International Institute for
Applied Systems Analysis
Schlossplatz 1
A-2361 Laxenburg, Austria

Tel: +43 2236 807 342
Fax: +43 2236 71313
E-mail: publications@iiasa.ac.at
Web: www.iiasa.ac.at

Interim Report **IR-07-009**

A Game of Natural Gas Suppliers in a Non-Market Economic Environment

Oleg I. Nikonov
Ural State Technical University, Ekaterinburg, Russia
(email: aspr@ustu.mail.ru)

Approved by

Arkady Kryazhimskiy (kryazhim@iiasa.ac.at)
Program Leader, Dynamic Systems Program

February 2007

Acknowledgments

This work was partially supported by RFBR, grants 06-01-00483a and 05-01-08034-ofi-p, and by RFH, grant 05-02-02118 a

Abstract

We consider a non-cooperative two-player game with payoff functions of a special type, for which standard existence theorems and algorithms for searching Nash equilibrium solutions are not applicable. The problem statement is motivated by situations arising in the process of determining a time for starting the construction of a new gas pipeline and a time of putting it into operation. The paper develops the approach suggested in [1]–[5].

Key words: *non-cooperative two-person game, best reply, Nash equilibrium, application to energy problems*

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Oleg I. Nikonov *
Ural State Technical University, Ekaterinburg, Russia
(email: aspr@ustu.mail.ru)

1. Introduction

We construct and analyse a game-theoretic model related to the process of making decisions on the design and commercialization of new gas pipelines. We consider a developing gas market with increasing demand, for which new pipelines delivering gas from different natural gas fields – and thus acting as competitors in a game – are being planned. Evidently, the appearance of every new player in the market leads to a decrease in the sales returns for the existing gas pipelines. Therefore, a reasonable argument is that the earlier a player enters the market, the greater total profit this player should receive. In the same time, the present value of the construction cost is decreasing, whereas gas demand and gas prices are expected to be increasing over time. Therefore, a reasonable delay in entering the market may be preferable. The above argument leads to a game-theoretic problem formulation, in which the points in time, at which the gas suppliers enter the market, act as crucial decisions.

In [1] an adequate game-theoretic model is proposed. The problem is formalized as a non-cooperative game in which the times of entering the market (commercialization times) play the role of control variables. The player's benefit is defined as the total profit gained during the pipeline's construction/operation period. The model includes a set of assumptions on the market price formation mechanism; the game is considered with the infinite time horizon; functions defining benefit rates and costruction costs are supposed to be monotonously decreasing. For the case of two players, an analytic solution was obtained and an algorithm for searching Nash equilibrium solutions was proposed. In [2, 3, 4] a computer realization of that algorithm, including such options as data approximation and generating forecasts was developed. An application to data on Turkey's gas market was suggested.

In subsequent research, attempts have been made to extend the developed approach, in particular, China's natural gas market has been considered. However, assumptions admissible for Turkey's gas market turned out to be unfit in the case of China's gas market. A key economic distinction was that in China the price formation mechanism could not be viewed as purely market-driven. In [5] a relevant modification of the model was described and results of data-based simulation of the operation of planned pipelines delivering gas from Russia to China were presented.

In the present paper we suggest a new mathematical model that takes into account the phenomena mentioned above. We use an approach similar to that developed in [1], however the assumptions we impose here are significantly different from – and sometimes opposite to – those adopted in [1].

The main features of our model are the following. The process of construction/operation of competing gas pipelines has a finite time horizon. The players' profit rate returns are monotonously increasing (not monotonically decreasing) over time. The construction cost is constant for each player. A players' payoff (to be minimized) is defined to be a function of the length of the period of the return of the investment cost (the payback period) and of the time, at which the player enters the market.

The results presented in this paper provide a theoretical basis for the elaboration of a decision support system applicable for planning energy infrastructures in situations where the price formation mechanisms may not be market-driven.

2. Problem Formulation: Notations and Assumptions

Our model assumes that two players (participants) develop their gas pipline projects for the same gas market. The model's main variables and parameters are the following.

Participants' benefit rates. The benefit rate for participant i $(i = 1, 2)$ is described by two functions, $\varphi_{i1}(t)$ and $\varphi_{i2}(t)$; $\varphi_{i1}(t)$ defines the benefit rate for participant i if participant i is a monopolist in the market, and $\varphi_{i1}(t)$ defines the benefit rate for participant i if participant i if both participants occupy the market. The presence of the opponent reduces the benefit rate for participant i, therefore $\varphi_{i1}(t) > \varphi_{i2}(t)$ for all t in a given time interval $[0, T]$ representing the life period for the participants' projects. Let us denote by t_1 and t_2 the points in time, at which participants 1 and 2 enter the market, respectively. Then the benefit rates for the participants are defined by

$$\varphi_1(t|t_2) = \begin{cases} \varphi_{11}(t) & \text{if } t < t_2 \\ \varphi_{12}(t) & \text{if } t \geq t_2 \end{cases};$$

$$\varphi_2(t|t_1) = \begin{cases} \varphi_{21}(t) & \text{if } t < t_1 \\ \varphi_{22}(t) & \text{if } t \geq t_1 \end{cases}.$$

The profits participant 1 and participant 2 gain on a time interval $[t_1, t_1+\delta]$ are $\int_{t_1}^{t_1+\delta} \varphi_1(t \mid t_2)dt$ and $\int_{t_1}^{t_1+\delta} \varphi_2(t \mid t_1)dt$, respectively. We assume the functions φ_{ij} to be differentiable, concave and monotonously increasing on $[0, T]$. The assumption that the benefit rates are increasing over time is motivated by modeling and forcasting results for China's natural gas market. This assumption is different from that suggested in [1].

Construction cost. Payback period. We assume that the construction costs are fixed and denoted by C_i, $i = 1, 2$. We also assume the interval $[0, T]$ to be so large that the construction costs are covered by the market sales:

$$\int_0^T \varphi_{i2}(t)dt > C_i. \tag{2.1}$$

Let us define times $\bar{t}_i, \bar{\bar{t}}_i, t_i^*, t_i^{**}$ as follows:

$$\bar{t}_i : \int_0^{\bar{t}_i} \varphi_{i1}(t)dt = C_i; \quad \bar{\bar{t}}_i : \int_{\bar{\bar{t}}_i}^T \varphi_{i2}(t)dt = C_i;$$

$$t_i^* : \int_0^{t_i^*} \varphi_{i2}(t)dt = C_i; \quad t_i^{**} : \int_{t_i^{**}}^T \varphi_{i1}(t)dt = C_i. \tag{2.2}$$

Clearly, $\bar{t}_i$ is the payback period for project i, provided participant i enters the market at time $t = 0$, while the the other participant never enters the market; $\bar{\bar{t}}_i, t_i^*, t_i^{**}$ are interpreted similarly.

In what follows we assume that the final time T is large enough in comparison with all time characteristics of the projects. Namely, the following relations are supposed to be true:

Assumption 1. *It holds that*

$$t_i^* < \bar{\bar{t}}_i, \ \ i = 1, 2. \tag{2.3}$$

Note that under Assumption 1 the next inequalities hold true: $0 < \bar{t}_i < t_i^* < \bar{\bar{t}}_i < t_i^{**} < T$. The value $\Delta_i = \Delta_i(t_1, t_2)$ defined by

$$\int_{t_i}^{t_i+\Delta_i} \varphi_i(t \mid t_j)dt = C_i, \tag{2.4}$$

will be called the payback period for project i; here $j = 1, 2, \ j \neq i$. In the next section we will describe properties of $\Delta_i(t_1, t_2)$.

Goals of control, payoff functions. The problem we consider in this paper, assumes that each participant tries to achieve two goals: to minimize his/her payback period $\Delta_i(t_1, t_2)$ and to minimize the commercialization time for his/her project – the time t_i, at which the project enters the market. The participants may have different priorities for these two criteria and thus choose different waits for them. The participants' control variables are – as in [1]–[3] – their commercialization times t_i. Thus, the payoff function participant i minimizes through the choice of his/her commercialization time t_i is

$$f_i(t_1, t_2 \mid \alpha_i) = \alpha_i t_i + \Delta_i(t_1, t_2); \tag{2.5}$$

here α_i is a weight coefficient, $0 \leq \alpha_i \leq 1$. With $\alpha_i = 1$ both criteria are equitable, in case $\alpha_i = 0$ one has the unique criterion – the payback period.

In what follows, we consider two problems. The first problem consists in optimization for one participant, while the choice of the other participant is fixed. We formulate this problem for participant 1 only.

Problem 1. *Construct the (generally, multi-valued) function* $t_1^0 = t_1^0(t_2 \mid \alpha_1)$ *such that*

$$f_1(t_1^0, t_2 \mid \alpha_1) = \min_{t_1} f_1(t_1, t_2 \mid \alpha_1). \tag{2.6}$$

In a standard terminology of game theory, $t_1^0 = t_1^0(t_2 \mid \alpha_1)$ is the best reply of participant 1 to strategy t_2 of participant 2.

Similarly, we introduce the best reply $t_2^0 = t_2^0(t_1 \mid \alpha_2)$ of participant 2 to strategy t_1 of participant 1.

The other problem consists in finding Nash equilibrium solutions in the corresponding two-player game. Using the introduced notations, we formulate it in the following way:

Problem 2. *Find pairs* $\{\hat{t}_1, \hat{t}_2\}$ *such that*

$$\begin{aligned} \hat{t}_1 \in \ & t_1^0(\hat{t}_2 \mid \alpha_1), \\ \hat{t}_2 \in \ & t_2^0(\hat{t}_1 \mid \alpha_2). \end{aligned} \tag{2.7}$$

3. Properties of Payback Periods

Let us study the function $\Delta_1 = \Delta_1(t_1, t_2)$. For $\Delta_2 = \Delta_2(t_1, t_2)$, similar results can be obtained through a change of the indices.

First of all, let us specify the domain of definition of $\Delta_1 = \Delta_1(t_1, t_2)$, i.e., determine the set $D_1 \subset [0, T] \times [0, T]$, in which equation (2.4) has a solution. For this purpose, introduce the following variable. Denote by $t_1'' = t_1''(t_2)$ the time such that

$$\int_{t_1''}^{T} \varphi_1(t \mid t_2) dt = C_1.$$

By Assumption 1 $t_1''(t_2) = \bar{\bar{t}}_1$ if $t_2 \le \bar{\bar{t}}_1$, and if t_2 varies from $\bar{\bar{t}}_1$ to T, time $t_1''(t_2)$ grows from $\bar{\bar{t}}_1$ to t_1^{**}. Thus, D_1 is defined by the inequalities $0 \le t_1 \le t_1''(t_2)$, $0 \le t_2 \le T$.

Introduce the following functions of $\tau \in [0, T]$:

$$g_0 = g_0(\tau) : \int_0^{g_0} \varphi_1(t \mid \tau) dt = C_1,$$

$$g_1 = g_1(\tau) : \int_{\tau}^{\tau + g_1} \varphi_{11}(t) dt = C_1$$

$$g_2 = g_2(\tau) : \int_{\tau}^{\tau + g_2} \varphi_{12}(t) dt = C_1,$$

$$g_3 = g_3(\tau) : g_3(\tau) = T - t_1''(\tau),$$

$$g_4 = T - \bar{\bar{t}}_1.$$

Note that $g_0(t_2) = \Delta_1(0, t_2)$, and this function decreases on $[0, \bar{t}_1]$ from t_1^* to $\bar{t}_1$. Then, as $\tau > \bar{t}_1 \quad g_0(\tau) \equiv \bar{t}_1$, the function $g_1(\tau) = \Delta_1(\tau, T)$ is defined and decreases on $[0, t_1^{**}]$ from $g_1(0) = \bar{t}_1$ to $T - t_1^{**}$. Similarly, $g_2(\tau) = \Delta_1(\tau, 0)$ is defined and decreases on $[0, \bar{\bar{t}}_1]$ from $g_2(0) = t_1^*$ to $T - \bar{\bar{t}}_1$. Finally, $g_3(t_2) \equiv T - \bar{\bar{t}}_1$ is defined on $[0, \bar{\bar{t}}_1]$ and decreases to $T - t_1^{**}$ on $[\bar{\bar{t}}_1, T]$.

In what follows, we consider the case where the following assumption is true.

Assumption 2. *For each admissible t_1 it holds that*

$$\varphi_{11}(t_1) < \varphi_{12}(t_1 + g_1(t_1)), \tag{3.1}$$

Assumption 2 imposes a relationship between the benefit rates $\varphi_{11}(t)$ and $\varphi_{12}(t)$ and the construction cost C_i. This relationship holds provided the payback period for project 1 is large compared to the variations of $\varphi_{1j}(t)$ in a neighborhood of t_1.

For a given t_2, define $t_1' = t_1'(t_2)$ by

$$\int_{t_1'}^{t_2} \varphi_{11}(t) dt = C_1. \tag{3.2}$$

Note that $t_1' = t_1'(t_2)$ is well defined and non-negative for $t_2 \ge \bar{t}_1$.

L e m m a 1. *In the set D_1, the function $\Delta_1 = \Delta_1(t_1, t_2)$ is defined correctly and is continuous. In each interior point (t_1, t_2) of D_1 such that $t_1 \neq t_1'(t_2)$ and $t_1 \neq t_2$, there exists the partial derivative $\frac{\partial \Delta_1(t_1,t_2)}{\partial t_1}$ and*

$$\frac{\partial \Delta_1(t_1,t_2)}{\partial t_1} = \begin{cases} -1+ \dfrac{\varphi_{11}(t_1)}{\varphi_{11}(t_1+g_1(t_1))}, & \text{if } 0 < t_1 < t_1'(t_2); \\ -1+ \dfrac{\varphi_{11}(t_1)}{\varphi_{12}(t_1+\Delta_1(t_1,t_2))}, & \text{if } t_1'(t_2) < t_1 < t_2; \\ -1+ \dfrac{\varphi_{12}(t_1)}{\varphi_{12}(t_1+g_2(t_1))}, & \text{if } t_2 < t_1 < T. \end{cases} \tag{3.3}$$

P r o o f. Since $\Delta_1 = \Delta_1(t_1, t_2)$ is defined as a solution of equation (2.4) with $i = 1$, the correctness of the definition of Δ_1 follows from the the definition of D_1 and properties of $\varphi_{ij}(t)$. Considering (2.4) as an equality defining an implicit function, we derive (3.3).

Lemma 1 allows us to describe the behaviour of $\Delta_1 = \Delta_1(t_1, t_2)$ in D_1. First, let us consider $\Delta_1(t_1, t_2)$ as a function of t_1, with t_2 fixed.

L e m m a 2. *Let Assumptions 1 and 2 be true. Then the payback period $\Delta_1(t_1, t_2)$, considered as a function of t_1, decreases on the interval $[0, T]$. The derivative $\frac{\partial \Delta_1(t_1,t_2)}{\partial t_1}$ is negative and continuous everywhere in $[0, T]$ except for two points, $t_1 = t_1'(t_2)$ and $t_1 = t_2$. At point $t_1 = t_1'(t_2)$ the derivative $\frac{\partial \Delta_1(t_1,t_2)}{\partial t_1}$ increases and at point $t_1 = t_2$ it decreases. For each $(t_1, t_2) \in D_1$ the next inequalities hold:*

$$g_1(t_1) \leq \Delta_1(t_1, t_2) \leq g_2(t_1). \tag{3.4}$$

P r o o f. Inequalities (3.4) follows directly from the definitions of function $\Delta_1(t_1, t_2)$ and functions $g_1(t_1)$ and $g_2(t_1)$. The other assertion of the lemma follows from an analysis of the sign of $\frac{\partial \Delta_1(t_1,t_2)}{\partial t_1}$ determined by (3.3) for various $(t_1, t_2) \in D_1$. For each of the cases listed in (3.3) we have $\frac{\partial \Delta_1(t_1,\, t_2)}{\partial t_1} < 0$ due to the inequalities $\varphi_{1j}(t_1) < \varphi_{1j}(t_1 + g_j(t_1))$, $j = 1, 2$ (following from the monotonicity assumption for $\varphi_{ij}(t)$), the inequality $\varphi_{i1}(t_1) > \varphi_{i2}(t_1)$, (3.1) and (3.4).

Fig. 1–4 provide graphical illustrations of the above assertions for an example considered in the last section. In Fig. 1 the graph of the benefit rate is shown for fixed $t_2 \in (\bar{t}_1, \bar{\bar{t}}_1)$. The area of the shaded figure is C_1, and its base is $\Delta_1(t_1, t_2) = 10.76$ for the arguments $t_1 = 15$, $t_2 = 20$. Fig. 2–4 show the graphs of function $\Delta_1(t_1, t_2)$ and its derivatives for two different values of t_2.

Closing this section, we characterize $\Delta_1(t_1, t_2)$ as a function of t_2.

L e m m a 3. *Let Assumptions 1 and 2 be true. Then the payback period $\Delta_1 = \Delta_1(t_1, t_2)$, considered as a function of t_2, has the following properties. Let a point $\mathbf{t}_1 \in [0, \bar{\bar{t}}_1]$ be fixed. Then $\Delta_1(\mathbf{t}_1, t_2)$ is constant in the interval $[0, \mathbf{t}_1]$: $\Delta_1(\mathbf{t}_1, t_2) \equiv g_2(\mathbf{t}_1)$; it decreases to $g_1(\mathbf{t}_1)$ in the interval $[\mathbf{t}_1, \mathbf{t}_1 + g_1(\mathbf{t}_1)]$; and it is constant again if $t_2 > \mathbf{t}_1 + g_1(\mathbf{t}_1)$: $\Delta_1(\mathbf{t}_1, t_2) \equiv g_1(\mathbf{t}_1)$. If $\mathbf{t}_1 \in [\bar{\bar{t}}_1, t_1^{**}]$, then $\Delta_1(\mathbf{t}_1, t_2)$ is defined only for t_2 such that $\int_{t_1}^{T} \varphi_1(t \mid t_2)dt \geq C_1$, it decreases for $t_2 < t_1^{**}$ and is constant on $[t_1^{**}, T]$.*

4. Best Reply Functions

As in the previous section we study the situation from the point of view of participant 1. The goal of this section is to to solve Problem 1, i.e., to construct the best reply $t_1^0(t_2 \mid \alpha_1)$ of participant 1 to a strategy t_2 of participant 2. Recall that $t_1^0(t_2 \mid \alpha_1)$ is defined in (2.6).

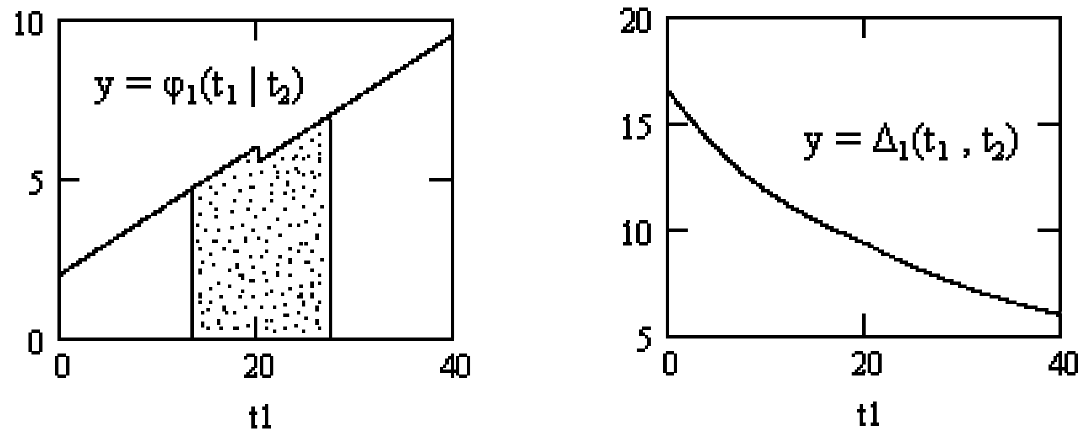

Fig.1-2. The benefit rate $\varphi_1(t_1|t_2)$ for $t_2 = 20 \in (\bar{t}_1, \bar{\bar{t}}_1)$ and the corresponding payback period $\Delta_1(t_1, t_2)$.

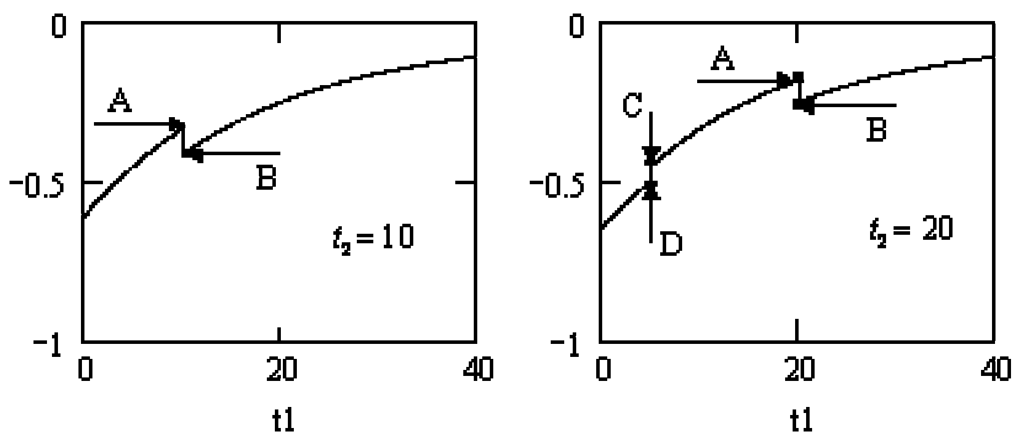

Fig.3-4. The derivative $\frac{\partial \Delta_1(t_1,\ t_2)}{\partial t_1}$ for two different values of t_2. A – D are the discontinuity points.

Let us introduce the following notations:

$$p_{11}(t) = \frac{\varphi_{11}(t)}{\varphi_{11}(t + g_1(t))}, \tag{4.1}$$

$$p_{12}(t) = \frac{\varphi_{12}(t)}{\varphi_{12}(t + g_2(t))}, \tag{4.2}$$

$$q_{11}(t) = \frac{\varphi_{11}(t)}{\varphi_{12}(t + g_1(t))}, \tag{4.3}$$

$$q_{12}(t) = \frac{\varphi_{11}(t)}{\varphi_{12}(t + g_2(t))}. \tag{4.4}$$

L e m m a 4. *Let Assumptions 1 and 2 be true. Then the functions $p_{11}(t), p_{12}(t)$, $q_{11}(t)$ and $q_{12}(t)$ are continuous and monotonously increasing in their domains of definition. Noreover, for each t the next inequalities hold:*

$$q_{11}(t) > q_{12}(t) > p_{12}(t), \;\; q_{11}(t) > p_{11}(t) \tag{4.5}$$

P r o o f. The inequalities (4.5) follow directly from the assumed inequalities $\varphi_{11}(t) > \varphi_{12}(t)$, $g_2(t) > g_1(t)$ and the monotonicity of $\varphi_{1j}(t)$. To prove that $p_{11}(t), p_{12}(t)$, $q_{11}(t)$ and $q_{12}(t)$ are increasing, let us estimate their derivatives. We have:

$$\frac{dp_{11}(t)}{dt} = \frac{\frac{d\varphi_{11}(t)}{dt}\varphi_{11}(t+g_1(t)) - \varphi_{11}(t)\frac{d\varphi_{11}}{dt}(t+g_1(t))(1+\frac{dg_1(t)}{dt})}{\varphi_{11}^2(t+g_1(t))} =$$

$$\frac{\frac{d\varphi_{11}(t)}{dt}\varphi_{11}(t+g_1(t)) - \varphi_{11}(t)\frac{d\varphi_{11}}{dt}(t+g_1(t)) + \varphi_{11}(t)\frac{d\varphi_{11}}{dt}(t+g_1(t))|\frac{dg_1(t)}{dt}|}{\varphi_{11}^2(t+g_1(t))}.$$

Due to the monotonicity and concavity of $\varphi_{11}(t)$ we have

$$\varphi_{11}(t+g_1(t)) > \varphi_{11}(t)$$

and

$$\frac{d\varphi_{11}(t)}{dt} > \frac{d\varphi_{11}}{dt}(t+g_1(t)).$$

Therefore,

$$\frac{dp_{11}(t)}{dt} > \frac{\varphi_{11}(t)\frac{d\varphi_{11}}{dt}(t+g_1(t))|\frac{dg_1(t)}{dt}|}{\varphi_{11}^2(t+g_1(t))} > 0.$$

The monotonicity of $p_{12}(t)$, $q_{11}(t)$ and $q_{12}(t)$ is stated similarly.

Let us fix an α_1 such that

$$q_{11}(0) < 1 - \alpha_1 < p_{1j}(t_1^{**}) \;\; (j = 1, 2) \tag{4.6}$$

and define points t_1^-, t_1^+ , t_1^q and t_2^q as solutions to the following equations

$$t_1^- : p_{11}(t) = 1 - \alpha_1,$$

$$t_1^+ : p_{12}(t) = 1 - \alpha_1,$$

$$t_1^q : q_{11}(t) = 1 - \alpha_1,$$

$$t_2^q : q_{12}(t) = 1 - \alpha_1.$$

Note that such an $\alpha_1 < 1$ exists and the roots of the above equations are defined uniquely. Based on α_1, we construct the function $t_1^0 = t_1^0(t_2 \mid \alpha_1)$ (which can be multi-valued at some points). Due to inequalities (4.5) we have $t_1^q < t_2^q < t_1^+$ and $t_1^q < t_1^-$. Generally, there are several opportunities for the location of t_1^- with respect to t_1^+ and t_2^q. In what follows, we deal with the case

$$t_1^q < t_2^q < t_1^- < t_1^+ \tag{4.7}$$

(other locations of t_1^- can be studied similarly).

Theorem 1. *Let Assumptions 1 and 2 be true.*

a) If $0 \le t_2 < t_2^q$, then $t_1^0(t_2 \mid \alpha_1) = t_1^+$.

b) If $t_2 \ge t_2^q$ and $t'(t_2) < t_1^q$, then the set $t_1^0(t_2 \mid \alpha_1)$ consists of one or two points, $\gamma_1(t_2)$ and t_1^+, where $\gamma_1(t_2)$ is the unique solution to the equation

$$\frac{\varphi_{11}(\gamma_1)}{\varphi_{11}(\gamma_1 + \Delta_1(\gamma_1, t_2))} = 1 - \alpha_1. \tag{4.8}$$

c) If $t_1'(t_2) = t_1^q$, then $\Delta_1(t_1, t_2) = g_1(t_1)$ and t_1^q is the root of equation (4.8). If $t_2^q < t_1'(t_2) < t_1^-$, then the set $t_1^0(t_2 \mid \alpha_1)$ contains the unique point $t_1'(t_2)$.

d) If t_2 is such that $t_1'(t_2) \ge t_1^-$, then the set $t_1^0(t_2 \mid \alpha_1)$ contains the unique point t_1^-.

Proof. From Assumptions 1 and 2 and Lemmas 1–4 it follows that for α_1 specified above the minimum of $f_1(t_1, t_2 \mid \alpha_1)$ with respect to t_1 is achieved only in the points where the derivative $\dfrac{\partial f_1(t_1, t_2 \mid \alpha_1)}{\partial t_1}$ changes its sign from "-" to "+" (the boundary points are excluded). The above derivative $\dfrac{\partial \Delta_1(t_1, t_2)}{\partial t_1} + \alpha_1 - 1$ has the same properties as $\dfrac{\partial \Delta_1(t_1, t_2)}{\partial t_1}$. Namely, it has a structure shown in Fig. 3–4. The curve located on the right to point B (see Fig. 3–4) coincides with the graph of function p_{12}, and while t_2 increases point B moves along this curve. In the same time, point A belongs to the graph of function q_{11} and moves along it. Point C lies on curve q_{12}. Finally, point D moves along curve p_{11}, and the part of the graph of $\dfrac{\partial \Delta_1(t_1, t_2)}{\partial t_1}$ which is located on the left to D, coincides with curve p_{11}. Note that for small t_2 points C and D vanish, while as t_2 approaches the final time T points A and D vanish.

The above properties and the relations between functions p_{1j} and q_{1j} allow us to prove the theorem. Let t_2 grow from zero to T. In case a) point A lies below the line $l : f \equiv 1 - \alpha_1$, and the signle minimum point is t_1^+. In case b) another possible minimum point appears: $\gamma_1(t_2)$. Graphically, it represents the intersection of line l and curve CA. Due to (3.1), starting from some value for t_2, point t_1^+ becomes no longer "suspicious" as a minimum point, and the unique solution is $\gamma_1(t_2)$ or $t_1'(t_2)$. At the latter point the function is discontinues (case c)). As point D reaches line l, we switch to curve p_{11} and find the unique solution t_1^- (case d)). This finalizes the proof.

Note that in the case under consideration the structure of function $t_1^0(t_2 \mid \alpha_1)$ is similar to the one presented in [1] but does not fully coincide with it. Namely, in [1] the domain of definition of function $t_1^0(t_2 \mid \alpha_1)$ consists of two intervals, in each of which the function is constant; in our situation, we also have two intervals and in one of them the function is not constant (it is constant only on a subinterval with the end point $t_2 : t_1'(t_2) \geq t_1^-$).

In section 6, we provide an example of a best reply function (Fig. 5).

5. Nash Equilibrium Solutions

Recall that the domain of definition of function $t_i^0(t_j \mid \alpha_i)$ $(i, j = 1, 2;\ i \neq j)$ contains an interval in which $t_i^0(t_j \mid \alpha_i)$ is not constant. This fact makes it difficult to state the existence of Nash equilibrium solutions straighforwardly and to define an algorithm for their construction.

Lemma 5. *A pair $\{\widehat{t}_1, \widehat{t}_2\}$ is a Nash equilibrium solution if and only if $\{\widehat{t}_1, \widehat{t}_2\}$ considered as a point on the (t_1, t_2) plane belongs to both graphs $t_1 = t_1^0(t_2 \mid \alpha_1)$ and $t_2 = t_2^0(t_1 \mid \alpha_2)$ plotted on the (t_1, t_2) plane.*

The lemma follows straitforwardly from the definition of Nash equilibrium solutions.

To find Nash equilibrium solutions, we use Theorem 1 and construct the curves indicated in Lemma 5. If these curves have common points, these points represent the sought Nash equilibrium solutions. Fig 6 shows an example illustrating the case of two Nash equilibrium solutions.

In the rest of this section we construct an algorithm for searching Nash equilibrium solutions in the case where the best reply functions are approximated by piecewise-constant functions. This case is meaningful for practice. A similar algorithm can also be applied to more general situations when each participant has more than two scenarios for benefit rates, which correspond to multiple operation modes for the gas pipelines. In what follows, the set of Nash equilibrium solutions will be denoted as NEP.

Let the best reply function for participant 1, $t_1^0 = t_1^0(t_2 \mid \alpha_1)$, take a finite number of values t_{1n}^0, $n = 1, \ldots, N$ in intervals (ξ_{n-1}, ξ_n), respectively. At points ξ_n , $n = 1, \ldots, N-1$ the set $t_1^0(t_2 \mid \alpha_1)$ consists of the two points t_{1n}^0 and $t_{1(n+1)}^0$. Similarly, for function $t_2^0 = t_2^0(t_1 \mid \alpha_2)$ we denote by t_{2m}^0 the constant values it takes on intervals (η_{m-1}, η_m) respectively, $m = 1, \ldots, M$. We set $\xi_0 = \eta_0 = 0$, $\xi_N = \eta_M = T$. We put the points t_{2m}^0 in the increasing order; note that the boundaries of the intervals, on which function $t_2^0 = t_2^0(t_1 \mid \alpha_2)$ takes values t_{2m}^0 are not ordered.

Let us describe a finite-step algorithm for finding the set NEP.

(A1) At step 1 mark points $t_{21}^0, \ldots, t_{2k_1}^0 \in [\xi_0, \xi_1]$. If there are no such points, we go to step 2. If such points exist, for each $m = 1, \ldots, k_1$ we check the relation

$$t_{1n}^0 \in [\eta_{m-1}, \eta_m]. \tag{5.1}$$

If this relation holds true, then the pair $\{t_{1n}^0, t_{2m}^0\}$ is attributed to the set NEP.

(A2) For an arbitrary step n, we observe the index k_{n-1} formed at the previous step; k_{n-1} corresponds to points t_{2m}^0 that have already been analyzed. If $k_{n-1} = M$, the algorithm stops. If $k_{n-1} < M$, new points $t_{2m}^0 \in (\xi_{n-1}, \xi_n]$, $m \geq k_{n-1} + 1$ are marked, and a new value for k_n is formed. Next, for $m = k_{n-1} + 1, \ldots, k_n$ one checks the relation (5.1). The pairs $\{t_{1n}^0, t_{2m}^0\}$, for which (5.1) holds true, are attributed to the set NEP. If $k_n < M$, one unit is added to n, and we go to step $n + 1$. If $k_n = M$, the algorithm stops.

Theorem 2. *In case of piecewise-constant best reply functions, algorithm (A1) – (A2) finds the set NEP of all Nash equilibrium solutions.*

6. Example

In this section we specify the above constructions for an example, in which the benefit rates are linear. Assume that

$$\varphi_{ij}(t) = a_{ij}t + b_{ij}, \tag{6.1}$$

where $0 < a_{i1} \le a_{i2}$, $0 < b_{i2} < b_{i1}$, $0 \le t \le T$; $i = 1, 2$. In this case Assumption 1 holds if

$$a_{i2}T^2 + 2b_{i2}T > 4C_i. \tag{6.2}$$

We easily find explicit formulas for points $t_i^*, \bar{t}_i, \bar{\bar{t}}_i$ and t_i^{**} and function $t_i'(t_j)$, $i \ne j$. In particular,

$$t_i^* = \frac{1}{a_{i2}}(-b_{i2} + \sqrt{b_{i2}^2 + 2a_{i2}C_i});$$

$$\bar{\bar{t}}_i = \frac{1}{a_{i2}}(-b_{i2} + \sqrt{b_{i2}^2 + a_{i2}(a_{i2}T^2 + 2b_{i2}T - 2C_i)}).$$

Let us also give formulas for $\Delta_i(t_1, t_2)$ and $\dfrac{\partial f_1}{\partial t_1}$ in the simplest case where $a_{11} = a_{12} = a_1$. We have:

$$\Delta_1(t_1, t_2) =$$

$$= \begin{cases} -\frac{1}{a_1}(a_1t_1 + b_{11} + \sqrt{(a_1t_1 + b_{11})^2 + 2a_1C_1}) & 0 \le t_1 \le t_1'(t_2) \\ -\frac{1}{a_1}(a_1t_1 + b_{12} + \sqrt{(a_1t_1 + b_{12})^2 + 2a_1C_1 - 2(b_{11} - b_{12})(t_2 - t_1)}) & t_1'(t_2) \le t_1 \le t_2 \\ -\frac{1}{a_1}(a_1t_1 + b_{12} + \sqrt{(a_1t_1 + b_{12})^2 + 2a_1C_1}) & t_2 \le t_1 \le t_1''(t_2). \end{cases}$$

$$\frac{\partial f_1}{\partial t_1} = \begin{cases} \alpha_1 - 1 + \dfrac{a_1t_1 + b_{11}}{\sqrt{(a_1t_1 + b_{11})^2 + 2a_1C_1}} & 0 < t_1 < t_1'(t_2) \\ \alpha_1 - 1 + \dfrac{a_1t_1 + b_{11}}{\sqrt{(a_1t_1 + b_{12})^2 + 2a_1C_1 - 2a_1(b_{11} - b_{12})(t_2 - t_1)}} & t_1'(t_2) < t_1 < t_2 \\ \alpha_1 - 1 + \dfrac{a_1t_1 + b_{12}}{\sqrt{(a_1t_1 + b_{12})^2 + 2a_1C_1}} & t_2 < t_1 < t_1''(t_2). \end{cases}$$

If $0 < \alpha_1 < 1$, the expressions for t_1^- and t_1^+ take the form:

$$t_1^- = \frac{1}{a_{11}}\left(-b_{11} + (1 - \alpha_1)\sqrt{\frac{2a_{11}C_1}{\alpha_1(2 - \alpha_1)}}\right),$$

$$t_1^+ = \frac{1}{a_{12}}\left(-b_{12} + (1 - \alpha_1)\sqrt{\frac{2a_{12}C_1}{\alpha_1(2 - \alpha_1)}}\right).$$

Assumption 2 imposes stronger constraints on parameter values. Note that these constraints are feasible (we omit a rigorous formulation involving a number of technical detailes).

We finalize the section by presenting some numerical results. Let both participants have same coefficients in equality (4.8), defined as follows:

$$a_{ij} = 0.2;\ b_{i1} = 2;\ b_{i2} = 1.5;\ i, j = 1, 2.$$

We set $\alpha_i = 0.5$, $T = 40$, $C_i = 60$. It is easy to find points (2.2). We have $t_i^* = 18.12$, $\bar{\bar{t}}_i = 33.20$, and Assumption 1 is true. Functions $\varphi_i(t_i|t_j)$ and $\Delta_i(t_i, t_j)$ are shown in Fig. 1–2. The derivative of $\Delta_i(t_i, t_j)$ is shown in Fig. 3–4. Finally, Fig. 5–6 present a graphical illustration of the best reply function and Nash equilibrium solutions. Note that in the considered situation (both participants have the same parameters) the Nash solutions are symmetrical: see (4.14), (6.64) and (6.64), (4.14).

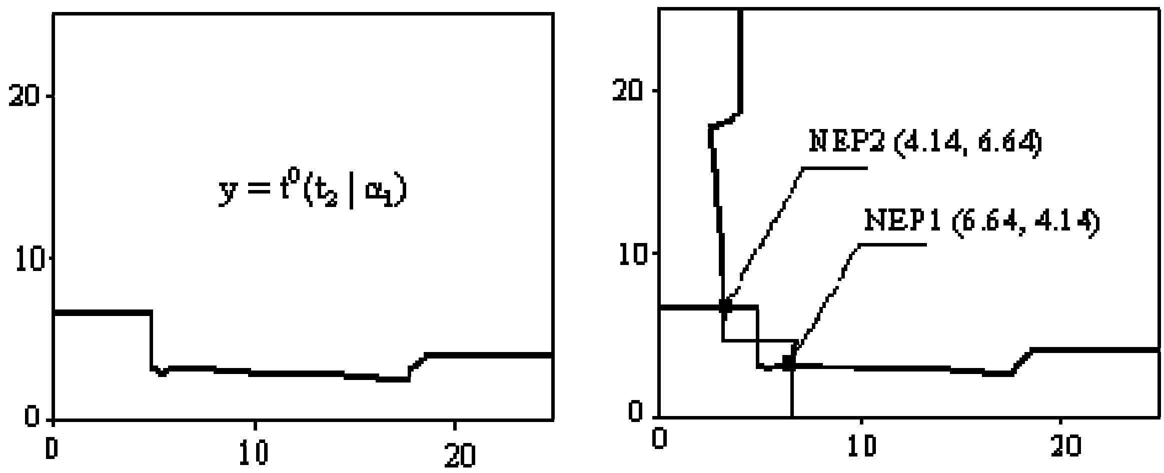

Fig.5-6. Best reply function and Nash eqilibrium points.

7. Conclusion

This paper is motivated by the issue of planning and putting into operation new gas pipeline systems. We proposed a new problem setting reflecting situations in which the price formation mechanism had not a purely market character. Mathematically, we formulated the problem as a non-cooperative two-person game. We analyzed the best reply functions and described an algorithm for finding Nash equlibrium solutions in the game.

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