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On Feedback Identification of Unknown Biochemical Characteristics in an Artificial Lake

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Abstract

The problem of dynamical identification of unknown characteristics (states/parameters) in a biochemical model of an artificial lake with only inflow and given observations of some states is considered. An algorithm that solves this simultaneous state and parameter estimation problem and that is stable with respect to bounded informational noises and computational errors is presented. The algorithm is based on the principle of auxiliary models with adaptive controls. Convergence of the algorithm is proven and a convergence rate is derived. The performance of the algorithm is illustrated to a typical single-species environmental example.

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Keywords

State/parameter estimation, bounded error, dynamical inversion, convergence, biochemical model, environment

Contents

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1 Introduction

The problem of dynamical identification on the basis of available information on a specific object is well known in engineering and scientific researches (*e.g.* [1]–[5]). For certain modern applications there is the necessity to reconstruct the unknown characteristics, system states and/or parameters, dynamically and preferably in real time (see *e.g.* [6]–[7], for biotechnological applications). Our goal now is to present an algorithm for the identification of unknown process characteristics using unknown-but-bounded data and to apply it to a typical single-species environmental system.

Let us describe our problem in some more detail. Let, therefore, a dynamical system be described by differential equations on a given bounded interval of time. Let, furthermore, some of the state trajectories of the system depend on a time-varying parameter, which in what follows is considered as an *input* with bounded non-homogeneity. *A priori* for both the state trajectories and input (time-varying parameter) only a *set* that contains admissible realizations is known. It is assumed that some of the system states are directly observed and that these observations are inaccurate. Hence, we are looking for an algorithm that approximately reconstructs the input (time-varying parameter) and unobservable states and which is both dynamical and stable. The algorithm is dynamical when the current values of input and states are produced on-line, *i.e.* in real time, so that the current values can be used for decision making during the process. The algorithm is stable when an approximation is as precise as one likes under sufficient accuracy of the observations.

The problem is treated within the class of inverse problems of dynamics of controlled systems. In a more general context, we can say that it is embedded in the theory of ill-posed problems. The modifications of this problem in *a posteriori* formulations have been solved in, for example [8]–[10]. The solution presented here follows the theory of stable dynamical inversion developed in [11]–[18], where a combination of the methods from the theory of ill-posed problems [10] and from the theory of positional control [19] was used. The essence of the approach applied here is that the state/parameter estimation algorithm is represented as a control algorithm of some artificial dynamical system (a model). Given current observations of the system the control input to the model is adapted such that

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its realization in time is subjected to some regularization principle, which guarantees the stability of the algorithm.

So, the essence of the problem of dynamical identification may be described in the following way. There is a dynamical system S functioning on a time interval $T = [t_0, \vartheta]$. Its trajectory $x(t) = x(t; x_0, \mu(t)) \in \mathbb{R}^q$, $t \in T$ depends on an unknown time-varying input $\mu(t) \in P$. Here $P \subset L_2(T; \mathbb{R}^N)$ is the set of admissible inputs. On the interval T, a uniform net $\Delta = \{\tau_i\}_{i=0}^m$ with a step δ is chosen, where $\tau_0 = t_0$, $\tau_{i+1} = \tau_i + \delta$, $\tau_m = \vartheta$. An output $y(t) = Cx(t)$ is measured at the time instants τ_i (C is an $r \times q$-dimensional matrix). The inaccurate measurement vector $\xi_i = \xi(\tau_i) \in \mathbb{R}^r$ satisfies the inequality

$$\|\xi_i - y(\tau_i)\| \leq h, \quad i \in [0 : m-1],$$

where h is the error bound. It is required to design an algorithm which allows us to reconstruct some pair $(\varphi(t), w(t))$ synchro with the process. This pair must be "close" to the pairs $(\mu(t), x(t))$ compatible with the output $y(t)$. In Fig.1 the scheme of solving algorithms, which is stable with respect to informational noises and computational errors, is shown. According to the scheme, the system S is accompanied by a certain artificial computer-modeled closed-loop control system (a model M). This model, functioning on the time interval T, has an unknown input (control) $\varphi^h(t)$ and an output $w^h(t)$. The model M can be given *a priori* or can be constructed. The process of synchronous feedback control of the systems S and M is organized on the interval T. This process is decomposed into $m-1$ identical steps. At the i-th step carried out on the time interval $\delta_i = [\tau_i, \tau_{i+1})$, the following actions are performed. First, at time instant τ_i, according to a chosen rule φ^h, the control $\varphi^h(t) = \varphi^h(\tau_i, \xi_0, \ldots, \xi_i, w^h(\tau_0), \ldots, w^h(\tau_i))$, $t \in [\tau_i, \tau_{i+1})$ is calculated. Then (till the moment τ_{i+1}) the control $\varphi^h(t)$, $\tau_i \leq t < \tau_{i+1}$, is put into the system M. The values of $\varphi^h(\tau_i)$ and $w^h(\tau_{i+1})$ result from the algorithm at the i-th step. Thus, all complexity of solving the problem is reduced to the appropriate choice of the model M and the function φ^h.

In essence, the procedure for solving the problem of dynamical identification is equivalent to the procedure for solving the following two problems:

1. the problem of choosing a model M
2. the problem of choosing some rule φ^h for forming a control in the model.

Note that a number of factors play an important role for solving problems (i) and (ii). For example, among these factors are the prior information on the structure of S (form of the equation(s), solution properties and so on), the properties of the set of admissible inputs P, the structure of output y (*e.g.*, the properties of matrix C) and so on.

A more specific implementation of the scheme described above has been developed in [11–18]. In particular, Chapter V of [11] is devoted to the investigation of the discussed problem of dynamical identification for one, wide enough, class of systems S described by a vector nonlinear ordinary differential equation of the form

$$\dot{x}(t) = f_1(t, x(t)) + f_2(t, x(t))\mu(t), \quad t \in [0, \vartheta],$$

$$x(0) = x_0, \quad x \in \mathbb{R}^q, \quad u \in \mathbb{R}^N.$$

In this chapter, procedures for choosing the model M and for forming the control φ^h, in the case where all or some coordinates of x are measured, have been developed. In [13] three more rules for choosing models and control laws have been suggested. Two of them allow us to find functions $\varphi^h(t)$ that are weakly convergent (in L_2-metrics) to an unknown input $\mu(t)$ as $h \to 0$. The algorithm provides strong convergence in the case where all states coordinates are measured. The papers [12, 14, 15, 17] have been devoted to the investigation (from the viewpoint of the approach under discussion) of dynamical identification problems for parabolic and hyperbolic distributed parameter systems. In particular, in [14, 15], given measurements of pollutant concentration at fixed domains, the problem of reconstructing point-wise sources intensities is considered. In these studies, the system is supposed to be described by diffusion type of equations. In [16], a problem of "compensation" of disturbances is solved on the base of controlled models. In particular, the solving algorithm is an "identification–control" algorithm with the synchro ("in real time") functioning blocks "dynamical identifier" and "controller". In [18], the problem of reconstructing a right-hand part of one system in a Hilbert space unsolved with respect to the derivative has been investigated.

In the present report, the approach presented in [11–18] is used for solving the identification problem of unknown characteristics (states/parameters) in a biochemical model of an artificial lake with only inflow and given bounded-noise observations of some states. Unfortunately, it is impossible to directly apply the algorithms from the studies cited above to solve our problem (with the exception of one, see Remark 1 in section 3). However, taking into account the specific form of the environmental system, in the next section we will show how to apply the scheme presented in Fig.1. In this case we consider a system of nonlinear ordinary differential equations of the third order with an unknown time-varying scalar input entering the right-hand side of two equations and where only the first state coordinate is inaccurately measured under bounded noise. The identification objective is to reconstruct the other two coordinates and the unknown input/time-varying parameter.

Alternatively, for the stochastic noise case an Extended Kalman Filtering (EKF) approach could have been chosen to solve this simultaneous state and parameter estimation problem, but it is well-known that the EKF will not guarantee convergence. As a solution to this, in the 90's, for specific classes of systems modifications based on regularization theory have been suggested (see *e.g.* [20] and [21]). In the last decade computational Bayesian methods, generally requiring a large computational effort, have also been introduced to solve the state/parameter estimation problem (see *e.g.* [23]). Most recently, a set-membership solution to this problem has been proposed [22].

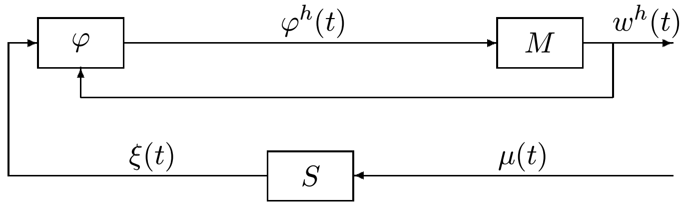

Figure 1: Scheme of solving algorithms for the dynamical identification problem.

In this paper the scheme of Fig. 1 is, in particular, applied to the dynamical identification of biochemical parameters and states in an artificial lake with only inflow on the basis of the model introduced in section 2. In section 3 the problem is further formulated in mathematical terms and the solution algorithm is described. Section 4 contains the proof of the convergence theorem via Lemmas 1-6 and a more general Theorem 1.2.1 from [12]. The algorithm's convergence rate is derived in section 5. The algorithm is further illustrated in section 6 by applying it to the environmental system introduced in section 2. Finally, the conclusions are stated in section 7.

2 A Dynamical Model of an Artificial Lake

The case study we wish to address here has a close relationship with the previous work by Vanrolleghem and van Daele [24] and the recent work by Stigter *et al.* [25], which both focussed on optimal experiment design in bioreactor modeling (see also [26] and [27] for further details). This paper can then be considered as a follow-up on this work and an extension towards environmental applications. Recall, once again, that (speaking in general terms) we want to determine the input/time-varying parameter $\mu(t)$ and some states of S (see Fig. 1) from experimental data with unknown-but-bounded error. Hereto we first define our system S.

Let us, without loss of generality, assume that the dissolved oxygen concentration in the inflow of the system is at saturation level, *i.e.* it is not affected by bacteria which are assumed not to be present in the inflow. The consumption of substrate by the bacteria in the system is aerobic and directly affects the dissolved oxygen concentration in the system. The following *non-linear* dynamic model describes the biochemical processes in S under ideally mixed conditions,

$$\begin{aligned}
\frac{\mathrm{d}\,C_{DO}(t)}{\mathrm{d}\,t} &= k_{La}\left[C_{sat}^{en} - C_{DO}(t)\right] - OUR(t) + \frac{F_{in}(t)}{V(t)}\left[C_{sat} - C_{DO}(t)\right], \\
\frac{\mathrm{d}\,C_X(t)}{\mathrm{d}\,t} &= \mu(C_S(t))C_X(t) - \frac{F_{in}(t)}{V(t)}C_X(t), \qquad (1)\\
\frac{\mathrm{d}\,C_S(t)}{\mathrm{d}\,t} &= -\frac{\mu(C_S(t))}{Y}C_X(t) + \frac{F_{in}(t)}{V(t)}(C_{S,in}(t) - C_S(t)),
\end{aligned}$$

where

$$\begin{aligned}
V(t) &= \int_0^t F_{in}(\tau)\,d\tau + V(t_0), \\
OUR(t) &= \frac{(1-Y)}{Y}\mu(C_S(t))C_X(t), \\
\mu(C_S(t)) &= \mu_{max}(t)\frac{C_S(t)}{K_S + C_S(t)}, \quad t \in T = [t_0, \vartheta).
\end{aligned}$$

Furthermore, k_{La} is the re-aeration coefficient, $V(t)$ is the volume of the lake, C_{sat}^{en} is the saturation concentration of dissolved oxygen, including a small (constant) correction for the endogenous respiration of the biomass, C_{sat} is the (normal) saturation concentration of dissolved oxygen in the inflow, $\mu_{max}(t)$ is the maximum specific growth rate, K_S is the half-saturation constant, Y is the yield coefficient of biomass on substrate, $OUR(t)$ is the oxygen uptake rate of the biomass in the lake, $C_{DO}(t)$ is the dissolved oxygen concentration in the lake, $C_X(t)$ is the biomass concentration, and $C_S(t)$ is the biomass growth rate. Notice that in (1) $\mu_{max}(t)$ is a time-varying parameter. Most often, it varies due to adaptation of the organisms, additional substrate limitations or, in general, to kinetic modeling errors.

For a further interpretation of our system we note that the first equation in (1) describes the dissolved oxygen concentration in the system, where the first term on the right-hand side presents the re-aeration, the second term the oxygen consumption by the aerobic biomass and the last term the inflow and dilution of dissolved oxygen. The second equation in (1) describes the biomass dynamics, where the first term on the right-hand side describes the biomass growth and the last term the dilution. This growth term with some yield coefficient (Y) can also be found in the third equation in (1), but then as a consumption term in the substrate balance. It is furthermore assumed that there are *dissolved oxygen data only* and no biomass nor substrate data. In the sequel we will thus focus on the simultaneous estimation of the parameter $\mu_{max}(t)$, the biomass concentration $C_X(t)$ and the biomass growth rate $C_S(t)$ through the measurements of the dissolved oxygen concentration $C_{DO}(t)$ with point-wise bounded error. Hence, we will assume that the parameters $V(t_0)$, Y, K_S, $C_X(t_0)$, $C_S(t_0)$, C_{sat}^{en}, C_{sat}, k_{La} and functions $F_{in}(t)$, $C_{S,in}(t)$ have already been estimated off-line or have been established from literature.

3 Problem Formulation and Description of Solution Algorithm

The problem in question can be more specifically formulated as follows. An unknown function $\mu_{max}(t)$ acts on the system (1) generating an unknown solution $C(t) = C(t; C_0, \mu_{max}) = (C_{DO}(t), C_X(t), C_S(t))$. Here $C_0 = (C_{DO}(t_0), C_X(t_0), C_S(t_0))$ is an initial state. The time interval T is put into parts by subintervals $[\tau_i, \tau_{i+1})$, $\tau_{i+1} = \tau_i + \delta$, $\delta > 0$, $i \in [0 : m]$, $\tau_0 = t_0$, $\tau_m = \vartheta$. At the time instants τ_i the elements $C_{DO}(\tau_i)$ are measured inaccurately, *i.e.* $\xi_i^h = \xi(\tau_i) \in \mathbb{R}$, such that

$$|C_{DO}(\tau_i) - \xi_i^h| \leq h \tag{2}$$

for $i = 1, \dots, m$, are given. Herein, the symbol $|x|$ denotes the absolute value of a number x. An algorithm calculating the function $v(t) = v^h(t)$ and the function $w^h(t) = \{w_1^h(t), w_2^h(t)\}$ being approximations of $\mu_{max}(t)$ and $C_X(t), C_S(t)$, respectively, has thus to be found.

Notice that the functions $C(t)$, $C_{DO}(t)$ and $\mu_{max}(t)$ correspond to the general functions $x(t)$, $y(t)$ and $\mu(t)$ in the Introduction. From now on, it is assumed that we know a real number $K \in (0, +\infty)$ such that the unknown functions $\mu_{max}(t)$ and $OUR(t)$ satisfies the following conditions:

$$OUR(t), \mu_{max}(t) \in \mathcal{L}_\infty(T; \mathbb{R}), \quad |OUR(t)| \leq K \text{ for almost all } t \in T. \tag{3}$$

Let the following condition be fulfilled.

Condition 1. *$b_0 \leq F_{in}(t) \leq b_1$ for almost all $t \in T$, $0 < b_0 \leq b_1$,*

$$Y \in (0,1), \quad V(t) \geq V_0 > 0, \quad C_{S,in}(t) \in \mathcal{C}^1(T; \mathbb{R}), \quad F_{in}(t) \in \mathcal{L}_\infty(T; \mathbb{R}),$$

$$C_X(t) \geq C_X > 0, \quad C_S(t) \geq C_S > 0.$$

Here, $\mathbb{R}$ denotes the set of all real numbers; $\mathcal{C}^1(T; \mathbb{R})$ is the space of continuously differentiable functions $x(t) : T \to \mathbb{R}$ with the norm

$$\|x(t)\|_{\mathcal{C}^1} = \max\{\max_{t \in T} |x(t)|, \max_{t \in T} |\dot{x}(t)|\};$$

$\mathcal{L}_\infty(T; \mathbb{R})$ is the space of Lebesque measurable functions $x(t) : T \to \mathbb{R}$ with the norm $\|x(t)\|_{\mathcal{L}_\infty} = \operatorname{vrai\,sup}_{t \in T} |x(t)|$. We assume that the initial states of the system $C_X(t_0)$, $C_S(t_0)$ and function $V(t)$ are known.

For solving our dynamic identification problem, we apply the adaptive control method proposed in [11]–[18] (see also the block scheme in Fig.1). Hereto, first a family

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}, \quad \tau_{h,0} = t_0, \quad \tau_{h,m_h} = \vartheta, \tag{4}$$

$$\tau_{h,i+1} = \tau_{h,i} + \delta(h), \quad m_h = (\vartheta - t_0)\delta^{-1}(h),$$

of partitions of the interval T with diameters

$$\delta(h) = \delta(\Delta_h), \quad \delta(h) \to 0 \quad \text{as} \quad h \to 0,$$

is defined, where the error bound h (as in what follows) is explicitly denoted in the indices. Furthermore, an auxiliary system M, functioning synchronically with the "real" system (1), is chosen. Noting that (i) $OUR(\tau_i) = \varphi_i^h$, (ii) $\frac{F_{in}(t)}{V(t)} = \frac{\dot V(t)}{V(t)} \approx \delta^{-1}\ln\left(\frac{V(\tau_{i+1})}{V(\tau_i)}\right)$, (iii) $\mu(C_S(\tau_i))C_X(\tau_i) = \psi_i^h$ and (iv) $F_{in}(\tau_i) \approx \frac{V_i - V_{i-1}}{\delta}$, the model M in Fig. 1 can be described by the *linear* system

$$\begin{aligned}
w_0^h(\tau_{i+1}) &= w_0^h(\tau_i) + \delta[k_{La}(C_{sat}^{en} - \xi_i^h) - \varphi_i^h] + \ln(V(\tau_{i+1})V^{-1}(\tau_i))(C_{sat} - \xi_i^h), \\
w_1^h(\tau_{i+1}) &= W(\tau_{i+1})C_X(t_0) + V^{-1}(\tau_{i+1})\delta\sum_{j=0}^{i} V(\tau_j)\psi_j^h, \\
w_2^h(\tau_{i+1}) &= W(\tau_{i+1})C_S(t_0) + C_{S,in}(\tau_{i+1}) - W(\tau_{i+1})C_{S,in}(t_0) - \\
&- V^{-1}(\tau_{i+1})\Big[\sum_{j=0}^{i}(C_{S,in}(\tau_{j+1}) - C_{S,in}(\tau_j))V(\tau_j) + \delta Y^{-1}\sum_{j=0}^{i} V(\tau_j)\psi_j^h\Big],
\end{aligned} \tag{5}$$

with the initial states

$$w_0^h(t_0) = \xi_0^h, \quad w_1^h(t_0) = C_X(t_0), \quad w_2^h(t_0) = C_S(t_0).$$

Here

$$\psi_i^h = Y\varphi_i^h/(1-Y), \quad \varphi_i^h = \varphi^h(\tau_i), \quad W(\tau_j) = V(t_0)/V(\tau_j)$$

and $w_0^h(t)$ is an approximation of $C_{DO}(t)$. Hence, the linear model M has a control input $\varphi^h(t)$ and output $w^h(t)$, *i.e.* where the vectors $w^h(\tau_i)$ are found from (5) and the rule for calculating φ_i^h is given below.

Let us describe the algorithm. Before time instant t_0, the value $h \in (0,1)$, the function

$$\alpha = \alpha(h) : (0,1) \to \mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$$

and the partition $\Delta = \Delta_h$ with diameter $\delta = \delta(\Delta_h)$ (see Eqn. (4)) and the model (5) are fixed. After that, a process of feedback control of the model M synchro with operation of system (1) is organized. This will constitute the essence of the identification algorithm. The work of the algorithm starting at time t_0 is decomposed into $m_h - 1$ steps. At the i-th step carried out during the time interval $\delta_i = [\tau_i, \tau_{i+1})$, the following actions take place. First, the control in the *linear* model M (*i.e.* the *non-linear* function $OUR(t)$ which contains the unknown time-varying parameter $\mu_{max}(t)$, see Introduction and text above)

$$\varphi^h(t) = \varphi_i^h, \quad t \in \delta_i \tag{6}$$

is calculated. This calculation is based on the following feedback principle (see Fig. 1) with tuning parameters $\alpha(h)$ and K,

$$\varphi_i^h = \varphi_i^h(\xi_i^h, w_0^h(\tau_i)) = \begin{cases} -s_i\alpha^{-1}(h), & \text{if} \quad |s_i| \le \alpha(h)K \\ -K\,\mathrm{sign}\, s_i, & \text{otherwise} \end{cases}$$

$$s_i = \xi_i^h - w_0^h(\tau_i).$$

After that, the phase state $w^h(\tau_{i+1})$ of the model at the moment τ_{i+1} is found by the equations in (5). At the same time, the function $v^h(t)$, an approximation of $\mu_{max}(t)$, defined by the rule

$$v^h(t) = v_i^h, \quad t \in [\tau_i, \tau_{i+1}),$$

with

$$v_i^h = v_i^h(\xi_i^h, \varphi_i^h, w^h(\tau_i)) = \frac{Y(K_S + w_2^h(\tau_i))\varphi_i^h}{(1-Y)w_1^h(\tau_i)w_2^h(\tau_i)},$$

is calculated using the expressions of $OUR(t)$ and $\mu(C_S(t))$, see Eqn. (1). The algorithm stops at time instant ϑ. The convergence of this algorithm is stated in Theorem 1, but first we introduce the following condition.

Condition 2. *There exist numbers $w_1 > 0$ and $w_2 > 0$, such that for all $h \in (0,1)$ and all $t \in T$ the following inequalities are true:*

$$w_1^h(t) \geq w_1, \quad w_2^h(t) \geq w_2.$$

Theorem 1. *Let Conditions 1, 2 and the convergence*

$$\alpha(h) \to 0, \quad (h + \delta(h))\alpha^{-1}(h) \to 0 \quad as \quad h \to 0 \tag{7}$$

hold. Then

$$\sup_{t\in T} |w_1^h(t) - C_X(t)| \to 0, \quad \sup_{t\in T} |w_2^h(t) - C_S(t)| \to 0,$$

$$\int_{t_0}^{\vartheta} |v^h(\tau) - \mu_{max}(\tau)|^2\, d\tau \to 0 \quad as \quad h \to 0.$$

As it is seen from this theorem, the function $v^h(t) = v^h(\xi(t), \varphi^h(t), w^h(t))$ can serve as "an appropriate" approximation of the unknown time-varying parameter $\mu_{max}(t)$ under a corresponding value of h, and the functions $w_1^h(t)$, $w_2^h(t)$ as the ones to $C_X(t)$ and $C_S(t)$, respectively. Furthermore, (7) defines required properties of the tuning parameter $\alpha(h)$.

Let us dwell on considerations underlying such a choice of the model and its control law. Model (5) is, in essence, a discrete approximation to system (1). Indeed, as can be seen from the proof given in the next section, in particular inequalities (12) (Lemma 4) and (16) (Lemma 6), if the control action $\varphi^h(t)$ (6) in the model is close to $OUR(t)$ in the mean square metric then the model trajectory $\{w_0^h(t), w_1^h(t), w_2^h(t)\}$ approximates the trajectory $\{C_{DO}(t), C_X(t), C_S(t)\}$ of system (1). In its turn, our way of choosing the control $\varphi^h(t)$ provides an insignificant increase of the functional $\varepsilon(t)$ (see Eqn. (40) in Appendix). This fact allows us to make a conclusion on the convergence of $\varphi(t)$ to $OUR(t)$ (Theorem 2).

Remark 1. *One of the methods for solving the problem of dynamical identification applicable to the case discussed above is given in [11, sections 17, 19]. Let us describe it briefly. On a finite time interval $T = [t_0, \vartheta]$, a system of ordinary differential equations*

$$\dot{y}(t) = f(t, y, z, v),$$

$$\dot{z}(t) = g(t, y, z),$$

where y, z and v are vectors of corresponding dimensions, is considered. At time instants $\tau_i \in T$ ($\tau_0 = t_0$, $\tau_{i+1} = \tau_i + \delta$, $\delta > 0$), the values of $z(\tau_i)$ are inaccurately measured. The resulting measurement vectors $\xi(\tau_i)$ satisfy the inequality

$$\|\xi(\tau_i) - z(\tau_i)\| \le h.$$

The objective is to reconstruct the unmeasured state coordinates $y = y(t)$ and the unknown input $v = v(t) \in Q$ (Q is a convex, bounded and closed set).

To solve this dynamical identification problem, the scheme described in the Introduction is applied, where the model is of the form

$$w(\tau_i) = w(\tau_{i-1}) + \delta u(\tau_i).$$

At each time step $\delta_i = [\tau_i, \tau_{i+1})$, the vectors $p_*(\tau_i)$, $u(\tau_i)$ and $\bar{v}(\tau_i)$ are found from

$$p_*(\tau_i) \in \arg\min\{|\delta^{-1}(\xi(\tau_i) - \xi(\tau_{i-1})) - g(\tau_i, w, \xi(\tau_i))| : w\},$$

$$u(\tau_i) = -\alpha^{-1}(p_*(\tau_i) - w(\tau_i)),$$

$$\bar{v}(\tau_i) \in \arg\min\{|u(\tau_i) - f(\tau_i, p_*(\tau_i), \xi(\tau_i), v)| : v \in Q\},$$

where α is an auxiliary parameter. As approximations to the unknown input $v(t)$ and coordinates $y(t)$, the functions

$$v^h(t) = \bar{v}(\tau_i), \quad y^h(t) = p_*(\tau_i) \text{ for } t \in [\tau_i, \tau_{i+1}),\ i \in [0 : m-1]$$

are taken. Hence, this shows that the algorithm from [11, sections 17, 19] is rather complicated to be applied for solving our problem, since at each ith step it is necessary to solve two nonlinear extremal problems (to find $p_*(\tau_i)$ and $\bar{v}(\tau_i)$). In our algorithm, there are no extremal problems to solve, all the values are found from explicit formulas.

4 Proof of algorithm convergence

Before we turn to the proof of Theorem 1 (convergence of the algorithm), let us adduce auxiliary statements related to the bounding of terms in (5) and estimation errors.

Let $c^{(0)} = 2F_{max}(C_{S,in}(t_0) + C_{max}(\vartheta - t_0))$ and $i(t) = \kappa((t - t_0)/\delta)$, where $\kappa(a)$ denotes the integer part of a real number a. Furthermore, we define

$$a(t) = d(\ln V(t))/dt.$$

Lemma 1. *Let $|\dot{C}_{S,in}(t)| \le C_{max}$, $0 < F_{in}(t) \le F_{max}$ for almost every $t \in T$. Then the inequality*

$$\Bigg| \int_{t_0}^{t} V(\tau)a(\tau)C_{S,in}(\tau)\, d\tau - \Big[C_{S,in}(\tau_{i(t)})V(\tau_{i(t)}) - C_{S,in}(t_0)V(t_0) - $$

$$- \sum_{i=0}^{i(t)-1} (C_{S,in}(\tau_{i+1}) - C_{S,in}(\tau_i))V(\tau_i) \Big] \Bigg| \le c^{(0)}\delta \quad \text{for } t \in T$$

holds.

Proof. From the inequality

$$|V(t) - V(t+\delta)| \le \int_t^{t+\delta} |F_{in}(\tau)|\, d\tau \le F_{max}\delta, \quad t, t+\delta \in T,$$

we have

$$\left| \int_{t_0}^{t} \dot{C}_{S,in}(\tau)(V(\tau) - V_\delta(\tau))\, d\tau \right| \le F_{max}(t - t_0)C_{max}\delta \tag{8}$$

where $V_\delta(t) = V(\tau_i)$ for $t \in [\tau_i, \tau_{i+1})$. Note that

$$\int_t^{t+\delta} |V(\tau)a(\tau)C_{S,in}(\tau)|\, d\tau \le F_{max}(C_{S,in}(t_0) + C_{max}(\vartheta - t_0))\delta.$$

Therefore we have for $t \in [\tau_i, \tau_{i+1})$

$$\left| \int_{t_0}^{\tau_i} V(\tau)a(\tau)C_{S,in}(\tau)\, d\tau - \int_{t_0}^{t} V(\tau)a(\tau)C_{S,in}(\tau)\, d\tau \right| \le F_{max}(C_{S,in}(t_0) + C_{max}(\vartheta - t_0))\delta \tag{9}$$

Then, applying integration by parts, we obtain

$$\int_{t_0}^{\tau_i} V(\tau)a(\tau)C_{S,in}(\tau)\, d\tau = V(\tau_i)C_{S,in}(\tau_i) - V(t_0)C_{S,in}(t_0) - \int_{t_0}^{\tau_i} V(\tau)\dot{C}_{S,in}(\tau)\, d\tau. \tag{10}$$

The desired inequality follows from (8),(9) and the previous equality. □

Lemma 2. *Let the conditions of Lemma 1 be fulfilled, $V(t) \ge V_0 > 0$, and*

$$\tilde{b}(\tau) = \tilde{b}_j \quad \textit{for} \quad \tau \in [\tau_j, \tau_{j+1}), \quad |\tilde{b}_j| \le d \quad \textit{for} \quad j \in [0 : m_h - 1].$$

Then the inequality

$$\left| V^{-1}(t) \int_{t_0}^{t} V(\tau)\tilde{b}(\tau)\, d\tau - V^{-1}(\tau_{i(t)})\delta \sum_{j=0}^{i(t)} V(\tau_j)\tilde{b}_j \right| \le c_1\delta \quad \textit{for } t \in T$$

holds.

Lemma 2 can be verified by simple algebraic transformations.

For what follows, let us introduce the system of differential equations

$$\begin{aligned} \dot{C}_{x\psi}(t) &= \psi(t) - a(t)C_{x\psi}(t) \\ \dot{C}_{S\psi}(t) &= -\frac{\psi(t)}{Y} + a(t)C_{S,in}(t) - a(t)C_{S\psi}(t), \quad t \in T \end{aligned} \tag{11}$$

with the initial conditions

$$C_{x\psi}(t_0) = C_X(t_0), \quad C_{S\psi}(t_0) = C_S(t_0)$$

and function $\psi(\cdot)$ of the form

$$\psi(t) = \psi_j^h = Y\varphi_j^h/(1-Y) \quad \text{for} \quad t \in [\tau_j, \tau_{j+1}).$$

Let us furthermore define the piecewise-continuous functions

$$w_1^h(t) = w_1^h(\tau_i), \quad w_2^h(t) = w_2^h(\tau_i) \quad \text{as} \quad t \in [\tau_i, \tau_{i+1}) \cap T.$$

Then, the following Lemmas can be stated.

Lemma 3. *Let the conditions of Lemmas 1 and 2 be fulfilled. Then the following inequalities*

$$|C_{x\psi}(t) - w_1^h(t)| \le c_2\delta, \quad |C_{S\psi}(t) - w_2^h(t)| \le c_3\delta \quad \textit{for} \quad t \in T$$

hold.

Proof. The equation

$$\dot{x}(t) = f(t) - a(t)x(t), \quad x(t_0) = x_0, \quad f(\cdot) \in \mathcal{L}_2(T;\mathbb{R}),$$

has a solution that can be found by the Cauchy integral formula:

$$x(t) = V^{-1}(t)V(t_0)x_0 + V^{-1}(t)\int\limits_{t_0}^{t} V(\tau)f(\tau)\,d\tau.$$

In this case, the solution to system (11) is being found by the formulas:

$$C_{x\psi}(t) = V^{-1}(t)V(t_0)C_X(t_0) + \int\limits_{t_0}^{t} V^{-1}(t)V(\tau)\psi(\tau)\,d\tau,$$

$$C_{S\psi}(t) = V^{-1}(t)V(t_0)C_S(t_0) + \int\limits_{t_0}^{t} V^{-1}(t)V(\tau)(a(\tau)C_{S,in}(\tau) - \psi(\tau)Y^{-1})\,d\tau.$$

Validity of the Lemma now follows from these equalities and from Lemmas 1 and 2. □

Lemma 4. *Let $Y \in (0,1)$ and the conditions of Lemma 3 be fulfilled. Then the following inequalities*

$$|C_X(t) - w_1^h(t)| \le c_4(\delta + \int\limits_{t_0}^{t} |\varphi^h(\tau) - OUR(\tau)|\,d\tau), \tag{12}$$

$$|C_S(t) - w_2^h(t)| \le c_5(\delta + \int\limits_{t_0}^{t} |\varphi^h(\tau) - OUR(\tau)|\,d\tau) \; \textit{for } t \in T$$

hold.

Proof. Let $\mu_1(t) = C_{x\psi}(t) - C_X(t)$, $\mu_2(t) = C_{S\psi}(t) - C_S(t)$. The second and the third equation of the system (1) can be written in the form

$$\frac{\mathrm{d}\, C_X(t)}{\mathrm{d}\, t} = \frac{Y}{1-Y} OUR(t) - a(t) C_X(t),$$

$$\frac{\mathrm{d}\, C_S(t)}{\mathrm{d}\, t} = -\frac{Y}{1-Y} OUR(t) + a(t)(C_{S,in}(t) - C_S(t)).$$

Then functions $\mu_1(t)$ and $\mu_2(t)$ are solutions to the equations:

$$\dot{\mu}_1(t) = \frac{Y}{1-Y}(\varphi^h(t) - OUR(t)) - a(t)\mu_1(t),$$

$$\dot{\mu}_2(t) = -\frac{1}{1-Y}(\varphi^h(t) - OUR(t)) - a(t)\mu_2(t),$$

with the initial conditions: $\mu_1(t_0) = \mu_2(t_0) = 0$. Using the Cauchy integral formula,

$$\mu_1(t) = \frac{Y}{1-Y} \int\limits_{t_0}^{t} \Phi(t,\tau)(\varphi^h(\tau) - OUR(\tau)) d\tau,$$

$$\mu_2(t) = -\frac{1}{1-Y} \int\limits_{t_0}^{t} \Phi(t,\tau)(\varphi^h(\tau) - OUR(\tau)) d\tau,$$

where $\Phi(t,\tau) = V^{-1}(t)V(\tau)$. In this case the following inequalities:

$$|\mu_1(t)| \le c_1 \int\limits_{t_0}^{t} |\varphi^h(\tau) - OUR(\tau)|\, d\tau \qquad (13)$$

$$|\mu_2(t)| \le c_2 \int\limits_{t_0}^{t} |\varphi^h(\tau) - OUR(\tau)| d\tau, \quad t \in T \qquad (14)$$

hold. The validity of the Lemma follows from inequality (13), (14) and from Lemma 3. □

Lemma 5. *Let the conditions of Lemma 4 be fulfilled and* $C_X(t) \ge C_X > 0$, $C_S(t) \ge C_S > 0$. *Then under fulfillment of Condition 2, there exists an* $h_* > 0$, *such that for all* $h \in (0, h_*)$ *and* $t \in [\tau_i, \tau_{i+1})$, $i \in [0 : m_h - 1]$ *the following inequality*

$$|v_i^h - \mu_{max}(t)| \le c_6(\delta + |\varphi_i^h - OUR(t)| + \int\limits_{t_0}^{t} |\varphi^h(\tau) - OUR(\tau)|\, d\tau) \qquad (15)$$

holds.

See the Appendix for the proof of Lemma 5.

Lemma 6. *Let Conditions 1 and 2 be fulfilled and let in system (5) a value φ_i^h be defined by formulas (6). Then the inequalities*

$$|w_0^h(\tau_i) - C_{DO}(\tau_i)|^2 \le C^{(0)}(h + \delta + \alpha), \tag{16}$$

$$\int_{t_0}^{\vartheta} |\varphi^h(\tau)|^2 \, d\tau \le \int_{t_0}^{\vartheta} |OUR(\tau)|^2 \, d\tau + C^{(1)}(h + \delta)\alpha^{-1} \tag{17}$$

hold.

See the Appendix for the proof of Lemma 6.

The monograph [12] contains Theorem 1.2.1 that for the considered case can be formally stated as follows.

Theorem 2. *Let Conditions 1, 2 and the inequalities*

$$\sup_{i \in [0:m_h]} |w_0^h(\tau_i) - C_{DO}(\tau_i)| \le \nu(h), \quad \int_{t_0}^{\vartheta} |\varphi^h(\tau)|^2 \, d\tau \le \int_{t_0}^{\vartheta} |OUR(\tau)|^2 \, d\tau + \nu_1(h),$$

where $\nu(h) \to 0+$, $\nu_1(h) \to 0+$ as $h \to 0+$, be fulfilled. Then the following convergence takes place:

$$\varphi^h(t) \to OUR(t) \quad in \quad L_2(T; R) \quad as \quad h \to 0,$$

i.e.,

$$\int_{t_0}^{\vartheta} |\varphi^h(\tau) - OUR(\tau)|^2 \, d\tau \to 0 \quad as \quad h \to 0.$$

In turn, Lemmas 4–6 and Theorem 2 imply Theorem 1. Consequently, convergence of the algorithm (see section 3) as $h \to 0$ has been proven. However, the question remains how fast it converges. In the next section an answer to this question is given.

5 Algorithm's convergence rate

The estimate of the algorithm's convergence rate can be specified under some additional conditions. Let us give these conditions.

Theorem 3. *Let function $\mu_{max}(t)$ be a function of bounded variation. Then the following estimate of the algorithm convergence rate can be found:*

$$\int_{t_0}^{\vartheta} |v^h(\tau) - \mu_{max}(\tau)|^2 \, d\tau \le C_1 \lambda(h, \delta, \alpha),$$

$$\sup_{t \in T} |w_1^h(t) - C_X(t)| \le C_2 \lambda^{1/2}(h, \delta, \alpha), \quad \sup_{t \in T} |w_2^h(t) - C_S(t)| \le C_3 \lambda^{1/2}(h, \delta, \alpha).$$

Here $\lambda(h, \delta, \alpha) = h + \delta + \alpha + (h + \delta)\alpha^{-1}$, c_j $(j = 1, 2, 3)$ are some constants which can be written explicitly.

Proof. It can be easily seen that for $t \in \delta_i = [\tau_i, \tau_{i+1})$ the estimates below are true

$$|\xi_i^h - C_{DO}(t)| \le c_1(h+\delta), \tag{18}$$

$$|C_{DO}(t_0) - w_0^h(t_0)| \le h, \tag{19}$$

$$|b(t) - b(\tau_i)| \le c_2\delta, \quad |C_{DO}(t) - C_{DO}(\tau_i)| \le c_3\delta, \tag{20}$$

where $b(t) = \ln V(t)$. In addition, we have

$$\left| \int\limits_{\tau_i}^{t} a(\tau)C_{DO}(\tau)d\tau - (b(\tau_{i+1}) - b(\tau_i))\xi_i^h \right| \le c_6(h+\delta^2). \tag{21}$$

Given inequality (16) we deduce

$$|w_0^h(t) - C_{DO}(t)| \le c_7(h+\delta+\alpha), \quad t \in T. \tag{22}$$

From (18)–(22) follows the inequality

$$\sup_{t\in T} \left| \int\limits_{t_0}^{t} (\varphi^h(\tau) - OUR(\tau))d\tau \right| \le c_8(h+\delta+\alpha) \tag{23}$$

Taking into account (17) we deduce

$$\int\limits_{t_0}^{\vartheta} |\varphi^h(\tau) - OUR(\tau)|^2\,d\tau = \int\limits_{t_0}^{\vartheta} |\varphi^h(\tau)|^2\,d\tau - 2\int\limits_{t_0}^{\vartheta} \varphi^h(\tau)OUR(\tau)\,d\tau + \int\limits_{t_0}^{\vartheta} |OUR(\tau)|^2\,d\tau \le \tag{24}$$

$$\le 2\int\limits_{t_0}^{\vartheta} (OUR(\tau) - \varphi^h(\tau))OUR(\tau)\,d\tau + c_9(h+\delta)\alpha^{-1}.$$

Due to the condition in the theorem, the function $\mu_{max}(\cdot)$ is a constrained variation function. Therefore, the function $OUR(t)$ is of the same kind as well. In this case from (23), (24) and Lemma 1.3.3 [12] we have

$$\int\limits_{t_0}^{\vartheta} |\varphi^h(\tau) - OUR(\tau)|^2\,d\tau \le c_{10}\lambda(h,\delta,\alpha). \tag{25}$$

The validity of the theorem follows from (25) and Lemmas 4 and 5. □

From Theorem 3 the following corollary can be deduced

Corollary 1. *Let $\delta(h) = h$, $\alpha(h) = h^{1/2}$ and function $\mu_{max}(t)$ be a function of bounded variation. Then the following estimate of algorithm convergence rate is true:*

$$\int\limits_{t_0}^{\vartheta} |v^h(\tau) - \mu_{max}(\tau)|^2\,d\tau \le c_1 h^{1/4},$$

$$\sup_{t\in T} |w_1^h(t) - C_X(t)| \le c_2 h^{1/8}, \quad \sup_{t\in T} |w_2^h(t) - C_S(t)| \le c_3 h^{1/8}.$$

So far, the analysis holds for a lake system with only inflow (1). The next corollary presents the result for an isolated lake without any in- and outflows (1), *i.e.* for $F_{in}(t) \equiv 0$.

Let, instead of Condition 1, the following one be fulfilled:

Condition 3.

$$F_{in}(t) = 0, \quad Y \in (0, 1),$$

$$C_X(t) \geq C_X > 0, \quad C_S(t) \geq C_S > 0 \quad \textit{(see Condition 1).}$$

In this case, the second and third equations of system (1) take the form

$$\frac{dC_X(t)}{dt} = \frac{Y}{1-Y} OUR(t),$$

$$\frac{dC_S(t)}{dt} = -\frac{1}{1-Y} OUR(t).$$

Therefore, as a system M, one should take

$$\begin{aligned} w_0^h(\tau_{i+1}) &= w_0^h(\tau_i) + \delta\{k_{La}(C_{sat}^{en} - \xi_i^h) - \varphi_i^h\}, \\ w_1^h(\tau_{i+1}) &= w_1^h(\tau_i) + \delta \frac{Y}{1-Y} \varphi_i^h, \\ w_2^h(\tau_{i+1}) &= w_2^h(\tau_i) - \delta \frac{1}{1-Y} \varphi_i^h. \end{aligned}$$

The next statement follows from the results presented above.

Corollary 2. *Let Conditions 2 and 3 be fulfilled, the system M be of the form (5) and the function $\varphi^h(t)$ (the control in M) be found from (6). Then, the assertions of Theorems 1–3 as well as of Corollary 1 are valid.*

6 Simulation results

The algorithm described in section 3 is tested on the lake system S, given by (1), with limited sensor availability. Let us assume that this system evolves on the time interval $[0, 3]$ d, so that the amount of information is limited which justifies the bounded-error approach. It is furthermore assumed that the maximum specific growth rate $\mu_{\max}(t)$ is equal to 0.047987 or $2.62 \cdot 10^{-4}$ $sin(t)$ $[1/d]$, so that both the time-invariant and the time-varying case are evaluated. Our starting point is that we do not know the behavior nor the magnitude of $\mu_{\max}(t)$. Recall that our aim is to recover $\mu_{\max}(t)$ and the states $C_X(t)$, $C_S(t)$ from a finite number of corrupted samples of the evolution of the dissolved oxygen concentration $C_{DO}(t)$ only. In particular, during the numerical experiments, at each ith step, we use $C_{DO}(\tau_i) + h$ instead of $C_{DO}(\tau_i)$.

The simulation results with $\mu_{max}(t)$ constant and for different noise levels are presented in Fig. 2–4. In these figures, the results of the computer modeling exercise of the dynamic inverse problem are presented for the following case (see Eqn. (1)):

$K_S = 1.0$ $[kTon/km^3]$,	$Y = 0.64$ $[g(C_X)/g(C_S)]$,
$C_{S,in} = 5.0 + 0.05\ \sin(t)$ $[kTon/km^3]$,	$V(t_0) = 1.0$ $[km^3]$,
$F_{in}(t) = 0.1 + 0.05\ \sin(t)$ $[km^3/d]$,	$k_{La} = 0.6$ $[1/d]$,
$C_{sat}^{en} = 7$ $[kTon/km^3]$,	$C_{sat} = 9$ $[kTon/km^3]$

The initial conditions for the system are as follows:

$$C_{D0}(t_0) = 7\ [kTon/km^3], \quad C_X(t_0) = 4000\ [kTon/m^3], \quad C_S(t_0) = 0.05\ [kTon/km^3]$$

The tuning parameters of the algorithm are chosen as:

$$\delta = 0.001\ [d], \qquad \alpha = 0.001\ [1/d], \qquad K = 2\ [kTon/km^3.d]$$

In Figs. 2–4 the thin lines represent the coefficient $\mu_{max}(t)$ and the states $C_X(t)$, $C_S(t)$ in mg/L (which is equal to $kTon/km^3$), while the bold lines represent the estimate $v^h(t)$ and the states of the auxiliary system, $w_1^h(t)$ and $w_2^h(t)$.

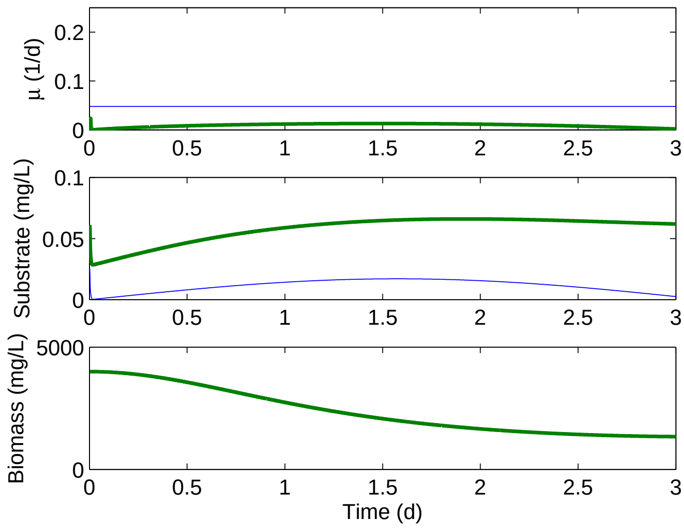

Figure 2: Identification results: $h = 10^{-2}$.

Fig. 2 corresponds to the case when $h = 10^{-2}$. In Fig. 3, $h = 10^{-3}$, while in Fig. 4, $h = 10^{-4}$. Equation (1) was solved using the Euler method with integration step δ. At the moment $t = \tau_i$ the value $\xi_i^h = C_{DO}(\tau_i) + h$ was measured. The results of the numerical experiments show that the mean-square convergence of $v^h(t)$ to $\mu_{\max}(t)$ and the uniform convergence of $w_1^h(t)$ to $C_X(t)$ and $w_2^h(t)$ to $C_S(t)$ take place under “reduction” of parameters α, h and δ or of one of them. For a further interpretation of the graphical results we recall that $v^h(t)$ converges to $\mu_{\max}(t)$ in the $\mathcal{L}_2[0,3]$ metric, not in the continuous metric.

In the next step, we consider a realistic case with (i) $\mu_{max}(t)$ time-varying, (ii) more variation in the inflow, *i.e.* $F_{in}(t) = sin(t)$ and (iii) relatively large error bound $h = 0.1$ on the dissolved oxygen concentration data. Fig. 5 indicates that, although a significant (constant) off-set in the estimate of $\mu_{max}(t)$ appears due to the relatively large error bound, the substrate concentration is reasonably recovered from the noisy data. Notice that, for all cases, the biomass concentration is estimated very well.

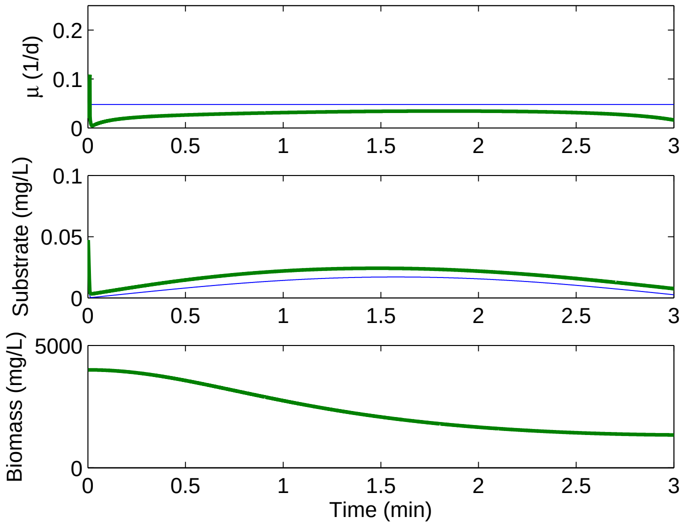

Figure 3: Identification results: $h = 10^{-3}$.

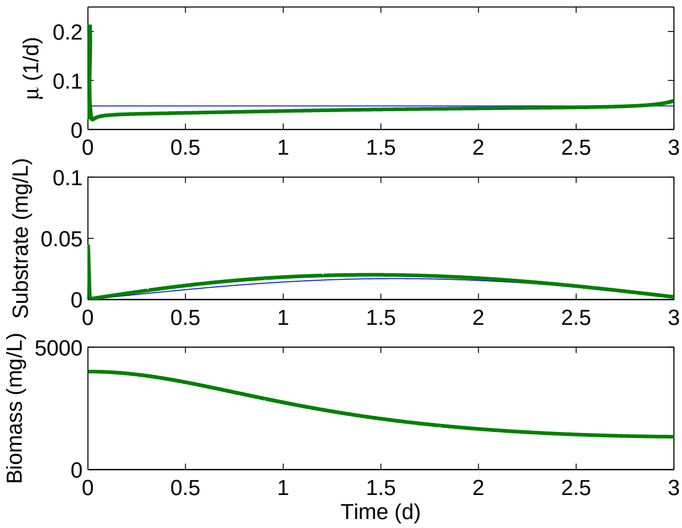

Figure 4: Identification results: $h = 10^{-4}$.

7 Conclusions

In this paper, given data with unknown-but-bounded error (2), a simultaneous state and parameter estimation algorithm, based on stable dynamical inversion using the scheme of Fig. 1 with M a *linear* approximation of S, has been proposed. In particular, the algorithm has been developed for a rather general class of single-species lake systems (1) with limited sensor availability, that is only the dissolved oxygen concentration is measured. Convergence of this algorithm for this specific case has been proven. Furthermore, convergence rates, as a function of the error bound, integration step and tuning parameter α, have been derived. Finally, the algorithm has been tested in simulation, showing stable and reliable results.

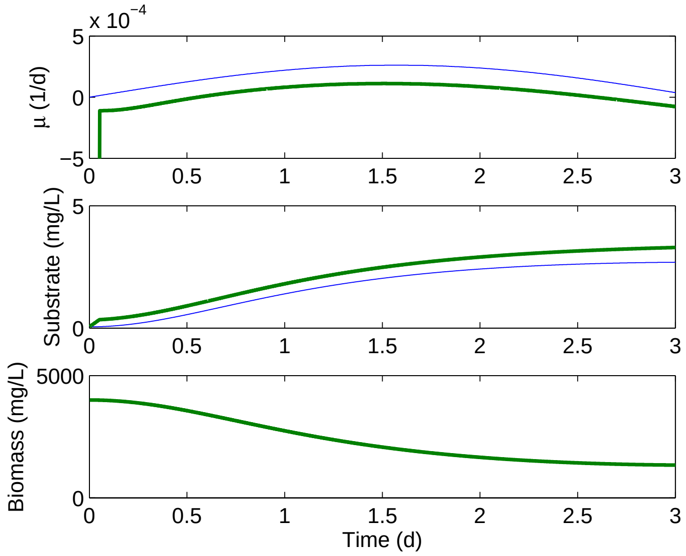

Figure 5: Identification results: $\mu_{max}(t)$ time-varying and $h = 10^{-1}$.

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A Appendix - Proofs of Lemma 5 and 6

Lemma 5. Using Lemma 3 and Condition 2, it is possible to specify an h_1 and δ_1 such that for all $h \in (0, h_1)$ and $\delta = \delta(h) \in (0, \delta_1)$ the following inequalities are true:

$$C_{x\psi(t)} \geq c_\psi^{(1)} > 0, \quad C_{S\psi}(t) \geq c_\psi^{(2)} > 0 \quad \text{for} \quad t \in T. \tag{26}$$

Define the value

$$\Delta_{i,t}^h \equiv \left| \frac{Y(K_S + w_2^h(\tau_i))\varphi_i^h}{(1-Y)w_1^h(\tau_i)w_2^h(\tau_i)} - \frac{Y(K_S + C_{S\psi}(t))OUR(t)}{(1-Y)C_{x\psi}(t)C_{S\psi}(t)} \right|. \tag{27}$$

Given Condition 2 and the inequalities (26), by using Lemma 3 it is easy to drive the estimates:

$$\Delta_{i,t}^h \leq c_3(\delta + |\varphi_i^h - OUR(t)|), \quad t \in [\tau_i, \tau_{i+1}) \tag{28}$$

uniform in all $i \in [0 : m_h - 1]$ and $h \in (0, h_1)$. In addition, the following inequalities take place:

$$\left| \frac{1}{C_X(t)C_S(t)} - \frac{1}{C_{x\psi}(t)C_{S\psi}(t)} \right| \leq c_4(|\mu_1(t)| + |\mu_2(t)|) \leq c_5 \int_{t_0}^{t} |\varphi^h(\tau) - OUR(\tau)|\, d\tau, \tag{29}$$

$$\left| \frac{1}{C_X(t)} - \frac{1}{C_{x\psi}(t)} \right| \leq c_6 \int_{t_0}^{t} |\varphi^h(\tau) - OUR(\tau)|\, d\tau. \tag{30}$$

Using (29) and (30), it is easy to determine:

$$\Delta_t \equiv \left| \frac{K_S + C_{S\psi}(t)}{C_{x\psi}(t)C_{S\psi}(t)} - \frac{K_S + C_S(t)}{C_X(t)C_S(t)} \right| \leq c_7 \int_{t_0}^{t} |\varphi^h(\tau) - OUR(\tau)|\, d\tau. \tag{31}$$

Further, we have

$$\mu_{max}(t) = \frac{\mu(C_S(t))(K_S + C_S(t))}{C_S(t)},$$

$$\mu(C_S(t)) = \frac{Y}{1-Y}\frac{OUR(t)}{C_X(t)}.$$

Therefore

$$\mu_{max}(t) = \frac{Y(K_S + C_S(t))OUR(t)}{(1-Y)C_X(t)C_S(t)},$$

and

$$|v_i^h - \mu_{max}(t)| = \left| \frac{Y(K_S + w_2^h(\tau_i))\varphi_i^h}{(1-Y)w_1^h(\tau_i)w_2^h(\tau_i)} - \frac{Y(K_S + C_S(t))OUR(t)}{(1-Y)C_X(t)C_S(t)} \right|.$$

Combining (28), (31), and taking into account the inequality

$$|v_i^h - \mu_{max}(t)| \leq \Delta_{i,t}^h + \frac{Y}{1-Y}|OUR(t)|\Delta_t, \quad t \in [\tau_i, \tau_{i+1}), \quad i \in [0 : m_h - 1],$$

we get Eqn. (15) in Lemma 5. □

Lemma 6. For the proof of the lemma we estimate the variation of the function

$$\varepsilon(t) = |\tilde{w}_0^h(t) - C_{DO}(t)|^2 + \alpha(h) \int_{t_0}^{t} \{|\varphi^h(\tau)|^2 - |OUR(\tau)|^2\}\, d\tau, \quad t \in T.$$

Here a function $\tilde{w}_0^h(t)$, $t \in [\tau_i, \tau_{i+1})$, $i \in [0 : m_h - 1]$, defines by the rule

$$\begin{aligned}
&\dot{\tilde{w}}_0^h(t) = k_{La}(C_{sat}^{en} - \xi_i^h) - \varphi_i^h + a(t)(C_{sat} - \xi_i^h), \quad t \in [\tau_i, \tau_{i+1}),\\
&\dot{\tilde{w}}_0^h(\tau_i) = w_0^h(\tau_i).
\end{aligned}$$

Note, that

$$\lim_{t \to \tau_{i+1} - 0} \tilde{w}_0^h(t) = w_0^h(\tau_{i+1}).$$

Let

$$\mu_i = 2(\tilde{w}_0^h(\tau_i) - C_{DO}(\tau_i)) \int_{\tau_i}^{\tau_{i+1}} (\dot{\tilde{w}}_0^h(t) - \dot{C}_{DO}(t))\, dt.$$

It is easily seen that the following inequality is true:

$$\varepsilon(\tau_{i+1}) \le \varepsilon(\tau_i) + \delta(h) \int_{\tau_i}^{\tau_{i+1}} |\dot{\tilde{w}}_0^h(\tau) - \dot{C}_{DO}(\tau)|^2\, d\tau + \mu_i + \alpha(h) \int_{\tau_i}^{\tau_{i+1}} \{|\varphi_i^h|^2 - |OUR(\tau)|^2\}\, d\tau. \tag{32}$$

Consider the value μ_i in the right-hand part of inequality (32). The following relation is fulfilled:

$$\mu_i = -2s_i^* \int_{\tau_i}^{\tau_{i+1}} \{k_{La}(C_{DO}(\tau) - \xi_i^h) + a(\tau)(C_{DO}(\tau) - \xi_i^h) + OUR(\tau) - \varphi_i^h)\}\, d\tau = \sum_{j=1}^{3} \lambda_{ji}, \tag{33}$$

where

$$\begin{aligned}
\lambda_{1i} &= 2k_{La}s_i^* \int_{\tau_i}^{\tau_{i+1}} (\xi_i^h - C_{DO}(\tau))\, d\tau,\\
\lambda_{2i} &= 2s_i^* \int_{\tau_i}^{\tau_{i+1}} a(\tau)(\xi_i^h - C_{DO}(\tau))\, d\tau,\\
\lambda_{3i} &= 2s_i^* \int_{\tau_i}^{\tau_{i+1}} (\varphi_i^h - OUR(\tau))\, d\tau,\\
s_i^* &= C_{DO}(\tau_i) - w_0^h(\tau_i).
\end{aligned}$$

Therefore, from (32), (33) we derive

$$\varepsilon(\tau_{i+1}) \le \varepsilon(\tau_i) + \sum_{j=1}^{3} \lambda_{ji} + \alpha(h) \int_{\tau_i}^{\tau_{i+1}} \{|\varphi_i^h|^2 - |OUR(\tau)|^2\}\, d\tau + \delta L_i^h, \tag{34}$$

where a value L_i^h is defined by the formula

$$L_i^h = \int\limits_{\tau_i}^{\tau_{i+1}} |k_{La}(\xi_i^h - C_{DO}(\tau)) + a(\tau)(\xi_i^h - C_{DO}(\tau)) + \varphi_i^h - OUR(\tau)|^2 \, d\tau.$$

Let us estimate each term in the right-hand part of inequality (34). From (3) (Sec. 3) and Condition 1 it follows that

$$\max\{ \sup_{t_0 \le t \le \vartheta} |C_{DO}(t)|, \sup_{t_0 \le t \le \vartheta} |C_X(t)|, \sup_{t_0 \le t \le \vartheta} |C_S(t)|\} \le d_0 < +\infty.$$

Consequently, using this inequality and the inequality $|\xi_i^h - C_{DO}(\tau_i)| \le h$, we deduce the estimates

$$\lambda_{1i} \le d_1(h + \delta)\delta, \tag{35}$$

$$\lambda_{2i} \le d_2(h + \delta)\delta, \tag{36}$$

$$\lambda_{3i} \le 2s_i \int\limits_{\tau_i}^{\tau_{i+1}} (\varphi_i^h - OUR(\tau)) \, d\tau + d_3 h\delta, \quad s_i = \xi_i^h - w_0^h(\tau_i), \tag{37}$$

$$\sum_{i=0}^{m_h - 1} L_i^h \le d_4. \tag{38}$$

Here, d_j, $j \in [0 : 4]$, are constants that can be explicitly written. Further, we use the obvious equality

$$\varphi_i^h = \arg\min\{2s_i u + \alpha(h)u^2 : -K \le u \le K\}.$$

Then, from (3), (6) (Sec. 3) and (37), the following inequality

$$\lambda_{3i} + \alpha(h) \int\limits_{\tau_i}^{\tau_{i+1}} \{|\varphi_i^h|^2 - |OUR(\tau)|^2\} \, d\tau \le \tag{39}$$

$$\le \int\limits_{\tau_i}^{\tau_{i+1}} \Big\{ \Big[2s_i\varphi_i^h + \alpha(h)|\varphi_i^h|^2\Big] - \Big[2s_i OUR(\tau) + \alpha(h)|OUR(\tau)|^2\Big] \Big\} \, d\tau + d_3 h\delta \le d_3 h\delta$$

holds. Taking into account (34)–(37), we have for all $i \in [1 : m_h]$ the following estimate

$$\varepsilon(\tau_i) \le d_5(h + \delta). \tag{40}$$

Inequalities (16) and (17) in Lemma 6 follow from (40). $\square$