EXPLOITABLE SURPLUS IN N-PERSON GAMES

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ABSTRACT

Any cooperative n-person game with transferable utility has a noncooperative mode in which the players sell out of their positions to an external market of entrepreneurial organizing agents. Assuming a market of price takers, this game of competitive self-valuation always has an equilibrium price solution. Every core imputation in the original game constitutes a set of equilibrium prices. If there is no core the entrepreneurs can exploit the coalitions for a profit, i.e., they realize a positive rent for their organizing function. Application is made to determining fair wages to labor, and finding equilibrium prices for legislators selling their votes.

In this paper we describe a new approach to the valuation of n-person cooperative games with transferable utility. The idea is that values are determined competitively by creating a "market" for the players (or for the players' positions). Specifically, if value in the game is transferable, then outside entrepreneurs will view potential combinations of players as a source of potential profits. In such an environment any proposed valuation of the players will be seen as a set of prices by the entrepreneurs, who can acquire control of coalitions by paying these prices or more. It is natural then to ask whether a given valuation is in equilibrium, i.e. whether, given the others' prices, a player could charge more (or less) and do better.

The conclusion is that, in the face of profit maximizing price-takers, an equilibrium in pure strategies always exists in which every player gets what he asks. These valuations are called "market values." It turns out that every core imputation is a market value. On the other hand if the core does not exist the players will not be able to divide the whole value of the game and the entrepreneurs realize a "rent" from their contribution as organizers. In other words, the nonexistence of the core means that in a sense the players can be "exploited" due to their inability to cooperate.

We now define these ideas more precisely and illustrate with two applications: the 'fair wage' problem, and 'political bribery'.

It is useful to think of a cooperative game with transferable utility as a *production process*. The players $\{1,2,\ldots,n\} = N$ are the *factors*, and their joint payoff is what they can produce. Then, the *production function* is simply the characteristic function of the game, v. We make the following assumptions on v:

(1) Free disposal: $v(S) \geq 0$ for all $S \subseteq N$ and $v(\phi) = 0$;

(2) Joint production: $v(S \cup T) \geq v(S) + v(T)$ whenever $S \cap T = \phi$.

Conversely, given any production function satisfying (1) and (2) on factor set N, v may be interpreted as a game by supposing that each factor i is *represented* by some agent who is a player. For the present we assume that distinct factors are identified with distinct players. However, it is also possible within this framework to treat the case where a player simultaneously represents several different factors (see the fair wage problem below).

Now suppose that there is a market of outside agents or *entrepreneurs* who are potential buyers: their role is to buy up sets of factors and cause them to produce effectively. The problem is to determine what constitutes a fair wage or *value* for the individual factors.

We propose the following answer. Let each player (i.e., factor representative) announce what he thinks he is worth: thus, each i quotes a price $p_i \geqq 0$. Now let the potential buyers arrive. Each of them perceives the same production function, v, and has an unlimited budget. We suppose that they arrive in some order and take the prices as given. The first buyer in line will then buy some set that maximizes his potential *profit*, $v(S) - \sum_S p_i$. Typically there will only be one such maximum profit set; however, in case of ties a specific tie-breaking rule must be used. We say that the tie-breaking rule is *efficient* if whenever T^* is the set of factors bought at prices $\underset{\sim}{p}$ then $v(T^*) \geqq v(T)$ for all maximum profit sets T.

Now define the *sell-out game* as follows: for strategies $\underset{\sim}{p} = (p_1, p_2, \ldots, p_n)$ the *payoff* to i is

$$\varphi_i(p) = \begin{cases} p_i & \text{if } i \text{ is bought} , \\ v(i) & \text{otherwise} . \end{cases} \tag{3}$$

A vector $\bar{\underset{\sim}{p}}$ is a *strong equilibrium* for this game if no collection of players can simultaneously change their strategies and all do better (assuming the others hold fast). It may then be shown [Young, 1978d]:

For any efficient tie-breaking rule, a strong equilibrium in pure strategies always exists. Moreover, there is always a strong equilibrium $\bar{p}$ in which each player receives what he asks.

Any such $\bar{p}$ is called a *market value* for v. The class of market values is the set of n-vectors $p \geqq 0$ with the following two properties [Young, 1978d]:

(4) N is a maximum profit set with respect to p ;

(5) no factor i is in every maximum profit set with respect to p .

A simple example will illustrate these ideas. Three laborers may be organized in different combinations to produce a divisible output. The outputs of the different combinations are shown below, where the larger combinations exhibit the advantages of a division of labor, and not all laborers are equally skilled.

$$v(\phi) = 0$$

$$v(1) = 6 \qquad v(1,2) = 27$$

$$v(2) = 7 \qquad v(1,3) = 29$$

$$v(3) = 8 \qquad v(2,3) = 32$$

$$v(1,2,3) = 40 \quad .$$

There is a unique vector $\bar{p}$ satisfying conditions (4) and (5), namely $\bar{p}_1 = 8$, $\bar{p}_2 = 11$, $\bar{p}_3 = 13$. These are the wages (in units of output) that one might expect to see if the laborers are unable to organize to produce by themselves, and if there are outside entrepreneurs who compete for control.

Notice that each laborer's wage is greater than the amount he can produce in isolation, as it should be. But the sum of all wages is less than the total output, meaning that the entrepreneur realizes a profit of eight units. At prices $\bar{p}$ there are several combinations of factors that are equally profitable: each of the sets {1,2}, {1,3}, {2,3}, and {1,2,3} would yield a profit of 8 units to an organizer. For equilibrium to hold, the tie must be broken efficiently, i.e., by employing the set with highest output, namely {1,2,3}. An explanatory mechanism for this outcome is to imagine that each of the laborers shades his asking price by a small amount ε; then {1,2,3} is the *unique* most profitable set. Thus an efficient tie-breaking rule has the property that it exhibits continuous behavior of the outcome as the equilibrium is approached from below.

There is an important relation between the class of market values and the core. In fact, every imputation in the core is a market value. To see this, consider the conditions for a core imputation: $\sum_S p_i \geqq v(S)$ for all $S \subset N$ and $\sum_N p_i = v(N)$. This says that no set is profitable and the set N yields zero profit. Thus condition (4) is satisfied. But so also is (5), since the *empty* set is also a maximum profit set in this case.

If the core is empty, however, then there are no strong equilibrium prices that permit the players to divide the whole value of the game. In this situation an outside entrepreneur will always be able to realize a surplus. This fact is illustrated in the following application.

A Fair Wage Problem

Let 1,2,...,n designate laborers who are available for hire by entrepreneurs. The laborers have different skills, and each combination $S \subseteq N$ has a potential productive value v(S) (in, say, units of output). We assume that joint production is possible, e.g., is not prevented by exogenous fixed factors of production.

Instead of trying to undercut each other, suppose the laborers form a union to set their wages jointly. Then the union representative has the problem of finding a *wage structure* $w_1, \ldots, w_n$ that maximizes the return to labor. The employer has the problem of hiring a set of laborers that will maximize his profits. If there is only one potential employer and the union is in a position to call a general strike then this is a bargaining problem. However, suppose instead that there are other potential employers, and that the union does not feel itself strong enough internally to risk calling a general strike. (This is likely to be the case if v has no core). The primary employer can then be expected to act as a price taker: faced with a set of wage demands $w_1, w_2, \ldots, w_n$ he employs some combination S yielding maximum profits and walks away from the rest. On the other hand the union representative must face the possibility that if wages are set too high, some laborers will go unemployed. The real wage of such unemployed laborers will then be whatever they are paid by the union as unemployment compensation. Moreover this compensation must come out of the other workers' wages. Hence the *real* wage structure $w_1, w_2, \ldots, w_n$ is only sustainable if all are employed at these wages, that is, only if N is a maximum profit set at wages $\underset{\sim}{w}$.

The union representative therefore solves the problem

$$(6) \qquad \max \sum_N w_i$$

subject to

$$v(N) - \sum_N w_i \geqq v(S) - \sum_S w_i \qquad \text{for all } S \subset N \ .$$

An optimal solution $\underset{\sim}{w}^*$ to (6) always exists. By definition, N is a maximum profit set under $\underset{\sim}{w}^*$. Moreover, if some factor i were in *every* maximum profit set, then w_i^* could be increased and N would still be a maximum profit set, a contradiction. Therefore every optimal solution satisfies conditions (4) and (5), hence is a market value for v. These are called the *core market values* for v.

A core market value $\underset{\sim}{w}^*$ represents a wage structure that yields the highest total return to the factors, and the least profit to the entrepreneurs. This profit, $\pi^* = v(N) - \sum_N w_i^*$ is called the *exploitable surplus* of the game v. A positive exploitable surplus exists if and only if v has no core. If v has a core then the set of core market values equals the core.

The meaning of (6) becomes clearer if we re-write it as follows:

(7) $\min \pi$

subject to

$$\sum_S w_i \geqq v(S) - \pi \quad \text{all } S \subset N$$

$$\sum_N w_i = v(N) - \pi \ .$$

This says that the exploitable surplus represents the least amount that must be skimmed off the value of *all* coalitions for the core to first appear, and the core market values are precisely the imputations in the core of the game that is "left over." While this notion bears a certain formal similarity to the "least core," the values it gives, and their interpretation, are quite different.

The existence of an exploitable surplus was predicated on the assumption that the union did not consider a general strike as a viable option. If this *were* an option, then it would appear that they could ask for any wages such that $\sum_N w_i = v(N)$ and, because of competition among the entrepreneurs, they will all be assured of employment. However this argument is only plausible if $\underset{\sim}{w}$ is in the core. If $\underset{\sim}{w}$ is not in the core, then for some S $v(S) - \sum_S w_i > 0$. But then an entrepreneur could bid away S by offering them higher wages and still make a profit, and the strike would collapse. Thus if the core does not exist, a strike is vulnerable and one can expect to observe exploitable surplus for the entrepreneur and a core market value for the wage structure. On the other hand, if the core does exist, the core market values coincide with the core.

Political Bribery

In the sell-out game (3), it was assumed in the definition of the payoff function that player i gets v(i) -- the amount he can "produce by himself" -- even if he is not bought. However, this hypothesis overlooks two points. The first is the possibility that v(i) does not represent value that i can obtain acting alone, but rather, is value that i's actions have to someone else. The second point is, that i may incur an opportunity cost by selling out; that is, there may be an inherent value to i in *not* selling out which is different from v(i). Both of these situations require an appropriate modification of the payoff function (3), and both arise in the following model of political bribery.

A legislature may be thought of as a production process in which the legislators are the factors, voting is the process, and legislation the output. This output is valuable, -- not generally to the legislators themselves -- but to outside *interest groups* having a stake in the legislation. Moreover it is not too far-fetched to say that there exist entrepreneurs who might try to organize the factors to produce in a certain way -- namely, lobbyists representing these interest groups.

Suppose a lobbyist proposes a special-interest bill having potential value M, and to pass it he will need to bribe a winning coalition of the legislature. The production function for this "legislative game" is easily given:

$$v(S) = \begin{cases} M \text{ if } S \text{ is a winning coalition} \\ 0 \text{ if } S \text{ is a losing coalition} \end{cases} .$$

Notice that value in this game does not accrue directly to the legislators. However, even though v(i) in such a game is typically zero, the opportunity cost to i of selling out may well be positive, since selling oneself may involve certain risks or perhaps even pangs of conscience.

Let p_i^0 represent the opportunity cost to legislator i of selling out, that is, the minimum price needed to get him to go along with the bill. If the legislators all have equal votes and are arranged in increasing order of p_i^0, then we have a monotone increasing "supply curve" for votes as shown in Figure 1.

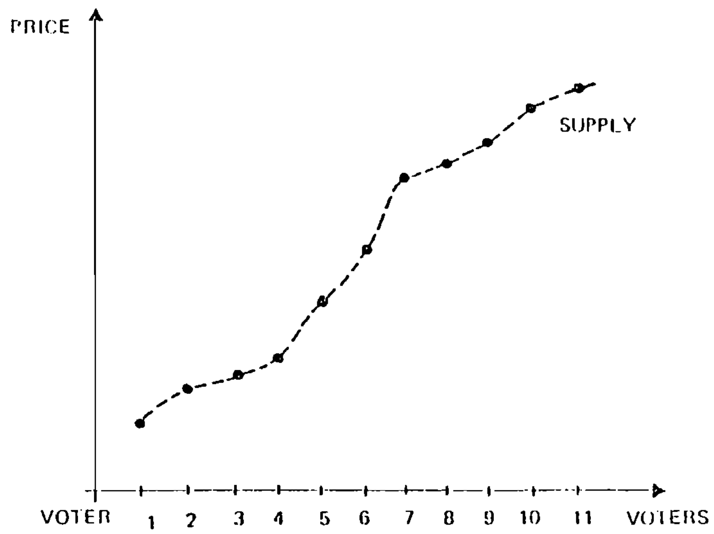

Figure 1.

If the lobbyist knew the supply curve he could engage in price discrimination and, moving up from the low end of the curve, pay just enough to each voter until he secured a majority. But in this context it may be difficult, if not impossible, for the lobbyist to gain much knowledge of the supply curve.[1)]

Suppose instead that he acts as a price taker. Then the payoff function for the sell-out game is the following modified form of (3): i gets his asking price p_i if he is bought, and p_i^0 otherwise.[2)] In this case the voters at the low end of the curve

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1) In addition, there may well be competition from other lobbyists who are proposing other bills for this same slot on the agenda.

2) In an earlier version of this model [Young,1978a], the payoff function was defined only in terms of *direct* payments to the players: thus i's payoff was p_i if i is bought and *zero* otherwise. Also, the value of the bill, M, was treated as infinite. These differences lead in some cases to slightly different equilibrium solutions than obtain in the present model. They also result in a distinction between "price" and "income" which is not necessary if opportunity costs are treated as indirect income. In the earlier version the term 'canonical equilibrium' was used instead of 'core market value'.

can strategically raise their prices, and command a surplus. If a majority of k is required to win, $\frac{n}{2} < k < n$, and M is sufficiently large ($M \geq k\, p^0_{k+1}$) then each of the first k players can raise his price to $p = p^0_{k+1}$, the opportunity cost of the $(k+1)^{st}$ player; moreover these prices, $(p, \ldots, p, p^0_{k+2}, \ldots, p^0_n)$, constitute the unique market value for the sell-out game. The lobbyist's demand curve is a "spike" of height M at voter k+1, and his profit of M-kp represents ordinary economic surplus. (Figure 2).

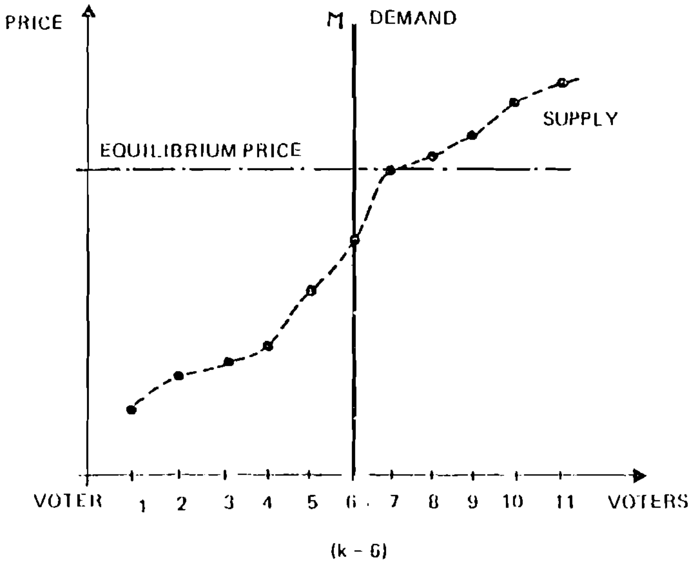

Figure 2.

This model of political bribery was first described in [Young, 1978a] where a theorem relating market value to marginal values is given for the case of weighted majority games. Various other approaches to competitive bribery may be found in [Young, 1978b, 1978c], [Shubik and Young, 1978], and [Shubik and Weber, 1978].

Both of the above examples illustrate the proposition that a game without a core may be exploited for profit. Moreover it is precisely this exploitation that introduces stability into the system, since the removal of surplus allows a core to exist on what is "left over." Put another way, such a game may be extended to include entrepreneurs, and this intended game always does have a core [Young , 1978d]. While the players in the original game do not split all the proceeds, thus violating Shapley's "efficiency" axiom [1953], this does *not* in fact imply that production is inefficient. On the contrary, full value v(N) is achieved, but outside entrepreneurs realize a surplus from their ability to exploit what might *otherwise* have been an inefficient solution.

Actually, a truly monopolistic agent would be able to realize a surplus of up to v(N). Here we have studied the case where there is a "primary" entrepreneur who is forced to be a price taker because of potential competition from other entrepreneurs standing behind him, or (as in the case of political bribery) because price discrimination may not be possible for lack of information. This approach gives a "conservative" estimate on how much surplus the entrepreneurs can skim off, (the "exploitable surplus") and the imputations in the core of what is left over (the "core market values") give the most optimistic picture of what the players in such a game can hope to achieve.

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