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A Dynamic Linear Programming Approach to National Settlement System Planning

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Preface

Interest in human settlement systems and policies has been a critical part of urban-related work at IIASA since its inception. Recently this interest has given rise to a concentrated research effort focusing on migration dynamics and settlement patterns. Four sub-tasks form the core of this research effort:

I. the study of spatial population dynamics;

II. the definition and elaboration of a new research area called demometrics and its application to migration analysis and spatial population forecasting;

III. the analysis and design of migration and settlement policy;

IV. a comparative study of national migration and settlement patterns and policies.

This paper, the fourth in the policy analysis series formulates the human settlement system planning problem as a dynamic linear programming problem. Dynamic linear programming has been a topic of interest in IIASA's System and Decision Sciences Area for some time. This paper is a joint product of the System and Decision Sciences Area and the Human Settlement and Services Area.

Related papers in the policy analysis series, and other publications of the migration and settlement study, are listed on the back page of this report.

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Abstract

The problem of human settlement system (HSS) planning is formulated as a dynamic linear programming (DLP) problem. In DLP large time-dependent linear programming problems are solved using both optimal control and linear programming techniques. A multi-regional population growth model forms the state equation of the DLP problem. Budget-, resources- and quality of life-constraints are considered. This introductory paper demonstrates the formalization of the HSS planning problem and indicates its solution, the realization of the solution and the interpretation of the dual relationship.

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INTRODUCTION

The development of human settlement systems is becoming a public concern in most countries. Countries all over the world are adopting policies to guide the growth and the distribution of their populations (for some details, see Willekens 1976a). This trend toward explicit national settlement policies is enhanced by the realization that land and environment are not free goods, but are scarce resources to be conserved. The task of settlement planning is to elaborate such control policies of population distribution over space and/or time to achieve desirable socio-economic goals (conservation of the environment, economic efficiency, etc.), taking into account a large number of factors and constraints (total population age and sex structure, birth-, death- and migration rates, scarceness of resources, educational constraints, etc.). An effective way to make optimal decisions when a very large number of variables and constraints are involved, is by applying mathematical programming. Most successful in dealing with large static problems has been linear programming. Dynamic decision problems, on the other hand, have been treated by using optimal control theory. National settlement systems are large scale and dynamic in nature, and problems of their planning can therefore be expressed as dynamic optimization problems; more particularly as dynamic linear programming problems (DLP). DLP comprise both static linear programming and control theory methods (for details, see Propoi 1976a).

The purpose of this paper is to discuss briefly the possibilities and perspectives of the DLP approach to national settlement system planning. It consists of two parts--the first part describes DLP models of national settlement system planning; the second is devoted to the application of DLP theory and methods in the solution of these models.

1. THE PLANNING PROBLEM

The purpose of this section is to describe in some detail the problem of national settlement system planning. The models we envisage are in the format of a DLP problem. A DLP problem consists of three components: the state equations, the constraints imposed on the system variables, and the performance index (objective function). The state equations describe the combined effect of internal systems dynamics and policy intervention on the population distribution. The internal dynamics are represented by the "laws of motion." External intervention will disturb the motion of the system. But the degree and the direction of the disturbance depend on the dynamic characteristics of the system.

To avoid counteractive and undesired effects of a settlement policy, we need to understand the internal dynamics governing a multiregional population system, that is, we need to understand the behavior of the system over time before applying control to it. The mechanism of spatial demographic growth has been studied by Rogers (1968, 1971, 1975). Some relevant aspects of his work will be reviewed in the first section.

To transform the growth model into a policy model we add a sequence of vectors, describing control actions distributed over time and space. A control vector defines the instruments of population distribution policy. A fundamental feature of population distribution policy is that it does not occur in a vacuum. In most instances, it is subordinate to social and economic policies. Frequently the goals of population redistribution are environmental and economic in nature. To achieve these non-demographic goals, use is made of non-demographic but economic and legal instruments. Although the focus is on population and its distribution, the policy implementation requires the consideration of socio-economic factors. The study of the interdependence between spatial population growth and the socio-economic system is the subject of demometrics. The first section of this paper shows how demometrics may contribute to the formulation of national settlement system planning models. In particular, it is relevant to the formulation of the complete state equation of the system, describing not only the internal dynamics, but also the influence of external intervention on the system.

Besides the state equations, there are the constraints. Relocating people or intervening in the residential location decision incurs a cost, both from economic and social points of view. The planning model must reflect these constraints. They will be treated in a second section. The third section discusses the objectives of the system planning and derives explicit expressions for the preference system of the policy-maker. In this paper, it is assumed that this preference system may adequately be described by linear functions.

1.1 The State Equations

The state equations describe the development of the multiregional population system over time. They appear as linear heterogenous equations. The homogeneous part of the equation system describes the behavior of the system undisturbed by outside influences. This behavior is described by a multiregional demographic growth model. The heterogenous part describes the impact of factors exogenous to the demographic system, such as policy intervention. Both components of the state equations will now be discussed in more detail.

1.1.a. The Homogeneous Part: The Multiregional Demographic Growth Model

The dynamics of multiregional population systems are governed by the interaction of fertility, mortality and migration. In recent years demographers, geographers, economists and planners have devoted their attention to model these dynamics in order to describe and explain the changes taking place in actual human settlement systems. The models that have been developed have a similar underlying structure. In most instances, they appear as a system of linear difference equations or they may be transformed into it. The general format of the models is the matrix equation

$$x(t + 1) = G(t)\ x(t) \tag{1.1}$$

where $x(t)$ is the population distribution at time t; $G(t)$ is the population growth matrix at time t, which in most cases is assumed to be constant over time: $G(t) = G$. This model does not consider exogenous contributions to population growth. They will be added later.

Depending on the aggregation level, $x(t)$ is the population by region, or the population by age and region. Matrix models of aggregate multiregional population change are, for example, the Markov chain model, the input-output model and the components-of-change model. Willekens (1977) shows how they relate to equation (1.1). The model of disaggregate multiregional population change is known as the multiregional cohort-survival model (Rogers, 1975, Chapter 5; see also Rees and Wilson, 1975). In this paper we review briefly the components-of-change model and the cohort-survival model. It is assumed that the multiregional population system is closed, i.e., no external migration is allowed for.

i. Components-of-Change Model

The components-of-change model of multiregional population growth has been described by Rogers (1966, 1968, 1971). Conceptually, it may be considered as an extension of the Markov model. Consider an ergodic Markov chain

$$x'(t + 1) = x'(t)\ P \tag{1.2}$$

or

$$x(t + 1) = P'x(t) \tag{1.3}$$

where P is the transition matrix. An element p_{ij} of this matrix denotes the probability that an individual in region i at time t will be in region j at time t + 1. In an ergodic Markov chain model, it is possible to move from an arbitrary state i to any other state in one or more steps. This implies that the row elements of P sum up to unity. In this pure migration model, natural increase is ignored.

The components-of-change model introduces fertility and mortality by premultiplying x(t) by a suitably constructed fertility and mortality matrix. Such matrices have in the principal diagonal the probabilities of dying and childbearing respectively. Let B and D be the fertility and mortality matrix. Then the components-of-change model becomes

$$x(t + 1) = [P' + B - D]\ x(t)$$

or

$$x(t + 1) = G\ x(t) \quad , \tag{1.4}$$

with G = P' + B - D being the growth matrix. The components-of-change model is in the form of (1.1). The assumptions underlying this model are analogous to those of the Markov model: Markov property, time homogeneity, no multiple transition. The column elements of G usually do not sum up to unity. The deviation is due to natural increase. If in each region the birth rate equals the death rate, then the components-of-change model reduces to the ergodic Markov chain model.

ii. Multiregional Cohort-Survival Model

The multiregional cohort-survival model describes the growth of multiregional population systems disaggregated by age (Rogers, 1975, Chapter 5). The basic format of the model once again is

$$x(t + 1) = G\ x(t) \quad . \tag{1.5}$$

But in this case,

$$x(t) = \begin{bmatrix} x^{(t)}(0) \\ x^{(t)}(5) \\ \vdots \\ x^{(t)}(z) \end{bmatrix} \qquad x^{(t)}(a) = \begin{bmatrix} x_1^{(t)}(a) \\ x_2^{(t)}(a) \\ \vdots \\ x_N^{(t)}(a) \end{bmatrix}$$

where $x^{(t)}(a)$ is the regional distribution of the population in age group a to a + 4, assuming an age interval of 5 years,

$x_i^{(t)}(a)$ is the population in age group a to a + 4 in region i at time t,

z is the highest age group (85 years and over, say), and

N is the number of regions.

The growth matrix G is of the form

$$G = \begin{bmatrix} 0 & 0 & B(\alpha-5) & \cdots & B(\beta-5) & 0 & \cdots & 0 & 0 \\ S(0) & 0 & & & & & & \vdots & \vdots \\ 0 & S(5) & \ddots & & & & & \vdots & \vdots \\ \vdots & & & \ddots & & & & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & S(z-5) & 0 \end{bmatrix} \qquad (1.6)$$

where α and β are, respectively, the youngest and oldest ages of the reproductive period. The matrix G is known as the generalized Leslie matrix, indicating that it is a generalization of the growth matrix of the single region cohort-survival model, described by Leslie in 1945. The elements B(a) of the first row describe the fertility behavior of the population and the migration and survival pattern of the just born. The subdiagonal elements S(a) denote the migration and survival pattern of the people aged a to a + 5. The submatrices B(a) and S(a) are computed from observed fertility rates and from the multiregional life table. For details, see Rogers (1975).

Both the components-of-change model and the multiregional cohort-survival model take the form of a system of homogeneous first order difference equations. They describe the dynamics of a closed multiregional system. The transformation of these models to open systems is straightforward. We add to the system (1.1) a vector s(t):

$$x(t + 1) = G(t)\ x(t) + s(t) \tag{1.1a}$$

which then describes the exogenous contributions to population growth, such as external migration. The inclusion of socio-economic policy variables affecting population growth in the models, needs some more discussion. This is the topic of the next section.

1.1.b. The Complete State Equation: Addition of Control Variables

In the components-of-change model and the cohort-survival model, population at time t and its regional and/or age distribution depends only on the population distribution in the previous time period. They are pure demographic models, since they do not include other socio-economic variables. In this closed system, the predetermined variables consist of lagged endogenous variables. The growth path of the system is completely determined by the growth matrix G and the initial condition.

To make the models more realistic, we extend the set of predetermined variables to include economic variables such as income, employment, housing stock, accessibility, several types of government expenditures, and so on. Some of the predetermined variables are controllable by the policy-maker, and are labeled policy variables, control variables, or instrument variables. Others are uncontrollable but are exogenously given.

The complete policy model may therefore be written, assuming linearity.[1]

$$x(t + 1) = G(t)\ x(t) + D(t)\ u(t) + E(t)\ w(t) + s(t) \qquad (1.7)$$

where $x(t)$ and $s(t)$ are as in (1.1), (1.1a),
$u(t)$ is the vector of controllable variables,
$w(t)$ is the vector of uncontrollable predetermined socio-economic variables,
$D(t)$ and $E(t)$ are matrix multipliers.

For simplicity, and without loss of generality, we delete again the uncontrollable predetermined variables. The model (1.7) then reduces to

$$x(t + 1) = G(t)\ x(t) + D(t)\ u(t) \quad . \qquad (1.8)$$

The control vector $u(t)$ consists of socio-economic instrument variables affecting the distribution of the population. The matrix multiplier $D(t)$ is important in this setting. An element $d_{ij}(t)$ denotes the impact on the population in region-age combination i of a unit change in the j-th instrument at the step t. In many cases the elements of this matrix are also assumed constant over time: $D(t) = D$. This implies that the effects of certain policies on the population distribution are independent of the time period when the policies are implemented. This is consistent with the Markovian assumption of time-homogeneity. The linearity of (1.8) implies that the effects of the various policies are additive.

Equation (1.8) is the state equation of a state-space model. How it may be derived from linear demometric models, describing the interdependence between demographic and socio-economic

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[1] The fact that (1.7) is a first-order difference equation is by no means restrictive. Higher-order difference equations may be converted into a system of first-order difference equations, (Zadeh and Desoer, 1963).

variables, is described in Willekens (1976b). The rationale for using the state-space model (1.8) as the analytical or numerical tool for population policy analysis, is that the homogeneous part of (1.8) is exactly the demographic growth model (components-of-change or cohort-survival), that describes the population growth without intervention. The logical extension of population growth models to policy models is therefore the addition of a heterogenous part to the growth model (see also Rogers, 1966; 1968, Chapter 6; 1971, pp. 98-108). The resulting model is a heterogenous system of linear first-order difference equations.

1.2 Constraints

Policy making is subject to constraints. The values that the control and state vectors u(t) and x(t) in (1.8) can take on are restricted by political, economic and social considerations. For example, let u(t) denote the number of inmigrants from outside the system, that have to move in, in order to achieve certain population distribution objectives. It is politically and socially unacceptable to relocate a very large number of people during a short time period. Therefore, there is an upper bound to the number of inmigrants during a unit time period (Evtushenko and MacKinnon, 1975, p. 5):

$$\sum_i u_i(t) \leq \bar{u}(t) \quad , \quad t = 0,1,\ldots,T-1 \tag{1.9}$$

where the scalar $\bar{u}(t)$ is the total inmigration pool available in the t-th time period.

Instead of restricting the control vector by defining a total inmigration pool, each element of u(t) may be required to lie within a lower and an upper bound:

$$\underline{u}_i(t) \leq u_i(t) \leq \bar{u}_i(t) \quad . \tag{1.10}$$

Population redistribution policy is not free. Imposing controls implies the incurrence of costs. It is therefore natural to assume a budget constraint limiting the action span of the policy maker. We distinguish between a budget constraint for each period:

$$c'(t)\ u(t) \leq C(t) \quad , \quad t = 0,1,\ldots,T-1 \tag{1.11}$$

and a global budget constraint:

$$\sum_{t=0}^{T-1} c'(t)\ u(t) \leq C \quad . \tag{1.12}$$

An element $c_i(t)$ of the cost vector $c(t)$ denotes the cost of transferring a person to region i in the t-th time period. The total budget available during period t is $C(t)$. The global budget is C.

Frequently, the population distribution itself is constrained in addition to the control vector. For example, in a pure redistribution policy, the total population of the system is held constant

$$\sum_{j=1}^{n} x_j(t) = X = \sum_{j=1}^{n} x_j(0) \quad , \quad t = 1,2,\ldots,T \quad . \tag{1.13}$$

As in the case of the control vector, the policy maker may want to put lower and upper bounds on the population in each region. This would avoid the excessive growth of some regions and the depopulation of others:

$$\underline{x}_j(t) \leq x_j(t) \leq \bar{x}_j(t) \quad , \quad t = 1,2,\ldots,T \quad . \tag{1.14}$$

A constraint receiving considerable attention in recent years is the resource constraint. Not only capital, but also raw materials, water, and environment all are scarce resources. As mentioned in the introduction to this paper, human settlement policies in most countries are directed toward the conservation of those resources. This commitment must be reflected in the planning model. Therefore, we introduce the resource constraint:

$$R(t)\ x(t) + Q(t)\ u(t) \leq f(t) \quad , \quad t = 0,1,\ldots,T \tag{1.15}$$

where f(t) is the vector of available resources in the t-th time period. The matrices R and Q are rectangular matrices. An element $r_{kj}(t)$, for example, denotes the amount of resource k required by an individual in region j during time period t. An element $q_{k\ell}(t)$ denotes the use of resource k per unit of control ℓ during period t. Note that (1.11) is a special case of (1.15) in which a single resource, capital, is considered associated with the control.

Another constraint relates to the quality of life or income levels. Let g(t) be the vector denoting the regional distribution of required quality of life-levels. The quality of life constraint is then

$$M(t)\ x(t) + N(t)\ u(t) \geq g(t) \quad . \tag{1.16}$$

An element $m_{ij}(t)$ of M(t) denotes the per capita level of the quality of life index i in region j at time t. An element $n_{i\ell}(t)$ of N(t) represents the impact of policy variable ℓ on the level of the quality of life index i.

A final restriction on the action span of the policy maker is represented by the boundary conditions. Since the planning of settlement systems starts from the current population distribution, we have the initial condition

$$x(0) = x_0 \quad . \tag{1.17}$$

On the other hand, the population distribution at the planning horizon x(T) may be fixed

$$x(T) = x_T \tag{1.18}$$

or may be kept free.

1.3 Performance Indices

The ultimate goal of national settlement system planning is to increase the quality of life. There is no agreement on the factors determining the quality of life, and even less on its quantitative measurement. For practical reasons, the quality of life goal is replaced by a single objective, involving monetary costs and benefits only. Such an objective function is given in (1.19). It is necessary to maximize the total benefit J(u):

$$J(u) = \sum_{t=0}^{T} \alpha'(t)\ x(t) + \beta'(t)\ u(t) \quad , \qquad (1.19)$$

where $\alpha(t)$ is the vector of unit benefit associated with the regional population levels at step t, and
$\beta(t)$ is the vector of unit benefit associated with the controls.

A performance index involving costs is shown in (1.20). The problem is to minimize

$$L(u) = \sum_{t=0}^{T} \gamma'(t)\ x(t) + \delta'(t)\ u(t) \qquad (1.20)$$

where $\gamma(t)$ is the vector of unit costs associated with the regional population levels at step t, and
$\delta(t)$ is the vector of unit costs associated with the controls.

In some instances, the policy maker may not want to minimize the costs associated with the settlement system and with the intervention in this system. Instead he may just want to bring the population distribution as close as possible to a desired distribution $\bar{x}(T)$ at the planning horizon. This problem has been treated by Willekens (1976b, pp. 66-85) for cases where explicit analytical solutions could be derived: the initial period control, and the linear feedback control problems.

In the case of DLP approach the performance index can be formulated as

$$J(u) = |x(T) - \bar{x}(T)| \rightarrow \min \tag{1.21}$$

where $|.|$ denotes the absolute value.

The goal of obtaining a desired population distribution at the end of the planning horizon can be formulated also in the following way. Given the positive numbers k_j, maximize the value

$$J(u) = \min_{1 \leq j \leq u} \left\{ \frac{x_j(T)}{k_j} \right\} \tag{1.22}$$

where numbers k_j define the desired proportions of the terminal distribution. It can be shown, that in this case the optimal distribution $\{x_j(T)\}$ possesses the following property (Kantorovitch, 1965):

$$\frac{x_1(T)}{k_1} = \cdots = \frac{x_j(T)}{k_j} = \cdots = \frac{x_n(T)}{k_n} \quad . \tag{1.23}$$

In some other cases the numerical analysis of the policy may be of interest which maximizes the performance index

$$J = \sum_{j=1}^{n} a_j(T)\ x_j(T) = a'(T)\ x(T) \tag{1.24}$$

where $a_j(T)$ is the weighting coefficient of a population group $x_j(T)$.

2. DYNAMIC LINEAR PROGRAMMING THEORY AND METHODS

The purpose of this part is to describe the DLP theory and methods in relation to problems of national settlement system planning.

The impact of linear programming models and methods on the practice of decision making is well known. (Dantzig, 1963; Kantorovitch, 1965). However, both the LP theory itself and the basic range of its applications are of one-stage, static nature. When the system to be optimized is developing, and its development is to be planned, a static approach is inadequate, and the problem of optimization becomes a dynamic multistage one.

It can be seen from the above, that the principal feature of settlement planning problems is their dynamic character. On the other hand, the basic relations and conditions in these problems are linear. Hence, DLP might be a very efficient approach for elaborating optimal policies in large-scale national settlement planning systems.

With a new quality of DLP, new problems arise. While for the static LP problems the basic question consists of determining the optimal decision, the realization of this decision (related to the questions of the feedback control of the optimal system, stability and sensitivity analysis of the optimal system, etc) is no less important for the dynamic problems.

This part consists of four sections. In the first section it is shown how demographic DLP problems can be reduced to a canonical form. This enables the development of a unified approach for a whole range of national settlement planning problems arising in practice.

The DLP theory is a base for obtaining the important properties of optimal demographic systems and for the development of computational methods for determining optimal policy in such systems. The DLP theory with emphasis on duality relations is given in the second section. The third section describes the DLP computational methods.

As has been mentioned before, the problems of realization of the optimal policy are very important for dynamic systems. These questions will be considered in the fourth section.

2.1 The DLP Canonical Form

Analysing models of multiregional population policy, which have been described in the first part, we can see, that all of them can be reduced to some canonical form. Before formulating DLP problems in a canonical form it is useful to single out and consider separately:

(i) state (development) equations of the systems with the distinct separation of state and control variables.

(ii) constraints imposed on these variables;

(iii) planning period T-the number of stages during which the system is considered;

(iv) performance index (objective function) which quantifies the quality of a control.

2.1.a. State Equations

State equations have the following form:

$$x(t+1) = G(t)x(t) + D(t)u(t) + s(t) \quad , \qquad (2.1)$$

where the vector $x(t) = \{x_1(t),\ldots,x_n(t)\}$ defines the state of the system at stage t in the state space X, which is supposed to be the n-dimension euclidean space; the vector $u(t) = \{u_1(t),\ldots,u_r(t)\} \varepsilon E^r$ (r-dimensional euclidean space) specifies the controlling action at stage t; the vector $s(t) = \{s_1(t),\ldots,s_n(t)\}$ defines the exogenous uncontrolled variable (known a priori in the deterministic models), for example, the exogenous part of equation (1.7) is $E(t)w(t) + s(t)$. $G(t)$ is the state transform matrix $(n \times n)$ (in the majority of demographic problems $G(t) = G$ is the growth matrix); $D(t)$ is the control transform matrix $(n \times r)$, which defines the influence of a control to the state of the system.

2.1.b. Constraints

In rather general form, constraints imposed on the state and control variables may be written as

$$R(t)x(t) + Q(t)u(t) \leq f(t) \quad , \qquad (2.2)$$

$$u(t) \geq 0 \quad , \qquad (2.3)$$

where $f(t) = \{f_1(t),\dots,f_m(t)\}$ is given vector, $R(t)$ and $Q(t)$ are $(m \times n)$ and $(m \times r)$ matrices.

2.1.c. Planning Period

The planning period T is supposed to be fixed. It is also assumed that the initial state of the system is given:

$$x(0) = x^0 \quad . \tag{2.4}$$

2.1.d. Performance Index

The performance index (which is to be maximized) has the following form

$$J_1(u) = \alpha'(T)x(T) + \sum_{t=0}^{T-1} [\alpha'(t)x(t)+\beta'(t)u(t)] \quad , \tag{2.5}$$

where $\alpha(t)$ $(t = 0,1,\dots,T)$ and $\beta(t)$ $(t = 0,1,\dots,T-1)$ are given weight coefficients (unit benefits, associated with $x(t)$ and $u(t)$).

2.1.e. Definitions

(i) The vector sequence $u = \{u(0),\dots,u(T-1)\}$ is <u>a control (policy)</u> of the system;

(ii) The vector sequence $x = \{x(0),\dots,x(T)\}$, which corresponds to control u from the state equations (2.1) with the initial state $x(0)$, is the <u>system's trajectory</u>;

(iii) The process $\{u,x\}$, which satisfies all the constraints of the problem (i.e. (2.1)-(2.4) in this case) is <u>feasible</u>;

(iv) The feasible process $\{u^*,x^*\}$ maximizing the performance index (2.5) is <u>optimal</u>.

Hence, the DLP problem in its canonical form is formulated as follows.

<u>Problem 1</u>: Given the initial population distribution

$$x(0) = x^0 \quad , \tag{2.6}$$

and the state equations:

$$x(t+1) = G(t)x(t) + D(t)u(t) + s(t) \quad , \tag{2.7}$$

where

$x(t)$ is the population distribution at time t (state of the systems);

$G(t)$ is the population growth matrix (usually constant over time);

$D(t) = \{d_{ij}(t)\} (i = 1,\ldots,m; j = 1,\ldots,r)$ denotes the impact on the population distribution $x_i(t)$ in region i by the control instrument $u_j(t)$;

$s(t)$ describes the exogenous contributions to population growth;

and the constraints

$$R(t)x(t) + Q(t)u(t) \leq f(t) \tag{2.8}$$

$$u(t) \geq 0 \tag{2.9}$$

where

$f(t) = \{f_1(t),\ldots,f_m(t)\}$ is the vector of available resources at time t;

the matrix $R(t) = \{r_{ki}(t)\} (k = 1,\ldots,m; i = 1,\ldots,n)$ denotes the amount of resource k required per individual in region i at step t;

the matrix $Q(t) = \{q_{ki}(t)\} (k = 1,\ldots,m; i = 1,\ldots,r)$ denotes the consumption of resource k per unit of control i at step t,

find a control (policy)

$$u = \{u(0),\ldots,u(T-1)\}$$

and corresponding state trajectory

$$x = \{x(0),\ldots,x(T)\}$$

which maximize the performance index

$$J_1(u) = \alpha'(T)x(T) + \sum_{t=0}^{T-1} [\alpha'(t)x(t) + \beta'(t)u(t)] \tag{2.10}$$

where

$\alpha(t)$ $(t = 0,\dots,T)$ is the n-vector of unit benefit, associated with the regional population distribution $x(t)$;
and $\beta(t)$ $(t = 0, T-1)$ is the r-vector of unit benefit associated with the control $u(t)$.

The choice of a canonical form of the problem is to some extent arbitrary, various modifications and particular cases of Problem 1 being possible. Some of them have been considered in the first part of this paper, a classification of these modifications is given in Table 1. In the table, state equations, for example, may include matrices A, B and/or vector s not depending on the number of stage t (I.2) or external disturbance $s(t)$ may vanish. (See (1.2)-(1.5)). Equations (I.3) are obtained, for example, from considering the difference approximation of the continuous analog of Problem 1.

An important class of DLP are the systems with delays in state and/or control variables (I.4), where $\{n_1,\dots,n_\nu\}$, $\{m_1,\dots,m_\mu\}$ are the sets of integers. They reflect the fact, that in a demographic system the state $x(t+1)$ at the step $t + 1$ may depend on certain previous states $x(t-n_1)$, $x(t-n_2),\dots,x(t-n_\nu)$ and certain previous control actions $u(t-m_1), u(t-m_2),\dots,u(t-m_\mu)$. In particular, when $\{n_1,\dots,n_\nu\} = \{0\}$, $\{m_1,\dots,m_\mu\} = \{0\}$, a conventional system (I.1) is obtained.

Constraints on the state and control variables can have the form of equalities (II.2), (see for example (1.13)) or be separate (II.3),(II.4), (examples are (1.9)-(1.14)). These variables can have additional restrictions on its sign (II.5), (II.6), (for example, the number of people cannot be negative). In some cases, the constraints should be considered in the summarized form (II.7) or (II.8) (see (1.12)).

It is useful to single out the constraints on the left and/or right side of the trajectory (boundary conditions). For example, the left and/or right side of the trajectory can be fixed (III.1), (III.3) or free (III.2), (III.4).

The number of steps T of the planning period can be fixed (IV.1) or may be defined by some conditions on the terminal state (i.e. (II.3), (II.5) for $t = T$). (Typical problem here: to bring a demographic system to a desired population distribution for minimal number of steps T).

The value of the performance index can depend only on the trajectory $\{x(t)\}$ (V.4) or on the control sequence $\{u(t)\}$ (V.3) or be even determined only by the terminal state $x(T)$ of the trajectory (V.2) (for example, see (1.19)-(1.22)).

In connection with Table 1, we can consider the patterns of Problem 1 modifications.

Problem 1a: (with terminal performance indices (1.19)-(1.22)). In this problem, the performance index (V.1) should be changed to (V.2).

Problem 1b: (with equality constraints). For this problem, the variable constraints are of equality form (II.2).

Problem 1c: (without state constraints). For this case, the problem has no constraints, or they (see (II.6)) may be imposed only on control variable (e.g. (1.9)-(1.12)). In case of a linear performance index, the problem is trivial. It is however of a significant interest, when the objective function is concave (in particular, nonpositive quadratic (Willekens, 1976b)).

Problem 1d: (nonfixed planning period). In this case, the number of stages T is not fixed but determined by the condition (III.3): $x(T) = x_T$.

Problem 1e: For this problem variable constraints are of the form (II.8) (e.g. budget constraint (1.12)).

Of course, Table 1 doesn't present the whole variety of modifications for Problem 1 and, naturally, Problems 1-1e do not present the total set of the possible DLP problems.

It should be noted that any problem stated above can be transferred into the other. For example, let us consider the Problems 1 and 1a with performance index (2.5). Introducing a new additional variable $x_0(t)\,(t = 0,\ldots,T)$, subject to $x_0(t+1) = x_0(t) + \alpha'(t)x(t) + \beta(t)u(t)$; $x_0(0) = 0$ one can see that

$$x_0(T) = \sum_{t=0}^{T-1} [\alpha'(t)x(t) + \beta'(t)u(t)] \quad .$$

So Problem 1 will have a form of Problem 1a with the performance index

$$J_1 = \tilde{\alpha}'(T)\tilde{x}(T)$$

and the state equations

$$\tilde{x}(t+1) = \tilde{G}(t)\tilde{x}(T) + \tilde{D}(t)u(t) + s(t) \quad ,$$

where

$$\tilde{\alpha}(T) = \{1,\alpha_1(T),\ldots,\alpha_n(T)\}; \quad \tilde{x}(t) = \{x_0(t),x_1(t),\ldots,x_n(t)\}$$

$$(t = 1,\ldots,T); \quad \tilde{x}(0) = \{0,x^0(0)\} \quad ,$$

$$\tilde{G}(t) = \begin{pmatrix} 1 & \alpha(t) \\ 0 & G(t) \end{pmatrix} \qquad \tilde{D}(t) = \begin{pmatrix} 0 & \beta(t) \\ 0 & D(t) \end{pmatrix} \quad .$$

Similarily performance indices (1.19), (1.20) can be reduced to (1.22). For example, the performance index (1.20) can be replaced by the problem

$$J(u) = \alpha \to \max$$

with additional terminal state constraints

$$x_j(T) \leq \alpha k_j \qquad (j = 1,\ldots,n) \quad .$$

If we consider Problem 1e with constraints (1.12) and introduce a variable $x_{n+1}(t)$, subject to state equation:

$$x_{n+1}(t) = x_{n+1}(t) + c'(t)u(t) \quad (t = 0,1,\ldots,T-1)$$

$$x_{n+1}(0) = 0$$

then we obtain Problem 1 with equations

$$\tilde{x}(t+1) = \tilde{G}(t)\tilde{x}(t) + \tilde{D}(t)u(t) + J(t)$$

where

$$\tilde{x}(t) = \{x_1(t),\ldots,x_n(t),x_{n+1}(t)\} \quad ;$$

$$\tilde{G}(t) = \begin{pmatrix} G(T) & 0 \\ 0 & I \end{pmatrix} \qquad \tilde{D}(t) = \begin{pmatrix} D(t) & 0 \\ c(T) & I \end{pmatrix}$$

and only one terminal condition

$$\tilde{x}(T) \leq \tilde{C}$$

where

$$\tilde{C} = \{0,\ldots,0,C\} \quad .$$

Here 0 and I are the zero and identity matrices of proper dimensions.

These reasonings show that it is sufficient to develop solution methods only for Problem 1 in order to obtain the solution methods for the whole set of DLP problems arising in case studies.

But before discussing these methods let us consider some important theoretical properties of the DLP problems.

2.2 DLP Theory

Problem 1 can be considered as an optimal control problem with state equation (2.6), initial condition (2.7), constraints on state and control variables (2.8), (2.9) and performance index (2.10). However, Problem 1 may be also considered as a certain "large" LP problem with constraints on variables in the form of equalities (2.6), (2.7) and inequalities (2.8), (2.9). In this case, Problem 1 turns out to be an LP problem with the staircase constraint matrix (Table 2).

For the numerical solution of Problem 1, one can therefore rely on a standard LP computer code. However, this straightforward approach to solving DLP problems is inefficient for two reasons. First, the "static" LP problem thus arrived at are so large in real cases that they cannot be solved even by using the most up-to-date computers.

The second reason is more important. Even if the optimal solution of the DLP Problem 1 should have been found by conventional means, the problems of the realization of this solution would still exist. These reasons provide the rationale for the development of dynamic LP methods. The methods must include: a theory (duality and optimality relations), numerical algorithms, and methods for the implementation of the solution.

The duality theory plays a key role in optimization methods. It permits the replacement of the original primal problem by some equivalent dual problem. It should be stressed that this equivalent dual problem can be interpreted in real terms for all real problems, thus enabling one to understand more deeply the original problem.

Analysing Problem 1, written in the form of Table 2, and applying to it LP duality theory, the following results can be obtained (Propoi, 1977).

Problem 2 (Dual): Find the dual control

$$\lambda = \{\lambda(T-1),\ldots,\lambda(1),\lambda(0)\}$$

and the associated dual trajectory

$$p = \{p(T),\ldots,p(1),p(0)\}$$

satisfying the co-state (dual) equation

$$p(t) = G'(t)p(t+1) - R'(t)\lambda(t) + \alpha(t) \qquad (2.11)$$

with the boundary condition

$$p(T) = \alpha(T) \qquad (2.12)$$

subject to the constraints

$$D'(t)p(t+1) - Q'(t)\lambda(t) \leq -\beta(t) \qquad (2.13)$$

$$\lambda(t) \geq 0 \qquad (2.13a)$$

and minimizing the performance index

$$J(\lambda) = p'(0)x^0 + \sum_{t=0}^{T-1} [p'(t+1)s(t) - f'(t)\lambda(t)] \quad . \qquad (2.14)$$

Here $p(t) = \{p_1(t),\ldots,p_n(t)\}$, $\lambda(t) = \{\lambda_1(t),\ldots,\lambda_m(t)\}$ $\lambda_i(t) \geq 0$ $(i=1,\ldots,m)$ are Lagrange multipliers for constraints (2.6), (2.7) and (2.8), (2.9) respectively.

The dual Problem 2 is also a control type problem as is the primal Problem 1. Here the variable $\lambda(t)$ is a dual control and $p(t)$ is a dual or a co-state of the system. Note, that we have reversed time in the dual Problem 2: $t = T - 1,\dots,1,0$.

For the pair of dual Problems 1 and 2 the following duality relations hold:

Theorem 1. (*The DLP global duality conditions*). *1) For any feasible controls u and λ, the inequality*

$$J_1(u) \le J_2(\lambda) \tag{2.15}$$

holds. 2) The solvability of either of Problem 1 or Problem 2 implies the solvability of the other, with

$$J_1(u^*) = J_2(\lambda^*) \quad . \tag{2.16}$$

where u and λ* are optimal controls of Problems 1 and 2.*

The equality (2.16) shows, that the solution of the primal Problem 1 can be replaced by the solution of the dual Problem 2, while the inequality (2.15) gives the upper bound of the Problem 1 performance index value.

The solution of the dual Problem 2 may be preferable from computational point of view for some cases; more important, that the duality relations can be effectively used for realization of optimal policy.

The duality relations can also be formulated in a decomposable way for each step t, $t = 0,1,\dots,T - 1$. For this purpose, let us introduce the Hamiltonian

$$H_1(p(t+1),u(t)) = \beta'(t)u(t) + p'(t+1)D(t)u(t) \tag{2.17}$$

for the primary Problem 1 and

$$H_2(x(t),\lambda(t)) = \lambda'(t)f(t) - \lambda'(t)R(t)x(t) \tag{2.18}$$

for the dual Problem 2.

<u>*Theorem 2*</u>. *(<u>The DLP local duality conditions</u>). 1) For any feasible processes $\{u,x\}$ and $\{\lambda,p\}$ the following inequalities hold:*

$$H_1(p(t+1),u(t)) \leq H_2(x(t),\lambda(t)) \qquad (t=0,\ldots,T-1) \quad .$$

2) For any feasible processes $\{u^*,x^*\}$ of the primal and $\{\lambda^*,p^*\}$ of the dual to be optimal it is necessary and sufficient that the values of Hamiltonians are equal:

$$H_1(p^*(t+1),u^*(t)) = H_2(x^*(t),\lambda^*(t)) \qquad (t=0,\ldots,T-1) \quad .$$

Theorem 2 shows that in order to investigate a pair of dual dynamic Problems 1 and 2 it is sufficient to consider a pair of dual "local" (static) problems of LP:

$$\begin{aligned} &\max H_1(p(t+1),u(t)) \\ &R(t)x(t) + Q(t)u(t) \leq f(t) \\ &\qquad\qquad u(t) \geq 0 \\ &\quad t = 0,.,\ldots,T-1 \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &\min H_2(x(t),\lambda(t)) \\ &D'(t)p(t+1) - Q'(t)\lambda(t) \leq -\beta(t) \\ &\qquad\qquad \lambda(t) \geq 0 \\ &\quad t = T-1,\ldots,1,0 \end{aligned} \tag{2.20}$$

So, any of the "static" duality relations or LP optimality conditions (Dantzig, 1963) for the pair of dual LP problems (2.19) and (2.20) linked by the state equations (2.6), (2.7) and (2.11), (2.12) determine the corresponding optimality conditions for the pair of dual DLP Problems 1 and 2. Such conditions have been formulated above; in a similar manner the following important optimality conditions are obtained (Propoi, 1977).

Theorem 3. (Maximum principle for primary Problem 1). For a control u^* to be optimal in the primary Problem 1, it is necessary and sufficient that there exists a feasible process $\{\lambda^*, p^*\}$ of the dual Problem 2, such that for $t = 0,1,\ldots,T-1$ the equality:

$$\max H_1(p^*(t+1), u(t)) = H_1(p^*(t+1), u^*(t))$$

holds, where the maximum is taken over all $u(t)$, satisfying the constraints (2.8), (2.9), and $\lambda^*(t)$ is the optimal dual variable in the LP problem (2.20).

Theorem 4. (Minimum principle for dual Problem 2). For a control λ^* to be optimal in the dual Problem 2 it is necessary and sufficient, that there exists a feasible process $\{u^*, x^*\}$ of the primary Problem 1, such that for $t = 0,1,\ldots,T-1$ the equality

$$\min H_2(x^*(t), \lambda(t)) = H_2(x^*(t), \lambda^*(t))$$

holds, where the minimum is taken over all $\lambda(t)$, satisfying the constraints (2.13), (2.13a) and $u^*(t)$ is the optimal primary variable in the LP problem (2.19).

These theorems can also be obtained by using the corresponding optimality conditions for discrete control systems (Propoi, 1973).

2.3 DLP Computational Methods

Simple DLP problems can be handled by standard LP codes. DLP problems of a realistic size require however, the development of special DLP methods. We shall distinguish finite and iterative methods.

DLP finite methods allow the finding of an optimal solution for a finite number of steps and are a further development of large-scale LP methods to dynamic problems. First of all, we mention the extension of the well-known simplex-method to DLP problems (Krivonozhko and Propoi, 1976). The dynamic simplex-method permits the obtaining of exact optimal solutions of DLP problems for a finite number of steps by treating at each step only the set of T local bases of dimension $m \times m$ (m is the number of constraint rows in the (2.2)) instead of handling with global basis of dimension $mT \times mT$ at the straightforward approach. The dynamic simplex-method is proved to be closely connected with the most effective large-scale LP methods based on factorization of the constraint matrix. These methods can also be used for the solution of DLP problems (Winkler, 1974; Chebotarev and Krivonozhko, 1976).

The second approach is based on decomposition methods of LP, especially on the Dantzig-Wolfe decomposition principle. For DLP problems this technique was used for example by Glassey (1970), Ho and Manne (1974) and Krivonozhko (1976).

Iterative methods do not produce exact solutions in a finite number of iteratives. But in many cases the approximate solution is quite adequate.

In addition, the iterative methods are characterized by simplicity of computer coding, low demands on computer memory and low sensitivity to the disturbances.

The most effective algorithms, however, combine the advantages of both the finite and the iterative methods. We mention here the finite-step algorithm, based on a penalty functions approach (Chebotarev, 1977) and the finite-step-algorithm, based on a Riccati equation solution (Propoi and Yadykin, 1975).

2.4 Implementation of Optimal Policies and Related Questions

Unlike for static LP, the realization of an optimal solution in dynamic problems is as important as its determination. One should mention here the questions of realization of the optimal solution as a program (i.e., in dependence of the numbers of state: $u^*(t)$ $(t=0,\ldots,T-1)$) or as a feedback control (i.e., in dependence on the current value of states: $u^*(t) = u_t^*(x(t))$ $(t=0,\ldots,T-1)$); stability and sensitivity of the optimal system, connection of optimal solutions for long- and short-range models, etc. These problems are a waiting solution. We shall mention only some of them here.

It is often necessary to determine in which way the performance index and/or the optimal control will behave when the parameters of the problem are changing (for example, "prices" $\alpha(t)$, $\beta(t)$, "resources" $f(t)$, "exogenous variables" $s(t)$ (parametric DLP). Solution methods in this case can be developed on the basis of static parametric LP (Dantzig, 1963).

In computing the optimal program, especially for a large T, it is very important to know how the inaccuracies in the coefficients of matrices $G(t)$, $D(t)$ and in other parameters of the system, influence the stability of the optimal program and the quality of control (sensitivity problem).

In many cases the most appropriate way of realizing an optimal policy can be reduced to the problem of finding the relations:

$$\delta u^*(t) = \Delta(t)\,\delta x^*(t) \qquad (t=0,1,\ldots,T-1)$$

where $\delta x^*(t) = x(t) - x^*(t)$ is the deviation of the current state $x(t)$ of the system from optimal state $x^*(t)$ and is supposed to be sufficiently small; $\delta u^*(t) = u(t) - u^*(t)$ is a required correction to the optimal program $\{u^*(t)\}$. This is the local feedback control of the optimal system.

Naturally, all the practical national settlement planning problems cannot be kept within the format of DLP. Here we should mention some directions of further DLP development.

In some cases the performance index is stated as quadratic or nonlinear (convex) function of state and control variables, (Willekens, 1976b). The extension of DLP methods to quadratic and convex DP problems can be developed in a way similar to the static methods (see, e.g. Hadley, 1964).

When the exogenous variables cannot be given a priori we come to DLP problems with uncertainty conditions. They can be formalized using stochastic optimization methods (Ermoljev, 1972) or max-min methods (Propoi and Yadykin, 1974). The solution of max-min DLP problems is of considerable practical interest when guaranteed control quality is to be obtained under the conditions of uncertainty, as well as for sensitivity analysis and related problems.

3. CONCLUSION

In this introductory paper we sketched the basic idea of the DLP approach to national settlement system planning. The approach might be a very effective tool for deriving and implementing optimal policies in demographic systems. However, additional work is required. It includes:

- development of a library of typical demographic policy models in DLP format;
- interpretation of the basic dual relations in demographic terms and the use of the DLP theory and methods for obtaining "qualitative" relations in demographic systems;
- numerical case studies of different DLP demographic models.

I. State Equations

(I.1) $x(t+1) = G(t)x(t) + D(t)u(t) + s(t)$

(I.2) $x(t+1) = Gx(t) + Du(t) + s$

(I.3) $x(t+1) = x(t) + G(t)x(t) + D(t)u(t) + s(t)$

(I.4) $x(t+1) = \sum_{i=1}^{\nu} G(t-n_i)x(t-n_i) + \sum_{j=1}^{\mu} D(t-m_j)u(t-m_j)$

II. Constraints

(II.1) $R(t)x(t) + Q(t)u(t) \leq f(t)$

(II.2) $R(t)x(t) + Q(t)u(t) = f(t)$

(II.3) $R(t)x(t) \leq f^{(1)}(t)$

(II.4) $Q(t)u(t) \leq f^{(2)}(t)$

(II.5) $x(t) \geq 0$

(II.6) $u(t) \geq 0$

(II.7) $\sum_{\tau=0}^{t-1} [R(\tau)x(\tau) + Q(\tau)u(\tau)] \leq f(t) \quad (t=1,\ldots,T)$

(II.8) $\sum_{t=0}^{T-1} [R(t)x(t) + Q(t)u(t)] \leq f$

III. Boundary Conditions

(III.1) $x(0) = x^{o}$

(III.2) $x(0)$ is free

(III.3) $x(T) = x_T$

(III.4) $x(T)$ is free

Table 1.

IV. Planning Period

(IV.1) T is fixed

(IV.2) T is free

V. Performance Indexes

(V.1) $$J_1(u) = \alpha'(T)\,x(T) + \sum_{t=0}^{T-1} [\alpha'(T)\,x(t) + \beta'(t)\,u(t)]$$

(V.2) $$J_1(u) = \alpha'(T)\,x(T)$$

(V.3) $$\alpha(t) = 0 \qquad (t = 0,\ldots,T)$$

(V.4) $$\beta(t) = 0 \qquad (t = 0,\ldots,T-1)$$

Variables											
u(0)	x(1)	...	x(t)	u(t)	x(t+1)	...	x(T-1)	u(T-1)	x(T)	Constraints	Right-hand side constants
-D(0)	I									=	s(0) + G(0)x^0
Q(0)		⋱								≤	f(0) + R(0)x^0
											⋮
			-G(t)	-D(t)	I					=	s(t)
			R(t)	Q(t)		⋱				≤	f(t)
											⋮
							-G(T-1)	-D(T-1)	I	=	s(T-1)
							R(T-1)	Q(T-1)		≤	f(T-1)
Performance Index Constants											
β(0)	α(0)	...	α(t)	β(t)	α(t+1)	...	α(T-1)	β(T-1)	α(T)	Max	

Table 2: Staircase Control Structure

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