# A TACTICAL LOBBYING GAME*

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## Preface

An economic approach to the problem of measuring the relative "power" or "value" of members in a decision making body is to ask how much the various members would be paid by outside agents trying to manipulate the outcome. A particular case is that in which there are two competing agents, or lobbyists, on opposite sides of the issue. A model of competition is developed that leads to a class of two-person zero sum games that are related to so-called Colonel Blotto games. These have potential application to a wide variety of problems in which two opponents compete for the control of certain targets.

Summary

Some basic solution properties and anomalies are investigated for a class of tactical games related to so-called Colonel Blotto games. In this model two agents compete for control of players in a given n-person simple game. It is shown that equilibrium solutions—even in mixed strategies—do not always exist. The case where the opponents have substantially unequal resources is solved and shown to attribute values to the players in the original n-person game that are in the least core. Some approximate values for particular cases where resources are equal are also cited.

A Tactical Lobbying Game

Two lobbyists, having equal budgets, approach a legislature (or a committee) with the idea of buying votes. We imagine that the lobbyists are on opposite sides of an issue: A wants the measure to pass, B wants it to fail. Lobbyist A offers amount $x_i$ to voter i and B offers $y_i$, subject to the budget constraints $\sum x_i \leqq a$ and $\sum y_i \leqq a$ $(a > 0)$. We assume that the budgets are completely divisible. If $x_i > y_i$ then A gains control of voter i, if $y_i > x_i$ then B gains control of i, and if $x_i = y_i$ there is a standoff or "tie" for that voter. Let the set of voters be designated by $N = \{1,2,\ldots,n\}$, and the voting procedure by the collection $\mathcal{S}$ of *winning sets*. We adopt the usual convention that

$$N \in \mathcal{S} \ , \quad \phi \notin \mathcal{S} \ ,$$

$$\text{and} \quad S \in \mathcal{S} \quad \text{and} \quad S \subseteq T \quad \text{implies} \quad T \in \mathcal{S} \ .$$

Then A *wins* (and B *loses*) if $\{i \in N : x_i > y_i\} \in \mathcal{S}$; similarly B *wins* (and A *loses*) if $\{i \in N : x_i \geqq y_i\} \notin \mathcal{S}$. We attach a value of +1 to winning, -1 to losing. If there is a tie for some voter, we may consider that he will go either way with a 50-50 probability. Thus, given payment vectors $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$ and $\underset{\sim}{y} = (y_1, y_2, \ldots, y_n)$ by A and B respectively, let $w_A$ be the number of *winning*, $w_B$ the number of *losing*, sets S such that $x_i \geqq y_i$ for all $i \in S$ and $x_i \leqq y_i$ for all $i \notin S$.† Then the expected payoffs to A and B will be

$$v(\underset{\sim}{x},\underset{\sim}{y}) = (w_A - w_B)/(w_A + w_B) \quad \text{and} \quad -v(\underset{\sim}{x},\underset{\sim}{y}) = (w_B - w_A)/(w_A + w_B) \quad .$$

Strictly speaking payoffs should be taken *net* of costs. However, in the context of lobbying, winning or losing is

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†The author wishes to thank R.J. Weber for pointing out a slip in an earlier version of this definition [8].

assumed to be of incomparably greater value than the prices paid, so that for most purposes the costs can be suppressed.

We call the games defined by the payoff function (1) *tactical lobbying games*. This class of games was introduced in [8] as a specific model of political lobbying; and also as a general model for imputing *values* to the voters in a voting game (i.e., a simple game). A related formulation of a lobbying game that applies to a whole legislative session rather than "bill by bill" is developed in [5]. When only one lobbyist attempts to buy votes a quite different model and a different concept of value result [6].

The purpose of this note is to point out several interesting features of tactical lobbying games, and to mention some unsolved problems regarding them.

The first example of a tactical lobbying game seems to have been considered by Borel [1], who formulated it as a problem of defense. In Borel's example, three "points" are to be defended against an "aggressor", each deploying the same number of forces. The aggressor's objective may be formulated in either of two ways: (i) maximize the expected number of points captured; or (ii) maximize the expectation that a majority of points are captured. For three points, and with equal budgets, these objectives amount essentially to the same thing, since (except for ties) each player will capture at least one but no more than two points.

Subsequently, this type of game was generalized, using the *first* objective only, by Tukey and others [2,3,4], to so-called "Colonel Blotto" games. In Colonel Blotto games, a weight $w_i$ is associated with each point and the objective is to maximize the total expected weight of the points captured. Gross [2] and Gross and Wagner [3] showed that such games always have an equilibrium solution (in mixed strategies) and gave various methods for constructing solutions. Unfortunately, except for special cases (as in that mentioned above of three points of equal weight) these solutions do not work for lobbying games.

Indeed, lobbying games may be viewed as a different way of generalizing Borel's original idea by using the *second* type of objective defined above, and their behavior appears to be quite different than that of Colonel Blotto games. The most essential difference from Colonel Blotto games is, as we shall presently show, that *an equilibrium solution does not always exist.* This situation can arise in certain cases where the roles of the "aggressor" and the "defender" are not symmetric. (Note that Colonel Blotto games are always symmetric for the two protagonists, given that they have equal resources.)

To illustrate what sort of behavior can be encountered in lobbying games, we will consider various simple games $\Gamma$ on three voters. Let the three voters be denoted 1,2,3. A *mixed strategy* of a lobbyist will be a probability distribution represented by a probability measure $\mu$ on the Borel sets in the simplex

$$\underset{\sim}{X} = \left\{ \underset{\sim}{x} = (x_1, x_2, x_3) \geqq 0 \,,\ \sum x_i = 1 \right\} \quad , \quad \mu(\underset{\sim}{X}) = 1 \quad .$$

The payoff function $v(\underset{\sim}{x},\underset{\sim}{y})$ for this class of games is bounded and Borel measurable so the integrals $\int_{\underset{\sim}{X}} v(\underset{\sim}{x},\underset{\sim}{y})d\mu$ are defined. We say $(\mu,\nu)$ is an *equilibrium pair* of mixed strategies if for every $\underset{\sim}{x}^0$ and $\underset{\sim}{y}^0 \in \underset{\sim}{X}$ we have

$$\int_{\underset{\sim}{X}} v(\underset{\sim}{x},\underset{\sim}{y}^0)\, d\mu \geqq \int_{\underset{\sim}{X} \times \underset{\sim}{X}} v(x,y)d(\mu \times \nu) \geqq \int_{\underset{\sim}{X}} v(\underset{\sim}{x}^0,\underset{\sim}{y})\, d\nu \quad .$$

If voter 1 is a *dictator*, i.e., if 1's assent is necessary and sufficient to pass a measure, then all the other voters are dummies and nothing can be gained by bribing them. Hence the unique equilibrium solution is for both lobbyists A and B to spend all their resources on voter 1, and the game has a solution in pure strategies. Clearly the same result holds whenever the voting game $\Gamma$ has a dictator.

Suppose on the other hand that the assent of all three voters is required to pass. Then lobbyist A has a much more difficult job to succeed than does B. Indeed, no matter what pure strategy A uses he will always lose at least one voter *unless* it happens that B is using exactly the *same* pure strategy. Therefore it is clear that B should employ a measure $\nu$ on $\underset{\sim}{X}$ such that the measure of any single point is zero. In that case any strategy of A will fail with probability 1, so that the game has an equilibrium solution (in fact, a multiplicity of equilibria) consisting of a mixed strategy for B and a pure strategy for A (namely $\underset{\sim}{x} = \underset{\sim}{0}$).

We say that a voter is a *veto player* if his assent is necessary to win, that is, if he is contained in *every* winning set. Generalizing from the above example, if there are two or more veto players, then the lobbying game has an equilibrium solution in which B wins with probability 1. The three-person game having two veto players and one dummy is handled in this way.

If $\Gamma$ is the game of simple majority rule on three voters, then the lobbying game is symmetric for A and B and there are an infinite number of different equilibrium solutions, as pointed out by Gross and Wagner [3]. One such solution is to erect a hemisphere on a circle inscribed in $\underset{\sim}{X}$ and to consider a uniform distribution on the hemisphere: the projection of this onto $\underset{\sim}{X}$ constitutes an equilibrium mixed strategy for both lobbyists (Figure 1). Other solutions exist, however, in which the subset of pure strategies used has an arbitrarily small area [3].

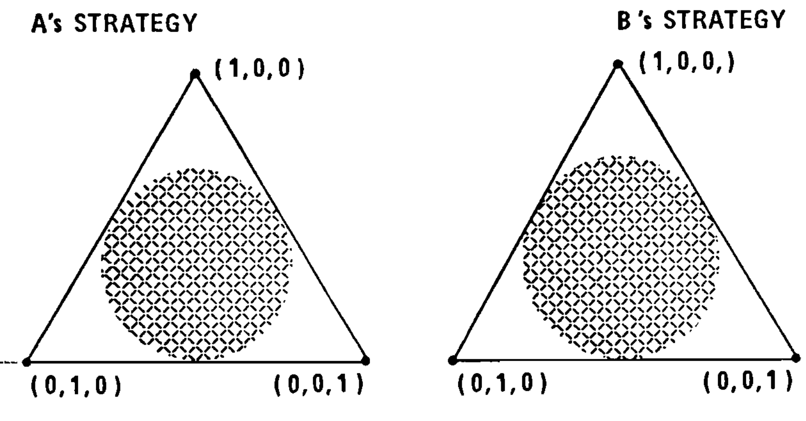


Figure 1

The only other case to consider for three-person games is that where the minimal winning sets have the form {1,2} and {1,3}. Then voter 1 is a veto player but not a dictator, and the associated lobbying game is not symmetric for A and B. In this case we assert that *no* equilibrium solution exists even in mixed strategies. The reason for this is, roughly speaking, that both B and A want to spend an arbitrarily large part of their resources on voter 1, but not *all* their resources. Suppose, in fact, that there is an equilibrium, and that A succeeds with probability p. For any pure strategy of B, such as $\underset{\sim}{y}^1$ in Figure 2, A wins only with strategies in the shaded area; hence A plays strategies in this area with probability at least p.

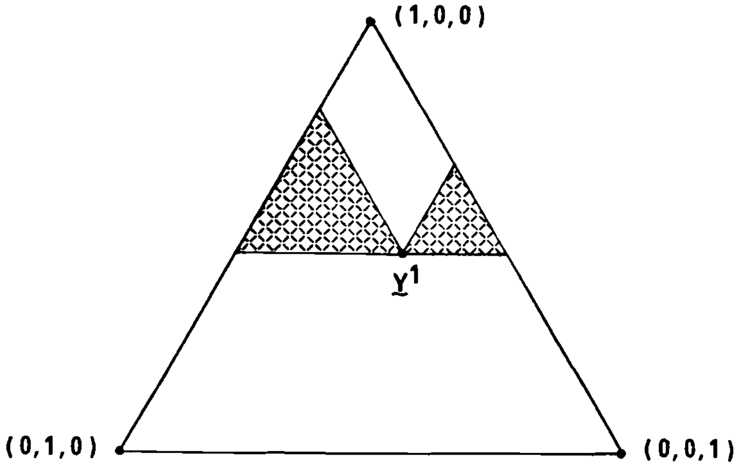


**Figure 2**

But a similar analysis holds for $\underset{\sim}{y}^2$ (Figure 3). In fact we may find an infinite sequence of points converging to (1,0,0) such that the corresponding shaded areas are *disjoint*.

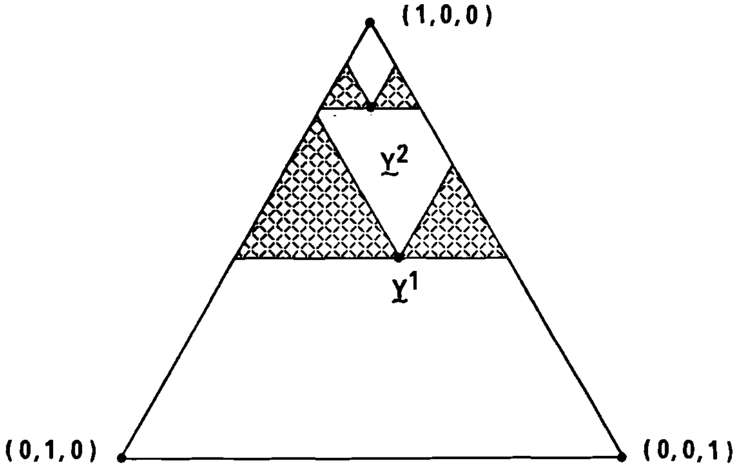


**Figure 3**

Thus p = 0. But for any mixed strategy of B, A always has an opposing strategy that *does* succeed with positive probability. Hence there can be no equilibrium. The general proof of non-existence when there is a single veto player who is not a dictator is given below.

Our observations may be summarized in the following theorem.

*Theorem 1* Let Γ be a simple game, G the associated tactical lobbying game.

(i) If Γ has a dictator i then $x_i = y_i = 1$ gives an equilibrium solution for G in pure strategies.

(ii) If Γ has a set V of at least two veto players then $(\underset{\sim}{0}, \nu)$ is an equilibrium pair for G for any ν satisfying $\nu(\{\underset{\sim}{y} \in \underset{\sim}{X} : y_i = 0 \text{ for all } i \notin V\}) = 1$ and $\nu(\underset{\sim}{y}) = 0$ for all $\underset{\sim}{y} \in \underset{\sim}{X}$.

(iii) If Γ has exactly one veto player who is not a dictator, then there is no equilibrium solution for G.

*Proof:* The truth of (i) and (ii) was noted above, so it remains only to establish (iii).

Let $\Gamma$ be a voting game on voters $\{1,2,...,n\}$ and suppose that voter 1 is a veto player but not a dictator. Suppose further, by way of contradiction, that $(\mu,\nu)$ is an equilibrium pair for the associated lobbying game G. Let p be A's expectation of success under this pair.[†] Suppose first that $p > 0$. For each $\varepsilon > 0$ define $y^{\varepsilon} = (1-\varepsilon, \varepsilon/(n-1), ..., \varepsilon/(n-1))$ and let $D_{\varepsilon}$ be the set of all $\underset{\sim}{x} \in \underset{\sim}{X}$ such that if A plays $\underset{\sim}{x}$ and B plays $\underset{\sim}{y}^{\varepsilon}$, then A has a positive expectation of success. Evidently,

$$(1) \qquad \mu(D_{\varepsilon}) \geqq p \quad .$$

Moreover, we must have for any such $\underset{\sim}{x}$ that

$$x_1 \geqq y_1^{\varepsilon}$$

and

$$x_j \geqq y_j^{\varepsilon} = \varepsilon/(n-1) \quad \text{for at least one } j \neq 1 \quad .$$

Since $\sum x_i = 1$, it follows that

$$x_1 \leqq 1 - \varepsilon/(n-1) < 1 - \varepsilon/n \quad \text{for all } \underset{\sim}{x} \in D_{\varepsilon} \quad .$$

In particular,

$$(2) \qquad D_{\varepsilon} \cap D_{\varepsilon/n} = \emptyset \quad .$$

But then by (1) the measure of the set $D_{\varepsilon} \cup D_{\varepsilon/n} \cup D_{\varepsilon/n^2} \cup \ldots$ is unbounded, a contradiction. Hence $p = 0$.

Now consider B's mixed strategy $\nu$. Let $E_{\varepsilon} = \{\underset{\sim}{y} : y_1 \leqq 1-\varepsilon\}$ and suppose that for some $\varepsilon > 0$,

$$(3) \qquad \nu(E_{\varepsilon}) > 0 \quad .$$

[†]This is generally different than A's expected numerical payoff.

Let $\mu^*$ correspond to the uniform density. Given any $\underset{\sim}{y} \in E_\varepsilon$ let $y_k$ be the maximum among all $y_i$, $i \neq 1$. Then $y_k \geqq \varepsilon/(n-1)$. Since 1 is the *unique* veto player, there is a winning set S *not* containing k. We may therefore construct various $\underset{\sim}{x}$ such that $x_i > y_i$ for all $i \in S$. In fact, let $\delta = \varepsilon/(n-1)^2$ and

$$Z_{\underset{\sim}{y}} = \left\{\underset{\sim}{x}: y_i < x_i \leqq y_i + \delta \quad \text{for all} \quad i \neq k\right\} .$$

Then $\mu^*(Z_{\underset{\sim}{y}}) \geqq c\,\delta^{n-1} > 0$ for some suitable constant $c > 0$. Thus if A plays instead the mixed strategy $\mu^*$, then A succeeds with probability at least $c\delta^{n-1} > 0$ against *any* $y \in E_\varepsilon$. Since by assumption B plays in $E_\varepsilon$ with probability $\nu(E_\varepsilon) > 0$, it follows that $(\mu^*,\nu)$ gives a positive probability of success to A, contradicting the fact that $(\mu,\nu)$ is an equilibrium.

Therefore $\nu(E_\varepsilon) = 0$ for all $\varepsilon > 0$. But then $\nu(1,0,\ldots,0) = 1$, so A can switch from $\mu$ and play the pure strategy $(1,0,\ldots,0)$ with a positive probability of success (because of our convention on resolving ties), again contradicting the fact that $(\mu,\nu)$ is an equilibrium. Hence no equilibrium exists.□

If an equilibrium solution $(\mu,\nu)$ to the tactical lobbying game *does* exist, values may be imputed to the voters in the voting game by considering the expected amounts they are offered,

$$\text{(4)} \qquad \text{value of } i = \int_{\underset{\sim}{X}} x_i \, d\mu + \int_{\underset{\sim}{X}} y_i \, d\nu \quad .$$

This value, when it exists, will be called a *noncooperative value* of the underlying voting game. Notice that when there is a dictator, all the value is ascribed to him. In the case of two or more veto players, however, the value is in some sense indeterminate. In the case of one veto player who is not a dictator, the value is undefined, although a case can be made that in some sense the veto player should receive an *arbitrarily large share* of the total value, though not all of it. Except for the case when there is a multiplicity of veto players the value of a dummy is zero, whenever the value is defined.

One problem with this value is that the explicit computation of equilibrium solutions for any but the smallest games seems to be difficult. A second problem, already noted with respect to games having a multiplicity of veto players, is that there may be several equilibrium solutions that yield different noncooperative values.

It is worthwhile to note that if $(v_1, v_2, \ldots, v_n) = \underset{\sim}{v}$ is a noncooperative value for $\Gamma$ as in (4), say $\underset{\sim}{v}$ is obtained from equilibrium pair $(\mu, \nu)$, and if $\underset{\sim}{v}'$ is another value obtained from $(\mu', \nu')$, then $(\lambda\mu + (1-\lambda)\mu', \lambda\nu + (1-\lambda)\nu')$ is also an equilibrium pair, so $\lambda\underset{\sim}{v} + (1-\lambda)\underset{\sim}{v}'$ is also a noncooperative value. Thus the collection of all noncooperative values for $\Gamma$ forms a *convex set*. As in the case of the core and other solution concepts, any imputation in this set could be considered in some sense as a plausible imputation of value to the players for the game $\Gamma$. However, notice that the set of noncooperative values may not be a *closed* convex set. This possibility is illustrated by the three-person voting game requiring unanimity for A, where (up to a multiple) *any* vector except those that ascribe all value to one player is a noncooperative value.

Instead of attempting to compute the noncooperative value explicitly by integration, a useful approach is to approximate the continuous strategy spaces by a finite grid. A natural way to do this is by allocating a large but finite number u of *indivisible* units to each lobbyist that may be distributed among the voters. This leads to a two-person, zero-sum matrix game whose equilibrium solutions were used to approximate the equilibrium solution to the infinite lobbying game.

For comparison with the Shapley-Shubik and the Banzhaf values two voting games were investigated: the weighted voting game on four players (2,1,1,1) with quota 3, and the weighted voting game on five players (3,1,1,1,1) with quota 4. In both, the minimal winning coalitions are the same as the minimal blocking coalitions, so the associated tactical lobbying game is in each case a symmetric two-person game.

The noncooperative value found for the game (2,1,1,1) is (1/2,1/6,1/6,1/6), which is the same as both the Banzhaf and the Shapley-Shubik values.†

For the game (3,1,1,1,1) the Shapley-Shubik value is (.60,.10,.10,.10,.10) and the Banzhaf value is (.64,.09,.09,.09,.09). For this game the noncooperative value is approximately (.56,.11,.11,.11,.11). There may, in fact, be more than one noncooperative value for this game, but whether any of them corresponds to the Shapley-Shubik or the Banzhaf values is not known.

In any case, the value that would be attributed to the players in a voting game by two lobbyists competing for their votes appears to be a new concept that differs from both the Shapley-Shubik and the Banzhaf values. The value is not defined for a game with a single veto player who is not a dictator. However, we conjecture that this is the *only* situation in which it does not exist. The value (or values) are difficult to compute precisely but techniques exist for finding very good approximations.

It is perhaps also worth pointing out that there is a situation, for any game $\Gamma$, in which a solution in *pure* strategies obtains -- namely, when one lobbyist has considerably more resources than the other. In fact, if B has sufficiently more resources than A that he can completely prevent A from succeeding, then in the limit his most "efficient" strategy is to allocate his resources such that

$$\sum y_i = \min$$

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†This finding disproves a conjecture made in [7], which is an earlier version of [8].

subject to

$$\sum_{i \in S} y_i \geqq a \qquad \text{for all } S \in \mathcal{S},$$

$$\underset{\sim}{y} \geqq 0 \quad .$$

This value turns out to be the least core [8].

For both of the weighted voting games studied above, the least core equals the nucleolus, which is *proportional* to the weights. Thus, it seems that as the relative resources of the two lobbyists change, the relative values attributed to the voters also change. In particular, for the above two games the value of the most powerful voter increases substantially relative to the others as the lobbyists' resources approach equality.

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