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Interactive Fuzzy Random Two-level Linear Programming Through Fractile Criterion Optimization

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Foreword

In this paper, assuming cooperative behavior of the decision makers, we consider solution methods for decision making problems in hierarchical organizations under fuzzy random environments. To deal with the formulated two-level linear programming problems involving fuzzy random variables, α-level sets of fuzzy random variables are introduced and an α-stochastic two-level linear programming problem is defined for guaranteeing the degree of realization of the problem. Taking into account vagueness of judgments of decision makers, fuzzy goals are introduced and the α-stochastic two-level linear programming problem is transformed into the problem to maximize the satisfaction degree for each fuzzy goal. Through the use of the fractile criterion optimization model, the transformed stochastic two-level programming problem can be reduced to a deterministic one. Interactive fuzzy programming to obtain a satisfactory solution for the decision maker at the upper level in consideration of the cooperative relation between decision makers is presented. It is significant to note here that all of the problems to be solved in the proposed interactive fuzzy programming can be easily solved by the simplex method, the sequential quadratic programming or the combined use of the bisection method and the sequential quadratic programming. An illustrative numerical example is provided to demonstrate the feasibility and efficiency of the proposed method.

Abstract

This paper considers two-level linear programming problems involving fuzzy random variables. Having introduced level sets of fuzzy random variables and fuzzy goals of decision makers, following fractile criterion optimization, fuzzy random two-level programming problems are transformed into deterministic ones. Interactive fuzzy programming is presented for deriving a satisfactory solution efficiently with considerations of overall satisfactory balance.

Keywords: Fuzzy programming; fuzzy random variables; interactive decision making; two-level linear programming problems; fractile criterion optimization; level sets.

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About the Authors

Masatoshi Sakawa joined the Integrated Modeling Environment in April 2009. His research and teaching activities are in the area of systems engineering, especially mathematical optimization, multiobjective decision making, fuzzy mathematical programming and game theory. In addition to over 300 articles in national and international journals, he is an author and coauthor of 5 books in English and 14 books in Japanese. At present Dr. Sakawa is a Professor at Hiroshima University, Japan and is working with the Department of Artificial Complex Systems Engineering. Dr. Sakawa received BEng, MEng, and DEng degrees in applied mathematics and physics at Kyoto University, in 1970, 1972, and 1975 respectively. From 1975 he was with Kobe University, where from 1981 he was an Associate Professor in the Department of Systems Engineering. From 1987 to 1990 he was Professor of the Department of Computer Science at Iwate University and from March to December 1991 he was an Honorary Visiting Professor at the University of Manchester Institute of Science and Technology (UMIST), Computation Department, sponsored by the Japan Society for the Promotion of Science (JSPS). He was also a Visiting Professor of the Institute of Economic Research, Kyoto University from April 1991 to March 1992. In 2002 Dr. Sakawa received the Georg Cantor Award of the International Society on Multiple Criteria Decision Making.

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Contents

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1 Introduction

Fuzzy random variables, first introduced by Kwakernaak [16], have been developing in various ways [15, 27, 21]. An overview of the developments of fuzzy random variables was found in the article of Gil, Lopez-Diaz and Ralescu [7]. Studies on linear programming problems with fuzzy random variable coefficients, called fuzzy random linear programming problems, were initiated by Wang and Qiao [45], Qaio, Zhang and Wang [28] as seeking the probability distribution of the optimal solution and optimal value. Optimization models for fuzzy random linear programming problems were first considered by Luhandjula et al. [22, 24] and further developed by Liu [19, 20] and Rommelfanger [30]. A brief survey of major fuzzy stochastic programming models was found in the paper by Luhandjula [23]. As we look at recent developments in the fields of fuzzy random programming, we can see continuing advances [9, 12, 10, 11, 14, 30, 13, 2, 47].

However, decision making problems in hierarchical managerial or public organizations are often formulated as two-level mathematical programming problems [34]. In the context of two-level programming, the decision maker at the upper level first specifies a strategy, and then the decision maker at the lower level specifies a strategy so as to optimize the objective with full knowledge of the action of the decision maker at the upper level. In conventional multi-level mathematical programming models employing the solution concept of Stackelberg equilibrium, it is assumed that there is no communication among decision makers, or they do not make any binding agreement even if there exists such communication [41, 3, 40, 25] . Compared with this, for decision making problems in such as decentralized large firms with divisional independence, it is quite natural to suppose that there exists communication and some cooperative relationship among the decision makers [34].

Lai [17] and Shih et al. [39] proposed solution concepts for two-level linear programming problems or multi-level ones such that decisions of decision makers in all levels are sequential and all of the decision makers essentially cooperate with each other. In their methods, the decision makers identify membership functions of the fuzzy goals for their objective functions, and in particular, the decision maker at the upper level also specifies those of the fuzzy goals for the decision variables. The decision maker at the lower

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level solves a fuzzy programming problem with a constraint with respect to a satisfactory degree of the decision maker at the upper level. Unfortunately, there is a possibility that their method leads a final solution to an undesirable one because of inconsistency between the fuzzy goals of the objective function and those of the decision variables. In order to overcome the problem in their methods, by eliminating the fuzzy goals for the decision variables, Sakawa et al. have proposed interactive fuzzy programming for two-level or multi-level linear programming problems to obtain a satisfactory solution for decision makers [35, 36]. The subsequent works on two-level or multi-level programming have been appearing [18, 32, 33, 37, 38, 42, 26, 1, 29, 34].

Under these circumstances, in this paper, assuming cooperative behavior of the decision makers, we consider solution methods for decision making problems in hierarchical organizations under fuzzy random environments. To deal with the formulated two-level linear programming problems involving fuzzy random variables, α-level sets of fuzzy random variables are introduced and an α-stochastic two-level linear programming problem is defined for guaranteeing the degree of realization of the problem. Taking into account vagueness of judgments of decision makers, fuzzy goals are introduced and the α-stochastic two-level linear programming problem is transformed into the problem to maximize the satisfaction degree for each fuzzy goal. Following the fractile criterion optimization model [8], the transformed stochastic two-level programming problem can be reduced to a deterministic one. Interactive fuzzy programming to obtain a satisfactory solution for the decision maker at the upper level in consideration of the cooperative relation between decision makers is presented. It is shown that all of the problems to be solved in the proposed interactive fuzzy programming can be easily solved by the simplex method, the sequential quadratic programming or the combined use of the bisection method and the sequential quadratic programming. An illustrative numerical example is provided to demonstrate the feasibility and efficiency of the proposed method.

2 Fuzzy random two-level linear programming problems

Fuzzy random variables, first introduced by Kwakernaak [16], have been defined in various ways [16, 27, 15, 21]. For example, as a special case of fuzzy random variables given by Kwakernaak, Kruse and Meyer [15] defined a fuzzy random variable as follows.

Definition 1 (Fuzzy random variable) *Let (Ω, B, P) be a probability space, $F(\mathcal{R})$ the set of fuzzy numbers with compact supports and X a measurable mapping $\Omega \to F(\mathcal{R})$. Then X is a fuzzy random variable if and only if given $\omega \in \Omega$, $X_\alpha(\omega)$ is a random interval for any $\alpha \in (0,1]$, where $X_\alpha(\omega)$ is an α-level set of the fuzzy set $X(\omega)$.*

Although there exist some minor differences in several definitions of fuzzy random variables, fuzzy random variables are considered to be random variables whose observed values are fuzzy sets.

In this paper, we deal with two-level linear programming problems involving fuzzy random variable coefficients in objective functions formulated as:

$$\left.\begin{array}{ll} \underset{\text{for DM1}}{\text{minimize}} & z_1(\boldsymbol{x}_1,\boldsymbol{x}_2) = \tilde{\bar{\boldsymbol{C}}}_{11}\boldsymbol{x}_1 + \tilde{\bar{\boldsymbol{C}}}_{12}\boldsymbol{x}_2 \\ \underset{\text{for DM2}}{\text{minimize}} & z_2(\boldsymbol{x}_1,\boldsymbol{x}_2) = \tilde{\bar{\boldsymbol{C}}}_{21}\boldsymbol{x}_1 + \tilde{\bar{\boldsymbol{C}}}_{22}\boldsymbol{x}_2 \\ \text{subject to} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \le \boldsymbol{b} \\ & \boldsymbol{x}_1 \ge \boldsymbol{0}\ ,\ \boldsymbol{x}_2 \ge \boldsymbol{0} \end{array}\right\}. \tag{1}$$

It should be emphasized here that randomness and fuzziness of the coefficients are denoted by the "dash above" and "wave above" i.e., "$^-$" and "$^\sim$", respectively. In (1), $\boldsymbol{x}_1$ is an n_1 dimensional decision variable column vector for the decision maker at the upper level (DM1), $\boldsymbol{x}_2$ is an n_2 dimensional decision variable column vector for the decision maker at the lower level (DM2), $z_1(\boldsymbol{x}_1,\boldsymbol{x}_2)$ is the objective function for DM1 and $z_2(\boldsymbol{x}_1,\boldsymbol{x}_2)$ is the objective function for DM2. Elements $\tilde{\bar{C}}_{ljk}$, $k = 1,2,\ldots,n_j$ of coefficient vectors $\tilde{\bar{\boldsymbol{C}}}_{lj}$, $l = 1,2$, $j = 1,2$ are fuzzy random variables characterized by the membership function

$$\mu_{\tilde{\bar{C}}_{ljk}}(\tau) = \begin{cases} L\left(\dfrac{\bar{d}_{ljk} - \tau}{\beta_{ljk}}\right) & , \quad \text{if } \tau \le \bar{d}_{ljk} \\ \\ R\left(\dfrac{\tau - \bar{d}_{ljk}}{\gamma_{ljk}}\right) & , \quad \text{otherwise,} \end{cases}$$

where the function $L(t) = \max\{0,\lambda(t)\}$ is a real-valued continuous function from $[0,\ \infty)$ to $[0,\ 1]$, and $\lambda(t)$ is a strictly decreasing continuous function satisfying $\lambda(0) = 1$. Also, $R(t) = \max\{0,\rho(t)\}$ satisfies the same conditions. The parameters β_{ljk} and γ_{ljk}, representing left and right spreads of $\mu_{\tilde{\bar{C}}_{ljk}}(\cdot)$, are positive numbers. The parameter $\bar{d}_{ljk}$ is a mean value of $\tilde{\bar{C}}_{ljk}$. Introducing a random variable $\bar{t}_l$, Katagiri et al. [12, 13] defined a random variable $\bar{d}_{ljk}$ as $\bar{d}_{ljk} = d^1_{ljk} + \bar{t}_l d^2_{ljk}$. This definition of random variables is one of the simplest randomization modeling of coefficients using dilation and translation of random variables, as discussed by Stancu-Minasian [44]. Using this definition, all fuzzy random variable coefficients of the l th objective function are strongly correlated with each other since they are affected by the common random variable $\bar{t}_l$.

In this paper, in order to consider more general situations, random variables are defined as distinct random variables. In view of ability to represent a wide class of random phenomena together with tractability, we adopt Gaussian random variables as $\bar{d}_{ljk}$. To be more specific, $\bar{\boldsymbol{d}}_l = (\bar{\boldsymbol{d}}_{l1},\bar{\boldsymbol{d}}_{l2})$, $l = 1,2$ are assumed to be (n_1+n_2) dimensional Gaussian random variable row vectors with mean vector $\boldsymbol{M}_l$ and positive-definite covariance matrix V_l, where $\bar{\boldsymbol{d}}_1$ and $\bar{\boldsymbol{d}}_2$ are mutually independent. Figure 1 illustrates an example of the membership function of a fuzzy random variable $\tilde{\bar{C}}_{ljk}$.

Since each coefficient $\tilde{\bar{C}}_{ljk}$ is a fuzzy random variable defined as a random variable whose realizations are L-R fuzzy numbers, each objective function $\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x} = \tilde{\bar{\boldsymbol{C}}}_{l1}\boldsymbol{x}_1 + \tilde{\bar{\boldsymbol{C}}}_{l2}\boldsymbol{x}_2$ is also a fuzzy random variable whose realizations are fuzzy numbers characterized by the

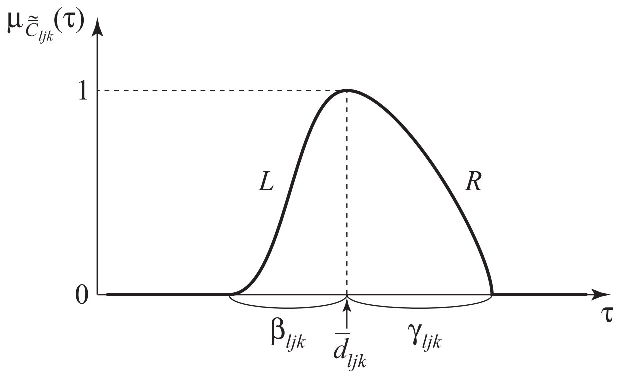

Figure 1: An example of the membership function $\mu_{\tilde{\bar{C}}_{ljk}}(\cdot)$ of a fuzzy random variable $\tilde{\bar{C}}_{ljk}$.

membership function

$$\mu_{\tilde{\bar{\boldsymbol{C}}}_l \boldsymbol{x}}(\upsilon) = \begin{cases} L\left(\dfrac{\bar{\boldsymbol{d}}_l \boldsymbol{x} - \upsilon}{\boldsymbol{\beta}_l \boldsymbol{x}}\right) & , \quad \text{if } \upsilon \leq \bar{\boldsymbol{d}}_l \boldsymbol{x} \\ R\left(\dfrac{\upsilon - \bar{\boldsymbol{d}}_l \boldsymbol{x}}{\boldsymbol{\gamma}_l \boldsymbol{x}}\right) & , \quad \text{otherwise.} \end{cases}$$

An example of the membership function of the objective function of DMl is shown in Figure 2.

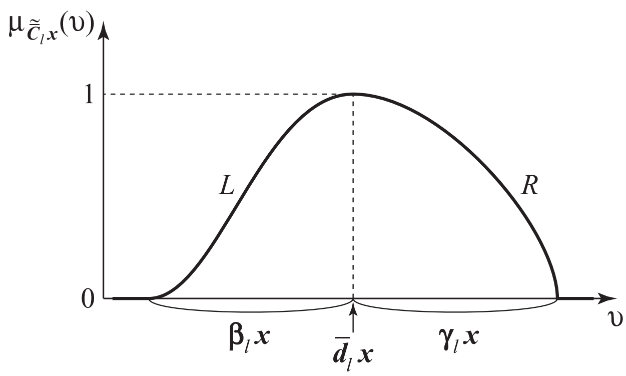

Figure 2: An example of the membership function $\mu_{\tilde{\bar{\boldsymbol{C}}}_l \boldsymbol{x}}(\cdot)$ of the objective function of DMl.

3 Level Sets and fuzzy goals

Observing that (1) involves fuzzy random variables in the objective functions, we first introduce the α-level set of the fuzzy random variables. The α-level set of the fuzzy random variables $\tilde{\bar{C}}_{ljk}$ is defined as a random interval for which the degree of their membership functions exceeds the level α:

$$\tilde{\bar{C}}_{ljk\alpha} = \{\tau \mid \mu_{\tilde{\bar{C}}_{ljk}}(\tau) \geq \alpha, \tau \in R\}, \quad j = 1, 2,\ k = 1, 2, \ldots, n_j.$$

For notational convenience, in the following, let $\tilde{\bar{\boldsymbol{C}}}_{l\alpha} = (\tilde{\bar{\boldsymbol{C}}}_{l1\alpha}, \tilde{\bar{\boldsymbol{C}}}_{l2\alpha})$, $l = 1, 2$ be an α-level set defined as the Cartesian product of α-level sets $\tilde{\bar{C}}_{ljk\alpha}$ of fuzzy random variables $\tilde{\bar{C}}_{ljk}$, $j = 1, 2$, $k = 1, 2, \ldots, n_j$.

Now suppose that DM1 decides that the degree of all of the membership functions of the fuzzy random variables involved in (1) should be greater than or equal to some value α. Then for such a degree α, (1) can be interpreted as the following stochastic two-level linear programming problem which depends on the coefficient vectors $(\bar{\boldsymbol{C}}_{11}, \bar{\boldsymbol{C}}_{12}) \in (\tilde{\bar{\boldsymbol{C}}}_{11\alpha}, \tilde{\bar{\boldsymbol{C}}}_{12\alpha})$ and $(\bar{\boldsymbol{C}}_{21}, \bar{\boldsymbol{C}}_{22}) \in (\tilde{\bar{\boldsymbol{C}}}_{21\alpha}, \tilde{\bar{\boldsymbol{C}}}_{22\alpha})$:

$$\left.\begin{array}{rl}
\underset{\text{for DM1}}{\text{minimize}} & z_1(\boldsymbol{x}_1, \boldsymbol{x}_2) = \bar{\boldsymbol{C}}_{11}\boldsymbol{x}_1 + \bar{\boldsymbol{C}}_{12}\boldsymbol{x}_2 \\
\underset{\text{for DM2}}{\text{minimize}} & z_2(\boldsymbol{x}_1, \boldsymbol{x}_2) = \bar{\boldsymbol{C}}_{21}\boldsymbol{x}_1 + \bar{\boldsymbol{C}}_{22}\boldsymbol{x}_2 \\
\text{subject to} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\
& \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0}
\end{array}\right\}. \tag{2}$$

Observe that there exists an infinite number of such problems depending on the coefficient vector $(\bar{\boldsymbol{C}}_{11}, \bar{\boldsymbol{C}}_{12}) \in (\tilde{\bar{\boldsymbol{C}}}_{11\alpha}, \tilde{\bar{\boldsymbol{C}}}_{12\alpha})$ and $(\bar{\boldsymbol{C}}_{21}, \bar{\boldsymbol{C}}_{22}) \in (\tilde{\bar{\boldsymbol{C}}}_{21\alpha}, \tilde{\bar{\boldsymbol{C}}}_{22\alpha})$, and the values of $(\bar{\boldsymbol{C}}_{11}, \bar{\boldsymbol{C}}_{12})$ and $(\bar{\boldsymbol{C}}_{21}, \bar{\boldsymbol{C}}_{22})$ are arbitrary for any $(\bar{\boldsymbol{C}}_{11}, \bar{\boldsymbol{C}}_{12}) \in (\tilde{\bar{\boldsymbol{C}}}_{11\alpha}, \tilde{\bar{\boldsymbol{C}}}_{12\alpha})$ and $(\bar{\boldsymbol{C}}_{21}, \bar{\boldsymbol{C}}_{22}) \in (\tilde{\bar{\boldsymbol{C}}}_{21\alpha}, \tilde{\bar{\boldsymbol{C}}}_{22\alpha})$ in the sense that the degree of all of the membership functions for the fuzzy random variables in (2) exceeds the level α. However, if possible, it would be desirable for DM1 to choose $(\bar{\boldsymbol{C}}_{11}, \bar{\boldsymbol{C}}_{12}) \in (\tilde{\bar{\boldsymbol{C}}}_{11\alpha}, \tilde{\bar{\boldsymbol{C}}}_{12\alpha})$ and $(\bar{\boldsymbol{C}}_{21}, \bar{\boldsymbol{C}}_{22}) \in (\tilde{\bar{\boldsymbol{C}}}_{21\alpha}, \tilde{\bar{\boldsymbol{C}}}_{22\alpha})$ in (2) to minimize the objective functions under the constraints. From such a point of view, for a certain degree α, it seems to be quite natural to have (2) reformulated as the following α-stochastic two-level linear programming problem:

$$\left.\begin{array}{rl}
\underset{\text{for DM1}}{\text{minimize}} & z_1(\boldsymbol{x}_1, \boldsymbol{x}_2) = \bar{\boldsymbol{C}}_{11}\boldsymbol{x}_1 + \bar{\boldsymbol{C}}_{12}\boldsymbol{x}_2 \\
\underset{\text{for DM2}}{\text{minimize}} & z_2(\boldsymbol{x}_1, \boldsymbol{x}_2) = \bar{\boldsymbol{C}}_{21}\boldsymbol{x}_1 + \bar{\boldsymbol{C}}_{22}\boldsymbol{x}_2 \\
\text{subject to} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\
& \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0} \\
& \bar{\boldsymbol{C}}_1 = (\bar{\boldsymbol{C}}_{11}, \bar{\boldsymbol{C}}_{12}) \in \tilde{\bar{\boldsymbol{C}}}_{1\alpha},\ \bar{\boldsymbol{C}}_2 = (\bar{\boldsymbol{C}}_{21}, \bar{\boldsymbol{C}}_{22}) \in \tilde{\bar{\boldsymbol{C}}}_{2\alpha}
\end{array}\right\}. \tag{3}$$

Considering vague natures of the decision makers' judgment, it is natural to assume that decision makers may have vague or fuzzy goals for each of the objective functions in the α-stochastic two-level linear programming problem (3). In a minimization problem, a goal stated by decision makers may be to achieve "substantially less than or equal to some value." This type of statement can be quantified by eliciting a corresponding membership function. Figure 3 illustrates a possible shape of a monotone decreasing membership function.

Having elicited the membership functions $\mu_l(\bar{\boldsymbol{C}}_l\boldsymbol{x})$, $l = 1, 2$ which well represent the

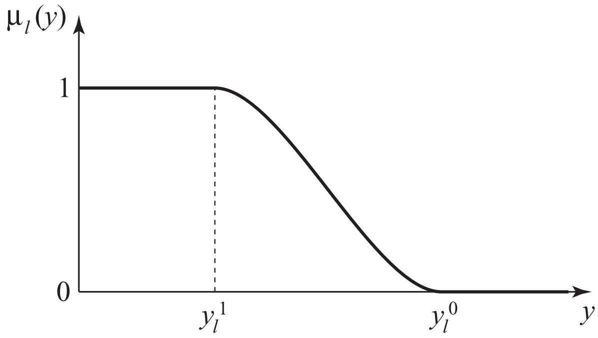

Figure 3: An example of a membership function $\mu_l(\cdot)$ of a fuzzy goal.

fuzzy goals of the decision makers at both levels, problem (3) can be transformed as:

$$\left.\begin{array}{ll} \underset{\text{for DM1}}{\text{maximize}} & \mu_1(\bar{\boldsymbol{C}}_1\boldsymbol{x}) \\ \underset{\text{for DM2}}{\text{maximize}} & \mu_2(\bar{\boldsymbol{C}}_2\boldsymbol{x}) \\ \text{subject to} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\ & \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0} \\ & \bar{\boldsymbol{C}}_1 \in \tilde{\bar{\boldsymbol{C}}}_{1\alpha},\ \bar{\boldsymbol{C}}_2 \in \tilde{\bar{\boldsymbol{C}}}_{2\alpha} \end{array}\right\}. \tag{4}$$

Observing $\bar{\boldsymbol{C}}_l\boldsymbol{x}$ and $\mu_l(\bar{\boldsymbol{C}}_l\boldsymbol{x})$ involve random variables, it is significant to note here (4) is a stochastic programming problem.

4 Fractile criterion optimization

Since (4) contains random variable coefficients, solution methods for ordinary deterministic two-level linear programming problems cannot be directly applied. In stochastic programming, expectation optimization, variance minimization, probability maximization and fractile criterion optimization [5, 6, 8, 43, 46, 4] are typical optimization models for objective functions involving random variables. For instance, let the objective function represent a profit. If the decision maker wishes to simply maximize the expected profit without caring about the fluctuation of the profit, the expectation optimization model [6] to optimize the expectation of the objective function is appropriate. On the other hand, if the decision maker hopes to decrease the fluctuation of the profit as little as possible from the viewpoint of the stability of the profit, the variance minimization model [6] to minimize the variance of the objective function is useful. In contrast to these two types of optimizing approaches, as satisficing approaches, the probability maximization model [6] and the fractile criterion optimization model or Kataoka's model [8] have been proposed. When the decision maker wants to maximize the probability that the profit is greater than or equal to a certain permissible level, probability maximization model [6] is recommended. In contrast, when the decision maker wishes to optimize such a permissible level as the probability that the profit is greater than or equal to the permissible level is greater than or equal to a certain threshold, the fractile criterion optimization model will be appropriate.

In this paper, assuming that the decision makers are interested in the probability that each objective function attains a goal value rather than the expectation or variance of each membership function, we adopt the fractile criterion optimization model [8] as a decision making model. Through fractile criterion optimization, problem (4) can be rewritten as:

$$\left.\begin{array}{ll} \underset{\text{for DM1}}{\text{maximize}} & h_1 \\ \underset{\text{for DM2}}{\text{maximize}} & h_2 \\ \text{subject to} & \Pr\left\{\mu_1(\bar{\boldsymbol{C}}_1\boldsymbol{x}) \geq h_1\right\} \geq \theta_1 \\ & \Pr\left\{\mu_2(\bar{\boldsymbol{C}}_2\boldsymbol{x}) \geq h_2\right\} \geq \theta_2 \\ & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\ & \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0} \\ & \bar{\boldsymbol{C}}_1 \in \tilde{\bar{\boldsymbol{C}}}_{1\alpha},\ \bar{\boldsymbol{C}}_2 \in \tilde{\bar{\boldsymbol{C}}}_{2\alpha} \end{array}\right\} \tag{5}$$

where h_l is regarded as a goal value for the membership function $\mu_l(\cdot)$ and θ_l is a probability level.

Now, let $\bar{C}^L_{ljk\alpha}$ and $\bar{C}^R_{ljk\alpha}$ be τ and τ' satisfying $L((\bar{d}_{ljk} - \tau)/\beta_{ljk}) = \alpha$ and $R((\tau' - \bar{d}_{ljk})/\gamma_{ljk}) = \alpha$, respectively. Then, the α-level set of $\tilde{\bar{C}}_{ljk}$ becomes a closed interval $[\bar{C}^L_{ljk\alpha}, \bar{C}^R_{ljk\alpha}]$ which varies randomly, as shown in Figure 4.

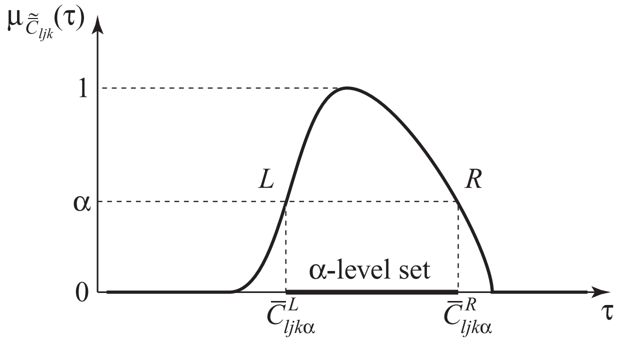

Figure 4: An example of the α-level set of a fuzzy random variable $\tilde{\bar{C}}_{ljk}$.

Hence, (5) can be rewritten as:

$$\left.\begin{array}{ll} \underset{\text{for DM1}}{\text{maximize}} & h_1 \\ \underset{\text{for DM2}}{\text{maximize}} & h_2 \\ \text{subject to} & \Pr\left\{\mu_1(\bar{\boldsymbol{C}}^L_{1\alpha}\boldsymbol{x}) \geq h_1\right\} \geq \theta_1 \\ & \Pr\left\{\mu_2(\bar{\boldsymbol{C}}^L_{2\alpha}\boldsymbol{x}) \geq h_2\right\} \geq \theta_2 \\ & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\ & \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0} \end{array}\right\}. \tag{6}$$

Since $\mu_l(\cdot)$, $l = 1, 2$ are monotone decreasing, (6) can be rewritten as:

$$\left.\begin{array}{ll} \underset{\text{for DM1}}{\text{maximize}} & h_1 \\ \underset{\text{for DM2}}{\text{maximize}} & h_2 \\ \text{subject to} & \Pr\left\{\bar{\boldsymbol{C}}^L_{1\alpha}\boldsymbol{x} \leq \mu^*_1(h_1)\right\} \geq \theta_1 \\ & \Pr\left\{\bar{\boldsymbol{C}}^L_{2\alpha}\boldsymbol{x} \leq \mu^*_2(h_2)\right\} \geq \theta_2 \\ & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\ & \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0} \end{array}\right\} \tag{7}$$

where $\mu_l^*(\cdot)$ is a pseudo-inverse function of $\mu_l(\cdot)$ defined by $\mu_l^*(h_l) = \sup\{y \mid \mu_l(y) \geq h_l\}$.

In view of $\alpha = L((\bar{d}_{ljk} - \bar{C}^L_{ljk\alpha})/\beta_{ljk})$ in (7), it holds that

$$\bar{C}^L_{ljk\alpha} = \bar{d}_{ljk} - L^*(\alpha) \cdot \beta_{ljk}$$

where $L^*(\cdot)$ is a pseudo-inverse function of $L(\cdot)$ defined by $L^*(\alpha) = \sup\{\tau \mid L(\tau) \geq \alpha\}$. From this result, the left side of the first and second constraint in (7) can be expressed as:

$$\Pr\left\{\bar{\boldsymbol{C}}^L_{l\alpha}\boldsymbol{x} \leq \mu_l^*(h_l)\right\} = \Pr\left\{(\bar{\boldsymbol{d}}_l - L^*(\alpha)\cdot\boldsymbol{\beta}_l)\boldsymbol{x} \leq \mu_l^*(h_l)\right\}.$$

Recalling the assumption that $\bar{\boldsymbol{d}}_l$ is an (n_1+n_2) dimensional Gaussian random variable row vector with mean vector $\boldsymbol{M}_l$ and positive-definite covariance matrix V_l, it holds that

$$\begin{aligned}
&\Pr\left\{(\bar{\boldsymbol{d}}_l - L^*(\alpha)\cdot\boldsymbol{\beta}_l)\boldsymbol{x} \leq \mu_l^*(h_l)\right\}\\
&= \Pr\left\{\bar{\boldsymbol{d}}_l\boldsymbol{x} \leq L^*(\alpha)\cdot\boldsymbol{\beta}_l\boldsymbol{x} + \mu_l^*(h_l)\right\}\\
&= \Pr\left\{\frac{\bar{\boldsymbol{d}}_l\boldsymbol{x} - \boldsymbol{M}_l\boldsymbol{x}}{\sqrt{\boldsymbol{x}^T V_l \boldsymbol{x}}} \leq \frac{L^*(\alpha)\cdot\boldsymbol{\beta}_l\boldsymbol{x} - \boldsymbol{M}_l\boldsymbol{x} + \mu_l^*(h_l)}{\sqrt{\boldsymbol{x}^T V_l \boldsymbol{x}}}\right\}\\
&= \Phi\left(\frac{(L^*(\alpha)\cdot\boldsymbol{\beta}_l - \boldsymbol{M}_l)\boldsymbol{x} + \mu_l^*(h_l)}{\sqrt{\boldsymbol{x}^T V_l \boldsymbol{x}}}\right)
\end{aligned}$$

where $\Phi(\cdot)$ is the probability distribution of a standard Gaussian distribution with mean 0 and variance 1. From the above results it can be shown that

$$\begin{aligned}
&\Phi\left(\frac{(L^*(\alpha)\cdot\boldsymbol{\beta}_l - \boldsymbol{M}_l)\boldsymbol{x} + \mu_l^*(h_l)}{\sqrt{\boldsymbol{x}^T V_l \boldsymbol{x}}}\right) \geq \theta_l\\
&\Leftrightarrow \frac{(L^*(\alpha)\cdot\boldsymbol{\beta}_l - \boldsymbol{M}_l)\boldsymbol{x} + \mu_l^*(h_l)}{\sqrt{\boldsymbol{x}^T V_l \boldsymbol{x}}} \geq \Phi_l^{-1}(\theta_l)\\
&\Leftrightarrow \mu_l^*(h_l) \geq (\boldsymbol{M}_l - L^*(\alpha)\cdot\boldsymbol{\beta}_l)\boldsymbol{x} + \Phi_l^{-1}(\theta_l)\sqrt{\boldsymbol{x}^T V_l \boldsymbol{x}}\\
&\Leftrightarrow h_l \leq \mu_l\left((\boldsymbol{M}_l - L^*(\alpha)\cdot\boldsymbol{\beta}_l)\boldsymbol{x} + \Phi_l^{-1}(\theta_l)\sqrt{\boldsymbol{x}^T V_l \boldsymbol{x}}\right)
\end{aligned}$$

where $\Phi_l^{-1}(\cdot)$ is the inverse function of $\Phi_l(\cdot)$.

In this way, (7) can be transformed as:

$$\left.\begin{aligned}
&\underset{\text{for DM1}}{\text{maximize}}\ h_1\\
&\underset{\text{for DM2}}{\text{maximize}}\ h_2\\
&\text{subject to}\ \mu_1\left(Z^F_{1\alpha}(\boldsymbol{x})\right) \geq h_1\\
&\qquad\qquad \mu_2\left(Z^F_{2\alpha}(\boldsymbol{x})\right) \geq h_2\\
&\qquad\qquad A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b}\\
&\qquad\qquad \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0}
\end{aligned}\right\} \tag{8}$$

equivalently,

$$\left.\begin{aligned}
&\underset{\text{for DM1}}{\text{maximize}}\ \mu_1\left(Z^F_{1\alpha}(\boldsymbol{x})\right)\\
&\underset{\text{for DM2}}{\text{maximize}}\ \mu_2\left(Z^F_{2\alpha}(\boldsymbol{x})\right)\\
&\text{subject to}\ A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b}\\
&\qquad\qquad \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0}
\end{aligned}\right\} \tag{9}$$

where

$$Z^F_{l\alpha}(\boldsymbol{x}) = (\boldsymbol{M}_l - L^*(\alpha)\cdot\boldsymbol{\beta}_l)\boldsymbol{x} + \Phi_l^{-1}(\theta_l)\sqrt{\boldsymbol{x}^T V_l \boldsymbol{x}}. \tag{10}$$

In this equation, recalling that the covariance matrix is assumed to be positive-definite, It is evident that $\sqrt{\boldsymbol{x}^T V_l \boldsymbol{x}}$ is convex and $Z^F_{l\alpha}(\boldsymbol{x})$ is also convex if $\Phi_l^{-1}(\theta_l) > 0$, i.e., $\theta_l > 0.5$, $l = 1, 2$.

5 Interactive fuzzy programming

Observing the transformed problem (9) is a deterministic two-level programming problem, we can now construct the interactive algorithm to derive a satisfactory solution for the decision maker at the upper level in consideration of the cooperative relationships between DM1 and DM2,

Interactive fuzzy programming

Step 1 In order to calculate the individual minimum and maximum of $\mathrm{E}\{z_l(\boldsymbol{x}_1, \boldsymbol{x}_2)\} = \boldsymbol{M}_l\boldsymbol{x}$, solve the following problems:

$$\left.\begin{array}{ll} \text{minimize} & \boldsymbol{M}_l\boldsymbol{x} \\ \text{subject to} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \le \boldsymbol{b} \\ & \boldsymbol{x}_1 \ge \boldsymbol{0},\ \boldsymbol{x}_2 \ge \boldsymbol{0} \end{array}\right\},\ l = 1, 2, \tag{11}$$

$$\left.\begin{array}{ll} \text{maximize} & \boldsymbol{M}_l\boldsymbol{x} \\ \text{subject to} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \le \boldsymbol{b} \\ & \boldsymbol{x}_1 \ge \boldsymbol{0},\ \boldsymbol{x}_2 \ge \boldsymbol{0} \end{array}\right\},\ l = 1, 2. \tag{12}$$

Let $z^E_{l,\min}$ and $z^E_{l,\max}$ be the the minimal objective function value to (11) and the maximal objective function value to (12), respectively. Observing that (11) and (12) are linear programming problems, they can be easily solved by some linear programming technique like the simplex method.

Step 2 Ask the decision makers to determine the membership functions $\mu_l(\cdot)$, $l = 1, 2$ by considering the obtained values of $z^E_{l,\min}$ and $z^E_{l,\max}$, $l = 1, 2$.

Step 3 Ask DM1 to specify the initial value of the degree of realization $\alpha \in (0, 1)$ and that of the probability level $\theta_l(> 0.5)$, $l = 1, 2$.

Step 4 For the specified values of α and θ_l, $l = 1, 2$, the following problem is solved for obtaining a solution which maximizes the smaller degree of satisfaction between those of the two decision makers:

$$\left.\begin{array}{ll} \text{maximize} & \min\{\mu_1\left(Z^F_{1\alpha}(\boldsymbol{x})\right), \mu_2\left(Z^F_{2\alpha}(\boldsymbol{x})\right)\} \\ \text{subject to} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \le \boldsymbol{b} \\ & \boldsymbol{x}_1 \ge \boldsymbol{0},\ \boldsymbol{x}_2 \ge \boldsymbol{0} \end{array}\right\} \tag{13}$$

equivalently,

$$\left.\begin{array}{ll} \text{maximize} & v \\ \text{subject to} & \mu_1\left(Z^F_{1\alpha}(\boldsymbol{x})\right) \ge v \\ & \mu_2\left(Z^F_{2\alpha}(\boldsymbol{x})\right) \ge v \\ & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \le \boldsymbol{b} \\ & \boldsymbol{x}_1 \ge \boldsymbol{0},\ \boldsymbol{x}_2 \ge \boldsymbol{0} \end{array}\right\}. \tag{14}$$

In view of (10), this problem is rewritten as:

$$\left.\begin{array}{ll} \text{maximize} & v \\ \text{subject to} & \mu_1\left((\boldsymbol{M}_1 - L^*(\alpha)\cdot\boldsymbol{\beta}_1)\boldsymbol{x} + \Phi_1^{-1}(\theta_1)\sqrt{\boldsymbol{x}^T V_1 \boldsymbol{x}}\right) \geq v \\ & \mu_2\left((\boldsymbol{M}_2 - L^*(\alpha)\cdot\boldsymbol{\beta}_2)\boldsymbol{x} + \Phi_2^{-1}(\theta_2)\sqrt{\boldsymbol{x}^T V_2 \boldsymbol{x}}\right) \geq v \\ & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\ & \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0} \end{array}\right\} \qquad (15)$$

equivalently,

$$\left.\begin{array}{ll} \text{maximize} & v \\ \text{subject to} & (\boldsymbol{M}_1 - L^*(\alpha)\cdot\boldsymbol{\beta}_1)\boldsymbol{x} + \Phi_1^{-1}(\theta_1)\sqrt{\boldsymbol{x}^T V_1 \boldsymbol{x}} \leq \mu_1^*(v) \\ & (\boldsymbol{M}_2 - L^*(\alpha)\cdot\boldsymbol{\beta}_2)\boldsymbol{x} + \Phi_2^{-1}(\theta_2)\sqrt{\boldsymbol{x}^T V_2 \boldsymbol{x}} \leq \mu_2^*(v) \\ & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\ & \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0} \end{array}\right\}. \qquad (16)$$

Obtaining the optimal value of v to this problem is equivalent to finding the maximum of v so that the set of feasible solutions to (16) is not empty. Although this problem is a nonlinear programming problem, we can easily find the maximum of v by the following algorithm on the basis of the bisection method and some convex programming technique like the sequential quadratic programming since the constraints of (16) are convex if v is fixed.

The combined use of the bisection method and the sequential quadratic programming

4-1 Set $l := 0$ and $v := 0$. Test whether the set of feasible solutions to (16) for $v = 0$ is empty or not using the sequential quadratic programming. If it is empty, the decision makers must reassess membership functions, α or θ_l. Otherwise, let $v_{\text{feasible}} := v$ and go to 4-2

4-2 Set $v := 1$. Test whether the set of feasible solutions to (16) for $v = 1$ is empty or not using the sequential quadratic programming . If it is not empty, $v = 1$ is the optimal value v^* to (16) and the algorithm is terminated. Otherwise, the maximum of v so that the set of feasible solutions to (16) is not empty exists between 0 and 1. Let $v_{\text{infeasible}} := v$ and go to 4-3.

4-3 Set $v := (v_{\text{feasible}} + v_{\text{infeasible}})/2$, $l := l + 1$ and go to 4-4.

4-4 Test whether the set of feasible solutions to (16) for v determined in 4-3 is empty or not using the sequential quadratic programming. If it is not empty and $(1/2)^l \leq \varepsilon$, the current value of v is regarded as the optimal value v^* to (16) and the algorithm is terminated. If it is not empty and $(1/2)^l > \varepsilon$, let $v_{\text{feasible}} := v$ and return to 4-3. On the other hand, if it is empty, let $v_{\text{infeasible}} := v$ and return to 4-3.

Then, for the obtained optimal value v^*, we can determine the corresponding optimal value $\boldsymbol{x}^*$ by solving the following convex programming problem:

$$\left.\begin{array}{ll} \text{minimize} & (\boldsymbol{M}_1 - L^*(\alpha)\cdot\boldsymbol{\beta}_1)\boldsymbol{x} + \Phi_1^{-1}(\theta_1)\sqrt{\boldsymbol{x}^T V_1 \boldsymbol{x}} \\ \text{subject to} & (\boldsymbol{M}_2 - L^*(\alpha)\cdot\boldsymbol{\beta}_2)\boldsymbol{x} + \Phi_2^{-1}(\theta_2)\sqrt{\boldsymbol{x}^T V_2 \boldsymbol{x}} \leq \mu_2^*(v^*) \\ & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\ & \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0} \end{array}\right\}. \qquad (17)$$

Step 5 The DM1 is supplied with the current values of $\mu_1\left(Z^F_{1\alpha}(\boldsymbol{x}^*)\right)$ and $\mu_2\left(Z^F_{2\alpha}(\boldsymbol{x}^*)\right)$ for the optimal solution $\boldsymbol{x}^*$ calculated in step 4. If DM1 is satisfied with the current membership function values, the interaction process is terminated. If DM1 is not satisfied and desires to update α and/or θ_l, $l = 1, 2$, ask DM1 to update α and/or θ_l and return to step 4. Otherwise, ask DM1 to specify the minimal satisfactory level $\hat{\delta}$ for $\mu_1\left(Z^F_{1\alpha}(\boldsymbol{x})\right)$ and the permissible range $[\Delta_{\min}, \Delta_{\max}]$ of the ratio of membership functions $\Delta = \mu_2\left(Z^F_{2\alpha}(\boldsymbol{x})\right)/\mu_1\left(Z^F_{1\alpha}(\boldsymbol{x})\right)$.

Observe that the larger the minimal satisfactory level is assessed, the smaller the DM2's satisfactory degree becomes. Consequently, in order to take account of the overall satisfactory balance between both decision makers, DM1 needs to compromise with DM2 on DM1's own minimal satisfactory level. To do so, the permissible range of the ratio of the satisfactory degree of DM2 to that of DM1 is helpful.

Step 6 For the specified value of $\hat{\delta}$, solve the following problem to maximize the DM2's membership function $\mu_2\left(Z^F_{2\alpha}(\boldsymbol{x})\right)$ considering the constraint that the DM1's membership function $\mu_1\left(Z^F_{1\alpha}(\boldsymbol{x})\right)$ must be greater than or equal to $\hat{\delta}$:

$$\left.\begin{array}{rl} \text{maximize} & \mu_2\left(Z^F_{2\alpha}(\boldsymbol{x})\right) \\ \text{subject to} & \mu_1\left(Z^F_{1\alpha}(\boldsymbol{x})\right) \geq \hat{\delta} \\ & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\ & \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0} \end{array}\right\}. \tag{18}$$

This problem can be rewritten as:

$$\left.\begin{array}{rl} \text{maximize} & \mu_2\left((\boldsymbol{M}_2 - L^*(\alpha)\cdot\boldsymbol{\beta}_2)\boldsymbol{x} + \Phi_2^{-1}(\theta_2)\sqrt{\boldsymbol{x}^T V_2 \boldsymbol{x}}\right) \\ \text{subject to} & \mu_1\left((\boldsymbol{M}_1 - L^*(\alpha)\cdot\boldsymbol{\beta}_1)\boldsymbol{x} + \Phi_1^{-1}(\theta_1)\sqrt{\boldsymbol{x}^T V_1 \boldsymbol{x}}\right) \geq \hat{\delta} \\ & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\ & \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0} \end{array}\right\} \tag{19}$$

equivalently,

$$\left.\begin{array}{rl} \text{minimize} & (\boldsymbol{M}_2 - L^*(\alpha)\cdot\boldsymbol{\beta}_2)\boldsymbol{x} + \Phi_2^{-1}(\theta_2)\sqrt{\boldsymbol{x}^T V_2 \boldsymbol{x}} \\ \text{subject to} & (\boldsymbol{M}_1 - L^*(\alpha)\cdot\boldsymbol{\beta}_1)\boldsymbol{x} + \Phi_1^{-1}(\theta_1)\sqrt{\boldsymbol{x}^T V_1 \boldsymbol{x}} \leq \mu_1^*(\hat{\delta}) \\ & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\ & \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0} \end{array}\right\}. \tag{20}$$

Observing that problem (20) is a convex programming problem, it can be solved by some convex programming technique like the sequential quadratic programming. For the optimal solution $\boldsymbol{x}^*$ to (18), calculate $\mu_1\left(Z^F_{1\alpha}(\boldsymbol{x}^*)\right)$, $\mu_2\left(Z^F_{2\alpha}(\boldsymbol{x}^*)\right)$ and Δ.

Step 7 The DM1 is supplied with the current values of $\mu_1\left(Z^F_{1\alpha}(\boldsymbol{x}^*)\right)$, $\mu_2\left(Z^F_{2\alpha}(\boldsymbol{x}^*)\right)$ and Δ calculated in step 6. If $\Delta \in [\Delta_{\min}, \Delta_{\max}]$ and DM1 is satisfied with the current membership function values for the optimal solution $\boldsymbol{x}^*$, the interaction process is terminated. Otherwise, ask DM1 to update the degree of realization α, the probability level θ_l, $l = 1, 2$ or the minimal satisfactory level $\hat{\delta}$, and return to step 6.

In the proposed algorithm, $\Delta_{\min}$ and $\Delta_{\max}$ are usually set to be less than 1 since $\mu_1\left(Z^F_{1\alpha}(\boldsymbol{x}^*)\right)$ should be greater than $\mu_2\left(Z^F_{2\alpha}(\boldsymbol{x}^*)\right)$ because of the priority of DM1. In step 6, if $\Delta < \Delta_{\min}$, i.e., $\mu_1\left(Z^F_{1\alpha}(\boldsymbol{x}^*)\right)$ is much greater than $\mu_2\left(Z^F_{2\alpha}(\boldsymbol{x}^*)\right)$, DM1 will decrease $\hat{\delta}$ to improve $\mu_2\left(Z^F_{2\alpha}(\boldsymbol{x}^*)\right)$ and increase Δ. Otherwise, if $\Delta_{\max} < \Delta$, i.e., $\mu_1\left(Z^F_{1\alpha}(\boldsymbol{x}^*)\right)$ is slightly greater or less than $\mu_2\left(Z^F_{2\alpha}(\boldsymbol{x}^*)\right)$, DM1 will increase $\hat{\delta}$ to improve $\mu_1\left(Z^F_{1\alpha}(\boldsymbol{x}^*)\right)$ and decrease Δ. On the other hand, if DM1 decreases (increases) α and/or θ_l, $l = 1, 2$, both $\mu_1\left(Z^F_{1\alpha}(\boldsymbol{x}^*)\right)$ and $\mu_2\left(Z^F_{2\alpha}(\boldsymbol{x}^*)\right)$ would increase (decrease). With this observation, it can be expected that desirable values of $\mu_1\left(Z^F_{1\alpha}(\boldsymbol{x}^*)\right)$, $\mu_2\left(Z^F_{2\alpha}(\boldsymbol{x}^*)\right)$ and Δ will be obtained through a series of update procedures of $\hat{\delta}$, α and/or θ_l, $l = 1, 2$ with DM1.

6 Numerical example

To demonstrate the feasibility and efficiency of the proposed method, consider the fuzzy random two-level linear programming problem formulated as:

$$\left.\begin{array}{ll}\underset{\text{for DM1}}{\text{minimize}} & z_1(\boldsymbol{x}_1,\boldsymbol{x}_2) = \tilde{\bar{\boldsymbol{C}}}_{11}\boldsymbol{x}_1 + \tilde{\bar{\boldsymbol{C}}}_{12}\boldsymbol{x}_2 \\ \underset{\text{for DM2}}{\text{minimize}} & z_2(\boldsymbol{x}_1,\boldsymbol{x}_2) = \tilde{\bar{\boldsymbol{C}}}_{21}\boldsymbol{x}_1 + \tilde{\bar{\boldsymbol{C}}}_{22}\boldsymbol{x}_2 \\ \text{subject to} & \boldsymbol{a}_{11}\boldsymbol{x}_1 + \boldsymbol{a}_{12}\boldsymbol{x}_2 \leq 100 \\ & \boldsymbol{a}_{21}\boldsymbol{x}_1 + \boldsymbol{a}_{22}\boldsymbol{x}_2 \leq 115 \\ & \boldsymbol{a}_{31}\boldsymbol{x}_1 + \boldsymbol{a}_{32}\boldsymbol{x}_2 \leq 155 \\ & \boldsymbol{a}_{41}\boldsymbol{x}_1 + \boldsymbol{a}_{42}\boldsymbol{x}_2 \leq 110 \\ & \boldsymbol{x}_1 = (x_{11}, x_{12}, x_{13}, x_{14})^T \geq \boldsymbol{0} \\ & \boldsymbol{x}_2 = (x_{21}, x_{22}, x_{23}, x_{24})^T \geq \boldsymbol{0} \end{array}\right\}, \tag{21}$$

where $\lambda(\cdot)$ and $\rho(\cdot)$ are defined as $\lambda(t) = \rho(t) = 1 - t$, $\bar{\boldsymbol{d}}_l$, $l = 1, 2$ are Gaussian random variable vectors with expectation $\boldsymbol{M}_l$ and positive-definite covariance matrix V_l.

Table 1 shows values of parameter vectors of fuzzy random variables $\boldsymbol{M}_l$, $\boldsymbol{\beta}_l$, $\boldsymbol{\gamma}_l$ $l = 1, 2$, and Table 2 shows values of coefficients of constraints $\boldsymbol{a}_i$, $i = 1, 2, 3, 4$.

Table 1: Value of each element of $\boldsymbol{d}^1_l$, $\boldsymbol{d}^2_l$, $\boldsymbol{\beta}_l$, $\boldsymbol{\gamma}_l$, $l = 1, 2$.

	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}
$\boldsymbol{M}_1$	−18.0	−6.0	−7.0	−15.0	−20.0	−14.0	−5.0	−16.0
$\boldsymbol{M}_2$	−7.0	−14.0	−16.0	−4.0	−15.0	−8.0	−18.0	−14.0
$\boldsymbol{\beta}_1$	2.2	3.2	2.0	4.8	1.6	0.8	3.0	1.2
$\boldsymbol{\beta}_2$	3.0	2.0	3.5	1.4	5.2	2.8	3.0	4.2
$\boldsymbol{\gamma}_1$	3.4	1.5	2.0	2.4	2.0	0.6	3.0	0.6
$\boldsymbol{\gamma}_2$	2.5	1.2	3.5	0.8	4.8	1.6	3.0	3.2

Table 2: Value of each element of $\boldsymbol{a}_i$, $i = 1, 2, 3, 4$.

	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}
$\boldsymbol{a}_1$	3.0	2.0	1.0	4.0	5.0	3.0	2.0	6.0
$\boldsymbol{a}_2$	2.0	1.0	2.0	3.0	5.0	2.0	4.0	4.0
$\boldsymbol{a}_3$	3.0	4.0	3.0	5.0	2.0	4.0	1.0	3.0
$\boldsymbol{a}_4$	1.0	3.0	2.0	2.0	5.0	1.0	3.0	2.0

Covariance matrices of Gaussian random variable vectors are:

$$V_1 = \begin{bmatrix} 9.00 & 3.00 & 2.80 & -1.50 & 1.30 & -3.00 & 2.00 & 1.40 \\ 3.00 & 4.00 & -1.20 & 0.20 & -1.50 & 2.40 & -0.50 & 2.00 \\ 2.80 & -1.20 & 4.00 & -2.00 & 0.50 & -1.80 & 1.20 & -2.10 \\ 1.50 & 0.20 & -2.00 & 16.00 & -2.00 & 2.10 & -2.20 & 2.80 \\ 1.30 & -1.50 & 0.50 & -2.00 & 25.00 & -0.70 & 0.80 & -2.00 \\ 3.00 & 2.40 & -1.80 & 2.10 & -0.70 & 16.00 & -1.50 & 0.60 \\ 2.00 & -0.50 & 1.20 & -2.20 & 0.80 & -1.50 & 4.00 & -3.30 \\ 1.40 & 2.00 & -2.10 & 2.80 & -2.00 & 0.60 & -3.30 & 25.00 \end{bmatrix},$$

$$V_2 = \begin{bmatrix} 4.00 & -1.40 & 0.80 & 0.20 & 1.60 & 1.00 & 1.20 & 2.00 \\ 1.40 & 4.00 & 0.20 & -1.00 & -2.20 & 0.80 & 0.90 & 1.80 \\ 0.80 & 0.20 & 9.00 & 0.20 & -1.50 & 1.50 & 1.00 & 0.60 \\ 0.20 & -1.00 & 0.20 & 1.00 & 0.80 & 0.40 & -1.50 & 0.70 \\ 1.60 & -2.20 & -1.50 & 0.80 & 25.00 & 1.20 & -0.20 & 2.00 \\ 1.00 & 0.80 & 1.50 & 0.40 & 1.20 & 4.00 & 0.50 & 1.40 \\ 1.20 & 0.90 & 1.00 & -1.50 & -0.20 & 0.50 & 9.00 & 0.80 \\ 2.00 & 1.80 & 0.60 & 0.70 & 2.00 & 1.40 & 0.80 & 16.00 \end{bmatrix}.$$

Through the use of this numerical example, it is now appropriate to illustrate the proposed interactive fuzzy programming.

Expectation optimization problems (11) and (12) are solved by the simplex method and the individual minimum $z^E_{1,\min} = -627.501$, $z^E_{2,\min} = -862.857$ and maximum $z^E_{1,\max} = 0.000$, $z^E_{2,\max} = 0.000$ are obtained. Here, in view of the linearity of the formulated problems, assume that the decision makers identify the linear membership function

$$\mu_l(y) = \begin{cases} 1 & , \text{ if } y \le y_l^1 \\ \dfrac{y - y_l^0}{y_l^1 - y_l^0} & , \text{ if } y_l^1 < y \le y_l^0 \\ 0 & , \text{ if } y > y_l^0 \end{cases}$$

whose parameter values are determined by the Zimmermann method [48]. Then, the parameter values characterizing membership functions become as $y_1^1 = -627.501$, $y_1^0 = -369.286$, $y_2^1 = -862.857$ and $y_2^0 = -609.167$.

Initial values of the degree of realization of the problem α and probability levels θ_l, $l = 1, 2$ are set as $\alpha = 0.8$, $\theta_1 = 0.7$ and $\theta_2 = 0.6$. For these initial values, (16) is solved by the combined use of the bisection method and the sequential quadratic programming. The obtained result is shown at the column labeled "1st" in Table 3. Considering that both of the membership function values are a little low, DM1 updates α from 0.8 to 0.7. For the updated value of α, the corresponding problem (16) is solved, and the obtained result is shown at the column labeled “2nd” in Table 3. DM1 is not satisfied with this solution, but he does not desire to update α, θ_1 or θ_2 since the membership function values are improved. Thus, DM1 determines the minimal satisfactory level $\hat{\delta} = 0.70$ to improve μ_1 (the satisfactory degree of DM1) at the expense of μ_2 (the satisfactory degree of DM2). Furthermore, DM1 specifies the upper bound $\Delta_{\max} = 0.85$ and the lower bound $\Delta_{\min} = 0.75$ for the ratio of membership functions $\Delta = \mu_2/\mu_1$. For the updated value of $\hat{\delta} = 0.70$, (20) is solved by the sequential quadratic programming. The obtained result is shown at the column labeled “3rd” in Table 3.

For the current values of μ_1, μ_2 and Δ, DM1 considers that μ_1 is improved but μ_2 is too bad, and Δ is less than $\Delta_{\min}$. Hence, DM1 is not satisfied with this solution and updates the minimal satisfactory level $\hat{\delta}$ from 0.70 to 0.60. For the updated value of $\hat{\delta}$, (20) is solved and the obtained result is shown at the column labeled “4th” in Table 3. Since μ_2 is improved but Δ is greater than $\Delta_{\max}$, DM1 is not satisfied with this solution and updates the minimal satisfactory level $\hat{\delta}$ from 0.60 to 0.65. For the updated value of $\hat{\delta}$, (20) is solved and the obtained result is shown at the column labeled “5th” in Table 3. In this example, since Δ exists in the interval $[\Delta_{\min}, \Delta_{\max}]$ and DM1 is satisfied with the overall satisfactory balance between μ_1 and μ_2, at the 5th iteration, the interactive algorithm is terminated.

Table 3: Interaction process.

Interaction	1st	2nd	3rd	4th	5th
α	0.800	0.700	0.700	0.700	0.700
θ_1	0.700	0.700	0.700	0.700	0.700
θ_2	0.600	0.600	0.600	0.600	0.600
$\hat{\delta}$	—	—	0.700	0.600	0.650
$\mu_1\left(Z^F_{1\alpha}(\boldsymbol{x})\right)$	0.525	0.579	0.700	0.600	0.650
$\mu_2\left(Z^F_{2\alpha}(\boldsymbol{x})\right)$	0.525	0.579	0.459	0.579	0.531
Δ	1.000	1.000	0.698	0.965	0.816

In the proposed interactive fuzzy programming, through a series of update procedures of the minimal satisfactory level $\hat{\delta}$, the degree of realization α and the probability level θ_l, $l = 1, 2$, it can be possible to obtain a satisfactory solution where the satisfactory degree of DM1 is guaranteed to be greater than or equal to the minimal satisfactory level $\hat{\delta}$ and is well balanced with that of DM2.

7 Conclusions

In this paper, assuming cooperative behavior of the decision makers, interactive decision making methods in hierarchical organizations under fuzzy random environments were considered. For the formulated fuzzy random two-level linear programming problems,

α-level sets of fuzzy random variables were introduced and an α-stochastic two-level linear programming problem was defined for guaranteeing the degree of realization of the problem. Considering the vague natures of decision makers' judgments, fuzzy goals were introduced and the α-stochastic two-level linear programming problem was transformed into the problem to maximize the satisfaction degree for each fuzzy goal. Through the fractile criterion optimization model, the transformed stochastic two-level programming problem was reduced to a deterministic one. Interactive fuzzy programming to obtain a satisfactory solution for the decision maker at the upper level in consideration of the cooperative relation between decision makers was presented. It should be emphasized here that all problems to be solved in the proposed interactive fuzzy programming can be easily solved by the simplex method, the sequential quadratic programming or the combined use of the bisection method and the sequential quadratic programming. An illustrative numerical example demonstrated the feasibility and efficiency of the proposed method. Extensions to other stochastic programming models will be considered elsewhere. Also extensions to fuzzy random two-level linear programming problems with two decision makers under noncooperative environments will be required in the near future.

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