{\sim}{m}(a) \, \underset{\sim}{\hat{\ell}}(a) \, da \tag{1.4}$$

where

$$\underset{\sim}{\Psi}(r) = \begin{bmatrix} {}_1\Psi_1(r) & {}_2\Psi_1(r) \\ {}_1\Psi_2(r) & {}_2\Psi_2(r) \end{bmatrix} .$$

An element ${}_i\Psi_j(r)$ denotes the discounted number of daughters born in region j to a mother born in region i. Note that $\underset{\sim}{\Psi}(r) = \underset{\sim}{R}(0)$ if the discount rate is zero. In fact, $\underset{\sim}{\Psi}(r)$ may be thought of as the Net Reproduction Matrix with discounting.

For the Slovenia-rest of Yugoslavia example, the expected number of offspring, discounted at a rate r = 0.006099 is

$$\underset{\sim}{\Psi}(0.0061) = \begin{bmatrix} 0.813686 & 0.008942 \\ 0.102414 & 0.994966 \end{bmatrix} .$$

$$\begin{matrix} \overline{0.916100} & \overline{1.003908} \end{matrix}$$

The discounted number of offspring of a woman born in Slovenia is less than unity. At first glance, this would mean that she does not pay back all of her debt, while a woman born in the rest of Yugoslavia pays back more than her debt. However, this is not so as will be shown in the following section.

1.2 The Concept of the Spatial Reproductive Value

To derive an interpretation for the discount rate r and for the reproductive value, it is useful to consider life and birth as a societal investment in individuals. The ultimate demographic objective the society pursues is to keep itself up and to expand at the highest rate possible.[2] To attain this goal it invests in lives. Each individual is given a life at birth. During the reproductive ages the individual has to repay the debt incurred at birth by producing offspring. The amount of offspring produced should assure the society a continuing growth at the highest rate possible.

This maximum growth rate is only feasible if the intrinsic rate of return to investment is as great as possible.[3] The rate of return is the discount rate which makes the present value of the net earnings stream equal to the present value of the costs incurred.

Suppose society invests in a single life. Let the present value of life at birth be equal to one, then the rate of return must be such that the present value of the offspring (earnings) produced by the baby girl equals one too. Hence

$$1 = \int_{\alpha}^{\beta} e^{-ra}\, m(a)\, \hat{\ell}(a)\, da \quad . \tag{1.5}$$

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[2]This statement is not new. Lotka (1956, p. 128) writes: "But, in the organic world at large also, there is presumably at least some tendency for the adjustment of the procreation factor so to take place as to make the rate of increase r a maximum under the existing conditions."

[3]This is true since each offspring is in turn a new investment.

Equation (1.5) is identical to the characteristic equation of single region mathematical demography from which the r implied by a net maternity function $m(a)\ \hat{\ell}(a)$ is calculated. The characteristic equation may therefore be given an alternative interpretation: it makes the investment costs equal to the present value of the net earnings stream. The stable growth rate r is the intrinsic rate of return to the investment.

The theory presented here differs in some respect from the theory of reproductive value of Fisher (1929) and Keyfitz (1975). Unlike both authors, we do not rely explicitly on the repayment-of-debt-interpretation. The focus is on the society as an investor and not on the individual in which a life has been invested. This approach enables us to explain why r takes the value it has. Moreover, it facilitates the interpretation of the reproductive value in the multiregional case, where not only the number of offspring are important, but also where they are born. In addition, it is speculated that this theory puts light on the convergence path of a population to stability. The observed and projected population growth rate is analogous to the rate of interest. It is not constant over time and space but fluctuates from one period to the other and between regions. In the long run, the rate of interest converges to the intrinsic rate of return. At equilibrium, i.e., in the stable population, both are equal.

To generalize the concept of reproductive value to a multiregional population system, suppose that the society invests not in a single life but in a portfolio, each element in the portfolio being a region. The society distributes its investment in lives between the regions in such a way as to support a maximum overall growth rate. Equivalently, the rate of return must be maximum. Denote the spatial distribution of births (i.e., investment in lives) by the vector $\{\underset{\sim}{Q}\}$. For a two-region system,

$$\{\underset{\sim}{Q}\} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} .$$

The total discounted number of offspring of Q_1 births in region 1 is ${}_1\Psi_{.}Q_1$. A portion ${}_1\Psi_1 Q_1$ of the births will be born in region 1 and ${}_1\Psi_2 Q_1$ will be born in region 2, due to the migration behavior of the girl before and during her reproductive ages. Analogously, the discounted value of all births in region 1 is

$$Q_1 = {}_1\Psi_1 Q_1 + {}_2\Psi_1 Q_2 \quad .$$

In general

$$\{\underset{\sim}{Q}\} = \underset{\sim}{\Psi}\{\underset{\sim}{Q}\} \quad . \qquad (1.6)$$

To maintain the growth potential of the population and to assure a maximum rate of return, the present value of the flow of births in region i must remain equal to the actual number of births or lives invested Q_i. This implies that <u>for each region, the ratio of the present value of the births in that region in each generation to the number of births at the beginning of that generation must be equal to one</u>, i.e.,

$$\frac{\sum_j {}_j\Psi_i Q_j}{Q_i} = 1 \quad . \qquad (1.7)$$

If the investment of one life in a region is seen as a debt of the region to society, then in equilibrium each region must repay its debt together with the interest, computed at a rate r. Furthermore, the relative spatial distribution of births must remain constant, and the actual number of births at time t is given by

$$e^{rt}\{\underset{\sim}{Q}\} \quad . \qquad (1.8)$$

By way of conclusion, <u>the distribution of births $\{\underset{\sim}{Q}\}$ in the stable population may be interpreted as the spatial distribution of investments in lives which enables the society as a whole to grow at the highest rate possible, or equivalently, which gives it the greatest rate of return to investment under the schedules</u>

<u>of fertility, mortality and migration. The distribution $\{\underset{\sim}{Q}\}$ is unique up to a scalar</u>.

Mathematically, $\{\underset{\sim}{Q}\}$ is the right eigenvector of $\underset{\sim}{\Psi}(r)$. For the system Slovenia-Rest of Yugoslavia,

$$\{\underset{\sim}{Q}\} = \begin{bmatrix} 1.000000 \\ \\ 20.823654 \end{bmatrix} \quad , \tag{1.9}$$

under the convention that the number of births in Slovenia is unity. This implies that at stability, 4.58% of the births occur in Slovenia and 95.42% in the rest of Yugoslavia.

The distribution of births pictures only one side of the coin. The other side is the value of a baby in a given region for the overall growth potential. Since society tries to sustain a maximal growth rate or rate of return, and since <u>the productive capacity</u> as shown in the fertility and mortality schedules is different from one region to another, it matters <u>where</u> the baby is born. Let the vector $\{\underset{\sim}{\nu}(0)\}$ denote the value of a baby or a 0-year-old girl by region. Note that it means how much a baby is worth for sustaining the maximum rate of growth. During the equilibrium, population grows at a maximum rate r and the relative value of a regional birth cannot be increased. Otherwise, a shifting of births could produce a higher r.[4]

<u>The value of a birth or a 0-year-old girl reflects her capacity to produce new lives</u>. If a 0-year-old in region 1 is worth $\nu_1(0)$, then the reproductive value of the discounted number of offspring must also be $\nu_1(0)$, i.e.,

$$\nu_1(0) = \nu_1(0)\,{}_1\Psi_1 + \nu_2(0)\,{}_1\Psi_2 \quad ,$$

[4]The spatial distribution of births $\{\underset{\sim}{Q}\}$ and of the reproductive value $\{\underset{\sim}{\nu}(0)\}$ may be interpreted, respectively, as the primal and the dual of mathematical programming. The specifics and the implications of this correspondence are being explored.

where ${}_i\Psi_j$ is the discounted number of offspring in region j of a girl born in region i. In general, we may write

$$\{\underset{\sim}{\nu}(0)\}' = \{\underset{\sim}{\nu}(0)\}' \; \underset{\sim}{\Psi}(r) \qquad (1.10)$$

where $\{\underset{\sim}{\nu}(0)\}$ is the vector of regional reproductive values of 0-year-old girls. It is constant up to a scalar. If the investment in one life in a region is seen as a debt of an individual to society, then in equilibrium, each individual must repay that debt together with the interest, computed at a rate r. The repayment does not have to take place in the region of origin. Partial repayment can occur in the other region, where a birth is worth more or less than in the region of origin.

Mathematically, $\{\underset{\sim}{\nu}(0)\}$ is the left eigenvector of $\underset{\sim}{\Psi}(r)$. For the system Slovenia-Rest of Yugoslavia,

$$\{\underset{\sim}{\nu}(0)\} = \begin{bmatrix} 1.000000 \\ \\ 1.818116 \end{bmatrix} \quad . \qquad (1.11)$$

For sustaining the maximum growth rate of 0.61%, a birth in the rest of Yugoslavia is worth almost twice as much as a birth in Slovenia. The reason is the high fertility level in the rest of Yugoslavia, combined with the low outmigration level to Slovenia where fertility is lower. If the fertility level of Slovenia would drop to 10% of its current value, keeping the profile constant, then the reproductive value of a birth in the rest of Yugoslavia would be 9 times that in Slovenia.

Because of the regional differences in the reproductive value of a 0-year-old, a baby girl born in Slovenia can repay her debt by giving life to only 0.916 offspring at the average, i.e., less than one daughter (discounted). The 0.102 daughters born in the rest of Yugoslavia have a higher value than an equivalent number of daughters born in Slovenia. The weighted discounted repayment is one:

$$1.000 = 1.000 * 0.814 + 1.818 * 0.102 \quad .$$

The above interpretation of the characteristic equations (1.6) and (1.10) and of the value of a 0-year-old girl, suggests asking how much the productive capacity is of a girl aged x.[5] The answer is the expected number of subsequent children discounted back to age x, and weighted by the region of birth:

$$\{\underset{\sim}{\nu}(x)\}' = \{\nu(0)\}'\left[\int_x^\beta e^{-r(a-x)} \underset{\sim}{m}(a)\ \hat{\underset{\sim}{\ell}}(a)\ da\right] \hat{\underset{\sim}{\ell}}^{-1}(x) \quad . \tag{1.12}$$

Denoting

$$\underset{\sim}{n}(x) = \int_x^\beta e^{-r(a-x)} \underset{\sim}{m}(a)\ \hat{\underset{\sim}{\ell}}(a)\ \hat{\underset{\sim}{\ell}}^{-1}(x)\ da \tag{1.13}$$

with

$$\underset{\sim}{n}(x) = \begin{bmatrix} {}_1n_1(x) & {}_2n_1(x) \\ {}_1n_2(x) & {}_2n_2(x) \end{bmatrix}$$

then we may write (1.12) as

$$\{\underset{\sim}{\nu}(x)\}' = \{\underset{\sim}{\nu}(0)\}'\ \underset{\sim}{n}(x) \quad . \tag{1.14}$$

The matrix $\underset{\sim}{n}(x)$ represents the expected number of offspring per woman aged x years, and discounted at age x. An element ${}_in_j(x)$ gives the number of children to be born in region j from a woman now x years of age and resident of region i, discounted back to age x. The vector $\{\underset{\sim}{\nu}(x)\}$ represents the reproductive value of an x-year-old girl by region of residence. Note that the values of the elements of $\{\underset{\sim}{\nu}(x)\}$ depend on the scaling inherent in $\{\underset{\sim}{\nu}(0)\}$. In the single region case, $\nu(0)$ is sometimes set equal to 1, i.e., the reproductive value of a child just born

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[5]Using the loan-and-repayment interpretation of the characteristic equation, the question would be how much of the debt is outstanding by the time a girl has reached age x.

is unity. In the multiregional case, the reproductive value of a child is affected by the region of birth. Equation (1.12) is the general formula of the spatial reproductive value. For $x = 0$, it reduces to (1.10) since $\hat{\underset{\sim}{\ell}}(0)$ is the identity matrix. Formula (1.12) may be derived equally well for the discrete model of population growth (see Appendix I).

The computation of the reproductive value of the total population is straightforward. The regional distribution of the total reproductive value, given by the vector $\{\underset{\sim}{v}\}$, is the sum of the age-specific reproductive values $\{\underset{\sim}{\nu}(x)\}$, weighted by the number of women in that age group and region in the base-year,

$$\{\underset{\sim}{v}\}' = \int_0^{\omega} \{\underset{\sim}{\nu}(x)\}' \; \underset{\sim}{k}(x) \; dx \tag{1.15}$$

where $\underset{\sim}{k}(x)$ is a diagonal matrix. The element $k_{ii}(x)$ denotes the number of women in region i and aged x years. Equation (1.15) may be written as

$$\{\underset{\sim}{v}\}' = \{\underset{\sim}{\nu}(0)\}' \int_0^{\omega} \underset{\sim}{n}(x) \; \underset{\sim}{k}(x) \; dx \quad . \tag{1.16}$$

The total reproductive value of the population is

$$v = \{\underset{\sim}{v}\}'\{\underset{\sim}{1}\} \quad . \tag{1.17}$$

Denote the discounted number of offspring of the total population, by place of residence of the population and by place of birth of the offspring, by the matrix $\underset{\sim}{nk}$:

$$\underset{\sim}{nk} = \int_0^{\omega} \underset{\sim}{n}(x) \; \underset{\sim}{k}(x) \; dx \quad . \tag{1.18}$$

Then

$$\{\underset{\sim}{n}k\} = \underset{\sim}{n}k\{\underset{\sim}{1}\} = \int_0^{\omega} \underset{\sim}{n}(x)\{\underset{\sim}{k}(x)\}\, dx \tag{1.19}$$

gives the discounted number of descendants by region in which they will be born. The quantity

$$\{\bar{\underset{\sim}{n}}k\}' = \{\underset{\sim}{1}\}' \underset{\sim}{n}k \tag{1.20}$$

gives the discounted number of descendants, by place of residence of the mothers. The total reproductive value of the whole system is

$$v = \{\underset{\sim}{\nu}(0)\}'\{\underset{\sim}{n}k\} = \{\underset{\sim}{v}\}'\{\underset{\sim}{1}\} \quad . \tag{1.21}$$

Note that in the single region case (1.16), (1.19), (1.20) and (1.21) coincide.

1.3 Numerical Evaluation of the Spatial Reproductive Value

The expression for $\{\underset{\sim}{\nu}(x)\}$ in (1.12) applies to exact age x. Approximation of the integral in (1.12) by a summation over 5-year age groups, yields a direct computable formula for the reproductive value at exact age x (Keyfitz, 1977, Chapter 6):

$$\{\underset{\sim}{\nu}(x)\}' = \{\underset{\sim}{\nu}(0)\}' \sum_{a=x}^{\beta-5} \left[e^{-(a+2.5-x)r}\ \underset{\sim}{M}(a)\ \underset{\sim}{L}(a)\right] [\hat{\underset{\sim}{\ell}}(x)]^{-1} \quad . \tag{1.22}$$

$$= \{\underset{\sim}{\nu}(0)\}'\ \bar{\underset{\sim}{n}}_x \tag{1.23}$$

where

$$\bar{\underset{\sim}{n}}_x = \sum_{a=x}^{\beta-5} \left[e^{-(a+2.5-x)r}\ \underset{\sim}{M}(a)\ \underset{\sim}{L}(a)\right] [\hat{\underset{\sim}{\ell}}(x)]^{-1} \tag{1.24}$$

is the numerical evaluation of $\underset{\sim}{n}(x)$ in equation (1.14).

The matrix $\bar{\underset{\sim}{n}}_x$ represents the discounted number of offspring per person <u>at exact age x</u>. An element ${}_j\bar{n}_i(x)$ gives the number of children born in region i, of a mother residing in region j and x years of age, discounted at age x. The first subscript therefore denotes the place of residence of the mother, while the second indicates the place of birth of the offspring.

The <u>average</u> spatial reproductive value for the age interval x to x + 4 at last birthday is denoted by $\{{}_5\underset{\sim}{V}_x\}$ and may be approximated by:

$$\{{}_5\underset{\sim}{V}_x\}' = \{\underset{\sim}{\nu}(0)\}' \frac{5}{2} \sum_{a=x}^{\beta-5} \left[e^{-r(a-x)} \underset{\sim}{M}(a) \underset{\sim}{L}(a) + e^{-r(a+5-x)} \underset{\sim}{M}(a+5) \underset{\sim}{L}(a+5)\right] \underset{\sim}{L}^{-1}(x)$$

$$= \{\underset{\sim}{\nu}(0)\}' \frac{5}{2} \sum_{a=x}^{\beta-5} [\underset{\sim}{M}(a) + e^{-5r} \underset{\sim}{M}(a+5) \underset{\sim}{S}(a)] e^{-r(a-x)} \underset{\sim}{L}(a) \underset{\sim}{L}^{-1}(x) \qquad (1.25)$$

$$= \{\underset{\sim}{\nu}(0)\}' \; {}_5\underset{\sim}{N}_x \qquad (1.26)$$

where

$${}_5\underset{\sim}{N}_x = \frac{5}{2} \sum_{x=a}^{\beta-5} [\underset{\sim}{M}(a) + e^{-5r} \underset{\sim}{M}(a+5) \underset{\sim}{S}(a)] e^{-r(a-x)} \underset{\sim}{L}(a) \underset{\sim}{L}^{-1}(x) \quad . \qquad (1.27)$$

Note that ${}_5\underset{\sim}{N}_x$ gives the discounted number of offspring per person <u>in age group x to x + 4</u>, and not the number per person at exact age x.

Equation (1.25) written for age group x + 5 to x + 9 gives:

$$\{{}_5\underset{\sim}{V}_{x+5}\}' = \{\underset{\sim}{\nu}(0)\}' \frac{5}{2} \sum_{a=x+5}^{\beta-5} [\underset{\sim}{M}(a) + e^{-5r} \underset{\sim}{M}(a+5) \underset{\sim}{S}(a)] e^{-r(a-x-5)} \underset{\sim}{L}(a) \underset{\sim}{L}^{-1}(x+5)$$

$$= \{\underset{\sim}{\nu}(0)\}' \; {}_5\underset{\sim}{N}_{x+5} \quad .$$

The vector $\{{}_5\underset{\sim}{V}_x\}'$ may now be written as

$$\{{}_5\underset{\sim}{V}_x\}' = \{\underset{\sim}{\nu}(0)\}' \; \frac{5}{2} \; [\underset{\sim}{M}(x) + e^{-5r} \; \underset{\sim}{M}(x+5) \; \underset{\sim}{S}(x)] \; \underset{\sim}{L}(x) \; \underset{\sim}{L}^{-1}(x)$$

$$+ \{\underset{\sim}{\nu}(0)\}' \; {}_5\underset{\sim}{N}_{x+5} \left[e^{-r(x+5)} \; \underset{\sim}{L}(x+5)\right]\left[e^{-rx} \; \underset{\sim}{L}(x)\right]^{-1}$$

$$= \{\underset{\sim}{\nu}(0)\}' \left[\frac{5}{2} \underset{\sim}{M}(x) + [\frac{5}{2} \underset{\sim}{M}(x+5) + {}_5\underset{\sim}{N}_{x+5}] \; e^{-5r} \; \underset{\sim}{S}(x)\right] \quad . \tag{1.28}$$

Equation (1.28) is analogous to the single region formula of ${}_5\underset{\sim}{N}_x$, derived by Keyfitz (1977).

Consider again the two-region system Slovenia-Rest of Yugoslavia. The elements of $\bar{\underset{\sim}{n}}_x$ are listed in Table 1. The discount rate is r = 0.006099. Note that $\bar{\underset{\sim}{n}}_0$ is identical to $\underset{\sim}{\Psi}(r)$, the characteristic matrix. The elements of ${}_5\underset{\sim}{N}_x$ are given in Table 2. The discounted number of female descendants of a woman living in Slovenia and 15 to 19 years old is 1.0076. A total of 0.9418 is expected to be born in Slovenia, while 0.0658 will be born in the Rest of Yugoslavia. On the other hand, a woman of the same age group in the Rest of Yugoslavia has an expected discounted number of daughters of 1.1943. A small fraction, namely 0.0067, will be born in Slovenia. This is mainly due to the low outmigration proportion of women in the Rest of Yugoslavia and to the low fertility in Slovenia. The curve of age-specific discounted number of offspring has a peak at age group 10-14 years.

The total discounted number of offspring of the observed population is given by (1.18) and numerically evaluated by the formula

$$\underset{\sim}{NK} = \sum_{x=0}^{\omega} {}_5\underset{\sim}{N}_x \; \underset{\sim}{K}(x) \quad , \tag{1.29}$$

with $\underset{\sim}{K}(x)$ being the diagonal matrix with the regional populations of age group x to x + 4 in the diagonal.

Table 1. Discounted number of offsprings per person at exact age x.

REGION OF RESIDENCE SLOVENIA

REGION OF BIRTH OF OFFSPRINGS

	TOTAL	SLOVENIA	R,YUGOS,
0	0,916100	0,813686	0,102414
5	0,971974	0,877292	0,094682
10	1,002009	0,916767	0,085242
15	1,032696	0,953190	0,079507
20	0,981935	0,929146	0,052788
25	0,652048	0,630542	0,021506
30	0,351120	0,344533	0,006587
35	0,153903	0,152558	0,001344
40	0,042952	0,042710	0,000242
45	0,004836	0,004807	0,000028
50	0,001423	0,001417	0,000006
55	0,000000	0,000000	0,000000
60	0,000000	0,000000	0,000000
65	0,000000	0,000000	0,000000
70	0,000000	0,000000	0,000000
75	0,000000	0,000000	0,000000
80	0,000000	0,000000	0,000000

REGION OF RESIDENCE R,YUGOS,

REGION OF BIRTH OF OFFSPRINGS

	TOTAL	SLOVENIA	R,YUGOS,
0	1,003908	0,008942	0,994966
5	1,158390	0,009091	1,149299
10	1,198425	0,008654	1,189771
15	1,238313	0,008204	1,230109
20	1,148223	0,005266	1,142957
25	0,743626	0,001747	0,741879
30	0,394423	0,000555	0,393868
35	0,184441	0,000131	0,184309
40	0,072078	0,000023	0,072054
45	0,013779	0,000002	0,013777
50	0,003460	0,000000	0,003460
55	0,000000	0,000000	0,000000
60	0,000000	0,000000	0,000000
65	0,000000	0,000000	0,000000
70	0,000000	0,000000	0,000000
75	0,000000	0,000000	0,000000
80	0,000000	0,000000	0,000000

Table 2. Discounted number of offsprings per person in age group x.

REGION OF RESIDENCE SLOVENIA

	REGION OF BIRTH OF OFFSPRINGS TOTAL	SLOVENIA	R,YUGOS.
0	0,943847	0,844677	0,099170
5	0,986890	0,896790	0,090100
10	1,017244	0,934792	0,082452
15	1,007583	0,941816	0,065767
20	0,819229	0,785372	0,033857
25	0,503857	0,492104	0,011753
30	0,254235	0,251223	0,003013
35	0,099454	0,098905	0,000549
40	0,024254	0,024192	0,000062
45	0,003166	0,003156	0,000010
50	0,000730	0,000730	0,000000
55	0,000000	0,000000	0,000000
60	0,000000	0,000000	0,000000
65	0,000000	0,000000	0,000000
70	0,000000	0,000000	0,000000
75	0,000000	0,000000	0,000000
80	0,000000	0,000000	0,000000

REGION OF RESIDENCE R,YUGOS,

	REGION OF BIRTH OF OFFSPRINGS TOTAL	SLOVENIA	R,YUGOS,
0	1,076650	0,009032	1,067618
5	1,178234	0,008880	1,169355
10	1,218204	0,008436	1,209768
15	1,194267	0,006720	1,187547
20	0,949871	0,003167	0,946704
25	0,572510	0,000973	0,571537
30	0,291552	0,000260	0,291292
35	0,129433	0,000047	0,129386
40	0,043597	0,000004	0,043593
45	0,008752	0,000001	0,008751
50	0,001785	0,000000	0,001785
55	0,000000	0,000000	0,000000
60	0,000000	0,000000	0,000000
65	0,000000	0,000000	0,000000
70	0,000000	0,000000	0,000000
75	0,000000	0,000000	0,000000
80	0,000000	0,000000	0,000000

The value of $\underset{\sim}{NK}$ is given in Table 3.

Table 3. Total discounted number of offspring of the observed population.

Region of birth of offsprings	Region of Residence of Mother		
	Slovenia	Rest of Yug.	Total ($\{\underset{\sim}{NK}\}$)
Slovenia	352,761	29,934	382,695
Rest of Yug.	26,333	5,119,601	5,145,934
Total ($\{\underset{\sim}{\bar{NK}}\}'$)	379,094	5,149,535	5,528,629

Table 3 shows that, under the 1961 regime of fertility, mortality and migration, the total discounted number of offspring of Yugoslavia is 5,528,629. Of them, 382,695 or 6.92% will be born in Slovenia. However, the female residents of Slovenia will account for only 379,094 or 6.86% of the total discounted number of births. Of the ultimate discounted 382,695 female children born in Slovenia, 29,934 can be attributed to women now residents of the Rest of Yugoslavia, and 352,761 to potential mothers now living in Slovenia. On the other hand, of the discounted 379,094 daughters born from the female population of Slovenia, 26,333 will be born in the Rest of Yugoslavia, and 352,761 in Slovenia.

To derive the reproductive value of the female population, we must weight the discounted number of offsprings for the region of birth. If we attach to a birth or a 0-year-old in Slovenia the reproductive value of unity, then a birth in the Rest of Yugoslavia is worth 1.818. Adopting this arbitrary scaling, the age-specific reproductive values by region of residence are given in Table 4. The elements are the weighted sums of the discounted number of offspring per woman by region of residence,

Table 4. Spatial reproductive value per person.

a. Person at exact age x.

	SLOVENIA	R.YUGOS.
0	1.000000	1.818116
5	1.049435	2.098650
10	1.071746	2.171795
15	1.097742	2.244685
20	1.025122	2.083295
25	0.669642	1.350569
30	0.356509	0.716653
35	0.155003	0.335227
40	0.043150	0.131027
45	0.004859	0.025051
50	0.001427	0.006291
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000

b. Person in age group x.

	SLOVENIA	R.YUGOS.
0	1.024980	1.950086
5	1.060603	2.134902
10	1.084699	2.207935
15	1.061388	2.165819
20	0.846928	1.724385
25	0.513472	1.040094
30	0.256700	0.529863
35	0.099904	0.235286
40	0.024305	0.079261
45	0.003175	0.015911
50	0.000730	0.003245
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000

the weights being the components of the $\{\underset{\sim}{v}(0)\}$ vector. The total reproductive value of the women in the two-region system is by Table 3,

$$\begin{bmatrix} 1.000000 & 1.818116 \end{bmatrix} \begin{bmatrix} 352{,}761 & 29{,}934 \\ 26{,}333 & 5{,}119{,}601 \end{bmatrix} = \begin{bmatrix} 400{,}638 \\ 9{,}337{,}963 \end{bmatrix} .$$

The total reproductive value of the whole system is

$$V = 400{,}638 + 9{,}337{,}963 = 9{,}738{,}601 \quad .$$

Note that the unit in which V is measured is the reproductive value of a 0-year-old in Slovenia. Using another unit would give V another value.

2. THE SPATIAL REPRODUCTIVE VALUE: APPLICATIONS

The concept of spatial reproductive value is a useful notion for the study of multiregional population dynamics. In this section, its contribution is illustrated for the definition of the birth trajectory and the population growth path. This topic is important since it is a first step to the study of the convergence path toward stability of a multiregional system, and since it enables a simple analytical expression for the stable birth trajectory. Moreover, it facilitates the sensitivity analysis of changes in observed rates and the analytical computation of the momentum of spatial zero population growth.

In the first section, we express the population and birth trajectory in terms of their spectral components. The spectral decomposition yields a simple expression for the growth path and illustrates the relevance of the reproductive value for spatial demographic analysis. The problems of norming, the numerical calculation, and some immediate applications are treated in subsequent sections.

2.1 Spectral Decomposition of the Population and Birth Trajectory

It can be shown that the continuous model of multiregional growth or the generalized Lotka model is equivalent to a system of linear homogeneous differential equations:

$$\{\dot{\underset{\sim}{k}}^{(t)}\} = \underset{\sim}{A}\{\underset{\sim}{k}^{(t)}\} \tag{2.1}$$

where $\{\underset{\sim}{k}^{(t)}\}$ is the population distribution by age and region at time t,

$$\{\underset{\sim}{k}^{(t)}\} = \begin{bmatrix} \{\underset{\sim}{k}^{(t)}(0)\} \\ \{\underset{\sim}{k}^{(t)}(dx)\} \\ \{\underset{\sim}{k}^{(t)}(2dx)\} \\ \vdots \\ \{\underset{\sim}{k}^{(t)}(\omega)\} \end{bmatrix}$$

with $\{\underset{\sim}{k}^{(t)}(x)\}$ denoting the regional distribution of the population in age group x to x + dx, and ω is the highest age,

$\{\dot{\underset{\sim}{k}}^{(t)}\} = \frac{d\{\underset{\sim}{k}^{(t)}\}}{dt}$ is the differential with respect to time,

$\underset{\sim}{A}$ is the transition matrix, assumed to be time-invariant.

The solution of the system of homogenous equation (2.1) is (Lancaster, 1969, pp. 189-190):

$$\{\underset{\sim}{k}^{(t)}\} = e^{\underset{\sim}{A}(t-t_0)}\left\{\underset{\sim}{k}^{(t_0)}\right\}$$

and

$$\{\underset{\sim}{k}^{(t)}\} = e^{\underset{\sim}{A}t}\{\underset{\sim}{k}^{(0)}\} \qquad (2.2)$$

where $\{\underset{\sim}{k}^{(0)}\} = \{{}_0\underset{\sim}{k}\}$ is the population distribution vector in the base year. The matrix $e^{\underset{\sim}{A}}$ is the population growth matrix of the continuous model, and is therefore analogous to the generalized Leslie matrix (Rogers, 1975, p. 123).[6]

The computation of $e^{\underset{\sim}{A}t}$ has received much attention in the literature, in particular in the engineering literature (see for example, Wolovich, 1974). One way of expressing $e^{\underset{\sim}{A}t}$ is by applying the spectral theorem.

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[6]Note that (2.3) is the generalization of the exponential growth model

$$K^{(t)} = e^{rt}K^{(0)} \quad , \qquad (2.3)$$

with $K^{(t)}$ being the total population at time t. Equation (2.3) is the solution of the differential equation $\dot{K}^{(t)} = rK^{(t)}$.

<u>Spectral theorem</u>: (Lancaster, 1969, p. 63).

Let $\underset{\sim}{A}$ be a simple N x N matrix, and f(•) a function. Let r_j be distinct eigenvalues of $\underset{\sim}{A}$ and let $\xi_1, \xi_2, \ldots, \xi_n$ and $\nu_1, \nu_2, \ldots, \nu_n$ be normalized right and left eigenvectors respectively, then

$$f(\underset{\sim}{A}) = \sum_{j=1}^{n} f(r_j)\ \underset{\sim}{B}_j \tag{2.4}$$

where $\underset{\sim}{B}_j = \{\underset{\sim}{\xi}_j\}\{\underset{\sim}{\nu}_j\}'$, j = 1,2,...,n. The matrices $\underset{\sim}{B}_j$ are called constituent matrices.[7]

The matrix $e^{\underset{\sim}{A}t}$ of the population growth model (2.2) generally satisfies the conditions set by the spectral theorem. If $f(A) = e^{\underset{\sim}{A}t}$, then $f(r_j) = e^{r_j t}$ (Lancaster, 1969, pp. 190-191). Therefore,

$$e^{\underset{\sim}{A}t} = \sum_{j=1}^{n} e^{r_j t}\{\underset{\sim}{\xi}_j\}\{\underset{\sim}{\nu}_j\}' \quad . \tag{2.5}$$

Substituting (2.5) into (2.3) yields

$$\{\underset{\sim}{k}^{(t)}\} = \sum_{j=1}^{n} e^{r_j t}\{\underset{\sim}{\xi}_j\}\{\underset{\sim}{\nu}_j\}'\{{}_o\underset{\sim}{k}\} \tag{2.7}$$

$$= \sum_{j=1}^{n} e^{r_j t}<\{\underset{\sim}{\nu}_j\}'\{{}_o\underset{\sim}{k}\}>\{\underset{\sim}{\xi}_j\} \quad , \tag{2.8}$$

where <•> denotes the inner product. Equation (2.8) expresses an arbitrary age and region distribution as a linear combination

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[7]The constituent matrix $\{\underset{\sim}{\xi}_j\}\{\underset{\sim}{\nu}_j\}'$ may be expressed in terms of the adjoint of $[\underset{\sim}{A} - r_j\underset{\sim}{I}] = \underset{\sim}{R}(r_j)$ (Lancaster, 1969, p. 175; Morgan, 1966):

$$\{\underset{\sim}{\xi}_j\}\{\underset{\sim}{\nu}_j\}' = [tr[\underset{\sim}{R}(r_j)]]^{-1}\ \underset{\sim}{R}(r_j) \quad , \tag{2.6}$$

where tr denotes the trace of a matrix.

of the right eigenvectors of $\underset{\sim}{A}$.[8] It is the solution of (2.2) in spectral form.

The eigenvectors $\{\underset{\sim}{\xi}_j\}$ and $\{\underset{\sim}{\nu}_j\}$ are normalized, i.e.,

$$\{\underset{\sim}{\nu}_i\}'\{\underset{\sim}{\xi}_j\} = \delta_{ij} \quad , \quad i,j = 1,\ldots,N$$

with δ_{ij} being the Kronecker delta, i.e., $\delta_{ij} = 1$ for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$. When the eigenvectors are not in normalized form, then (2.8) becomes

$$\{\underset{\sim}{k}^{(t)}\} = \sum_{j=1}^{n} \frac{1}{\{\underset{\sim}{\nu}_j\}'\{\underset{\sim}{\xi}_j\}} e^{r_j t} <\{\underset{\sim}{\nu}_j\}'\{{}_o\underset{\sim}{k}\}>\{\underset{\sim}{\xi}_j\} \ , \qquad (2.9)$$

where $\frac{1}{\{\underset{\sim}{\nu}_j\}'\{\underset{\sim}{\xi}_j\}}$ is the normalizing factor. Therefore (2.9) is valid for any scaling of the left and right eigenvectors of $\underset{\sim}{A}$.[9] The problem is to find a convenient expression for the normalizing factor. This is the topic of the next section.

Equation (2.9) describes the growth path of a population, the internal dynamics of which is described by the matrix $\underset{\sim}{A}$ and the initial condition by $\{{}_o\underset{\sim}{k}\}$. The matrix $e^{\underset{\sim}{A}}$ is a nonnegative indecomposable, primitive square matrix. According to the Perron-Frobenius theorem, it has a dominant eigenvalue e^{r_1}, which is unique, real and positive, and larger in absolute value than any other eigenvalue of the matrix. Therefore, as t becomes large, the population sequence is increasingly dominated by the maximal real root e^{r_1} (see also Rogers, 1975, p. 97). We may write for $t \to \infty$,

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[8] Note that if r_j is an eigenvalue of $\underset{\sim}{A}$, then e^{r_j} is an eigenvalue of $e^{\underset{\sim}{A}}$. The eigenvectors of $\underset{\sim}{A}$ and $e^{\underset{\sim}{A}}$ are identical.

[9] The eigenvector of a matrix is fixed up to a scalar. Therefore, if $\{\underset{\sim}{\xi}_j\}$ is an eigenvector of $\underset{\sim}{A}$, then also $c\{\underset{\sim}{\xi}_j\}$ is an eigenvector. It is the choice of c that determines the scaling.

$$\{\underset{\sim}{k}^{(t)}\} \doteq \frac{1}{\{\underset{\sim}{\nu}_1\}'\{\underset{\sim}{\xi}_1\}} e^{r_1 t} \langle\{\underset{\sim}{\nu}_1\}'\{{}_0\underset{\sim}{k}\}\rangle\{\underset{\sim}{\xi}_1\} \quad . \tag{2.10}$$

Equation (2.10) describes the growth path of the multi-regional stable population. The quantity r_1 is the intrinsic growth rate of the stable population. The left eigenvector $\{\underset{\sim}{\nu}_1\}$ and the right eigenvector $\{\underset{\sim}{\xi}_1\}$ associated with the dominant root e^{r_1} of $e^{\underset{\sim}{A}}$ have particularly interesting demographic interpretations. They represent the reproductive values by region and age, and the stable population distribution by region and age respectively. Therefore, the absolute number of people by age and region in the stable population is a function of the initial population distribution, the distribution of the stable population, the distribution of the reproductive values and of the stable growth rate. The stable population distribution $\{\underset{\sim}{\xi}_1\}$ remains constant, and the total population grows at a rate r_1. To convert the relative distribution vector $\{\underset{\sim}{\xi}_1\}$ into the vector of absolute amounts $\{\underset{\sim}{k}^{(t)}\}$, we must compute the scaling factor

$$d = \frac{1}{\{\underset{\sim}{\nu}_1\}'\{\underset{\sim}{\xi}_1\}} \langle\{\underset{\sim}{\nu}_1\}'\{{}_0\underset{\sim}{k}\}\rangle \quad . \tag{2.11}$$

The first element of d is the normalizing factor. The second element may be written as

$$v = \langle\{\underset{\sim}{\nu}_1\}'\{{}_0\underset{\sim}{k}\}\rangle = \int_0^{\omega} \{\underset{\sim}{\nu}(x)\}'\{{}_0\underset{\sim}{k}(x)\}\, dx \tag{2.12}$$

where $\{\underset{\sim}{\nu}(x)\}$ is the regional distribution of the reproductive value of individuals aged x years, and $\{{}_0\underset{\sim}{k}(x)\}$ is the initial regional distribution of people of age x. Expression (2.12) gives the total reproductive value of the initial population (in units determined by the scaling of $\{\underset{\sim}{\nu}_1\}$).

Assuming that $\{\underset{\sim}{\nu}_1\}$ and $\{\underset{\sim}{\xi}_1\}$ are normalized eigenvectors, then the time path of the stable population is given by the simple expression

$$\{\underset{\sim}{k}^{(t)}\} = ve^{r_1 t}\{\underset{\sim}{\xi}_1\} \quad . \tag{2.13}$$

The absolute number of people in each region and age group is its relative share, as measured by the normalized $\{\underset{\sim}{\xi}_1\}$, times the total reproductive value of the initial population, and discounted at time t at a rate r_1. The stable equivalent to the original population is[10]

$$\{\underset{\sim}{\bar{k}}^{(0)}\} = v\{\underset{\sim}{\xi}_1\} \quad .$$

The stable equivalent population in age group x is

$$\{\underset{\sim}{\bar{k}}^{(0)}(x)\} = v\{\underset{\sim}{\xi}_1(x)\} \quad . \tag{2.14}$$

2.2 The Determination of the Normalizing Factor

The growth path of the stable population is given by equation (2.10). The purpose of this section is to derive a usable and demographically meaningful expression for $\{\underset{\sim}{\nu}_1\}'\{\underset{\sim}{\xi}_1\}$. This would enable the use of (2.10) for any arbitrary scaling of $\{\underset{\sim}{\nu}_1\}$ and $\{\underset{\sim}{\xi}_1\}$.

The inner product $\{\underset{\sim}{\nu}_1\}'\{\underset{\sim}{\xi}_1\}$ may be written as[11]

$$\{\underset{\sim}{\nu}\}'\{\underset{\sim}{\xi}\} = \int_0^{\omega} \{\underset{\sim}{\nu}(x)\}'\{\underset{\sim}{\xi}(x)\}\, dx \tag{2.15}$$

where $\{\underset{\sim}{\nu}(x)\}$ represents the regional distribution of reproductive values at age x, and $\{\underset{\sim}{\xi}(x)\}$ dx is the regional distribution of people of age x to x + dx at stability. As mentioned before,

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[10] Note that v is measured in units consistent with the normalized $\{\underset{\sim}{\nu}_1\}$ vector.

[11] The subscript is dropped for convenience.

the units in which the reproductive value and the population distribution are measured are arbitrary. It is assumed that the unit is determined by the first element of $\{\underset{\sim}{\nu}(0)\}$ and the first element of $\{\underset{\sim}{\xi}(0)\}$.

By (1.12), $\{\underset{\sim}{\nu}(x)\}$ may be expressed in terms of $\{\underset{\sim}{\nu}(0)\}$:

$$\{\underset{\sim}{\nu}(x)\}' = \{\underset{\sim}{\nu}(0)\}' \int_x^\beta [e^{-ra}\, \underset{\sim}{m}(a)\, \hat{\underset{\sim}{\ell}}(a)\, da][e^{-rx}\, \hat{\underset{\sim}{\ell}}(x)]^{-1} \quad . \tag{2.16}$$

On the other hand, it can easily be shown that the age composition of the population at stability is equal to:

$$\{\underset{\sim}{\xi}(x)\} = e^{-rx}\, \hat{\underset{\sim}{\ell}}(x)\, \{\underset{\sim}{\xi}(0)\} \quad . \tag{2.17}$$

Therefore, we may write

$$\begin{aligned} \{\underset{\sim}{\nu}(x)\}'\{\underset{\sim}{\xi}(x)\} &= \{\underset{\sim}{\nu}(0)\}' \int_x^\beta [e^{-ra}\, \underset{\sim}{m}(a)\, \hat{\underset{\sim}{\ell}}(a)\, da] \\ &\quad \cdot [e^{-rx}\, \hat{\underset{\sim}{\ell}}(x)]^{-1} [e^{-rx}\, \hat{\underset{\sim}{\ell}}(x)]\{\underset{\sim}{\xi}(0)\} \qquad (2.18) \\ &= \{\underset{\sim}{\nu}(0)\}' \int_x^\beta e^{-ra}\, \underset{\sim}{m}(a)\, \hat{\underset{\sim}{\ell}}(a)\, da\, \{\underset{\sim}{\xi}(0)\} \quad . \end{aligned}$$

But

$$e^{-ra}\, \hat{\underset{\sim}{\ell}}(a)\{\underset{\sim}{\xi}(0)\} = \{\underset{\sim}{\xi}(a)\} \quad . \tag{2.19}$$

Hence

$$\{\underset{\sim}{\nu}(x)\}'\{\underset{\sim}{\xi}(x)\} = \{\underset{\sim}{\nu}(0)\}' \int_x^\beta \underset{\sim}{m}(a)\{\underset{\sim}{\xi}(a)\}\, da \quad . \tag{2.20}$$

Substituting (2.20) in (2.15), and noting that $\underset{\sim}{m}(a) = \underset{\sim}{0}$ for $a < \alpha$ and $a > \beta$, gives

$$\int_{\alpha}^{\beta} \{\underset{\sim}{\nu}(x)\}'\{\underset{\sim}{\xi}(x)\}\, dx = \{\underset{\sim}{\nu}(0)\}' \int_{0}^{\omega} \int_{x}^{\omega} \underset{\sim}{m}(a)\{\underset{\sim}{\xi}(a)\}\, da\, dx \quad . \tag{2.21}$$

The double integral at the right hand side may be written as

$$\left[\int_{0}^{\omega} \int_{x}^{\omega} e^{-ra}\, \underset{\sim}{m}(a)\, \hat{\underset{\sim}{\ell}}(a)\, da\, dx\right] \{\underset{\sim}{\xi}(0)\} \quad . \tag{2.22}$$

Denoting

$$\underset{\sim}{f}(x) = \int_{x}^{\omega} e^{-ra}\, \underset{\sim}{m}(a)\, \hat{\underset{\sim}{\ell}}(a)\, da \quad ,$$

the solution to the double integral is

$$\begin{aligned} \int_{0}^{\omega} \underset{\sim}{f}(x)\, dx &= [\underset{\sim}{f}(x)\, x]\Big|_{0}^{\omega} - \int_{0}^{\omega} x\, d\underset{\sim}{f}(x) \\ &= \underset{\sim}{0} - \int_{0}^{\omega} x\, \underset{\sim}{f}'(x)\, dx \\ &= \int_{0}^{\omega} x\, e^{-rx}\, \underset{\sim}{m}(x)\, \hat{\underset{\sim}{\ell}}(x)\, dx \quad . \end{aligned} \tag{2.23}$$

Introducing (2.23) into (2.22) and into (2.21) yields

$$\int_{0}^{\omega} \{\underset{\sim}{\nu}(x)\}'\{\underset{\sim}{\xi}(x)\}\, dx = \{\underset{\sim}{\nu}(0)\}'\left[\int_{0}^{\omega} x\, e^{-rx}\, \underset{\sim}{m}(x)\, \hat{\underset{\sim}{\ell}}(x)\, dx\right] \{\underset{\sim}{\xi}(0)\} \quad . \tag{2.24}$$

At stability, $\{\underset{\sim}{\xi}(0)\}$ obeys the following relationship (Rogers, 1975, p. 93):

$$\{\underset{\sim}{\xi}(0)\} = \underset{\sim}{\Psi}(r)\{\underset{\sim}{\xi}(0)\}$$

$$= \left[\int_0^{\omega} e^{-rx} \underset{\sim}{m}(x)\ \hat{\underset{\sim}{\ell}}(x)\ dx\right]\{\underset{\sim}{\xi}(0)\} \quad . \tag{2.25}$$

[12]

Equivalently, we may write for $\underset{\sim}{\Psi}(r)$ nonsingular,

$$\{\underset{\sim}{\xi}(0)\} = \left[\int_0^{\omega} e^{-rx} \underset{\sim}{m}(x)\ \hat{\underset{\sim}{\ell}}(x)\ dx\right]^{-1} \{\underset{\sim}{\xi}(0)\} \quad . \tag{2.26}$$

Substituting (2.26) into (2.24) gives

$$\int_0^{\omega} \{\underset{\sim}{\nu}(x)\}'\{\underset{\sim}{\xi}(x)\}\ dx = \{\underset{\sim}{\nu}(0)\}' \left[\int_0^{\omega} xe^{-rx} \underset{\sim}{m}(x)\ \hat{\underset{\sim}{\ell}}(x)\ dx\right] \cdot \left[\int_0^{\omega} e^{-rx} \underset{\sim}{m}(x)\ \hat{\underset{\sim}{\ell}}(x)\ dx\right]^{-1} \{\underset{\sim}{\xi}(0)\} \quad . \tag{2.27}$$

Note that

$$\int_0^{\omega} xe^{-rx} \underset{\sim}{m}(x)\ \hat{\underset{\sim}{\ell}}(x)\ dx = \underset{\sim}{R}^{(r)}(1)$$

is the first derivative of the multiregional characteristic matrix with respect to r, and that

$$\int_0^{\omega} e^{-rx} \underset{\sim}{m}(x)\ \hat{\underset{\sim}{\ell}}(x)\ dx = \underset{\sim}{\Psi}(r) = \underset{\sim}{R}^{(r)}(0)$$

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[12] Note that by (2.17), (2.25) is equivalent to

$$\{\underset{\sim}{\xi}(0)\} = \int_0^{\omega} \underset{\sim}{m}(x)\{\underset{\sim}{\xi}(x)\}\ dx \quad .$$

is the multiregional characteristic matrix itself. If we define the matrix of mean ages at childbearing in the stable population as

$$\underset{\sim}{\kappa} = \underset{\sim}{R}^{(r)}(1)[\underset{\sim}{R}^{(r)}(0)]^{-1} \quad , \tag{2.28}$$

then (2.27) becomes

$$\int_0^{\omega} \{\underset{\sim}{v}(x)\}'\{\underset{\sim}{\xi}(x)\}\, dx = \{\underset{\sim}{v}(0)\}' \underset{\sim}{\kappa}\{\underset{\sim}{\xi}(0)\} \quad . \tag{2.29}$$

The normalizing factor is

$$\frac{1}{\{\underset{\sim}{v}\}'\{\underset{\sim}{\xi}\}} = \frac{1}{\{\underset{\sim}{v}(0)\}'\underset{\sim}{\kappa}\{\underset{\sim}{\xi}(0)\}} \quad . \tag{2.30}$$

It is a weighted average of the mean ages at childbearing in the stable population. In the single region case, the normalizing factor reduces to $\frac{1}{\kappa}$, where κ is the mean age of childbearing in the stable population (Goodman, 1969, p. 665).

2.3 Stable Population Analysis

The expression of the normalizing factor in a demographically meaningful manner may now be used to gain new insights into several features of the stable population.

From (2.10), (2.12) and (2.30), we see that, no matter which scaling is used for the eigenvectors $\{\underset{\sim}{v}\}$ and $\{\underset{\sim}{\xi}\}$, the multiregional population trajectory for t large is:

$$\{\underset{\sim}{k}^{(t)}\} = \frac{V}{\{\underset{\sim}{v}(0)\}'\underset{\sim}{\kappa}\{\underset{\sim}{\xi}(0)\}} \; e^{rt}\{\underset{\sim}{\xi}\} \quad . \tag{2.31}$$

The ultimate trajectory of births is directly derived from (2.31). Considering only the population at age 0, we may write:

$$\{\underset{\sim}{Q}^{(t)}\} = \frac{V}{\{\underset{\sim}{v}(0)\}'\underset{\sim}{\kappa}\{\underset{\sim}{\xi}(0)\}} \; e^{rt}\{\underset{\sim}{\xi}(0)\} \quad , \tag{2.32}$$

or

$$\{\underset{\sim}{Q}^{(t)}\} = \frac{1}{\{\underset{\sim}{\nu}(0)\}'\underset{\sim}{\kappa}\{\underset{\sim}{\xi}(0)\}} e^{rt}\left[\int_0^{\omega} \{\underset{\sim}{\nu}(x)\}'\{{}_o\underset{\sim}{k}(x)\}\, dx\right]\{\underset{\sim}{\xi}(0)\} \tag{2.33}$$

where $\{\underset{\sim}{Q}^{(t)}\}$ gives the stable number of births, in absolute terms, in each region at time t associated with an arbitrary initial population distribution $\{{}_o\underset{\sim}{k}\}$.[13] The stable equivalent of the births is immediate:

$$\{\underset{\sim}{Q}\} = \frac{V}{\{\underset{\sim}{\nu}(0)\}'\underset{\sim}{\kappa}\{\underset{\sim}{\xi}(0)\}} \{\underset{\sim}{\xi}(0)\} \quad . \tag{2.34}$$

An equivalent expression for $\{\underset{\sim}{Q}\}$ may be even more meaningful:

$$\{\underset{\sim}{Q}\} = \frac{V\{\underset{\sim}{1}\}'\{\underset{\sim}{\xi}(0)\}}{\{\underset{\sim}{\nu}(0)\}'\underset{\sim}{\kappa}\{\underset{\sim}{\xi}(0)\}} \{\underset{\sim}{\bar{\xi}}(0)\} = Q\{\underset{\sim}{\bar{\xi}}(0)\} \tag{2.35}$$

where the elements of $\{\underset{\sim}{\bar{\xi}}(0)\}$ sum up to one. The quantity Q is the total number of births in the whole system. These births are divided between the regions according to $\{\underset{\sim}{\bar{\xi}}(0)\}$.

The ultimate birth trajectory, resulting from one girl at age x in region 1 is

$$\frac{1}{\{\underset{\sim}{\nu}(0)\}'\underset{\sim}{\kappa}\{\underset{\sim}{\xi}(0)\}} e^{rt}\, \nu_1(x)\ \{\underset{\sim}{\xi}(0)\} \quad , \tag{2.36}$$

which is analogous to the single-region formula $\nu(x)\ e^{rt}/\kappa$. Similarly, the ultimate birth trajectory resulting from one girl at age x in region 2 is

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[13] The single-region analogue of (2.33) is (Keyfitz, 1975, p. 591):

$$\underset{\sim}{Q}^{(t)} = \frac{1}{\kappa} e^{rt} \int_0^{\omega} \nu(x)\ {}_ok(x)\ dx \quad .$$

$$\frac{1}{\{\underset{\sim}{\nu}(0)\}'\underset{\sim}{\kappa}\{\underset{\sim}{\xi}(0)\}} \; e^{rt} \; \nu_2(x) \; \{\underset{\sim}{\xi}(0)\} \quad . \tag{2.37}$$

It becomes clear from (2.36) and (2.37) that the absolute number of births in the stable population depends on the reproductive value of the girl at age x and consequently on her region of residence ($\nu_1(x) \neq \nu_2(x)$). The relative distribution of the births over the regions is independent of her region of residence, but is solely determined by $\{\underset{\sim}{\xi}(0)\}$. This is consistent with the observation that the ultimate regional distribution of births is independent of the current population distribution.

The ultimate birth trajectory is exponential. This can easily be seen from (2.32) and (2.33). To derive an expression for the time path of the total reproductive value v, consider (2.32) and (2.35),

$$\{\underset{\sim}{Q}^{(t)}\} = e^{rt} \; \{\underset{\sim}{Q}\} = e^{rt} \; Q\{\underset{\sim}{\bar{\xi}}(0)\} \quad . \tag{2.38}$$

Let

$$Q^{(t)} = e^{rt} \; Q \quad , \tag{2.39}$$

then

$$Q^{(t)} = \epsilon^{rt} \; hv \tag{2.40}$$

where

$$h = \frac{\{\underset{\sim}{1}\}'\{\underset{\sim}{\xi}(0)\}}{\{\underset{\sim}{\nu}(0)\}'\underset{\sim}{\kappa}\{\underset{\sim}{\xi}(0)\}} \quad . \tag{2.41}$$

By (2.12)

$$Q^{(t)} = e^{rt} \; h \; \langle\{\underset{\sim}{\nu}_1\}'\{{}_o\underset{\sim}{k}\}\rangle \quad . \tag{2.42}$$

Similarly,

$$Q^{(t)} = e^{r(t-1)} \; h <\{\underset{\sim}{\nu}_1\}'\{\underset{\sim}{k}^{(1)}\}> \quad , \tag{2.43}$$

where $\{\underset{\sim}{k}^{(1)}\}$ is the population distribution one year after the base period.

The equalities (2.42) and (2.43) only hold if

$$e^r<\{\underset{\sim}{\nu}_1\}'\{_o\underset{\sim}{k}\}> = <\{\underset{\sim}{\nu}_1\}'\{\underset{\sim}{k}^{(1)}\}>$$

$$e^r \; v_0 = v_1 \tag{2.44}$$

where v_0 and v_1 are the total reproductive value of the population in the base year and in the first year respectively. Equation (2.44) shows that the reproductive value <u>immediately</u> follows an exponential trajectory. This is in contrast to the birth trajectory, which is exponential only when stability is reached.

Before proceeding to the numerical calculation of $\{\underset{\approx}{Q}\}$, we derive two more useful expressions. The first relates $\{\underset{\approx}{Q}\}$ to $\{\underset{\sim}{nk}\}$. The second expresses the stable equivalent population $\{\underset{\sim}{Y}\}$ as a function of $\{\underset{\approx}{Q}\}$.

Combining (2.34) and (1.21) gives

$$\{\underset{\approx}{Q}\} = \frac{1}{c} \{\underset{\sim}{\nu}(0)\}'\{\underset{\sim}{nk}\}\{\underset{\approx}{\xi}(0)\} \tag{2.45}$$

where

$$c = \{\underset{\sim}{\nu}(0)\}'\underset{\sim}{\kappa}\{\underset{\approx}{\xi}(0)\} \quad . \tag{2.46}$$

Equation (2.45) may be written as

$$\{\underset{\approx}{Q}\} = \frac{1}{c} \{\underset{\approx}{\xi}(0)\}\{\underset{\sim}{\nu}(0)\}'\{\underset{\sim}{nk}\} \quad . \tag{2.47}$$

Let

$$\underset{\sim}{C} = \{\underset{\sim}{\xi}(0)\}\{\underset{\sim}{\nu}(0)\}' \tag{2.48}$$

then

$$\{\underset{\sim}{Q}\} = \frac{1}{c} \underset{\sim}{C}\{\underset{\sim}{nk}\} \tag{2.49}$$

or

$$\{\underset{\sim}{Q}\} = \frac{1}{c} \underset{\sim}{C} \int_0^{\omega} \underset{\sim}{n}(x)\{{}_0\underset{\sim}{k}(x)\}\, dx \quad . \tag{2.50}$$

The stable number of births in each region is a linear combination of the discounted number of descendants by region of birth. Equation (2.49) is basically equivalent to (4.29) of Rogers (1975, p. 99):

$$\{\underset{\sim}{Q}\} = \underset{\sim}{B}\{\underset{\sim}{nk}\}$$

where $\underset{\sim}{B} = [\underset{\sim}{D} - (\underset{\sim}{I} - \underset{\sim}{DD}^{+})\ \underset{\sim}{\Psi}(r)]^{-1}\ [\underset{\sim}{I} - \underset{\sim}{DD}^{+}]$, and where $\underset{\sim}{D}^{+}$ denotes the generalized inverse of $\underset{\sim}{D}$, with $\underset{\sim}{D} = [\underset{\sim}{I} - \underset{\sim}{\Psi}(r)]$. The feature of (2.49) is that it does not require the computation of the generalized inverse, and that it relates $\{\underset{\sim}{Q}\}$ to easily interpretable demographic measures.

Formula (2.50) is particularly useful to study the impact on the ultimate birth trajectory of a change in the current population distribution. A change in the spatial distribution of females of age x by $\Delta\{{}_0\underset{\sim}{k}(x)\}$ changes the stable equivalent of births by

$$\Delta\{\underset{\sim}{Q}\} = \frac{1}{c} \underset{\sim}{C}\ \underset{\sim}{n}(x)\ \Delta\{{}_0\underset{\sim}{k}(x)\}$$

and the ultimate distribution of births at t by

$$\Delta\{\underset{\sim}{Q}(t)\} = e^{rt}\, \frac{1}{c} \underset{\sim}{C}\ \underset{\sim}{n}(x)\ \Delta\{{}_0\underset{\sim}{k}(x)\} \quad . \tag{2.51}$$

The total stable equivalent population is

$$\{\underset{\sim}{Y}\} = \int_0^\omega \{\underset{\sim}{\bar{k}}^{(0)}(x)\}\, dx \tag{2.52}$$

where $\{\underset{\sim}{\bar{k}}^{(0)}(x)\}$ is given by (2.14). For not normalized eigenvectors, (2.52) may be written as

$$\{\underset{\sim}{Y}\} = \frac{1}{c}\; v \int_0^\omega \{\underset{\sim}{\xi}(x)\}\, dx \quad . \tag{2.53}$$

But by (2.17), (2.53) becomes

$$\{\underset{\sim}{Y}\} = \frac{1}{c}\; v \left[\int_0^\omega e^{-rx}\, \underset{\sim}{\hat{\ell}}(x)\, dx\right] \{\underset{\sim}{\xi}(0)\} \quad . \tag{2.54}$$

Applying (2.34) gives[14]

$$\{\underset{\sim}{Y}\} = \left[\int_0^\omega e^{-rx}\, \underset{\sim}{\hat{\ell}}(x)\, dx\right] \{\underset{\sim}{Q}\} \quad . \tag{2.55}$$

Formula (2.55) looks very similar to the stable equivalent formula of the stationary population (Rogers and Willekens, 1976a, p. 16):

$$\{\hat{\underset{\sim}{Y}}\} = \underset{\sim}{e}(0)\{\hat{\underset{\sim}{Q}}\} = \left[\int_0^\omega \underset{\sim}{\hat{\ell}}(x)\, dx\right] \{\hat{\underset{\sim}{Q}}\} \tag{2.56}$$

where $\underset{\sim}{e}(0)$ is the life expectancy matrix, and $\{\hat{\underset{\sim}{Q}}\}$ is the distribution of births in the stationary population.

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[14]Note that the stable equivalent population between ages x and x + dx is

$$\{\underset{\sim}{\bar{k}}^{(0)}(x)\} = e^{-rx}\, \underset{\sim}{\hat{\ell}}(x)\, \{\underset{\sim}{Q}\}\, dx \quad .$$

The difference is that in (2.55) the intrinsic growth rate r is not zero.

By analogy to our approach to $\underset{\sim}{\Psi}(r)$ in the first section of this paper, we consider r in (2.55) as a rate of discount. The quantity $e^{-rx}\ {}_i\hat{\ell}_j(x)$ denotes the number of people born in region i and alive in region j at age x, discounted at age zero, and divided by the number of people born in region i. Similarly, $e^{-rx}\ {}_i\hat{\ell}_j(x)$ may be thought of as a discounted probability.[15] Integration of $e^{-rx}\ \underset{\sim}{\hat{\ell}}(x)$ over all ages yields the <u>matrix of discounted life expectancies $\underset{\sim}{e}^{(r)}(0)$</u>:

$$\underset{\sim}{e}^{(r)}(0) = \int_0^{\omega} e^{-rx}\ \underset{\sim}{\hat{\ell}}(x)\ dx \quad . \tag{2.57}$$

An element ${}_ie_j^{(r)}(0)$ can be given a dual interpretation, similar to the elements ${}_iL_j(x)$ of the life table. First, ${}_ie_j^{(r)}(0)$ may denote the number of years lived in region j by an individual born in region i, discounted at birth. Second, it may represent the discounted number of people living in region j and born in region i, per unit born in i.

Consistent with this argument, the stable equivalent of the population measures the discounted number of people by place of residence or the present value (value at birth) of the total population. This is a better indicator of the reproductive capacity than the number of heads in the population (see also Fisher, 1929, p. 30 and Keyfitz, 1969). Introducing (2.57) into (2.55) gives:

$$\{\underset{\sim}{Y}\} = \underset{\sim}{e}^{(r)}(0)\ \underset{\sim}{Q} \tag{2.58}$$

or for a two-region case:

$$Y_1 = {}_1e_1^{(r)}(0)\ Q_1 + {}_2e_1^{(r)}(0)\ Q_2$$

$$Y_2 = {}_1e_2^{(r)}(0)\ Q_1 + {}_2e_2^{(r)}(0)\ Q_2 \quad .$$

---

[15] Compare this with the analysis of Markov chains including discounting and fees (see e.g., Cinlar, 1975, p. 222).

Note that the stable equivalent of births is equal to

$$\{\underset{\sim}{Q}\} = \left[\int_0^{\omega} e^{-rx}\ \hat{\underset{\sim}{\ell}}(x)\ dx\right]^{-1} \{\underset{\sim}{Y}\}$$

or by (2.25):

$$\{\underset{\sim}{Q}\} = \left[\int_0^{\omega} e^{-rx}\ \underset{\sim}{m}(x)\ \hat{\underset{\sim}{\ell}}(x)\ dx\right]\left[\int_0^{\omega} e^{-rx}\ \hat{\underset{\sim}{\ell}}(x)\ dx\right]^{-1} \{\underset{\sim}{Y}\} \tag{2.59}$$

which is similar to the single-region formula

$$Q = bY$$

$$= \frac{\int_0^{\omega} e^{-rx}\ m(x)\ \hat{\ell}(x)\ dx}{\int_0^{\omega} e^{-rx}\ \ell(x)\ dx} \cdot Y \ .$$

2.4 Numerical Evaluations

The ultimate birth trajectory is given by (2.32). It depends on the growth rate r, the total reproductive value of the population v, the normalizing factor and the relative stable distribution of births. For the two-region system Slovenia-Rest of Yugoslavia, the total reproductive value is computed in the previous section and is equal to 9,738,601. The growth rate r is 0.006099. To compute the normalizing factor, we must know the stable mean ages matrix (2.28):

$$\underset{\sim}{\kappa} = \underset{\sim}{R}^{(r)}(1)\,[\underset{\sim}{R}^{(r)}(0)]^{-1}$$

where $\underset{\sim}{R}^{(r)}(1)$ and $\underset{\sim}{R}^{(r)}(0)$ are approximated by the following expressions:

$$\underset{\sim}{R}^{(r)}(1) = \sum_{x=\alpha-5}^{\beta-5} (x + 2.5)\ e^{-r(x+2.5)}\ \underset{\sim}{M}(x)\ \underset{\sim}{L}(x) \qquad (2.60)$$

$$\underset{\sim}{R}^{(r)}(0) = \sum_{x=\alpha-5}^{\beta-5} e^{-r(x+2.5)}\ \underset{\sim}{M}(x)\ \underset{\sim}{L}(x) \quad . \qquad (2.61)$$

The stable mean ages matrix in our illustration is

$$\underset{\sim}{K} = \begin{bmatrix} 27.310211 & 0.016201 \\ 0.243574 & 27.104595 \end{bmatrix} .$$

The vectors $\{\underset{\sim}{\xi}(0)\}$ and $\{\underset{\sim}{\nu}(0)\}$ are given by (1.9) and (1.11) respectively. The normalizing factor (2.30) is therefore

$$\left[\begin{bmatrix} 1.000000 \\ 1.818116 \end{bmatrix}' \begin{bmatrix} 27.310211 & 0.016201 \\ 0.243574 & 27.104595 \end{bmatrix} \begin{bmatrix} 1.000000 \\ 20.823654 \end{bmatrix}\right]^{-1} = [1054.266]^{-1} \quad .$$

The stable equivalent of births is by (2.34):

$$\{\underset{\sim}{Q}\} = \frac{9,738,601}{1054.266} \begin{bmatrix} 1.000000 \\ 20.823654 \end{bmatrix} = \begin{bmatrix} 9,237 \\ 192,355 \end{bmatrix}$$

or

$$\{\underset{\sim}{Q}\} = Q\{\underset{\sim}{\bar{\xi}}(0)\} = 201,592 \begin{bmatrix} 0.045820 \\ 0.954180 \end{bmatrix} .$$

The total number of births in the stable population is 201,592. Of these births, 4.58% occur in Slovenia, and 95.42% in the Rest of Yugoslavia.

To derive the stable equivalent of the population $\{\underset{\sim}{Y}\}$, we first compute the matrix of discounted life expectancies:

$$\underset{\sim}{e}^{(r)}(0) = \sum_{x=0}^{z} e^{-r(x+2.5)} \underset{\sim}{L}(x) \quad , \tag{2.62}$$

which, in the two-region example, is

$$\underset{\sim}{e}^{(r)}(0) = \begin{bmatrix} 52.227608 & 0.599635 \\ 5.629009 & 52.336800 \end{bmatrix} \quad .$$
$$\qquad\qquad\quad \overline{57.856617} \qquad \overline{52.936435}$$

The stable equivalent of the population is

$$\{\underset{\sim}{Y}\} = \underset{\sim}{e}^{(r)}(0)\{\underset{\sim}{Q}\} = \begin{bmatrix} 597,769 \\ 10,119,240 \end{bmatrix} \quad .$$

Of the 597,769 persons in Slovenia in the stable population, ${}_1e_1^{(r)}(0)\ Q_1 = 482,426$ are born in Slovenia, and ${}_2e_1^{(r)}(0)\ Q_2 =$ 115,342 in the Rest of Yugoslavia.

3. CONCLUSION

This paper has developed the concept of spatial reproductive value and has shown how it contributes to a better understanding of the dynamics of multiregional demographic growth.

Life is considered as an investment by the society in individuals. The stable population growth rate is the intrinsic rate of return to this investment. This rate is the maximum growth rate possible, given the existing fertility, mortality and migration schedules. Because of the regional differences in these schedules, the contribution of a birth to the overall growth rate of the population depends on the region of birth.

The contribution is measured by the reproductive value at birth. The allocation of the births to the regions conforms with the regional reproductive values. Regions with high reproductive values get a greater share of the total number of births. It has been shown that the distribution of births and of reproductive values is such that each individual, as well as each region, pays back the debt incurred in receiving a life from society.

Mathematically the stable population growth rate or the intrinsic rate of return to investment is the rate which gives to the multiregional characteristic matrix $\underset{\sim}{\Psi}(r)$ a dominant eigenvalue of unity. The associated left eigenvector represents the regional distribution of the reproductive values at birth, whereas the right eigenvector denotes the regional allocation of births. It has been shown that the regional distribution of the reproductive value and of the population at age x follows directly from the distribution of the reproductive value at birth and of the births respectively. In fact, the reproductive value of a woman is the weighted discounted value of her offspring. It depends not only on the number of offspring but also on when and where they will be born.

The reproductive value concept facilitates the analysis of multiregional stable populations. For example, a simple expression has been derived for the stable equivalents of the births and the population. Other applications such as sensitivity analysis and the study of the momentum of zero population growth will be reported elsewhere (Rogers and Willekens, 1976b).

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APPENDIX I

Derivation of the Spatial Reproductive Value from the Discrete Demographic Growth Model

The purpose of this Appendix is to derive the formula of the spatial reproductive value for the discrete model of population growth. Recall that the discrete growth model may be written as (Rogers, 1975, p. 123)

$$\{\underset{\sim}{K}^{(t+1)}\} = \underset{\sim}{G}\{\underset{\sim}{K}^{(t)}\}$$

where $\{\underset{\sim}{K}^{(t)}\}$ is the vector of the population distribution by region and age, and $\underset{\sim}{G}$ is the generalized Leslie matrix.

The vector of reproductive values is the left eigenvector of the growth matrix $\underset{\sim}{G}$

$$\{\underset{\sim}{v}\}'\underset{\sim}{G} = \lambda\{\underset{\sim}{v}\}' \qquad \text{(A1)}$$

or, for 5-year age groups,

$$[\{\underset{\sim}{v}(0)\}'\{\underset{\sim}{v}(5)\}'\ldots\{\underset{\sim}{v}(z)\}'] \cdot \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{B}(10) & \underset{\sim}{B}(15) & \cdots & \underset{\sim}{B}(\beta-5) & \cdots & \underset{\sim}{0} \\ \underset{\sim}{S}(0) & \underset{\sim}{0} & & & & \vdots & & \\ \underset{\sim}{0} & \underset{\sim}{S}(5) & & & & \vdots & & \\ \vdots & & \ddots & & & \vdots & & \\ \vdots & & & & \ddots & \underset{\sim}{S}(\beta-5) & & \\ \vdots & & & & & & \ddots & \\ \vdots & & & & & & \underset{\sim}{S}(z-5) & \underset{\sim}{0} \end{bmatrix}$$

$$= \lambda[\{\underset{\sim}{v}(0)\}'\{\underset{\sim}{v}(5)\}'\ldots\{\underset{\sim}{v}(z)\}'] \quad . \qquad \text{(A2]}$$

The scalar λ is the dominant eigenvalue of $\underset{\sim}{G}$. It can be seen that $\{\underset{\sim}{v}(x)\} = \{\underset{\sim}{0}\}$ for $x > \beta - 5$. At age group $[\beta - 5, \beta]$, we have

$$\{\underset{\sim}{v}(0)\}'\underset{\sim}{B}(\beta - 5) + \{\underset{\sim}{v}(\beta)\}'\underset{\sim}{S}(\beta - 5) = \lambda\{\underset{\sim}{v}(\beta - 5)\}'$$

but $\{\underset{\sim}{\nu}(\beta)\} = \{\underset{\sim}{0}\}$.

For age group $[\beta - 10, \beta - 5]$, we write

$$\{\underset{\sim}{\nu}(0)\}'\underset{\sim}{B}(\beta - 10) + \{\underset{\sim}{\nu}(\beta - 5)\}'\underset{\sim}{S}(\beta - 10) = \lambda\{\underset{\sim}{\nu}(\beta - 10)\}' .$$

Since

$$\{\underset{\sim}{\nu}(\beta - 5)\}' = \frac{1}{\lambda} \{\underset{\sim}{\nu}(0)\}'\underset{\sim}{B}(\beta - 5)$$

we have

$$\{\underset{\sim}{\nu}(0)\}'\underset{\sim}{B}(\beta - 10) + \frac{1}{\lambda} \{\underset{\sim}{\nu}(0)\}'\underset{\sim}{B}(\beta - 5)\underset{\sim}{S}(\beta - 10) = \lambda\{\underset{\sim}{\nu}(\beta - 10)\}' .$$

For age group $[\beta - 15, \beta - 10]$:

$$\{\underset{\sim}{\nu}(0)\}'\underset{\sim}{B}(\beta - 15) + \{\underset{\sim}{\nu}(\beta - 10)\}'\underset{\sim}{S}(\beta - 15) = \lambda\{\underset{\sim}{\nu}(\beta - 15)\}'$$

with

$$\{\underset{\sim}{\nu}(\beta - 10)\}' = \frac{1}{\lambda} \left[\{\underset{\sim}{\nu}(0)\}'\underset{\sim}{B}(\beta - 10) + \frac{1}{\lambda} \{\underset{\sim}{\nu}(0)\}'\underset{\sim}{B}(\beta - 5)\underset{\sim}{S}(\beta - 10)\right] .$$

Hence

$$\{\underset{\sim}{\nu}(0)\}'\underset{\sim}{B}(\beta - 15) + \frac{1}{\lambda} \{\underset{\sim}{\nu}(0)\}'\underset{\sim}{B}(\beta - 10)\underset{\sim}{S}(\beta - 15)$$
$$+ \frac{1}{\lambda^2} \{\underset{\sim}{\nu}(0)\}'\underset{\sim}{B}(\beta - 5)\underset{\sim}{S}(\beta - 10)\underset{\sim}{S}(\beta - 15) = \lambda\{\underset{\sim}{\nu}(\beta - 15)\}'$$

or

$$\{\underset{\sim}{\nu}(\beta - 15)\}' = \frac{1}{\lambda} \{\underset{\sim}{\nu}(0)\}'\underset{\sim}{B}(\beta - 15) + \frac{1}{\lambda^2} \{\underset{\sim}{\nu}(0)\}'\underset{\sim}{B}(\beta - 10)\underset{\sim}{S}(\beta - 15)$$
$$+ \frac{1}{\lambda^3} \{\underset{\sim}{\nu}(0)\}'\underset{\sim}{B}(\beta - 5)\underset{\sim}{S}(\beta - 10)\underset{\sim}{S}(\beta - 15)$$
$$= \{\underset{\sim}{\nu}(0)\}' \left[\sum_{\alpha=\beta-15}^{\beta-5} \lambda^{-\left(\frac{a-(\beta-15)}{5} + 1\right)} \underset{\sim}{B}(a)\underset{\sim}{A}(a)\underset{\sim}{A}^{-1}(\beta-15) \right] \qquad \text{(A3)}$$

where $\underset{\sim}{A}(a)$ is defined as

$$\underset{\sim}{A}(a) = \prod_{i=a-5}^{0} \underset{\sim}{S}(i) \quad , \qquad \text{for } a = 5,10,\ldots,z$$

$$\underset{\sim}{A}(a) = \underset{\sim}{I} \quad , \qquad \text{for } a = 0 \quad . \tag{A4}$$

In general, we have

$$\{\underset{\sim}{\nu}(x)\}' = \{\underset{\sim}{\nu}(0)\}' \left[\sum_{a=x}^{\beta-5} \lambda^{-(\frac{a-x}{5}+1)} \underset{\sim}{B}(a)\underset{\sim}{A}(a) \right] \underset{\sim}{A}^{-1}(x) \quad . \tag{A5}$$

Equation (A5) may also be written as

$$\{\underset{\sim}{\nu}(x)\}' = \{\underset{\sim}{\nu}(0)\}' \left[\sum_{a=x}^{\beta-5} \lambda^{-(\frac{a-x}{5}+1)} \underset{\sim}{B}(a)\underset{\sim}{L}(a) \right] \underset{\sim}{L}^{-1}(x) \tag{A6}$$

which is the discrete analogue of (1.12).

Note, however, that $\{\underset{\sim}{\nu}(x)\}$ does not refer to people at exact age x, but to people in age group [x,x + 4]. Hence $\{\underset{\sim}{\nu}(0)\}$ represents the spatial reproductive value of the people in the first age group, and not the spatial reproductive value at birth (or exact age 0).

APPENDIX II

Glossary of Mathematical Symbols

Symbol	Interpretation	Defined by equation or on page
$\underset{\sim}{m}(a)$	Diagonal matrix of regional fertility rates of exact age a.	p. 2
$\underset{\sim}{M}(a)$	Diagonal matrix of regional fertility rates of age group a to a + 4.	p. 3
$\underset{\sim}{\hat{\ell}}(a)$	Probability matrix. The element ${}_i\hat{\ell}_j(a)$ denotes the probability that an individual born in i will survive to be in j at exact age a.	p. 2
$\underset{\sim}{L}(a)$	Matrix of person-years lived or, equivalently, of the number of people in age group a to a + 4. The element ${}_iL_j(a)$ denotes the person-years lived in region j between ages a and a + 5 by an individual born in region i. Equivalently, it denotes the number of people born in i and living in j in age group a to a + 4, per unit born in i.	p. 3
$\underset{\sim}{S}(a)$	Survivorship matrix. The element $s_{ij}(a)$ denotes the proportion of people aged a to a + 4 in region i, surviving to be in region j and x + 5 to x + 9 years old 5 years later.	p. 13
$\underset{\sim}{R}(0)$	Net Reproduction Matrix. The element ${}_iR_j(0)$ denotes the expected number of offspring born in region j to a woman who is born herself in region i.	(1.3)
$\underset{\sim}{\Psi}(r)$	Multiregional Characteristic Matrix. The element ${}_i\Psi_j(r)$ denotes the <u>discounted</u> number of offspring born in region j to a mother born in region i.	(1.4)

Symbol	Interpretation	Defined by equation or on page
$\underset{\sim}{R}^{(r)}(k)$	k-th derivative of the multiregional characteristic matrix.	p. 28
$\underset{\sim}{\kappa}$	Matrix of mean age at childbearing.	(2.28)
$\{\underset{\sim}{\xi}(0)\}$	Right eigenvector of $\underset{\sim}{\Psi}(r)$. Stable regional distribution of births (population at exact age 0).	(1.6)
$\{\underset{\sim}{\nu}(0)\}$	Left eigenvector of $\underset{\sim}{\Psi}(r)$. Stable regional distribution of the reproductive value at birth.	(1.10)
$\{\underset{\sim}{Q}^{(t)}\}$	Regional number of births in the stable population at time t.	(2.32)
$Q^{(t)}$	Total number of births in the stable population at time t (all regions).	(2.42)
$\{\underset{\sim}{Q}\}$	Stable equivalent of the observed (base-year) regional number of births.	(2.34)
$e^{\underset{\sim}{A}}$	Multiregional population growth matrix of the continuous growth model.	(2.2)
$\{\underset{\sim}{\xi}\}$	Right eigenvector of $e^{\underset{\sim}{A}}$, associated with the dominant eigenvalue. Stable population distribution by age and region (fixed up to a scalar). $\{\underset{\sim}{\xi}\}' = [\{\underset{\sim}{\xi}(0)\}'\{\underset{\sim}{\xi}(dx)\}'\{\underset{\sim}{\xi}(2dx)\}'\cdots\{\underset{\sim}{\xi}(x)\}'\cdots]$	p. 22
$\{\underset{\sim}{\xi}(x)\}$	Regional distribution of the population at exact age x.	p. 25
$\{\underset{\sim}{\nu}\}$	Left eigenvector of $e^{\underset{\sim}{A}}$, associated with the dominant eigenvalue. Distribution of reproductive values by age and region (fixed up to a scalar). $\{\underset{\sim}{\nu}\}' = [\{\underset{\sim}{\nu}(0)\}'\{\underset{\sim}{\nu}(dx)\}'\{\underset{\sim}{\nu}(2dx)\}'\cdots\{\underset{\sim}{\nu}(x)\}'\cdots]$	p. 22

Symbol	Interpretation	Defined by equation or on page
$\{\underset{\sim}{v}(x)\}$	Regional distribution of the reproductive value at exact age x.	(1.12)
$\{\underset{\sim}{v}\}$	Regional distribution of the total reproductive value.	(1.15)
v	Total reproductive value (all regions)	(1.21)
$\{{}_5\underset{\sim}{V}_x\}$	Distribution of the average spatial reproductive value for the age interval x to x + 4.	(1.25)
$\{\underset{\sim}{k}^{(t)}\}$	Number of people by age and region at time t.	p. 21

$$\{\underset{\sim}{k}^{(t)}\}' = [\{\underset{\sim}{k}^{(t)}(0)\}'\{\underset{\sim}{k}^{(t)}(dx)\}'\{\underset{\sim}{k}^{(t)}(2dx)\}'\cdots$$

$$\cdots\{\underset{\sim}{k}^{(t)}(x)\}'\cdots]$$

Symbol	Interpretation	Defined by equation or on page
$\{{}_o\underset{\sim}{k}\} = \{\underset{\sim}{k}^{(0)}\}$	Number of people by age and region in the base year (t = 0). $[\{{}_o\underset{\sim}{k}(x)\} = \{\underset{\sim}{k}(x)\} = \underset{\sim}{k}(x)\{\underset{\sim}{1}\}$ on page 11-12].	p. 21
$\underset{\sim}{K}(x)$	Diagonal matrix containing the number of people in age group x to x + 4 by region. $[\underset{\sim}{K}(x)\{\underset{\sim}{1}\} = \{\underset{\sim}{K}(x)\}]$	p. 14
$\underset{\sim}{n}(x)$	Matrix of expected, discounted number of offspring per woman aged x years. The element ${}_in_j(x)$ denotes the number of children to be born in region j to a woman now x years of age and resident of region i, discounted back to age x.	(1.13)
$\underset{\sim}{nk}$	Matrix of discounted number of offspring of the total population, by place of residence of the mothers and by place of birth of the offspring.	(1.18)
$\{\underset{\sim}{nk}\}$	Discounted number of offspring of the total population by region of birth.	(1.19)

Symbol	Interpretation	Defined by equation or on page
$\{\bar{\underset{\sim}{n}}k\}$	Discounted number of offspring of the total population, by region of residence of the mothers.	(1.20)
$\bar{\underset{\sim}{n}}_x$	Matrix of computed discounted number of offspring per woman at exact age x (numerical approximation of $\underset{\sim}{n}(x)$).	(1.24)
${}_5\underset{\sim}{N}_x$	Matrix of the average discounted number of offspring per woman in age group x to x + 4.	(1.27)
$\{\underset{\sim}{NK}\}$	Numerical approximation of $\{\underset{\sim}{n}k\}$.	p. 17
$\{\tilde{\underset{\sim}{N}}K\}$	Numerical approximation of $\{\tilde{\underset{\sim}{n}}k\}$.	p. 17
$\underset{\sim}{N}K$	Numerical approximation of $\underset{\sim}{n}k$.	p. 17
c	Reciprocal of the normalizing factor.	(2.46)
$\underset{\sim}{C}$	Outer product of the eigenvectors of $\underset{\sim}{\Psi}(r)$.	(2.48)
$\{\underset{\sim}{Y}\}$	Stable equivalent of the observed total number of people by region.	(2.52)
$\underset{\sim}{e}^{(r)}(0)$	Matrix of discounted life expectancies.	(2.57)

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January 1977

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