RM-77-7

A PRACTICAL APPROACH TO CHOOSING ALTERNATE SOLUTIONS TO COMPLEX OPTIMIZATION PROBLEMS UNDER UNCERTAINTY

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January 1977

PREFACE

This paper describes methods for solving optimization problems under uncertainty--when for non-deterministic (stochastic) input data distribution functions are not precisely known or are not known at all. The methods have been elaborated for large-scale static and dynamic problems. They generalize certain known approaches which deal with decision-making under uncertainty.

Since under uncertain conditions the decision maker plays the decisive role, the methods here described are not strictly mathematical. Rather, they provide a general scheme for problem solution and possible ways to implement individual stages of a solution.

The paper generalizes the results of studies carried out in the USSR (mainly at the Siberian Power Institute of the Siberian Department of the USSR Academy of Sciences) in the last decade and also contains new information. Though there are some other approaches to these problems and they, as well as the approach described herein, might continuously be improved, this paper may nevertheless be considered as a completed study, ready for practical use. In particular, this approach is tried in the study of long-term prospective energy development that is currently being made by the IIASA Energy program for some typical regions of the world, taking into account global conditions and constraints.

SUMMARY

Many practical optimization problems are solved under uncertain conditions--when probability distributions are not known exactly or are completely unknown for a portion of the input data. Such uncertainty in input data leads to uncertainty in decision-making and makes it difficult to solve a problem.

Special methods for dealing with such situations are described in this paper. They focus on quite complex (including continuous and dynamic) large-scale problems and have been developed under the assumption that such problems have one main objective which can be quantified.

The methods described are aimed at the greatest possible formalization of the solution process, at the correct evaluation of consequences associated with this or that decision and at the elaboration of recommendations for decision-makers. The application of these methods allows one to reduce negative consequences (damage, over-expenditure, risk) which are conditioned by our inexact knowledge of the future. They are not able, however, to avoid these consequences or uncertainty in decision-making completely In general, some risk does inevitably exist and mathematical methods cannot provide the single optimal solution. One can only determine a set of rational variants which are good in this or that sense, but the final choice from among them inevitably has to be made by man (decision-maker) himself.

As to the general approach, the methods considered are based on already known methods of decision-making under uncertainty. They assume the calculation of a pay-off matrix and the use of special criteria (Wald's, Laplace's, Savage's, Hurwitz's, ...). But these methods extend this approach to complex optimization problems.

The basic ideas concern the "discretization" of continuous problems and the process for distinguishing the "first step" (priority or urgent decision) for dynamic problems. This allows one to deal with complex optimization problems.

Solution of a problem is divided into several operations (stages): statement of the problem; selection of a representative set of nature states; search for and preliminary analysis of feasible solutions; calculation of the pay-off matrix; analysis

of the pay-off matrix and choice of rational variants; final choice of the variant that is to be taken for realization. The methods for implementation of each operation have been proposed and are described in the paper.

In particular, four possible statements of dynamic problems are considered here. Most of these take into account the fact that, as a rule, we are really interested only in the first part of the whole time period under consideration. Just for this part (the "first step") our urgent decision must be made. Other parts of the period have to be studied mainly to take into proper account the consequences of this or that action undertaken in the "first step". The four statements differ from one another according to the assumptions made about a system's development (or operation) in the "afteraction" period.

For the selection of representative nature states and the search for our possible actions intuitive ways as well as certain formal methods described in the paper can be used. Some of the latter are based on the uniform or regular (in some sense) distribution of a given number of points in an n-dimensional domain of optimizing parameters or non-deterministic input data. The search for competing variants (actions) can also be made by optimization for several specific nature states selected in a previous operation. The optimal variants obtained in this way are dominant ones and can be considered as candidates for a problem solution.

The pay-off matrix is obtained by evaluating the effects (or consequences) for each competing solution variant under all selected states of nature. These are, as a rule, economic values (for example expenditures) but might be of other kinds as well.

The pay-off matrix represents the basic information for the following analysis and choice of rational variants. The proper procedures for this operation are described in detail in the paper. They involve decision criteria, available information about probabilities of nature states and the exclusion of non-dominant is sequentially variants. Thus the number of competing variants are sequentially reduced and if at some stage of the analysis all criteria point to one and the same optimal variant, then a problem is solved.

But in general (when uncertainty in input data still leads to uncertainty in decision-making) such an analysis identifies several rational variants which are optimal according to different criteria. In this case they have to be passed to the decision-maker together with their main quantitative characteristics, and the final choice among these remaining variants must be made by the decision-maker himself on the basis of his experience and intuition. Here additional objectives may be taken into account which were not considered during the earlier solution of a problem; use may be made of estimation by experts, etc. This operation is not considered here. But in spite of a "subjective" (heuristic) choice at this final stage, the preceding analysis guarantees a choice of only rational ("good") variants and insures against gross errors.

An example of the use of the methods described is also given in the paper.

CONTENTS

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I. INTRODUCTION

Uncertainty of input data affects very adversely the choice of optimal variants in many practical systems development problems and hinders the design of systems elements and their operation. If some input data are not known exactly, then there will be different optimal variants depending on their values as used in calculations. This leads to the uncertainty in decision-making. Neglecting uncertainties in data which really exist and consideration of just one combination of input data can lead to wrong choices and serious damage to real-life systems, often in the economic area.

This paper describes possible methods for taking into account uncertainty of input data. Such methods allow a reduction of negative consequences conditioned by our inexact knowledge of the future. However, it is impossible to avoid these consequences fully. Some risk does inevitably exist to the extent that there remains uncertainty in intial information.

The methods discussed below were developed to solve concrete optimization problems linked to substantiation of certain decisions. To be more specific, the methods are directed to the solution of the following class of problems:

1. Large-scale ones with big numbers of parameters x to be optimized and of uncertain input data y, both of which might be of continuous nature. Such problems may be both linear and nonlinear as well as dynamic ones.
2. Problems with uncertain input data whose distribution functions are not known at all or are not known exactly (uncertainty with regard to parameters of probabilistic distribution).

3. Problems with one main objective. Other minor objectives, if they cannot be expressed as constraints, may be taken into account in the last stage of the problem solution--when the final decision is chosen.

Such problems are usual for optimization of prospective systems development when we have rather complicated tasks, uncertain input data and mainly economical objectives.

Because of these features the methods under consideration differ from methods of game theory where conflict situations or collective decisions are investigated, as well as from methods of stochastic programming which deal with input data having an exact or sequentially improved probabilistic description. The methods considered in the paper are aimed at the highest possible degree of formalization for the solution process, at achieving the correct evaluation of consequences tied to this or that decision and to the elaboration of recommendations for the decision-maker. The latter must be specially noted, because uncertainty of input data leads in general to uncertainty in the final choice of decision. Therefore, we cannot expect mathematical methods to achieve a single optimal variant when there are uncertain conditions. One may only determine "good" (rational, intelligent, reasonable) variants and the final choice among them inevitably has to be made by the decision-maker himself.

The approach used is in general based on decision theory but it has certain differences compared with other methods of choice under uncertainty. This will be discussed in the next section.

II. GENERAL REMARKS ON METHODOLOGY

II.1. Initial Assumptions

Solutions of problems under uncertainty always require some assumptions. Those taken for considering methods are following:

- Formal solution of problems under uncertainty cannot in general identify single optimal variants. Only several rational variants which are good in this or that sense may be determined.

The final choice from among them has to be made by the decision-maker.

- Special criteria proposed for uncertain condition (criteria of Wald, Laplace, Savage, Hurwicz, ...) must be used for selection of rational variants. But none of them inspires full confidence and no single criterion can be used for the final choice of decision. Therefore in general, these criteria cannot eliminate uncertainty in decision-making.

- Subjective probabilities for various states of nature cannot be determined precisely. Only several possible probabilistic distributions can be subjectively obtained, and uncertainty of the situation is not eliminated in this way.

- Other objectives which may exist are taken into consideration only when a problem was solved according to the main objective--after identifying economically rational variants when the final choice from among them is made by the decision-maker.

- If a problem is a continuous one in respect to the continuous nature of optimizing parameters x and uncertain input data y then certain ways of discretization of the problem have to be used.

- If a problem is dynamic, then the "first step"--the first time interval or the most urgent (priority) decisions--is distinguished. The rational variants are identified just for this "first step" although a considerably longer period of time is investigated during the problem solution.

Under such assumptions the proposed approach is a variety of cost-benefit analysis [1] which uses economic data and monetary units. It includes the applications of special criteria of decision theory [2] as well as the assessment of subjective probability distribution [3]. It is also implied that expert estimates such as for example Delphi techniques [4] and if necessary multiobjective analysis (see for example [5]) have to be used. However, this approach differs from others [5,6,7] in certain rather essential points.

The utility function concept is not used here. This is mainly due to the fact that in a planned economy (of the USSR for example) the "worth" of money does not depend on its sum--any sum economized in some sector of the economy might be effectively used in other sectors and its efficiency to be assumed equal in all sectors.

According to assumption (3) above, subjective assessment of probability distribution can determine not single but several subjective distribution functions. This corresponds better to the real state of affairs. In fact one can identify possible probability distributions on the basis of knowledge and intuition but not in only one way. Therefore the requirement to determine several subjective distributions seems to be more realistic and appropriate.

Multiobjective analysis and expert estimates are applied only in the final stage of problem solution after identifying decision variants as rational according to the major objective. This is appropriate of course only for problems where a main objective really exists, but there are very many such problems especially concerned with systems development. The majority of methods of mathematical programming (linear, nonlinear, dynamic, stochastic) operates with just such mono-objective problems.

In short terms, with this approach one tries to analyze a situation in the most impartial way and to the greatest possible extent according to the main objective. During this analysis one does not attempt to eliminate the uncertainty of a situation but on the contrary intends to carry it honestly to the decision-maker in the form of several "economically equal" decision variants. And this is the goal of the methods described below. Then a decision-maker may choose the final variant to be implemented from among these "economically equal" ones by his experience and intuition or can draw in expert estimates and additional objectives. This last part of decision-maker actions is not considered in the paper. The special techniques mentioned above [4,5] can be applied there.

II.2. Ways of Describing Non-Deterministic Input Data

The following kinds of input data may be distinguished depending on their properties and our degree of knowledge:

a. Deterministic data, whose values are assumed to be known exactly.

b. Stochastic-definite data, for which sufficiently trustworthy probabilistic characteristics may be provided on the basis of statistical exercises using past observations.

c. Partly uncertain data, whose distribution functions cannot be determined well enough. Several possible distribution functions might be determined for such a case. This is the case of uncertainty about probabilistic description of stochastic input variables.

d. Uncertain data, whose distribution functions are not known at all (because of their nature or the absence of a suitable analogue in the past). Such data may be given only by means of intervals or variants of their possible values (without indicating the probability of individual values in these intervals).

The description of non-deterministic data (for continuous ones) is graphically shown in Fig. 1. For discrete data, instead of distribution functions, distribution series (or rows) can be used and intervals of possible values can be replaced simply by a set of possible values.

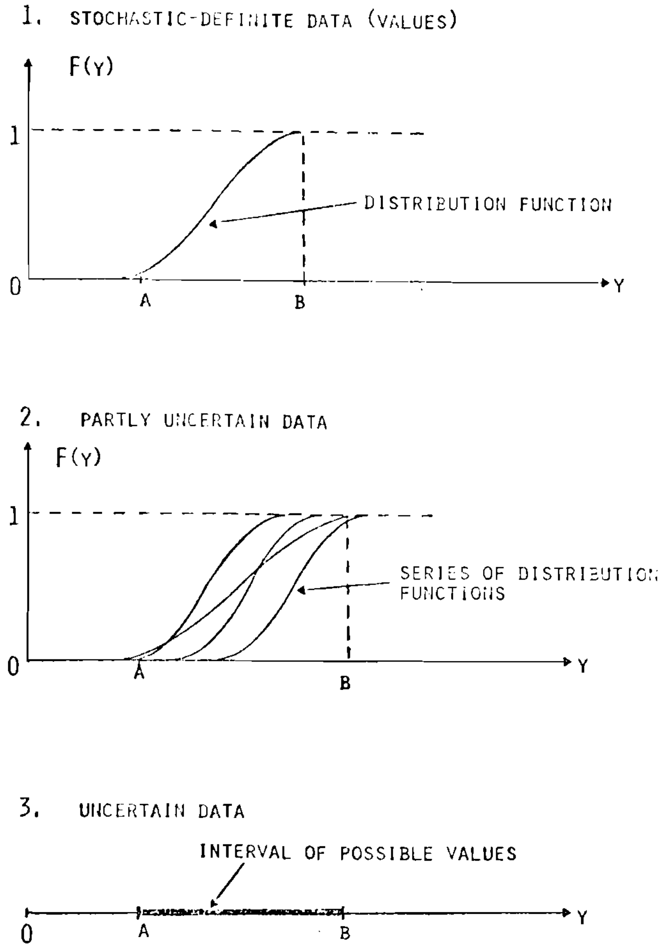


Fig.1. The forms of quantitative description of non-deterministic data.

In general we have all these kinds of input data while a problem is being solved. But uncertainty in decision-making is conditioned by the last two kinds only. If we have only deterministic and stochastic-definite data then it is possible, in principle, to find a single optimal variant with the aid of special stochastic methods, using mathematical expectation of objective function as the criterion of optimality. Therefore, the methods considered here are intended for general cases and for the last two kinds of data in particular.

II.3. The Principle of Making Priority Decisions

The further in the future the given time period is (from the moment of decision-making) the less correct is the information about this period. Therefore every decision concerning the development or functioning of systems has to be made as late as possible--as a rule, just before beginning of its implementation. Thus each time the "freshest" information with the smallest degree of uncertainty should be utilized.

In practice this means that final decisions should as a rule relate only to the nearest time interval, to priority construction projects etc. When possible, decisions should be divided into stages, for example: separately for the start of a project's design stage and the start of its construction stage; separately for the design of new equipment, for the production of prototypes and for the commercial production of new equipment. Thus, while the first stages are being realized, technical-economic indicators for projects and equipment are made more precise, and during completion of initial stages, other external conditions undergo refinement.

Some kinds of decisions are of course concerned with more distant time periods and require more advance planning. It is important only that such decisions be made no earlier than necessary--only when they become sufficiently mature and cannot be delayed.

This principle, besides laying a general framework for decision-making, plays an important role in the mathematical statement of dynamic problems.

II.4. Comparability of Decisions

Because of information uncertainty one has to consider decision variants for several possible future conditions. This is typical for all stochastic problems. To properly compare the consequences linked to every decision variant, one should keep the following rule: each variant should be considered under a set of possible future conditions which must be the same for all variants. In other words, the consequences, economic and otherwise, for each of our decision variants (our actions) have to be all determined in light of the same set of conditions (nature states).

In fact one needs to calculate a table (pay-off matrix) where the rows correspond to available variants and the columns to possible nature states. Such a table gives the possibility to compare decision variants. The figures in a single column are comparable since they are concerned with the same combination of input values.

II.5. Impacts of Given Constraints

One of the principal difficulties in solving the problems under consideration (and stochastic problems as well) is determining appropriate procedures for satisfying constraints (balance equations, resource limitations, etc.) or for assessing the consequences of not satisfying constraints. When competing decision variants are being considered (see previous point II.4) under different nature states, it may happen that certain constraints are not satisfied. And different constraints may not be satisfied for individual variants under different nature states.

The refinement of possible procedures for taking into account existing constraints requires special attention. These procedures have to correspond to reality as much as possible. Sometimes they may reflect additional undertakings which can be realized quickly (the acceleration of operation starts for system elements, the construction of temporary projects, importation of scarce resources, etc.). The overexpenditure due to such undertakings must be included in calculations.

But in many cases it is impossible to satisfy all constraints because of technical or economic reasons. Then the damage due to their not having been satisfied must be assessed. As a rule this is quite difficult. One possible approach that may be recommended is to consider such damage as uncertain input data together with other analogous data.

II.6. About Statement of Problems

As a rule, the statement of a problem involving uncertain input data is not at all an easy matter, especially when the problem is a dynamic one. We must establish:

- the concrete meaning of an action undertaken and the composition of parameters (components of vector x) which characterize the action;
- the composition of non-deterministic input data which are causes of uncertainty (the composition of components of vector y characterizing nature states);
- the procedures for satisfying constraints for assessing the results of their not being satisfied;
- the concrete form of an evaluation function $E(x,y)$, that will estimate the effect (say, expenditures) of different actions x under different nature states y.

Confusion is possible regarding parameters tied with our action x and parameters related to nature state y as well as parameters which must be included among components of vector x characterizing our main action and parameters tied with additional undertakings aimed at satisfying constraints.

It is difficult also to determine the meaning of our actions when solving dynamic problems.

II.7. "Discretization" of Continuous Problems

Most practical tasks involve continuous variables describing both optimizing parameters as well as input data. Thus there really are an infinite number of possible actions and nature states and we are not able to consider them all. Therefore it is necessary in some way to select a limited number of actions and nature states to consider while the problem is being solved.

In this respect it is important not to overlook potentially rational decision variants that might occur, and also to include in the number of nature states considered not only some mean (or middle) ones but also some extreme conditions (both sufficiently favorable and unfavorable but really possible).

In a mathematical sense we are speaking here of selections of vector x values x_i $(i=1,\dots,I)$ and vector y values $y_s (s=1,\dots,S)$, where I and S are the whole numbers of considered values of vectors x and y. In this operation the experience and intuition of specialists play a big role, and some formal methods can be used as well.

II.8. Mathematical Tools

In general, the methods described below are based on the assumptions mentioned above, and they are in fact an attempt of an extrapolation of known methods of decision-making under uncertainty (see for instance [2]) into the area of complicated optimization problems. These methods assume the use of a "pay-off matrix" and the application of special criteria appropriate for uncertain conditions (Wald, Laplace, Savage, Hurwicz, et al.).

But the important questions here are how to state a complex problem and how to calculate its pay-off matrix. While these questions are being solved for individual tasks there might be a need to use various mathematical methods (optimization and simulation, linear and non-linear, static and dynamic) depending on peculiarities of a given task.

III. GENERAL SCHEME FOR SOLVING OPTIMIZATION PROBLEMS UNDER UNCERTAINTY

As was already said, we assume that a problem has a main objective (as a rule an economic one) which can be quantitatively evaluated. Thus we can write an evaluating function (for a static problem):

$$E\ (x,\ y)\ , \qquad (1\text{-III})$$

where $x \varepsilon X$ is the m-dimensional vector, characterizing our choosing actions, and $y \varepsilon Y$ is the n-dimensional vector whose components are non-deterministic input data (deterministic data are reflected in an expression of function E). X and Y are domains where possible values of vectors x and y exist.

As a rule there may also be constraints in the form of equalities and inequalities.

The function $E(x,y)$ is not an objective function in the usual sense (as for deterministic or pure stochastic problems). We are not able to speak about, say, its minimization (if it represents expenditures) since its minimum will be obtained for different nature states y with various values of vector x. Since probabilities of different nature states are not known, we cannot use the mathematical expectation of function E(x,y) as the criterion of optimality. We must only strive, therefore, to make its values as small as possible.

The scheme for the solution of such problems supposes the sequence of operations shown in Fig. 2.

Fig. 2. The Sequence of Operations for Solving Optimization Problems under Uncertainty.

1. STATEMENT OF THE PROBLEM
2. SELECTION OF REPRESENTATIVE SET OF NATURE STATES
3. SEARCH FOR AND PRELIMINARY ANALYSIS OF VARIANTS FOR PROBLEM SOLUTION (OR ACTIONS)
4. CALCULATION OF PAY-OFF MATRIX
5. ANALYSIS OF PAY-OFF MATRIX AND CHOICE OF RATIONAL ACTIONS
6. FINAL CHOICE OF ACTION IMPLEMENTED

General remarks about the statement of problems have already been made (see point II.6.). It will be considered in the next section in more detail .

The selection of representative sets of nature states is in fact the "discretization" of the problem with regard to input data. In the continuous set Y it is necessary somehow to choose a finite number (S) of points which characterize sufficiently well the set as a whole. As a rule, the number S should not be large since a large number of calculations should be made for each nature state $Y_s (s=1,...,S)$. The size of this number must be established by taking into consideration the peculiarities of a given problem and computer availabilities. Possible methods for making this selection will be discussed in detail. This operation requires great attention as the completeness and reliability of subsequent analysis depends on how properly this choice was made.

The next operation is the search for and preliminary analysis of solution variants $x_i (i=1,...,I)$ which in principle might be rational ones and which have to be considered while a problem is being solved. Such variants are called "dominant" ones [2]. In this operation, on the one hand, the determination of our possible actions is made. If they are not evident and if we deal with a continuous set of optimizing parameters X, then this operation is a "discretization" of the problem (in regard to optimizing parameters) like the previous operation. But here the methods of selection of "representatives" are somewhat different.

On the other hand, in this operation, the question is examined whether there really is uncertainty in decision-making in the considered situation or not. If for all nature states $y_s (s=1,...,S)$ chosen in previous operations there is one and the same optimal variant x^o, then it means that uncertainty of input data will not lead to uncertainty in choice of action and we may surely implement variant x^o. But if this is not the case, then analysis will have to be made to select competing variants (actions) x_i $(i=1,...,I)$ which will be considered in the next operations.

The calculation of pay-off matrix $||E_{is}||$ may be one of the most laborious operations. The economic (or other) effect for every considered variant $x_i (i=1,...,I)$ is evaluated here for all selected future conditions $y_s (s=1,...,S)$. In other words, IxS values of function $E(x,y)$ are determined here to obtain the whole matrix $||E_{is}||$:

$$E_{is} = E(x_i, y_s) \qquad \begin{matrix} i=1,...,I \\ s=1,...,S \end{matrix} \qquad (2\text{-III})$$

Evidently, computers have to be used for such calculations if a problem is complex enough.

The fifth operation--the analysis of a pay-off matrix and the choice of rational actions--gives us in fact the final results of the formal solution of a problem. If there really is uncertainty in decision-making then the formal methods can point to several rational actions which might be considered as recommendations for a decision-maker. The special rules and criteria mentioned above are used in this operation.

Sometimes, when decision uncertainty is not great, the recommendations by different criteria may coincide. In this case the indicated single rational variant can be implemented with a great degree of confidence. But in general, various criteria point to different rational actions and uncertainty in decision-making remains, since none of the criteria inspires complete confidence.

In such cases it is desirable to take into account the partial knowledge of possible probabilities for considered nature states. Often enough this may eliminate the decision uncertainty.

If all the analysis undertaken in this operation (its peculiarities will be described in a special paragraph of the paper) is not able to identify the single optimal variant, then several rational variants are passed on to the next operation--for the final choice of decision. This last operation will not be considered in the paper in detail. As was noted above, the final choice has to be made by the decision-maker himself on the base of his experience and intuition. Additional non-economic objectives (of environmental, sociological, political character) may be taken into account here. Use may be also made of estimation by experts (for example the Delphi method). The way this operation is carried out depends heavily on peculiarities of individual decisions to be made. It should be only noted that in spite of a subjective (heuristic) choice at this final stage the preceding analysis guarantees a choice from only rational variants and insures against gross errors.

Below the methods for the five first operations will be described in detail. These methods have been published in a series of books and reports [8-12 and others].

IV. STATEMENT OF PROBLEMS

IV.1. Some Peculiarities of Static Problem Statement

Static problems usually involve a choice of parameters for systems projects or their equipment. Once chosen, the parameters do not change during the whole life of a project or equipment (while the operational conditions might change in unknown ways).

During statement of static problems it is necessary to establish precisely the composition of the x and y vector components and the specific form of the evaluating function $E(x,y)$.

The most difficult points concern the elucidation of procedures for satisfying constraints. As a rule there may be distinguished two kinds of constraints. The first kind includes constraints whose satisfaction is mandatory due to technical reasons (destruction of constructions and so on). Such constraints must be satisfied fully--if they are not satisfied for some variant x_i even under one of the considered nature states y_s, then this variant x_i must be merely excluded from the following solution of the problem. Constraints of the second kind cause only economic damages if they are violated. It is not necessary to satisfy them fully. But damage owing to their violation must be included in the evaluating function $E(x,y)$. Such damage as a rule concerns external systems or another part of the given system. It is difficult therefore to calculate such damage precisely and sometimes it has to be considered as an uncertain input data. Thus the composition of the x and y vector components, as well as the concrete form of function $E(x,y)$, have to be established simultaneously with procedures for satisfying constraints.

IV.2. Statement of Dynamic Problems

The statement of dynamic problems (and it is in the form of dynamic problems that most systems development tasks present themselves) is markedly different from the statement of static ones. Here the principle of making priority decisions (see point II.3) has to be correctly observed.

As a rule we are not equally interested in all parts of the whole given time period T for which a dynamic problem is to be solved. Our real interest is only in the nearest part ("first step") of the given period. The other part (the "afteraction" period) must be taken into account to see the consequences of these priority decisions. However, final decisions regarding the system's development (or operation) in this period can be made later. Therefore in dynamic problems we may (or must) distinguish the "first step" and "afteraction" periods.

For such a statement of dynamic problems the question of formulation of system development variants (trajectories) for the "afteraction" period is quite complex. Apparently, such variants have to correspond on the one hand to actions undertaken in the "first step" (the subsequent development of the system depends on the choice of primary projects) and on the other hand to concrete future conditions of the system's development (later decisions would be made based upon situations that have in fact occurred). However taking into account that subsequent decisions will continue to be made on the basis of uncertain information and therefore the final choice will be made by men--intuitively, heuristically--it is impossible to foresee such decisions and definitely select the variants of a system's development for the "afteraction" period. Therefore various approximate methods can be used to determine these variants, and different statements of dynamic problems correspond to them [9,11]. Some possible statements will be considered below.

Economic (or other) effects in dynamic problems (solved as a rule by dividing the time period studied into discrete intervals) can be estimated using the functional:

$$E = \sum_{t=1}^{T} E_t(x_{t-1}, x_t, y_t), \qquad (1\text{-IV})$$

where:

E_t = function of expenditures at time interval t;

T = total number of time intervals ("steps");

x_{t-1} and x_t = vectors of optimizing parameters at the beginning and at the end of time interval t;

y_t = vector of indefinite variables that characterizes nature states at time interval t.

The first statement is the most simple (and the most rough), by reducing the dynamic problem to a static one. This is simple enough when the nature state y_s (s = number of nature state) is taken as a concrete realization of vector y for the whole period T:

$$y_s = (y_{1s}, \dots, y_{ts}, \dots, y_{Ts}) \qquad (2\text{-IV})$$

and our possible action x_i (i = number of the action or variant) is the choice of a single-valued deterministic trajectory of the system's development for the whole period T:

$$x_i = (x_{1i}, \dots, x_{ti}, \dots, x_{Ti}) \qquad (3\text{-IV})$$

With such a statement of the problem the general sequence of its solution can be the same as for static problems. Some realizations (2-IV) and several possible actions (3-IV) are chosen. This is done by formal methods [9] or heuristic means, as will be described in the next sections.

For each action x_i and nature state y_s we estimate the expenditures E_{is} by functional (I-IV). This results in a "pay-off matrix" $\|E_{is}\|$, and on the basis of its analysis rational variants are chosen.

The shortcomings of this statement are obvious.

The second statement (and all following ones) supposes that the aim of the dynamic problem's solution is the choice of an expedient action for the nearest time period ("first step") only. One of the possible ways of solving such a problem is [13]:

(a) several possible actions are planned at the first step

$$x_i = x_{1i};$$

(b) a series of realizations (2-IV) for the whole studied period are chosen as before;

(c) for each planned action at the first step and the chosen nature state, deterministic optimization calculations are carried out for all steps beginning with the second one; this gives a pay-off matrix $\|E_{is}\|$, where E_{is} = value of the objective functional for the i-th action and the s-th nature state is determined as

$$E_{is} = E_1 (x_i, y_{1s}) + \min_{x_t} \sum_{t=2}^{T} E_t(x_{t-1}, x_t, y_{ts}) \qquad (4\text{-IV})$$

The first term in (4-IV) is equal to the function value in the first interval for fixed x_i and y_{1s}; the second summand is an extreme value of the functional in the period of afteraction.

Rational actions at the "first step" are chosen on the base of this matrix.

The merit of such a statement is the flexible and rather logical adaptation of various decisions to different states of nature. But it is laborious and does not quite correspond to real situations. The "suboptimization" of a system development at the second and following steps for differing conditions agrees with the assumption that further on (after the first step) we shall know precisely the forthcoming conditions and therefore shall be able to act optimally. In reality, making decisions at subsequent steps we shall continue to be under uncertainty and so not be able to act optimally.

Taking into account the second circumstance we may sometimes not demand such strict "suboptimization" of a system's development for the period of "afteraction" and may opt for a simplified third statement. Here some possible actions at the first step and several nature states (2-IV) for the whole studied period are also planned.

The difference is in how we take into account the "afteraction" period and make the calculation of the pay-off matrix. For each decision in the first step several (two to five) variants of the system's possible development in the "afteraction" period are planned (numbers of these variants are designated by j):

$$(x_{2ij}, \ldots, x_{tij}, \ldots, x_{Tij})\, j = 1,2,\ldots \qquad (5\text{-IV})$$

Further, for all variants j the functional (I-IV) is calculated at each chosen nature state. These calculations become not optimization but only evaluating (at fixed values x_{tij}) calculations unlike in the above mentioned second statement. Such a calculation determines the expenditure value:

$$E_{isj} = E_1 (x_i, y_{1s}) + \sum_{t=2}^{T} E_t(x_{t-1ij}, x_{tij}, y_{ts}) \qquad (6\text{-IV})$$

Now we suggest taking into account the adaptation of the system's development in the period of afteraction under diverse nature states by the choice of such a variant (from those mentioned) whereby the expenditures at the given nature state will be minimal:

$$E_{is} = \min_{j} E_{isj} \qquad (7\text{-IV})$$

This agrees with the assumption that at the second and following steps we shall choose this or that variant of the system's further development depending upon actual conditions, but this circumstance is taken into account here in simplified form. The values E_{is} obtained by relation (7-IV) are used for filling the pay-off matrix $\| E_{is} \|$. Its subsequent analysis and choice of rational decisions are carried out according to the usual order.

The fourth statement differs from the previous ones as follows: each rational action at the first step (optimal by the corresponding criterion) is determined on the assumption that at all steps of the "afteraction" period the choice is implemented according to this criterion. In other words, we choose a certain criterion, for instance that of Wald, and optimize the system's development for the whole period under study. Optimal action obtained for the first step belongs to the rational actions. Then we optimize the system's development (also for the whole period under study) by another criterion (for example the Savage criterion) and we get an additional rational action at the first step, etc.

Having fixed the criterion of optimality K we come to the problem similar to stochastic dynamic problems (with known probabilistic descriptions), when the extremes of the mathematical expectation of the functional is sought. It can be solved using ideas and methods of dynamic programming. For instance, if we take the Wald criterion (minimax expenditures) then for each t-th time interval the following functional equation must be solved (starting from the end):

$$\overset{o}{K}{}_t^T (x_{t-i}) = \min_{x_t} \max_{y_t} [E_t(x_{t-1}, x_t, y_t) + \overset{o}{K}{}_{t+1}^T(x_t)] \qquad (8\text{-IV})$$

where:

$^{o}K_t^T$ = minimal possible criterion value for the period from the beginning of the t-th interval to the end of the period being studied (it depends upon the vector value x_{t-1}).

The solution of such a problem with continuous values of vectors x and y, especially of large dimension, involves great (possibly even unsolvable) computational difficulties. But, if a finite and not too large number of vector x_t values, characterizing diverse system states, and also a limited number of nature states is taken (making the problem discrete), then the solution becomes possible in practice. In [9] there is the algorithm of a problem's solution using such a statement on the base of which the computer program has been worked out.

The suitable application of one or the other of the above mentioned statements depends upon the peculiarities of the given problem: its general laboriousness (computer time required), the time available for solution and the solution periodicity, etc. For each problem these questions have to be specified. The second statement is the one most widely used at the present time.

In [9] there are examples of solutions for problems relating to the management of power systems which take into account uncertainty in initial information. Also the example of solution of a dynamic problem is given in the last section of this paper.

V. SELECTION OF A REPRESENTATIVE SET OF NATURE STATES

V.1. Preliminary Analysis of Available Input Data

One of the main aims here is to clarify the properties of non-deterministic data with regard to their degree of uncertainty. One clarifies: really possible intervals (diapazones) of each data value; mutual dependences (correlations) between individual indicators; possibilities to obtain a probabilistic description for this or that data and the accuracy of such description; the continuous or discontinuous nature of individual input data; their most characteristic values, etc. Such an analysis is needed for quantitative description of input data and subsequent choice of their specific values.

The second aim of preliminary analysis is to specify the most valuable indicators, whose uncertainty leads most directly to the uncertainty of a situation. Those indicators whose uncertainty does not influence the results of a problem's solution (they are as a rule in the majority) may be given deterministically by their middle, or most probable, or some other characteristic values. Only main stochastic or uncertain input data whose uncertainty really leads to uncertainty in decision-making have to be included among vector y components. To decrease the dimensions of a problem and the laboriousness of its calculations it is desirable that the number of these components be as small as possible. Sometimes a special, rather involved investigation has to be made for these purposes.

The preliminary analysis of input data here under consideration must in fact be done before or simultaneously with the statement of a problem since a specific form of evaluating function $E(x,y)$ and general procedure of a problem's solution depend on the composition and properties of the vector y components.

V.2. Recommendations for the Selection

As to the selection of representative nature states, the first question is the determination of their total number S. As was mentioned in section III, this number must not be too large and is determined from the practical viewpoint of laboriousness of subsequent computations. Usually it is sufficient to consider $10 \div 30$ different nature states. If in the process of a problem's solution it becomes necessary to consider certain complementary nature states, they can be included additionally with corresponding supplemental calculations.

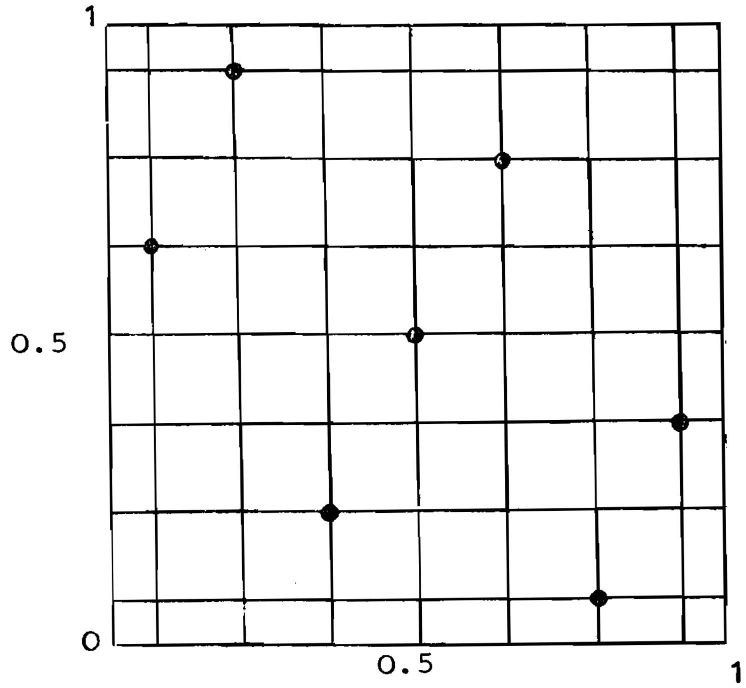
As a rule there are not many ($2 \div 5$, sometimes a few more) important indicators whose uncertainty most strongly influences results. Let us call these the first group of uncertain input data. Special attention must be paid to these data. Usually it is inadmissable to use formalized methods for the selection of probable combinations of their values.

The following way may be recommended for the selection of first group input data combinations. 2 ÷3 probable values are taken for each such indicator (for example, "optimistic", "pessimistic", "middle"). If there are no interdependences between these indicators and simultaneous appearance of any of their value combinations (for example, only "pessimistic" or only "optimistic" for all the indicators at once) is possible, then one must consider a complete set of combinations of their taken values. As the number of such indicators included in the first group is not large (2 ÷ 5) the general number of their value combinations will also not be large (5÷30). And it could be decreased additionally by clarifying and excluding those combinations that are close to other ones as regards their influence on a problem's solution. But if there are mutual interdependences among first group input data then one can exclude (from a complete set of combinations) the combinations which seem to be non-real. Thus a relatively small number (S_1 = 5 ÷20) of value combinations to be considered during the problem's solution is obtained for the first group of input data.

Such an approach cannot be used for the remaining non-deterministic input data (second group) as their number might be too big (tens or even hundreds). The number of their values combinations becomes too unwieldy and it is very difficult (or simply impossible) to select a limited number of "characteristic" combinations by an intuitive analysis. Formal methods for such a selection might be recommended here. As a rule they are based on a regular (in some sense) distribution of a given number of points in an n-dimensional parallelepiped or single cube (in a continuous domain Y of possible vector y values).

In particular, methods have been proposed [9] using linear code theory for choosing points evenly or uniformly distributed on a grid or in the centers of spheres having equal and maximum possible diameters. Fig. 3 illustrates such selection within a two-dimensional single cube. In Fig. 3a

a) on a grid



b) in the center of spheres

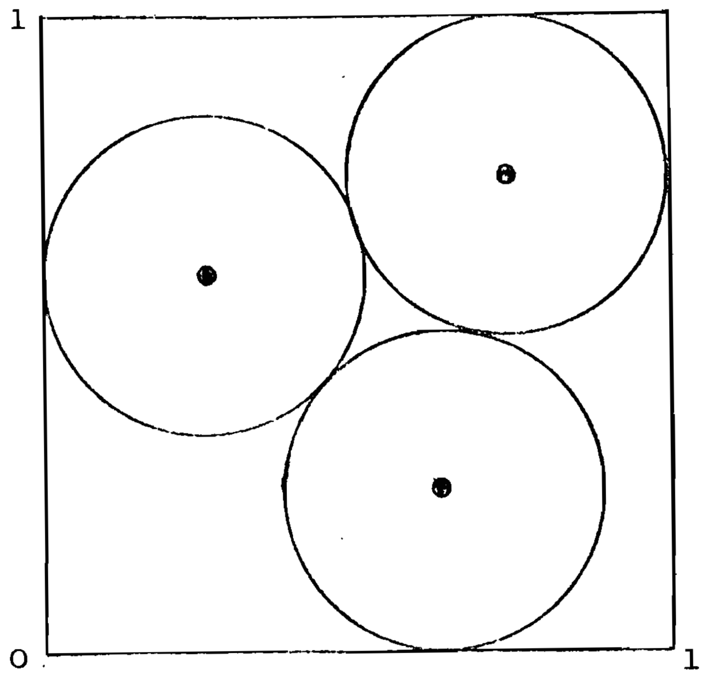


Fig. 3. Selection of Uniformly Situated Points.

a selection of seven points on a regular grid is shown. Fig.3b shows the selection of 3 points in the centers of spheres. Evenness is understood here as providing for the maximum possible distances between points (with a given total number S). Selection on a regular grid seems to be better for a big n (large dimensional vector y) than selection using the centers of spheres. The latter method uses the Euclid distance as a measure of nearness of points. But in a large dimensional space the Euclid distance becomes bad for such a measure. For example the main diagonal of the single n-dimensional cube strives to infinity if $n \to \infty$ though the length of each rib of the cube remains equal to one. Therefore with a big dimension of the single cube the centers of such spheres concentrate in the central part of the cube and when $n \to \infty$ they even merge with its central point (all having coordinates 0.5).

Selection using a regular grid has no such disadvantage. But nowadays the problem of such a choice has been solved for a partial case only [9], namely, for the case when the given total number S of selecting points is a simple number (1, 2, 3, 5, 7, 11, 13 and so on). This is not a great disadvantage however as we can anyway take a certain simple number nearest to the desired total number of nature states under consideration. One additional advantage of this method is the consideration of S values for each component of vector y (see Fig. 3a). Extrapolation of the points from a single cube into a real space Y (as a rule into an n-dimensional parallelepiped) may be simply done as a linear extrapolation.

Hence using such formal methods we can obtain a certain number (S_2) of value combinations for the second group (for the rest) of non-deterministic input data.

Now the representative set of nature states may be obtained if we combine with each other the S_1 combinations of first group input data values and the S_2 combinations of the second group. In practice this means that for each combination of first group input data values all S_2 combinations of the second group have to be considered. The general number of nature states under consideration will then be $S = S_1 \times S_2$. To ensure that this number corresponds approximately to the number outlined earlier one has to choose the numbers S_1 and S_2 in a proper way. For example, one can distribute non-deterministic input data suitably between first and second groups, find the minimum possible number (S_1) of combinations for the first group and determine a proper number (S_2) of combinations for the second group (which corresponds to the desired total number S and this first number S_1), etc.

If the general number (n) of non-deterministic input data (components of vector y) is not so big, then one may of course not divide them into two groups and may select the set of representatives for all such data simultaneously (as was recommended for the first group).

As a result of the whole operation we just obtain the set of nature states

$$y_s \quad (s = 1,\ldots,S) \quad , \qquad (1\text{-V})$$

which will be considered during solution of the problem. With dynamic problems this operation may be done either for the whole period studied (perhaps with the corresponding increase of the dimension of the vector y by a multiplication of the number of non-deterministic input data by the number of time intervals) or for each time interval individually. Then we obtain either S realizations (2-IV) or T sets (1-V) for individual intervals.

VI. SEARCH FOR AND PRELIMINARY ANALYSIS OF VARIANTS FOR PROBLEM SOLUTION

VI.1. Search for Solution Variants under Consideration

The aim of this search is the selection of potentially rational solution variants which should be considered during subsequent solutions of a problem. For continuous problems this is also in a certain sense a "discretization" of the problem.

In some problems (especially those with discrete parameters to be optimized) the competing variants $x_i (i=1,...,I)$ among which rational variants have to be determined are evident or may be outlined by use of intuition. That depends to a certain extent on the general number (I) of variants which may be practically considered taking into account the laboriousness of the following computations (during which the pay-off matrix $||E_{is}||$ would be calculated). If the number of variants which intuitively are interesting do not exceed this number I and if there is sufficient confidence that I variants, outlined by intuition, in fact include all variants (that might prove to be rational ones), then the operation considered would be relatively simple. This may be done in an intuitive manner.

For more complicated problems special methods have to be used. The main approach recommended here is the making of optimization calculations for several deterministically given conditions of a system's development or operation. A "locally" optimal variant obtained under some really possible conditions evidently would be dominant and might prove to be a rational one or might even be later taken for implementation.

This approach is rather laborious as it demands a series of optimization calculations. Therefore, the possibility of its application and the number of calculations to be made will depend on computers available, established terms for problem solution and so on. If there is an opportunity, the search for such "locally" optimal variants has to be made for all S conditions (nature states) selected in the previous operation. But in principle the number of conditions under which the deterministic optimization calculations are made might differ from the number of nature states S selected earlier and be either less or more.

Thus, as a result of such optimization calculations a multitude of possible variants for a problem's solution would be sought for the ensuing engineering analysis. It has to be pointed out that the "locally" optimal variants do not cover the entire set of dominant variants. In principle there may exist such variants which while not optimal under any conditions are in fact better than "locally" optimal ones for the totality of conditions, and may therefore prove to be rational ones. But there seems to be no way to determine such dominant variants except through intuition.

VI.2. Preliminary Analysis of the Solution Variants

The first aim of the preliminary analysis is to examine whether there really is uncertainty of choice or not; in other words, whether the "locally" optimal variants sought differ from each other from the viewpoint of our action or happen to be the same.

As to static problems, this examination relates to only the main parameters which characterize our action. Parameters related to additional "undertakings", which satisfy constraints, naturally may not be taken into consideration. If a problem is dynamic, then it is important to test a coincidence of priority projects for construction or values of main optimizing parameters in the first (nearest) time interval.

If the coincidence takes place then a problem solution can be finished and the single optimal variant sought has to be recommended for implementation.

The aim of the following analysis of "locally" optimal variants (if quite a few of them occur) is to select from them such a number (I) of competing variants as we can in practice consider during subsequent solution of a given problem. Then it is necessary to identify peculiarities which distinguish such variants from one another. This allows one to select variants which are most characteristic and interesting in this or that sense and to outline some additional variants which would combine features of different "locally" optimal variants and better correspond to various (not only to one) conditions of a system's development or operation. Such an analysis will have its own particularities for each problem and it is difficult to recommend some unified scheme.

As the final result of this operation, I solution variants x_i $(i = 1, \ldots, I)$ are obtained and then the pay-off matrix $||E_{is}||$ must be calculated.

VII. CALCULATION OF THE PAY-OFF MATRIX

The pay-off matrix is a table consisting of I lines and S columns. Into squares of the table are written the values E_{is} of an evaluating function (1-III) or (1-IV) obtained for solution variant x_i under nature state y_s.

The calculations of values E_{is} have to be made according to the chosen statement of a problem and the established form of an evaluating function or functional. It is assumed that the cost of additional "undertakings" in order to satisfy constraints and the damage which results from their not being satisfied are accounted for here (in values E_{is}). These calculations with dynamic problems will be in fact optimization ones if, for example, the second statement of such problems is used (see paragraph IV.2)--with fixed vector x values in the first time interval, the values of vector x in the "after-action" period are optimized. A general view of a matrix $||E_{is}||$ is represented in Fig. 4 (left part of the table). The values of expenditures (let an evaluation function be, for example, expenditures) written in some line give a (non-single) economic estimation of the corresponding solution variant (under deterministic conditions there would be only one column and there would be one estimation for each variant). If the probabilities for different nature states were known (that would correspond to the existence of deterministic and stochastic-definite input data only) then for each action (line) it would be possible to find a mathematical expectation of expenditures. Such an estimation would allow one to compare variants under consideration with each other with some degree of confidence and to choose the variant which is optimal "on the average". But for uncertain conditions such a possibility does not exist and economic estimations of variants x_i are non-single.

Therefore some characteristic values of an evaluating function can only be obtained for each variant x_i and used during the following analysis of the pay-off matrix. The most interesting characteristic estimations are:

$E_{is} = E\,(x_i,\ y_s)$ $\qquad$ $i = 1, \ldots, I;\ \ s = 1, \ldots, S.$

X \ Y	Y_1	Y_2	...	Y_s	...	Y_S	E_i^{MAX}	E_i^{MIN}	$\bar{E}_i$	R_i^{MAX}
X_1	E_{11}	E_{12}		E_{1s}		E_{1S}	E_1^{MAX}	E_1^{MIN}	$\bar{E}_1$	R_1^{MAX}
X_2	E_{21}	E_{22}		E_{2s}		E_{2S}	E_2^{MAX}	E_2^{MIN}	$\bar{E}_2$	R_2^{MAX}
⋮										
X_i	E_{i1}	E_{i2}		E_{is}		E_{iS}	E_i^{MAX}	E_i^{MIN}	$\bar{E}_i$	R_i^{MAX}
⋮										
X_I	E_{I1}	E_{I2}		E_{Is}		E_{IS}	E_I^{MAX}	E_I^{MIN}	$\bar{E}_I$	R_I^{MAX}
E_s^{MIN}	E_1^{MIN}	E_2^{MIN}		E_s^{MIN}		E_S^{MIN}				

$$E_i^{MAX} = \max_s E_{is}; \quad E_i^{MIN} = \min_s E_{is}; \quad \bar{E}_i = \frac{1}{S}\sum_{s=1}^{S} E_{is};$$

$$R_i^{MAX} = \max_s R_{is}, \ \text{WHERE } R_{is} = E_{is} - \min_{j\in\overline{1,I}} E_{js}$$

Fig. 4. The Pay-Off Matrix and Its Characteristic Values.

a) Arithmetic average of expenditures.

$$\bar{E}_i = \frac{1}{S} \sum_{s=1}^{S} E_{is}. \qquad (1\text{-VII})$$

This estimation has formal similarity with the mathematical expectation of expenditure--it coincides with the latter if the uniform low of probabilistic distrubtion (equal probabilities) is assumed for considering nature states y_s. But such an assumption is, as a rule, far from reality.

b) Maximum value of expenditures for a given variant.

$$E_i^{max} = \max_s \ E_{is} \qquad (2\text{-VII})$$

This characterizes the worst that might happen if this variant were taken for implementation. This is the more pessimistic estimation.

c) Minimum value of expenditures.

$$E_i^{min} = \min_s E_{is} \,. \quad (3\text{-VII})$$

This is the more optimistic estimation.

Each of these estimations characterizes the situation only in a one-side manner. Therefore none of them can be recommended for the final judging of variants. As a whole they still give a non-single estimation of variants and do not eliminate in general the uncertainty in decision-making. But each of these estimations gives certain characteristics of variants and their use allows us to formalize and simplify the process of pay-off matrix analysis. It is appropriate to add these characteristic values to the right side of the corresponding lines of a pay-off matrix (see Fig. 4).

Added to them is a maximum risk value which can be found from the pay-off matrix (see below, paragraph VIII.3) and which also characterizes the situation to a certain extent. To determine risk values R_{is} the minimal (i.e. in the columns) values E_s^{min} are needed. They are shown in the lowest line of the table. As the calculations for the pay-off matrix are made only in cases where uncertainty in decision-making has been cleared up (see paragraph VI.2) and ensuing analysis of the pay-off matrix surely will be made, the characteristic values of the evaluating function have to be determined just after (or at the same time as) the pay-off matrix itself is obtained.

VIII. ANALYSIS OF THE PAY-OFF MATRIX AND CHOICE OF RATIONAL ACTIONS

VIII.1. The General Scheme of the Analysis

This analysis includes several stages. Movement to a following stage is necessary only if a preceding stage does not occur successfully (does not identify the single optimal variant).

The sequence of the stages is given in Fig. 5. Detailed elucidation for each stage will be carried out below.

Here it should be mentioned that after this whole operation (if uncertainty of decision remains) the last operation (final choice of action implemented) follows and that last operations will not be considered in this paper.

VIII.2. Selection of Dominant Variants

Analysis of pay-off matrix $\|E_{is}\|$ begins with the search for dominant variants. Each of these is better than any other dominant variant under at least one (not mandatorily one and the same) nature state. Therefore all dominant variants have to be considered as potential candidates for rational variants and for implementation.

It is easier to select the set of dominant variants by an identification of non-dominant ones, which are worse than some dominant variants under all nature states considered. The following inequality takes place for non-dominant variant j (while it is being compared by pairs with dominant variant i):

$$E_{js} \geq E_{is} \quad \text{for all} \quad s=1,\ldots,S \tag{1-VIII}$$

and with at least one s there is strict inequality. Obviously such variants are bad and can be excluded out of the following consideration.

After removing non-dominant variants the set of dominant variants remains; we continue to denote the total number by I. The "locally" optimal variants sought during the third operation (see paragraph VI.1) of course would be conserved in this set since they are better than all the remaining ones under corresponding nature states. But there also might occur some of the variants singled out by intuition. And this would be good. Now only dominant variants are included in the pay-off matrix which are being passed to the next stage of analysis.

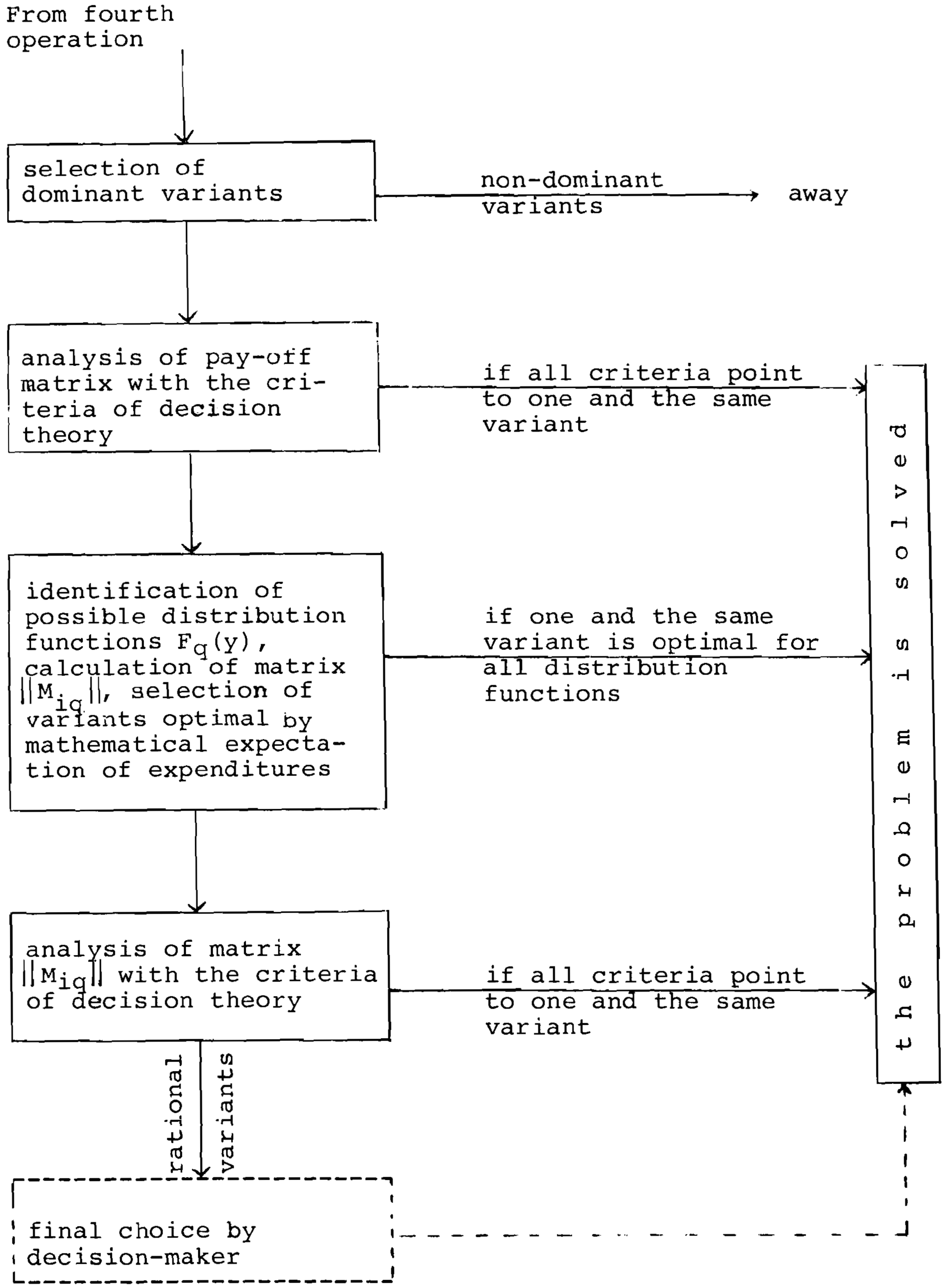


Fig. 5. The General Scheme of the Analysis and Choice of Rational Variants

VIII.3. Use of Decision Theory Criteria for Analysis of the Pay-Off Matrix

As was already said, the special criteria proposed for uncertain conditions [2] have to be used for the choice of rational variants (actions, decisions). The majority of such criteria use the characteristic values of the pay-off matrix described above (see Fig. 4). The most interesting criteria, it seems, are the following:

a) Laplace's criterion (minimum of arithmetic average of expenditures) which uses the estimation $\bar{E}_i$ (1-VII). It recommends choice of the variant x_L^o which gives the minimum $\bar{E}_i$:

$$\min_i \bar{E}_i = \min_i \frac{1}{S} \sum_{s=1}^{S} E_{is} \longrightarrow x_L^o \ . \qquad (2\text{-VIII})$$

This criterion originated from the principle of "insufficient reason"--we have no special reason to distinguish or prefer any one nature state when compared with others and assume that they are all equally probable. But this of course does not usually correspond to reality.

b) Wald's criterion (of minimax expenditures), which shows the variants x_W^o with the minimum of maximal expenditures E_i^{max} (2-VIII):

$$\min_i E_i^{max} = \min_i \max_s E_{is} \longrightarrow x_W^o \ . \qquad (3\text{-VIII})$$

c) Hurwicz's criterion (of pessimism-optimism), which minimizes a linear combination of maximal E_i^{max} and minimal E_i^{min} (3-VII) expenditures:

$$\min_i \left[\alpha E_i^{max} + (1-\alpha)\ E_i^{min}\right] \longrightarrow x_G^o \ , \qquad (4\text{-VIII})$$

where α is an indicator of "pessimism-optimism" ($0 \leq \alpha \leq 1$). The value of indicator α is chosen by the investigator himself depending on which estimation (maximal or minimal) he prefers. This therefore is a very subjective criterion.

d) Savage's criterion (of minimax risk) is analogous to Wald's criterion but uses risk values:

$$\min_i R_i^{max} = \min_i \max_s R_{is} \longrightarrow x_S^o \quad (5\text{-VIII})$$

Risk R_{is} is an overexpenditure which would take place under nature state y_s if the variant x_i were chosen instead of a variant which is "locally" optimal under these conditions y_s. Such risks give a certain characteristic of a situation--they show the relative difference of expenditures linked to the choice of one variant instead of another. In fact they characterize (though in a non-single manner) the value of the damage (risk) conditioned by uncertainty of a situation (input data).

To determine risks R_{is} one has to find the minimum of expenditures for each selected condition y_s (for each column of the pay-off matrix). Apart from values E_i^{min} (3-VII), the minimum of expenditures is here sought for in a column (not in a line):

$$E_s^{min} = \min_i E_{is} \; (s=1,\ldots,S). \quad (6\text{-VIII})$$

These values represent those minimal expenditures that we should have if we were to know beforehand what nature state will occur in the future and chose the variant optimal for these conditions.

Now the value of risk R_{is} for some variant x_i and conditions y_s is determined as the difference:

$$R_{is} = E_{is} - E_s^{min} \;, \quad (7\text{-VIII})$$

where E_{is} is taken from a corresponding square of the pay-off matrix All IxS values R_{is} give us the risk matrix $||R_{is}||$ which is analogous to the matrix $||E_{is}||$. There will be at least one zero element in each column of the matrix $||R_{is}||$ (for the variant which is optimal under conditions y_s).

Values of risks do not give a single estimation of overexpenditures; nor did absolute values of expenditures E_{is}. For each variant x_i values of risks may vary from zero to a maximum value R_i^{max} determined as

$$R_i^{max} = \max_s R_{is} \; (i=1,\ldots,I) \quad (8\text{-VIII})$$

These values give one more characteristic estimation of each variant compared. Therefore they have to be added to the characteristic values of the pay-off matrix obtained earlier (see Fig. 4) and they are used in one of the decision criteria (Savage's criterion). It may be noted here that an arithmetic average of risk ($\bar{R}_i$) also is a certain characteristic estimation. But while the competing variants x_i will be compared, the use of values $\bar{R}_i$ will give the same result as values $\bar{E}_i$(1-VII), that are used by Laplace's criterion, will.

All the above criteria (2-VIII÷5-VIII) have their own advantages and disadvantages considered in detail in [2]. None of them can be admitted as being the best, but they allow us to select rational variants, each of which is good in this or that respect. These variants obviously require attentive consideration while the final choice is being made of the variant to be implemented.

It is quite possible that there might exist certain variants which are not optimal by any of the described criteria but are good if we consider all of them at once. Such variants can be found with the aid of certain generalized criteria. Such a criterion (K) can be compiled [9] by the use of characteristic values of the pay-off matrix (together with value R_i^{max}):

$$\min_i K_i = \min_i [\alpha_1\bar{E}_i + \alpha_2 E_i^{max} + \alpha_3 E_i^{min} + \alpha_4 R_i^{max}] \longrightarrow x_K^o$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1; \quad 0 \leq \alpha_1, \alpha_2, \alpha_3, \alpha_4 \leq 1. \qquad (9\text{-VIII})$$

Depending on the values of coefficients α accepted by the investigator one may pass from this criterion to each of the four preceding ones or get this or that combination of them. Of course, this criterion has all the disadvantages of previous ones and does not remove uncertainty in decision-making (perhaps, conversely, it multiplies the number of rational variants). But it does allow one to search for variants interesting from the viewpoint of the totality of the preceding criteria.
And such variants also have to be considered while the final decision is made.

If all described criteria point to one and the same variant a given problem might be considered solved (see Fig. 5). Just this variant may be recommended for implementation. Such cases are rare but happen sometimes. But in general there will be several (say J) rational variants (recommendations of different criteria do not coincide completely) and one has to pass to the next stage of the analysis (see Fig.6).

VIII.4. Accounting Possible Probabilities of Different Future Conditions

It is very important to utilize available information about probabilities of different input data values. Very often this allows liquidating the uncertainty in decision-making. Such information concerns stochastic - definite and partly uncertain input data.

At this stage one attempts to identify possible probabilities of the appearance of different nature states considered using the probabilistic characteristics of individual input indicators, and then to determine a mathematical expectation for each rational variant.

Since we assume that vector y includes all kinds of non-deterministic input data, there exist indicators with distribution functions that are not known exactly or are not known at all. Therefore it is completely impossible to obtain a single distribution of probabilities for selected nature states y_s. But one can identify several (Q) subjective distribution functions (series):

$$F_q(y) \quad q=1,\ldots,Q, \qquad (10\text{-VIII})$$

which give probabilities P_{sq} for individual nature states y_s, where

$$\sum_{s=1}^{S} P_{sq} = 1.$$

Experts' estimations based on experience and intuition inevitably have to be used when such combinations $F_q(y)$ are specified. Of cours there is not a single way for such assessments [3,14]. But even such intuitive characteristics will be helpful. And the use of several

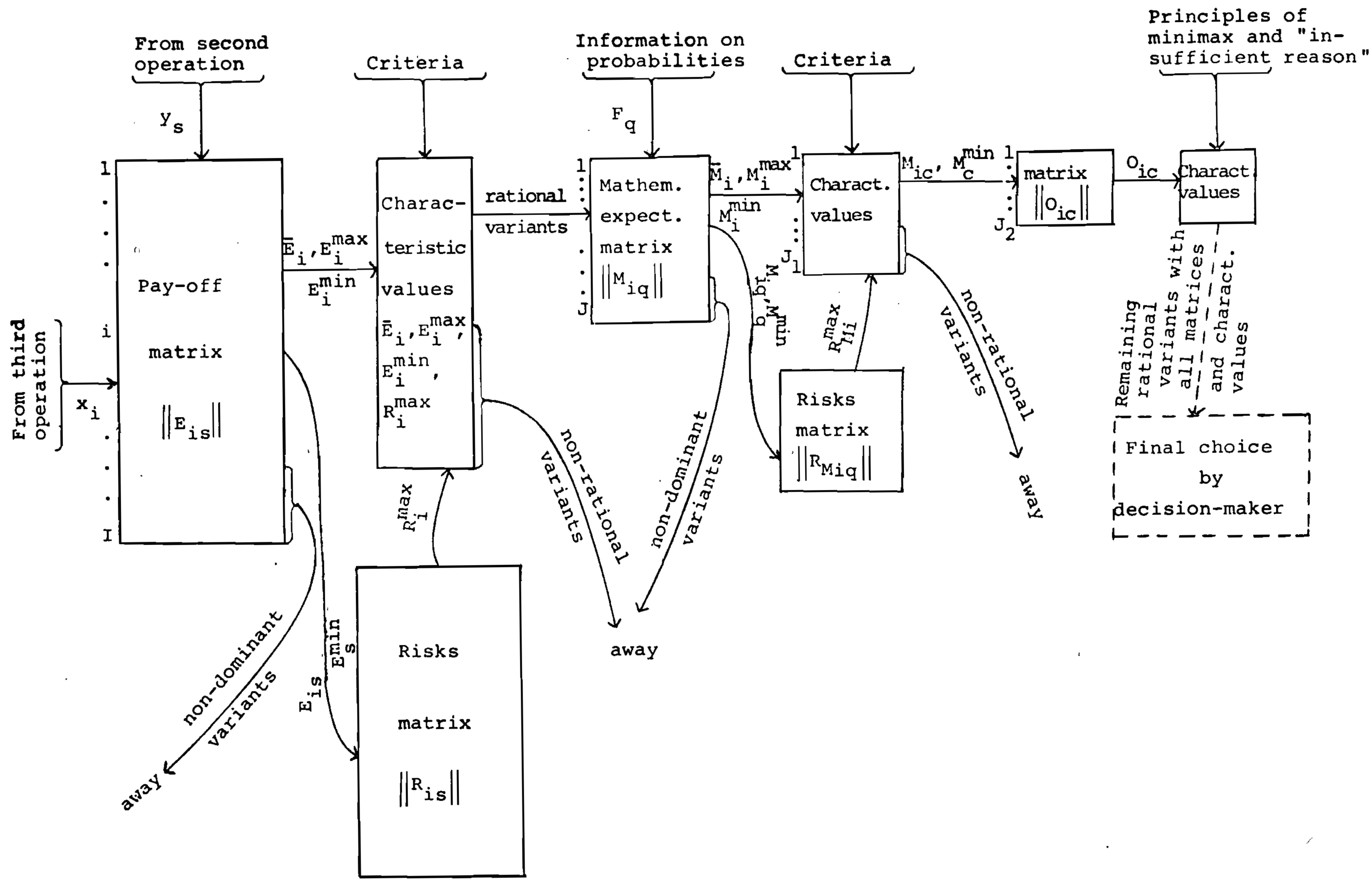


Fig. 6. Sequence of the Analysis for Choosing Rational Variants.

characteristics instead of one permits us to study more fully a field of possible probabilities for different nature states and guarantees against gross errors and arbitrariness.

Now for each distribution row $F_q(y)$ we can determine mathematical expectations of expenditures linked to a choice of various rational variants x_i:

$$M_{iq} = \sum_{s=1}^{S} p_{sq} E_{is} \quad , \qquad \text{(11-VIII)}$$

where E_{is} are obtained from the pay-off matrix.

For all variants x_i $(i=1,\ldots,J)$ and all rows $F_q(y)$ we just obtain matrix $||M_{iq}||$ (of mathematical expectations of expenditures).

The lines of this matrix correspond to the rational actions (just as in the pay-off matrix, but the number of lines would be less--$J \leq I$). The column in this case represents not individual nature states but the whole set with fixed combination of their probabilities ("mixed strategies" of nature). The values M_{iq} in individual columns can be compared with each other and this will identify the variant x_q^o which gives the minimum for the mathematical expectations of expenditure for this distribution series $F_q(y)$:

$$\min_i M_{iq} \longrightarrow x_q^o \quad . \qquad \text{(12-VIII)}.$$

If with all $F_q (q=1,\ldots,Q)$ there is obtained one and the same optimal variant then a problem is solved (see Fig. 5)--the consideration of the information about possible (probable) distribution laws has eliminated the uncertainty of the final choice. In opposite cases it would still be necessary to continue the analysis, but as a rule with a smaller "zone of uncertainty" (a smaller number J_1 of rational variants which remained).

VIII.5. Analysis of Mathematical Expectations Matrix $||M_{iq}||$

The matrix $||M_{iq}||$ with the remaining rational variants x_i $(i=1,\ldots,J_1)$ has to be analyzed with the same decision criteria as were used earlier for the pay-off matrix $||E_{is}||$. To do this the values E_{is} in formulas (2-VIII) ÷ (7-VIII) are changed in M_{iq} and index 2 in q (now uncertainty is considered with regard not to individual nature states y_s but to the distribution rows F_q). Also a special index has to be used (say M) for distinguishing risks R_{Miq} otained with matrix $||M_{iq}||$ from risks R_{is}. The whole scheme of matrix $||M_{iq}||$ analysis is similar to that of matrix $||E_{is}||$ (see Fig. 6).

One result of this analysis might possibly be the coincidental recommendations of different criteria. Then a problem is solved (see Fig. 5). If this has not happened then one may try to make the most detailed analysis of remaining rational variants in the light of matrix $||M_{iq}||$. In particular it might be useful to estimate "overexpenditures" for characteristic values used by different criteria which take place with this or that rational variant (comparing with the variants which are optimal by corresponding criteria) [9]. Let us denote some applicable criterion with index "c" and the total number of criteria with "C" (c=1,...,C). Then the minimal value of the mathematical expectation of expenditures according to the criterion c will be

$$M_c^{min} = \min_i M_{ic} , \qquad (13\text{-VIII})$$

where $M_{ic} = \bar{M}_i$ for Laplace's criterion, $M_{ic} = M_i^{max}$ for Wald's criterion, $M_{ic} = R_{Mi}^{max}$ for Savage's criterion, etc. If some variant is not optimal by the criterion c, then it will have some "overexpenditures" O_{ic} in value M_{ic} compared with the variant which is optimal by this criterion (as it is with risk R):

$$O_{ic} = M_{ic} - M_c^{min} . \qquad (14\text{-VIII})$$

Determining these "overexpenditures" for all remaining rational variants and for all criteria applied we shall obtain the matrix $||O_{ic}||$ which is similar to risks matrix $||R_{is}||$ or $||R_{Miq}||$ but expresses our possible losses from the viewpoint of different criteria. This matrix characterizes the relative effectiveness of various rational variants in the measures used by different criteria. Again we have a non-single estimation here due to the uncertainty of a situation and the possibility of the comparison of variants only in partial (individual) aspects.

While we have the matrix $||O_{ic}||$, we can use at least two sensitive principles which permit an additional comparison of variants, namely, the principle of "insufficient reason":

$$\min_i \bar{O}_i = \min_i \frac{1}{C} \sum_{c=1}^{C} O_{ic} \longrightarrow x^o_{ir} , \qquad (15\text{-VIII})$$

and the minimax principle:

$$\min_i O_i^{max} = \min_i \max_c O_{ic} \longrightarrow x^o_{mm} . \qquad (16\text{-VIII})$$

It is in principle possible also to weigh (subjectively of course) the relative significance of various criteria and to compile some generalized criterion similar to criterion (9-VIII). But this has a sense only if none of the weights is equal to zero. By the way, the principle of "insufficient reason" (15-VIII) assumes equal weights for all criteria.

It should be underlined here that such a procedure cannot and must not have the aim of determining the single optimal variant (theoretically it is possible to repeat such a procedure several times--to the values $\bar{O}_i$ and O_i^{max} obtaining "overoverexpenditure", etc.--till a single optimal variant is achieved). This procedure is only intended for more comprehensive analysis and for more substantive recommendations concerning the composition of rational variants to be presented to a decision maker. The latter is very important--not to miss a variant which might in fact be a good one.

An example of the procedure for determining the matrix $||O_{ic}||$ and of values $\bar{O}_i$ and O_i^{max} is shown in Fig. 7. Three criteria (Laplace, Wald and Savage) are applied here. Five variants x_i remained after analysis of the pay-off matrix $||E_{is}||$ and four distribution rows F_q were considered when matrix $||M_{iq}||$ was calculated. Two variants (x_1 and x_3) proved to be optimal by mathematical expectations of expenditures. (Their values are indicated by squares

		Matrix $\|M_{iq}\|$				Characteristic values used by criteria			Matrix of "over-expenditures" $\|O_{ic}\|$			Values minimized by principles	
		F_1	F_2	F_3	F_4	$\bar{M}_i$	M_i^{max}	R_{Mi}^{max}	O_{iL}	O_{iW}	$\bar{O}_{iS}$	$\bar{O}_i$	O_i^{max}
Rational Variants	x_1	40	50	60	90	60	90	55	0	0	10	3,3	10
	x_2	55	70	95	50	67,5	95	45	7,5	5	0	4,2	7,5
	x_3	90	105	50	35	70	105	55	non-rational variant				
	x_4	70	55	70	110	non-dominant variants							
	x_5	50	80	80	100								
minimal values		40	50	50	35	60	90	45					

Fig. 7. Example of the Analysis of Matrix $\|M_{iq}\|$.

and also shown in the lowest line since they are used for the estimation of risk matrix $||R_{Miq}||$ which is not shown in Fig. 7 but was used for the determination of R_{Mi}^{max} values.) Two variants (x_4 and x_5) turned out to be non-dominant (the variant x_1 is better for all F_q) and therefore were not considered further.

The use of these three criteria has identified two rational variants (x_1 and x_2). Variant x_3 is non-rational (since two preceding ones were dominant over it) and was excluded, although the decision-maker might feel that it deserves attention. The minimal values (13-VIII) of the criteria are noted below the table. They are used in determining "overexpenditures" O_{iL}, O_{iw} and O_{is} (14-VIII) and matrix $||O_{ic}||$ which has only two lines here. The principles (15-VIII) and (16-VIII) were applied and both variant x_1 and variant x_2 as well still remained rational (the first by the principle of "insufficient reason", the second by the minimax principle). But the economical consequences of each variant are now more evident and only two of them remain.

Returning to the general line of analysis, it seems now that all possibilities of formal methods are exhausted (see Fig. 5 and 6). As was said in the beginning of this paper we are not considering here the procedures of experts' estimation and application of other objectives which the final choice by decision-makers (the last operation) would require. We have analyzed a situation to the greatest possible extent. We eliminated many variants and identified the rational ones in the light of our single objective (economi-) cal or other). (If there had been a need to maximize (not minimize) some objective then all schemes for finding a solution would be the same, only instead of a minimization, a maximization should be made in appropriate places.) If the whole analysis did not provide the single rational (optimal in that case) variant, then the uncertainty of input data has indeed conditioned the uncertainty of decision-making, and we have to pass the identified rational variants along with all results of the solution (all matrices and characteristic values) to a decision-maker (for the completing operation which is not considered in this paper as it requires another special method [4,5]).

The example of a problem's solution by means of the given scheme will be given in the next section.

IX. THE EXAMPLE OF A PROBLEM SOLUTION

IX.1. General Description of the Problem

The example considered below is borrowed from [9]*. The sense of the problem consists in the choice of priority power plants which constructions would have to start immediately after those plants already under construction. But for correct choice of these priority plants we must consider certain additional time periods ("afteraction" period) and therefore the problem is a dynamic one. In the example the North-West united electric power system (UEPS) of the USSR is studied. It consists of three power nodes (Southern, Central and Northern), and several new plants might be constructed in each node. The selection of priority plants is associated with a specific node and the problem consists in fact not only in the determination of priority plants but also in their distribution over the UEPS region.

A 12-year future time period with yearly intervals was taken for the study of the UEPS development. For greater concretization and simplification, the problem was stated as a choice of only the first (one) priority power plant (it is assumed that the solution of such a problem might be repeated after a while for the choice of the next new plant), and the sense of the decision to be made is just the choice of the first new plant placed into one of the nodes. This decision is considered as the action at the "first step". The latter is not linked to a fixed duration

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*See the article "The Task of Determining the Priority Electric Power Plants into United Electric Power Systems" in [9].

of time. The duration of the "first step" in this case depends on the type of priority plant chosen which might require more or less time for its construction and might provide the increase of energy demand for longer or shorter time periods. Therefore the duration of a complementary "afteraction" period might also differ.

An experience of earlier studies of such problems shows that the input data whose uncertainty most strongly influenced the results are: values of the electricity consumption and the peaks of daily load (for the whole UEPS and its individual nodes), and the technical-economic indicators for new power plants and fuels. Therefore, the following input data were taken as uncertain ones while the problem was being solved:

-- electricity consumption

-- daily consumer-load peaks

-- fuel costs (per capita) for different kinds of power plants

-- per capita investments for new plants.

These data were given by intervals of their possible (probable) values. The rest of the input data were considered as deterministic. As a whole the vector of uncertain input data (y) characterizing state of nature had 32 components in this example.

IX.2. The Statement of the Problem and Scheme Used for Solution

The third statement of dynamic problems (see formulas (2-IV), (5-IV), (6-IV), (7-IV) in paragraph IV.2) was taken for the problem considered. That means that for each priority power plant sought, several variants of a following development of the UEPS in the "afteraction" period were outlined and "evaluating" (not optimization) calculations were made while the pay-off matrix was obtained.

23 states of nature were selected for the solution. This number was established from the viewpoint of the laboriousness of computations. Specific values for these 23 combinations of non-deterministic input data were selected with the use of the method considered in paragraph V.2 for the choice of a given number of points on a regular grid. Dominant variants for a priority construction were sought by deterministic optimization calculations for all 23 selected nature states. The non-linear discrete optimization model of electric power systems (see [15] was applied for this aim. The majority of priority (first) power plants happens to coincide and only 3 different priority plants were identified in 23 computations made. They are:

1. Gas-turbine station (GTS) in Southern node;
2. Hydro-accumulating power station (HAPS) in Central node;
3. HAPS in Northern node.

Such a result evidences that manoeuverable equipment for operation during daily load peaks is insufficient in the UEPS. The priority constructions of these 3 power plants were taken as possible actions at the "first step" and we shall call them the first, second and third actions accordingly.

Three "subactions"--variants of the UEPS development or sequences of power plant construction in the "afteraction" period--were outlined for the first action. They were selected from among "locally" optimal variants of the system development obtained for the considered states. But for the second and third actions there was considered only one development variant in "afteraction" period (one for each of these two actions).

All 5 of these "subactions" or sequences of power plant constructions (three for the first action and one each for the second and third) have been computed under all 23 nature states with the use of a special "evaluating"model which consists of certain (not all) blocks of the optimization model mentioned. As has been written in paragraph II.5, special "undertakings" have to be provided for the adaptation of a variant of system development to different nature states (mainly for the satisfaction of given constraints). In this specific problem the adaptation is needed primarily owing to different demands for electricity (and capacity) in various nature states. Such an adaptation in the example under consideration was provided by changing start times of power plant operation (and construction) according to whether the rate of demand increase was more or less is in this or that nature state considered. Therefore changeable data of starts for power plant operation (or construction) are in fact the main "undertakings" for the adaptation considered in this problem. Because of this, the outlined 5 sequences ("subactions") were lengthened, when necessary, to such an extent that the whole capacity of plants was sufficient even in the most unfavorable nature state (having the biggest demand).

Expenditures obtained for outlined actions and "subactions" under various nature states are represented in Fig.8a. In the line marked "min E_1" are shown the expenditures, which are minimal among three "subactions" taken for the first action. These expenditures will further characterize the first action, and the three lower lines of Fig. 8a are the pay-off matrix for the problem considered. In the two rightmost columns the values E_i^{max} and $\bar{E}_i$ which are minimized

Fig.8. **Expenditures for different variants under various nature states .**

a. Pay-off matrix (10^6 rubles)

actions	"sub-actions"	Combination of input data (nature states)																							E_i^{max}	$\bar{E}_i$
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23		
1st	1	9295	9662	11087	11076	10712	10169	11343	10988	10429	10619	11361	10077	10913	9807	10391	12406	10406	10416	11226	9402	7937	10361	12381	-	-
1st	2	9940	6839	10380	11037	10182	9139	10656	9927	8079	10046	10568	9389	10890	7895	9641	11428	7395	9179	10084	7539	7333	10180	11967	-	-
1st	3	9891	6831	10395	11048	9957	9118	10802	9939	9478	10097	10515	9171	10897	7912	8913	11430	7480	9152	10137	7558	7430	10040	12179	-	-
1st	$minE_1$	9295	6831	10380	11037	9957	9118	10656	9927	8078	10046	10515	9171	10890	7895	8913	11428	7395	9152	10084	7539	7233	10040	11967	11967	9459
2nd	-	11037	9029	11485	10971	10689	10141	10956	10859	10477	10437	11762	10151	11014	9958	10595	12302	10150	10365	11097	9560	7997	10344	12765	12765	10610
3rd	-	10843	9672	11075	11117	10604	10210	11054	10263	10587	10312	9944	9548	10855	10297	10616	11511	9777	10099	9783	9741	7836	11072	11639	11639	10360

b Risks matrix (10^6 rubles)

actions	nature states																							R^{max}
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	
1st	0	0	0	66	0	0	0	0	0	0	571	0	35	0	0	0	0	0	301	0	0	0	328	571
2nd	1742	2198	1105	0	732	1023	300	932	2395	391	1818	980	159	2063	1622	874	2755	1213	1314	2021	764	304	1126	2755
3rd	1548	2841	695	146	647	1092	398	336	2509	257	0	377	0	2402	1703	83	2382	947	0	2202	603	1032	0	2841

by Wald's and Laplace's criteria are written for all three actions. The risks matrix calculated on the basis of the pay-off matrix is given in Fig. 8b, where the maximum risk values for each action are also shown in the rightmost column.

During the subsequent solution of the problem the figures of these two tables were used for the comparison of the actions and the choice of rational priority power plants.

IX.3. Analysis of the Pay-Off Matrix and Determination of Rational Actions

The application of decision criteria to the pay-off matrix shows that the first action is optimal with Laplace's and Savage's criteria and the third with Wald's criterion (see the rightmost columns of Fig.8a and 8b). The second action is non-rational. It is worse than the first action by all indicators used ($\bar{E}_i$, E_i^{max}, R_i^{max}).

Since there happen to be two rational actions, an additional analysis has to be made. There had been no particular information about probabilities of different nature states. Therefore the analysis linked to calculation of the matrix of mathematical expecations of expenditure was not made. But the "overexpenditures" O_{iL}, O_{iW}, O_{iS} by the corresponding criteria have been calculated. They were determined accordingly with the formulae (13-VIII) and (14-VIII) but instead of mathematical expectations M_{ic} the characteristic values of expenditures ($\bar{E}_i$, E_i^{max}, R_i^{max}) were used. The matrix of "overexpenditures" obtained has the following figures:

Actions	$\bar{E}_i$	E_i^{max}	R_i^{max}	O_{iL}	O_{iW}	O_{iS}	$\bar{O}_i$	O_i^{max}
1st	[9459]	11967	[571]	0	328	0	[109]	[328]
3rd	10360	[11639]	2841	901	0	2270	1057	2270

One can see that the first action proved optimal by using the "overexpenditure" matrix for both the principles of "insufficient reason" (15-VIII) and minimax (16-VIII) (see last two columns). Therefore this action might be recommended for implementation with some confidence. There are also two more reasons for such a recommendation. First, this first action is optimal under a majority of nature states---under 18 from 23 (see zeros in Fig. 8b). And second, one can see in Fig.8a that all maximum values of expenditures (E_i^{max}) take place under the 23rd nature state which is the most unfavorable one. Therefore orientation toward Wald's criterion according to which the third action is optimal, means in fact orientation toward the worst conditions; and this seems to be too cautious (conservative) a decision.

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