VALUATION OF GAMES AND OTHER DISCRETE PRODUCTION PROCESSES BY COMPETITIVE BIDDING*

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ABSTRACT

The values of the players in an n-person cooperative game are analyzed by considering a simple auction model in which outside "entrepreneurs" bid to acquire control of the players. This bidding procedure always has a Nash equilibrium in pure strategies, thus yielding a concept of "market values" for the players. This class of values is easy to characterize and contains the core of the game. The model applies to various valuation problems (such as estate auctions, the setting of wage structures for laborers, or the valuation of divisions of a corporation) in which indivisible factors are present and there may be increasing returns to scale.

An Estate Valuation Problem

Uncle Rufus, a bachelor, dies leaving his estate to his three maiden sisters, Irma, Ora, and Mildred. The estate consists of three farm properties plus cash in the bank. Rufus's will specifies that Irma is to get the hill farm, Ora the prairie land, and Mildred the main house and surrounding pasture. However the will also specifies that each sister is to receive an equal share of the total estate, any differences in the value of the parcels being made up by a suitable distribution of the cash.

Unfortunately, a dispute arises concerning the value of the three properties. An appraiser is called in, and gives the following values:

	HF	\$130,000
(1)	MH	\$148,000
	PL	\$155,000 .

But Irma maintains that the appraisal underestimates the value of her sisters' shares: the main house and prairie land *together* are worth much more than \$303,000, since they can be farmed as a unit, realizing economies of scale. Irma contends she is being treated unfairly because of the way the property was divided. But Ora notes that the same argument can be made against the valuation of Irma's and Mildred's shares, since the hill farm could also be combined advantageously with the main house and pasture. Mildred has a similar complaint.
The fact is that the property kept all together as Rufus had it would be considerably more valuable than the appraised values of its parts.

So the appraiser is recalled and gives the following estimates:

	HF & MH	\$310,000
	HF & PL	\$320,000
(2)	PL & MH	\$420,000
	HF & PL & MH	\$550,000 .

The greatest loss in efficiency is in dividing the main house from the prairie land, but all combinations of two properties realize some economies of scale. The conjunction of the hill farm with the main house/prairie land combination does not produce any further economies of scale however. The problem is how to determine the relative values of the three individual parcels specified in Uncle Rufus's will.

This anecdote illustrates a very general problem: how to put a value on productive factors *individually*, given that these factors have value *jointly*.

In this paper we propose a way of valuing a collection of indivisible factors used to produce a single divisible output. A particular application is the valuation of players in cooperative n-person games with transferable utility.

Production Functions and Games

Let the numbers $\{1,2,\ldots,n\} = N$ denote the available *factors* of production, which may be completely indivisible. The *production function* is a function v on all possible subsets of N. We assume that

(3) $$v(\phi) = 0 \quad \text{and} \quad v(S) \geqq 0 \quad \text{for all } S \subseteq N \quad .$$

(4) $$v(S \cup T) \geqq v(S) + v(T) \quad \text{for all } S, T \subseteq N \quad .$$

(1) asserts that production is valuable but we need not produce, (2) asserts the possibility of joint production. This format is quite general, and allows for production functions of virtually any degree of combinatorial complexity, and for any amount of factor differentiation.

v may also be interpreted as the characteristic function of a cooperative n-person game with transferable utility. In this case the players are the factors, and the output of a coalition is the total utility it can achieve, which is assumed to be completely transferable--not only among the players, but also to agents outside of the game.

Valuation by Auction

We propose to determine the fair value of the factors by holding a simultaneous competitive auction for them. By "simultaneous" is meant that all factors are bid for simultaneously; by competitive is meant that there are at least two buyers. Each of the m buyers ($m \geqq 2$) is assumed to perceive exactly the same production function v, i.e. the "true" value of the possible combinations. The nontriviality of the auction results from the fact that the values this implies for the individual factors may not be at all obvious. The competitive bidding process establishes bounds on what those values should be.

The bidding procedure is the natural one: each bidder k names the amounts $p_1^k, \ldots, p_n^k$ that he is willing to pay for each of the factors; he is also free not to bid, or to bid for only some of the factors. Thus a *bid* is a vector $\underset{\sim}{p}^k$ consisting of nonnegative real numbers p_i^k and blanks (for no bid). The *prices* resulting from a collection of bids $\{\underset{\sim}{p}^1, \ldots, \underset{\sim}{p}^m\}$ is the vector of real numbers $\underset{\sim}{p}$ defined by

$$p_i = \max_{1 \leqq k \leqq m} \{p_i^k, 0\} \quad .$$

Thus the price of i is the maximum bid for i, or, if nothing is bid, then it is zero.

In order to allow for the efficient resolution of ties, we assume that the bidders line up in order of the total amount bid. Thus if k_1 is first in line, k_2 second, and so forth then $|\underset{\sim}{p}^{k_1}| \geqq |\underset{\sim}{p}^{k_2}| \geqq \cdots \geqq |\underset{\sim}{p}^{k_m}|$. Ties in the total size of bid are assumed to be resolved by some initially given "alphabetical" ordering of the bidders.

Given all bids and the resulting line-up, the first bidder in line takes all the factors for which he bid highest, the next bidder all the *remaining* factors for which he bid highest, and so on. This arrangement allows the bidders to acquire control of *combinations* of factors, which is of course their objective. This arrangement will be called the *pure bidding model*.

Equilibrium Prices and the Core

We now formally describe the bidding process as a game. Assume that the bidders are indexed such that 1 is first in line, 2 second, and so forth. Given bids $p^1, p^2, \ldots, p^m$, and associated prices p, the set of factors acquired by bidder 1 is $T_1 = \{i \in N : p_i^1 = p_i\}$. Similarly $T_2 = \{i \in N - T_1 : p_i^2 = p_i\}$ is the set of factors acquired by bidder 2, and so forth. This defines a disjoint collection of sets $T_1, T_2, \ldots, T_m$, where $\bigcup_k T_k$ is the set of all factors for which some bid was made.

The *payoff* to each bidder k, given bids $p^1, p^2, \ldots, p^m$, is evidently

$$\varphi_k(p^1, \ldots, p^m) = v(T_k) - \sum_{T_k} p_i \quad .$$

If $T_k = \phi$ the payoff is zero. Bids are assumed to be announced publicly, and bidders can revise their bids in light of the others'. After a time, we might expect that the bids reach an *equilibrium*, in the sense that no bidder can change his bid and increase his payoff, assuming that all other bidders hold fast. Any price vector p arising from such a set of equilibrium bids will be called an *equilibrium price vector*.

The *core* of the production function is the set of all vectors $p \geqq 0$ such that

$$\sum_S p_i \geqq v(S) \qquad \text{for all } S \subset N \quad ,$$

and

$$\sum_N p_i = v(N) \quad .$$

Theorem 1. For the pure bidding model, p *is an equilibrium price vector for* v *if and only if* p *is in the core of* v.

The proof is given in the Appendix.

At equilibrium prices p, no bidder buys a set of factors for more than they are jointly worth, since he could always simply not bid and thereby avoid a loss. On the other hand, since such a p is in the core, no bidder pays *less* than the factors are jointly worth. Hence any set T of factors bought by a given bidder has the property that $\sum_{T} p_i = v(T)$. In other words, everyone's payoff is zero.

The specification of those combinations of factors actually bought by the individual bidders at prices $\underset{\sim}{p}$ is called a *sale*; a sale is thus a family of m disjoint sets $T_1,\ldots,T_m$, and in equilibrium, $v(T_k) = \sum_{T_k} p_i$ for every k. Some of the T_k may, of course, be empty.

In the case of Uncle Rufus's farm, the core exists and consists of all 3-vectors summing to $550,000 that lie on the solid line segment shown in Figure 1. These vectors represent the possible equilibrium prices for the properties that could result from a simultaneous auction.

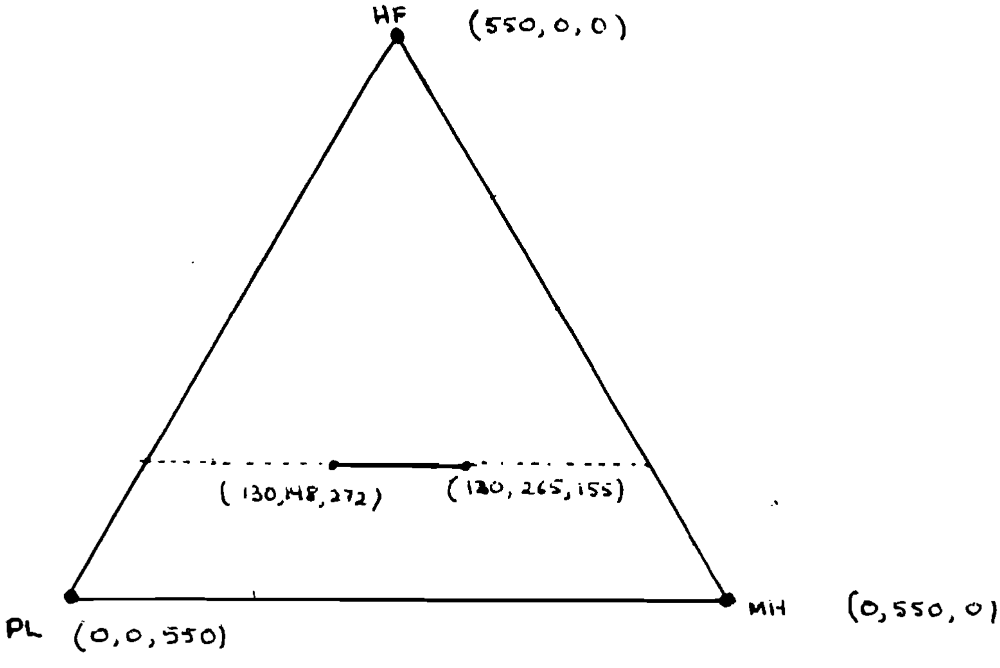

Figure 1.

A given equilibrium price vector such as $\bar{p} = (130,230,190)$ could arise in many different ways. For example, with three bidders $\bar{p}$ results from the equilibrium bids (to mention but two possibilities)

first bidder	→	(120,140,180)
second bidder	→	(0,230,190)
third bidder	→	(130,75,190)

as well as from

first bidder	→	(130,230,190)
second bidder	→	(130,230,190)
third bidder	→	(130,230,190).

In the first case the first bidder gets nothing; the second bidder gets the main house plus prairie land and the third bidder gets the hill farm. In the second case the first bidder takes all three properties.

Nonexistence of the Core and Option Holders

Theorem 1 says that, if the core exists, then a valuation of the factors by an outside competitive market results in a vector in the core. This is a pleasant result, since this is precisely the answer one would expect if the factor representatives (e.g. the three maiden ladies) try to reach an agreement among themselves without appealing to an outside market.

However, what happens if the numbers are different and the core does not exist? In that case it would appear much more difficult to predict either the result of an internal agreement or the outcome of an auctioning procedure. Nevertheless there is a straightforward procedure for determining value in this case.

The key point is to recognize that the ladies may not really wish to sell their inheritances, but only get the market to establish fair values for them. In fact, estate auctions are often held for exactly this purpose, and the device frequently used to retain the property is for the trustee of the estate to hold an *option* that allows him to match any and all other *bona fide* offers.

In this procedure, called the *bidding with option model*, there are two or more bidders and, in addition, an "insider" who holds an option to meet any and all bid prices established by the others. The bidding procedure is exactly as before, with precedence in line determined by total size of bid. But the payoff structure is different: given a set of bids and the associated factor prices $\underset{\sim}{p}$, the option holder is entitled to buy any set of factors he wants at these prices. Evidently his best response is to buy some set T such that $v(T) - \sum_T p_i$ is a maximum. This leaves the factors $N - T$, which must be purchased by those who bid highest for them, the bidders being called up in order. The fact that the highest bidders have to accept what the option holder rejects prevents bidders from inflating prices merely to "spite" the option holder. (The option holder need not buy anything, of course.)

Formally, let $\underset{\sim}{p}^1, \underset{\sim}{p}^2, \ldots, \underset{\sim}{p}^m$ be a set of bids, where we may assume that $\underset{\sim}{p}^k$ represents the kth highest total bid. (A bid may contain no entries for some factors, meaning "no bid.") The factor *prices* associated with these bids are $p_i = \max\{p_i^k, 0\}$, where $p_i = 0$ if no bids for i were made.

Given prices $\underset{\sim}{p}$ the *response* of the option holder is to buy some maximum profit set. The choice of this set is assumed to depend only on $\underset{\sim}{p}$, not on the individual bids generating $\underset{\sim}{p}$. For most $\underset{\sim}{p}$ there is only one such set, but to deal with ties some specific tie-breaking rule or *response function* g must be assumed. Thus for every $\underset{\sim}{p}$, $g(\underset{\sim}{p})$ is some set T^* maximizing $v(T) - \sum_T p_i$ over all $T \subseteq N$. We say that g is *efficient* if, moreover, $g(\underset{\sim}{p})$ has highest *value* among all maximum profit sets, that is, $v(g(\underset{\sim}{p})) \geqq v(T)$ for all maximum profit sets T under prices $\underset{\sim}{p}$.

Given $\underset{\sim}{p}^1, \underset{\sim}{p}^2, \ldots, \underset{\sim}{p}^m$ and prices $\underset{\sim}{p}$, define as before the sets $T_1 = \{i \in N : p_i^1 = p_i\}$, $T_2 = \{i \in N - T_1 : p_i^2 = p_i\}$, and so forth; these are the factor sets the bidders would get if the option were not exercised. The sets actually obtained are $U_k = T_k - S$, where $S = g(\underset{\sim}{p})$ and the payoff to the kth bidder in line is $v(U_k) - \sum_{U_k} p_i$.

Now S is a maximum profit set with profit $v(S) - \sum_S p_i$; thus by superadditivity

$$[v(S) - \sum_S p_i] + [v(U_k) - \sum_{U_k} p_i] \leqq v(S \cup U_k) - \sum_{S \cup U_k} p_i \leqq v(S) - \sum_S p_i$$

whence

$$v(U_k) - \sum_{U_k} p_i \leqq 0 \qquad \text{for all } k.$$

Hence for *any* bids the payoffs to all bidders will be nonpositive. What then is the incentive to bid?

The answer must be the common-sense one that, in practice, bidders imagine that there is a small chance that the option holder will forfeit by dropping dead, going bankrupt, or suffering some other act of God. Moreover, there is no need to engage in calculations involving the probability of such an event; it seems safe to assume that each of the bidders will act so as to suffer no loss if the option holder *does* exercise his right, but will at the same time work to increase the prospect of his payoff just in case the option holder does *not* exercise his right.

The payoff to the kth bidder in line if the option holder forfeits is the *forfeit payoff*

$$\varphi_k(g; \underset{\sim}{p}^1, \ldots, \underset{\sim}{p}^m) = v(T_k) - \sum_{T_k} p_i \quad ,$$

whereas if he does not forfeit it is the *no-forfeit payoff*

$$\nu_k(g; \underset{\sim}{p}^1, \ldots, \underset{\sim}{p}^m) = v(U_k) - \sum_{U_k} p_i \quad .$$

A set of bids is in *equilibrium* for the bidding with option game, and $\underset{\sim}{p}$ is a set of *equilibrium factor prices*, if the no-forfeit payoff for all bidders is zero, and no single bidder can change his bid such that his forfeit payoff increases while his no-forfeit payoff remains non-negative.

The introduction of the option holder is not only natural in the context of the problem, it turns out to guarantee that equilibrium prices always exist.

Before stating this theorem, let us consider the previous example, now somewhat modified: suppose that the value of the hill farm together with the main house is $390,000 instead of $310,000. Consider two bidders A, B, and suppose that A takes precedence in line over B in case of a tie. Let us further suppose that they have both read a text on game theory, and seeing that the core does not exist, begin by both bidding the nucleolus:

$$A \rightarrow (140,240,170)$$
$$B \rightarrow (140,240,170) \quad .$$

At these prices the most profitable sets of properties are all combinations of two: each such combination has a value of $10,000 in excess of its price, whereas all three properties cost together as much as they are worth.

Say the option holder takes the hill farm and main house. Then A must make good his offer of $170,000 for the prairie land. Therefore he suffers a loss, since the prairie land is only worth $155,000. A similar conclusion holds no matter which most profitable set the option holder decides to take. The conclusion is that these prices are not in equilibrium -- they are too high. Bidder A would do better to lower his prices, since then he at least does not risk taking a loss.

Next suppose that A decreases his bid as follows and B stays fixed:

$$B \rightarrow (140,240,170)$$
$$A \rightarrow (130,148,155) \quad .$$

B is now first and suffers a loss if the option is exercised.

Next let B change his bid to obtain

$$B \rightarrow (130,230,160)$$

$$A \rightarrow (130,148,155) \quad .$$

B is still first, prices are the same as B's bid, and the most profitable sets are {HF,MH}, {HF,PL}, {MH,PL}, and {HF,MH,PL}, each yielding a profit of $30,000. If at these prices the option holder buys {MH,PL}, then B gets HF but his payoff is zero; A also gets nothing. However, if the option holder forfeits, then B takes {HF,MH,PL} at a profit of $30,000. Again A gets nothing. But this is not an equilibrium, since A can raise his bid to match B's and because of the tie breaking rule, A goes to the fore:

$$A \rightarrow (130,230,160)$$

$$B \rightarrow (130,230,160) \quad .$$

Now A and B get no-forfeit payoffs of 0, and forfeit payments of $30,000 and 0 respectively. At this point equilibrium has been reached. For if B outbids A in total he will have to pay too much for some properties and the option holder could leave him with a loss, while A cannot lower his bid without losing his desirable place in line.

<u>Theorem 2</u>. *Under bidding with option, an equilibrium price vector exists for any efficient response function, and* $\underset{\sim}{p} \geqq \underset{\sim}{0}$ *is a set of equilibrium prices if and only if*

(i) N is a maximum profit set,

(ii) no factor is in every maximum profit set.

The existence of a $\underset{\sim}{p}$ satisfying the conditions is easily seen: beginning with $\underset{\sim}{p} = \underset{\sim}{0}$ choose any player i and raise his price just until he is no longer in every maximum profit set. At these prices if there remains any player j who is in every maximum profit set raise his price just until he is not in every maximum profit set. Continue raising prices of the players successively in this manner. The process must terminate, since maximum profitability can never be less than zero (by virtue of the empty set) and at termination N is still a maximum profit set. For a proof of Theorem 2 see the more general Theorem 3 below and its proof in the Appendix.

Any vector p satisfying conditions (i) and (ii) above is called a *market value* for v.

We note that, for any market value p, $p_i \geqq v(i)$ for all i. Indeed by condition (ii) there is a maximum profit set S not containing i, but $p_i < v(i)$ would imply, by superadditivity, that $S \cup \{i\}$ were more profitable than S, a contradiction.

If the core of v exists, then every vector in the core is also a market value. Indeed, for any p in the core, maximum profitability is zero, which is achieved both on N and on the empty set, so (i) and (ii) are both satisfied. Thus the notion of equilibrium price vector generalizes the concept of the core in a natural way. Moreover, the competitive bidding model shows how the core of v arises as a *noncooperative* solution of a "supergame" played for v. But the core--even if it exists for a given v--need not constitute the only market values for v: some factors may be able to get more than any imputation in the core would give them.

Opportunity Costs

In general, items offered at auction often carry *reservation bids or floor prices*, which effectively establish lower bounds on acceptable offers. From the point of the owner of a factor, the floor price represents either an estimate of the inherent worth of the factor to the owner, or an *opportunity cost*, i.e. its potential value in some context other than the auction. This value is not generally an "output", hence it is advisable to maintain a distinction between floor prices and the quantities v(i).

Given floor prices $(p_1^0, p_2^0, \ldots, p_n^0) = p^0$, a *bid* is now a vector p^k consisting of real numbers $p_i^k \geqq p_i^0$ and blanks (meaning no bid). The *factor prices* determined by a family of bids $p^1, p^2, \ldots, p^m$ is the vector p defined by

$$p_i = \max_k \{p_i^k, p_i^0\}, \text{ where } p_i = p_i^0 \text{ if no bids are made for } i.$$

A bidder's place in line is now determined by the total *surplus* he offers above the floor prices: that is, the bidders are ordered by the numbers $\pi_k = \sum_i (p_i^k - p_i^0)$, the sum over all i such that k bids for i.

It turns out that for any floor prices $\underset{\sim}{p}^0 \geqq \underset{\sim}{0}$, the bidding with option game always has a strong equilibrium. To describe these equilibria we define a new game $\tilde{v}$ as follows

$$\tilde{v}(S) = \max_{T \subseteq S} [v(T) + \sum_{S-T} p_i^0] \quad , \text{ for each } S \subseteq N \quad .$$

The value $\tilde{v}(S)$ can be interpreted as the maximum amount that could be obtained by collecting opportunity costs for some of the factors in S and using the rest to produce.

A *critical set* is any set S whose *productive surplus* $v(S) - \sum_S p_i^0$ is a maximum, and i is a *critical factor* if i is contained in every critical set. Note that $\tilde{v}(N) = v(S) + \sum_{N-S} p_i^0$ for any critical set S.

The response function $g(\underset{\sim}{p})$ is *efficient* if for every $\underset{\sim}{p} \geqq \underset{\sim}{p}^0$ $T = g(\underset{\sim}{p})$ is some maximum profit set having highest productive surplus. Theorem 2 now generalizes as follows.

Theorem 3. Given floor prices $\underset{\sim}{p}^0$, an equilibrium price vector exists for any efficient response function, and $\underset{\sim}{p}$ is an equilibrium price vector if and only if $\underset{\sim}{p}$ is a market value for $\tilde{v}$.

Theorem 2 is obtained from Theorem 3 by setting $\underset{\sim}{p}^0 = \underset{\sim}{0}$, in which case $\tilde{v} = v$. For the proof of Theorem 3, see the Appendix.

Auctioning the Estate

Irma, Ora, and Mildred decide to part with the properties after all, provided they can get enough for them. Irma is a little sentimental about the hill farm and wouldn't consider less than \$143,000 for it; Ora is wild about the main house and won't let it go for less than \$200,000, while Mildred could care less about the prairie and would settle for any price. Thus $\underset{\sim}{p}^0 =$ (143,000 , 200,000 , 0). The values of the properties are assumed to be modified earlier, with the hill farm and main house together worth $390,000.

The neighbor, meanwhile has got wind of their intention to sell, and being an old family friend, has secured an option to match any and all prices that the parcels might fetch at auction. In this case the auction would yield the *unique* equilibrium factor prices

HF	$143,000
MH	$243,000
PL	$173,000 .

At these prices any singleton set results in a loss for a buyer. The buyers' profits on the other combinations are

HF & MH	$4,000
HF & PL	$4,000
MH & PL	$4,000
HF & MH & PL	-$9,000 .

The unique critical set is the main house plus prairie land, and these are the two critical factors. In equilibrium, the neighbor will buy the main house and prairie land, and get them at a bargain: $4,000 less than their appraised value. Moreover, in equilibrium the hill farm will receive no bids, so Irma will keep it after all.

In terms of economic efficiency this solution makes sense, since by assumption the addition of the hill farm to the other two properties does not further increase productive efficiency while its utility to Irma is higher than to any outside buyer.

Conclusion

In summary, two related auction procedures for discrete productive factors have been described. The bidding procedure is the natural one: prices are formed simultaneously and competitively by taking the highest bid for each factor. The use of a precedence scheme for the buyers is a convention that allows for the equitable resolution of ties. In reality, of course, buyers will assess the value of the factors somewhat differently and ties would be a rarity.

Without an option holder an equilibrium only exists if the core of the production function exists, and in this case the possible equilibrium factor prices coincide with the imputations in the core. In the presence of an option holder -- a situation frequently encountered in practice -- equilibrium factor prices always do exist.

It might be said that this way of valuing the factors depends very much on the particular bidding procedure chosen. However, there is a strong argument that this valuation is the correct one quite independently of the bidding procedure since *exactly the same valuations result if the factor owners, rather than the buyers, set prices.*

Suppose indeed that the factor owners cannot agree on what auction procedure or other valuation method would be fair. The simplest solution is then for each factor owner to simply announce to a market of prospective buyers what he thinks he is worth. Given announced prices $\underset{\sim}{p} = (p_1, p_2, \ldots, p_n)$, the buyers arrive. The first in line helps himself to a maximum profit set (which may of course, be the empty set), leaving at best profitless sets for the others. Now let $g(\underset{\sim}{p})$ denote the set of *all* factors bought. This will always be a maximum profit set.

The factor owners are engaged in the following non-cooperative game: the strategy of factor i's owner is to name his price $p_i \geq p_i^0$ where p_i^0 is his opportunity cost, and the payoff to i given choices $\underset{\sim}{p} = (p_1, p_2, \ldots, p_n)$ is

$$p_i \quad \text{if} \quad i \in g(\underset{\sim}{p}) \quad ,$$

$$p_i^0 \quad \text{if} \quad i \notin g(\underset{\sim}{p}) \quad .$$

If g is efficient, a strong equilibrium in pure strategies always exists for this game. An appropriate measure of the *value* of players are their payoffs at such an equilibrium. As shown in [1] these equilibrium payoffs are precisely the market values for the game $\tilde{v}$.

Thus, whether prices are set by the owners of the factors in the face of a market of buyers, or by the buyers in a competitive bidding process, the outcomes are the same -- the class of market values.

APPENDIX

Proof of Theorem 1

Let 1,2,...,m be the "alphabetical" ordering, and let $\underset{\sim}{p}^1,\ldots,\underset{\sim}{p}^m$ be a set of equilibrium bids in the pure bidding model, $\underset{\sim}{p}$ the resulting set of factor prices. If T_k is the set of factors acquired by bidder k, the payoff to k is $q_k = v(T_k) - \sum_{T_k} p_i \geqq 0$.

Suppose, by way of contradiction, that some subset of N yields a positive profit at prices $\underset{\sim}{p}$. Say S, with profit $q = v(S) - \sum_S p_i > 0$ is a maximum profit set. If $q_h, q_k \geqq q$ for distinct h and k, then by superadditivity $T_h \cup T_k$ would have profit of at least 2q, contradicting the choice of q. Thus $q > q_k \geqq 0$ for some k. But then let k change his bid and offer $p_i + \varepsilon$ for all $i \in S$ and no bid for the other factors. Then k acquires precisely the set S, and for sufficiently small ε the payoff is larger than q_k, a contradiction. Therefore $q = 0$, that is

$$\sum_S p_i \geqq v(S) \quad \text{for all } S \subseteq N \quad .$$

Further, $q_k = 0$ for all k. Now the sale procedure implies that $p_i = 0$ for all $i \notin T = \bigcup_k T_k$. Hence

$$0 \geqq v(N) - \sum_N p_i = v(T) - \sum_T p_i \geqq \sum_k q_k = 0 \quad ,$$

the right-hand inequality by superadditivity. Thus $v(N) = \sum_N p_i$ and $\underset{\sim}{p}$ is in the core of v.

Conversely, let $\underset{\sim}{p}$ be in the core of v. Let all bidders bid $\underset{\sim}{p}$. The first bidder in line buys the set N, and the payoff to every bidder is zero. Let some bidder change his bid (in fact up to m - 1 bidders may change) and let the resulting factor prices be $\underset{\sim}{p}'$. Clearly $\underset{\sim}{p}' \geqq \underset{\sim}{p}$ so no set yields a positive payoff and no bidder is better off. □

Proof of Theorem 3

The market values for $\tilde{v}$ may be characterized in terms of v and $\underset{\sim}{p}^0$ as follows (the verification is left to the reader):

(5) all critical sets are maximum profit sets in v (relative to $\underset{\sim}{p}$).

(6) $p_i = p_i^0$ for every noncritical factor i.

(7) no factor is contained in every maximum profit set in v (relative to $\underset{\sim}{p}$).

In the subsequent proof, all statements about prices and profits are made with respect to v.

To prove Theorem 3, let $\underset{\sim}{p}$ be a market value for $\tilde{v}$, i.e. let $\underset{\sim}{p}$ satisfy (5) - (7), and let g be any efficient response function. Then by (5), $g(\underset{\sim}{p}) = S$ is a critical set. Let *all* bidders bid p_i for each $i \in S$ and not bid for all other factors. We will show that these bids are in equilibrium.

By condition (6) the resulting set of prices seen by the option holder is precisely $\underset{\sim}{p}$ (since no bid implies $p_i = p_i^0$). The no-forfeit payoff to all bidders is zero, while the forfeit payoff to the first bidder in line is $v(S) - \sum_S p_i = q$ and zero to the others.

Now suppose exactly one bidder b changes his bid. Suppose also that as a result of this change the factor prices are *still* $\underset{\sim}{p}$. If b was not first in line he is therefore still not first; so in case of forfeit the first bidder takes the set S, leaving at most a zero payoff to b, while under no-forfeit b's payoff is at most zero. If on the other hand b was first in line, then under the new bids his no-forfeit payoff is at most zero, while his forfeit payoff can be at most q. Therefore in any case b is no better off by changing. Hence the new factor prices $\underset{\sim}{p}'$ must be *different* from $\underset{\sim}{p}$. Since all bidders other than b stick to their original bids, b must now be outbidding the others on at least one factor.

Let $I = \{i \in N : p_i' > p_i\}$. b has to acquire all factors in I not acquired by the option holder. For fixed $j \in I$, condition (7) guarantees that there is a set S_j such that $j \notin S_j$ and S_j is maximum profit with respect to $\underset{\sim}{p}$. Under $\underset{\sim}{p}'$ this set is strictly *more* profitable than any set containing I.

Now let $S' = g(\underset{\sim}{p}')$ and let T be the set of factors b acquires if the option is exercised. Since S' is a most profitable set under $\underset{\sim}{p}'$ and $S' \cup T$ contains I, the above remark implies

$$v(S' \cup T) - \sum_{S' \cup T} p_i' < v(S') - \sum_{S'} p_i' \quad .$$

By superadditivity of v it follows that $v(T) - \sum_T p_i' < 0$, so b now suffers a loss. Therefore the original bids were in equilibrium.

To prove the necessity of the conditions (5)-(7), let $\underset{\sim}{p}^1, \underset{\sim}{p}^2, \ldots \underset{\sim}{p}^m$ be an equilibrium set of strategies for response function g. The factor prices are $p_i = \max_{1 \leq k \leq m} \{p_i^k, p_i^o\}$. If the option is exercised, the option holder acquires the maximum profit set $\bar{S} = g(\underset{\sim}{p})$, and the bidders acquire $U_1, \ldots, U_m$. Since in equilibrium all bidders' payoffs are zero (if no-forfeit), $v(U_k) - \sum_{U_k} p_i = 0$ for all k. Therefore by superadditivity the set $S^* = \bar{S} \cup U_1 \cup U_2 \cup \ldots \cup U_m$ is also a maximum profit set under $\underset{\sim}{p}$. The sale procedure implies that $p_i = p_i^o$ for all $i \notin S^*$. Hence S^* is also a most profitable set under $\underset{\sim}{p}^o$, that is, S^* is critical. Now for any critical set S

$$(8) \qquad v(S) - \sum_S p_i \leqq v(S^*) - \sum_{S^*} p_i \quad ,$$

whereas

$$v(S) - \sum_S p_i^o = v(S^*) - \sum_{S^*} p_i^o \quad ,$$

whence

$$(9) \qquad \sum_S (p_i - p_i^o) \geqq \sum_{S^*} (p_i - p_i^o) \quad .$$

Since $p_i = p_i^o$ for all $i \notin S^*$, equality holds in (8) and (9), and $p_i = p_i^o$ for all $i \notin S$. Since S was an arbitrary critical set, this establishes conditions (5) and (6) above for $\underset{\sim}{p}$.

It remains only to prove (7). For this, several facts need to be established concerning payoffs in equilibrium.

Consider the first bidder in line. The option holder (if he does not forfeit) acquires a maximum profit set S having profit $q = v(S) - \sum_S p_i$ and the first bidder acquires $T_1 - S = U_1$ having profit zero, T_1 being the set of all factors for which he bids highest. By superadditivity, $S \cup T_1$ is also a maximum profit set. Suppose then that the first bidder extends his bid by bidding p_i for all $i \in S - T_1$. He will get the same forfeit payoff, and a no-forfeit payoff of q. Since by hypothesis the bids were in equilibrium, this change cannot improve 1's position, hence 1's no-forfeit payoff was already q, i.e., T_1 was a maximum profit set.

(10) Thus in equilibrium the forfeit payoffs are q for the first bidder and zero for the others.

Consider now the last (mth) bidder in line, and let T_m be the set of factors acquired by m if the option is forfeited. If $p_i > p_i^o$ for some $i \in T_m$ then the auctioning procedure implies that $p_i = p_i^m > p_i^k$ for all $k < m$ such that p_i^k is defined. Therefore the last bidder could lower his bid on factor i, not lose his place in line, and still acquire precisely the set T_m if the option is forfeited but realize a higher profit. Moreover his no-forfeit payoff will be no lower than before. This situation contradicts equilibrium. Hence $p_i = p_i^m = p_i^o$ for all $i \in T_m$. Moreover for all $i \notin T_m$ either no one bids for i and $p_i = p_i^0$, or else $p_i = p_i^k$ for some $k < m$.

(11) Hence the last bidder in line can drop out and the factor prices will remain unchanged.

Suppose now that condition (7) does not hold, and let factor j be in every maximum profit set. Since $j \notin \phi$, ϕ could not be a maximum profit set; hence profits q are positive.

Let $\underset{\sim}{p}^{\varepsilon}$ be the price vector such that $p_i^{\varepsilon} = p_i$ for $i \neq j$ and $p_j^{\varepsilon} = p_j + \varepsilon$. Choose $\varepsilon > 0$ such that under prices $\underset{\sim}{p}^{\varepsilon}$, j is still in every maximum profit set. Let $S_{\varepsilon} = g(\underset{\sim}{p}^{\varepsilon})$, so $j \in S_{\varepsilon}$, and let $\hat{S}_{\varepsilon} = \{i \in N - S_{\varepsilon} : p_i > p_i^0\}$.

Now let the last bidder in line, b, change his bid as follows: for each $i \in S_{\varepsilon}$ he bids p_i^{ε}, and for each i in $\hat{S}_{\varepsilon}$ he bids $p_i - \delta$, for sufficiently small $\delta > 0$. These are all the factors he bids for. His total surplus bid then exceeds the first bidder's, so he goes to the head of the line. Moreover, by (11) the new factor prices are precisely $\underset{\sim}{p}^{\varepsilon}$. Since bidder b is only high bidder on the set S_{ε}, his payoff is $q - \varepsilon > 0$ if the option holder forfeits and 0 if he does not. Thus by (10) b is better off than before, a contradiction. This completes the proof of Theorem 3.
□

References

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