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On a boundedly rational Pareto-optimal trade in emission reduction

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Abstract

We consider the emission reduction process involving several countries, in which the countries negotiate, in steps, frequently enough, on small, local emission reductions and implement their decisions right away. In every step, the countries either find a mutually acceptable local emission reduction vector and use it as a local emission reduction plan, or terminate the emission reduction process. We prove that the process necessarily terminates in some step and the final total emission reduction vector lies in a small neighborhood of a certain Pareto maximum point in the underlying emission reduction game. We use examples to illustrate some features of the proposed decision making scheme and discuss a way to organize negotiations in every step of the emission reduction process.

Key words: emission reduction, Pareto maximum, repeated games, boundedly rational decisions, environmental negotiations

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1 Introduction

It has been recognized that emission reduction has been a common problem for all countries in a region. A country's industrial pollutants travel across borders and make neighboring countries suffer from contamination. The understanding that the emission reduction process involves multiple decision makers whose interests are interconnected but not identical has initiated a series of game-theoretic studies.

Today's practice in planning and controlling emission reductions is based on international agreements; accordingly, a significant part of research focuses on countries' incentives to participate in conventions, and on issues of formation and stability of coalitions (see Barrett, 1994, 2003; Finus, 2001). A considerable research effort concentrates on developing procedures that may lead the parties to an equilibrium solution and, in result, to a specification of emission reduction commitments. Part of the procedures proposed assumes that the parties use money transfers to compensate for cleaning up (see Maeler, 1990; Chander and Tulkens, 1992). Another approach suggests that the international agreements could be formed based on reciprocal emission reduction trade (see Hoel, 1991; Nentjes, 1993, 1994; Pethig, 1982); an analogous theoretical framework has been developed in Ehtamo and Hamalainen (1993). Kryazhimskiy *et al.* (2001) interpret environmental negotiations as a "trade" between the governments, in which emission reductions act as the "goods" traded. Martin *et al.* (1993) analyze a multi-agent dynamic game whose equilibrium solution may justify the countries' emission reduction plans.

The majority of the game-theoretic studies addressing the issue of emission reduction assume that every party has good knowledge on its own utility function – its overall gain due to emission reduction – and uses that knowledge in the negotiations leading to an international environmental agreement. That assumption natural from the standpoint of game theory, can however be criticized as an unrealistic one. Indeed, a country's utility has two components, the cost for national emission reduction (a negative component) and the ecological benefit from the emission reduction performed by all countries (a positive component). Even if we assume that a country's government is able to construct its cost function, based on economic considerations[1], we should admit that it can hardly estimate in advance, with an

[1]This assumption is however not so obvious; one can argue against it by saying that future changes in prices, unforeseeable today, will ruin today's cost estimates for high emission reduction values unreachable in the short run.

acceptable precision, the sizes of the country's ecological benefits for all future emission reduction values. This uncertainty makes one view negotiation patterns, in which the countries use full information on their global utility functions, as useful but rather theoretical constructions.

In this paper, we study decisions on reducing emission in the situation where each country has limited information on its global utility function. Namely, we assume that given the actual state of the countries in the emission reduction process, i.e., the actual values of the countries' total emission reductions, every country is able to reconstruct its marginal cost and benefit functions, i.e., the growth rates for its global cost and benefit functions in small neighborhoods of the actual state. Moreover, each country has no information on the utility functions of the other countries.

In this situation, it is hardly possible to provide a classical game-theoretic basis for shaping, today, a long-term agreement on substantial emission reduction[2]. A realistic operational mode is "myopic" planning and "myopic" implementation. In the "myopic" mode, instead of fixing a long-term agreement, the countries negotiate, in steps, frequently enough, on small, local emission reductions and implement their decisions right away. In every negotiation step, each country uses its current marginal utility to understand if a proposed local emission reduction vector meets the country's local utility growth criterion, i.e., increases, locally, the value of the country's global utility function. The countries' goal is to identify an acceptable local emission reduction vector satisfying all local utility growth criteria. The identified acceptable emission reduction vector defines the countries' cooperative local emission reduction plan. If the countries fail to find an acceptable emission reduction vector, the negations are terminated and the latest total emission reduction vector is agreed to be the outcome of the emission reduction process. The described decision making scheme follows the approach of theory of repeated games (see, e.g., Brown, 1951; Robinson, 1951; Axelrod, 1984; Smale, 1980; Fudenberg and Kreps, 1993; Weibull, 1995; for examples of economic applications see, e.g., Friedman, 1991; Kryazhimskiy *et al.*, 2001; Kryazhimskiy *et al.*, 2002).

In section 2 we introduce technical assumptions and describe the emission reduction process. In section 3 we prove that the process necessarily terminates in some step and its outcome lies in a small neighborhood of a certain Pareto maximum point in the emission reduction game; the radius of the neighborhood tends to zero together with the length of the time period between the points of decision making. In other words, we state that the proposed "myopic" decision making scheme allows the countries to find an equilibrium solution with an arbitrarily high precision. In section 4 we discuss our solvability statement using two examples. One example shows that the statement may fail to hold if the countries' network is not fully connected in the sense that there are at least two countries such that pollution produced by one country is not transported to the other one. The other example shows that the set of all Pareto maximum points, which are reachable via the proposed emission reduction process, can be considerably smaller than the set of all Pareto maximum points in the emission reduction game. In section 5 we discuss a possible way to

[2]This does not mean that the agreement is not reachable in principle; a reasonable decision can be found using, for example, political and general environmental considerations.

organize negotiations bringing the countries to a common decision in each step of the emission reduction process.

2 Emission reduction process

We consider an emission reduction process involving n countries, numbered $1, \ldots, n$, in which each country, i, controls its emission reduction value, $x_i \geq 0$, gradually increasing it over time. The process starts at time 0 with the zero emission reduction values and develops onward. In the long run, each country, i, is interested in maximizing its utility function, w_i:

$$w_i(x) = -c_i(x_i) + b_i\left(\sum_{j=1}^{n} a_{ji}x_j\right). \tag{2.1}$$

Here $x = (x_1, \ldots, x_n)$ is the full emission reduction vector; $c_i(x_i)$ is the cost paid by country i for the emission reduction x_i; $b_i(y)$ is the ecological benefit gained by country i thanks to the reduction of the total pollution load to its territory, $y = \sum_{j=1}^{n} a_{ji}x_j$; and a_{ji} is a proportion of emission from country j, which is transported to country i (a transport coeffcient). Clearly, $\sum_{i=1}^{n} a_{ji} = 1$ $(j = 1, \ldots, n)$. We assume that the countries' network is fully interconnected in the sense that each country pollutes itself and every other country, implying

$$a_{ji} > 0 \quad (j, i = 1, \ldots, n). \tag{2.2}$$

We call a vector $x = (x_1, \ldots, x_n)$ *positive* if $x_i > 0$ $(i = 1, \ldots, n)$.

Our technical assumptions are the following.

(A1) The cost functions, c_i $(i = 1, \ldots, n)$, defined on $[0, \infty)$ are continuously differentiable, convex, strictly monotonically increasing, positive-valued at all points except 0, and vanish at 0.

(A2) The benefit functions, b_i $(i = 1, \ldots, n)$, defined on $[0, \infty)$ are continuously differentiable, strictly concave, strictly monotonically increasing, positive-valued at all points except 0, and vanish at 0; moreover, the benefit functions are bounded from above, implying, in particular, that

$$b_i'(y) \to 0 \quad \text{as} \quad y \to \infty \quad (i = 1, \ldots, n). \tag{2.3}$$

(A3) The utility functions, w_i $(i = 1, \ldots, n)$, take positive values for all positive emission vectors belonging to a certain neighborhood of the origin (in this manner we exclude a trivial situation, in which some of the countries are not interested in emission reduction, since their utilities are maximized at the zero emission reduction vector).

The emission reduction process develops in steps. A step k is performed over a time interval $[t_k, t_{k+1}]$ where $t_k = k\delta$ with a given small $\delta > 0$ $(k = 0, 1, \ldots)$. For every country, i, we denote by $x_i(t_k)$ its total emission reduction value at the starting time of each step k, t_k. In step 0 the countries start with the zero emission reductions:

$$x_i(t_0) = x_i(0) = 0 \quad (i = 1, \ldots, n). \tag{2.4}$$

In each step, k, every country, i, plans an extra local emission reduction, $\Delta x_i(t_k) \geq 0$; at time t_{k+1} the country completes the planned local emission reduction process bringing its total emission reduction value to a new state, $x_i(t_{k+1}) = x_i(t_k) + \Delta x_i(t_k)$. Introducing notations for the initial emission reduction vector in step k, $x(t_k) = (x_1(t_k), \ldots, x_n(t_k))$, and for the local emission reduction vector in step k,

$$\Delta x(t_k) = (\Delta x_1(t_k), \ldots, \Delta x_n(t_{k+1})), \tag{2.5}$$

we represent the transformation of the emission reduction vector in step k as

$$x(t_{k+1}) = x(t_k) + \Delta x(t_k). \tag{2.6}$$

Prior to considering the rules for choosing $\Delta x_i(t_k)$, we assume that information available for each country, i, *a priori* is the collection of the transport coefficients a_{ji} $(j = 1, \ldots, n)$ only. Therefore, *a priori* each country may have no knowledge on the cost and benefit functions of the other countries and no knowledge on its own cost and benefit functions.

In each step, k, country i chooses $\Delta x_i(t_k)$ using the following additional information: the country's current emission reduction value, $x_i(t_k)$; the current value of the total reduction of the pollution load to its territory,

$$y_i(t_k) = \sum_{j=1}^{n} a_{ji} x_j(t_k); \tag{2.7}$$

and its marginal cost and benefit functions at points $x_i(t_k)$ and $y_i(t_k)$, respectively. The country constructs its marginal cost function at point $x_i(t_k)$ as a linear approximation to the virtual increment in its cost value, $c_i(x_i(t_k) + h) - c_i(x_i(t_k))$, corresponding to every small virtual positive increment in the emission reduction value, h; that linear approximation can be represented as $c_i'(x_i(t_k))h$. Similarly, the country constructs its marginal cost function at point $y_i(t_k)$ as a linear approximation to the virtual increment in its benefit value, $b_i(y_i(t_k) + h) - b_i(y_i(t_k))$, corresponding to a small virtual positive increment in the total emission reduction value, h; that linear approximation can be represented as $b_i'(y_i(t_k))h$.

Thus, we assume that in each step, k, the country is able to reconstruct, in linear approximation, the local structure of its cost and benefit functions in small neighborhoods of the actual emission reduction value, $x_i(t_k)$, and actual total pollution reduction value, $y_i(t_k)$, respectively. In more formal terms, we assume that in each step, k, every country, i, is able to reconstruct the derivatives $c_i'(x_i(t_k))$ and $b_i'(y_i(t_k))$.

While choosing a positive $\Delta x_i(t_k)$, country i negotiates with the other countries. In the negotiations, country i trades on exchanging its local emission reduction value, $\Delta x_i(t_m)$, to the local reduction of the total pollution load to its territory, which is due to the current efforts of the other countries, $\Delta y_i^0(t_k)$. Clearly, $\Delta y_i^0(t_k)$ is the sum of the local emission reductions of all the countries, except of country i, weighted with the corresponding transportation coefficients:

$$\Delta y_i^0(t_k) = \sum_{j=1,\ldots,n,\ j \neq i} a_{ji} \Delta x_j(t_k). \tag{2.8}$$

To each value of $\Delta y_i^0(t_k)$ emerging in the negotiations, country i responds with an emission reduction value $\Delta x_i(t_k)$ that can be exchanged to $\Delta y_i^0(t_k)$. The country's goal in the negotiations is to form a set of the local emission reduction values, $\Delta x_j(t_k)$ $(j = 1, \ldots, n)$, that would locally increase the country's utility, i.e., ensure

$$w_i(x(t_k) + \Delta x(t_k)) > w_i(x(t_k)). \tag{2.9}$$

Thus, in each round the country acts as a boundedly rational agent (see, e.g., Rubinstein, 1998).

Recall that in step k the country's knowledge about its cost and benefit functions, c_i and b_i, is restricted to the values $c_i'(x_i(t_k))$ and $b_i'(y_i(t_k))$. Using these values and referring to (2.1) and (2.7), country i reconstructs the partial derivatives

$$\frac{\partial w_i(x(t_k))}{\partial x_j} = a_{ji} b_i'(y_i(t_k)) \quad (j = 1, \ldots, n,\ j \neq i), \tag{2.10}$$

$$\frac{\partial w_i(x(t_k))}{\partial x_i} = a_{ii} b_i'(y_i(t_k)) - c_i'(x_i(t_k)), \tag{2.11}$$

which give it its marginal utility at point $x(t_k)$, i.e., a linear approximation to the increment $w_i(x(t_k) + h) - w_i(x(t_k))$ as a function of h. The necessity to use the marginal utility at point $x(t_k)$ instead of w_i makes the country consider a linear approximation to the original criterion (2.9):

$$\sum_{j=1,\ldots,n,\ j\neq i} \frac{\partial w_i(x(t_k))}{\partial x_j} \Delta x_j(t_k) + \frac{\partial w_i(x(t_k))}{\partial x_i} \Delta x_i(t_k) > 0. \tag{2.12}$$

The substitution of (2.11) and use of (2.8) transform (2.12) into

$$b_i'(y_i(t_k)) \Delta y_i^0(t_k) + [a_{ii} b_i'(y_i(t_k)) - c_i'(x_i(t_k))] \Delta x_i(t_k) > 0$$

or

$$\Delta y_i^0(t_k) > \lambda_i(t_k) \Delta x_i(t_k) \tag{2.13}$$

where

$$\lambda_i(t_k) = \frac{c_i'(x_i(t_k))}{b_i'(y_i(t_k))} - a_{ii}. \tag{2.14}$$

We call (2.13) the *local utility growth criterion* for country i in step k.

Let us give several definitions. We call a positive emission reduction vector $\Delta x(t_k)$ (2.5) *acceptable* in step k if for every country, i, the values $\Delta y_i^0(t_k)$ given by (2.8) and $\Delta x_i(t_k)$ satisfy the country's local utility growth criterion (2.13) in step k. Every step k, in which there exists an acceptable emission reduction vector, will be said to be *nondegenerate*; every step that is not nondegenerate will be called *degenerate*.

We introduce two assumptions, (A4) and (A5), characterizing the abilities and outcomes of the negotiations.

(A4) In the negotiations taking place in a nondegenerate step k, the countries find a positive emission reduction vector acceptable in step k. In the negotiations taking place in a degenerate step k, the countries identify that step k is degenerate. (A possible negotiation pattern is presented Section 5.)

Let us fix a $p > 0$. In what follows, $|\cdot|$ is a given norm in the n-dimensional linear space.

(A5) After finding, though negotiations, a positive emission reduction vector $\bar{\Delta}x(t_k) = (\bar{\Delta}x_1(t_k), \ldots, \bar{\Delta}x_n(t_k))$ acceptable in a nondegenerate step k, the countries form the final positive local emission reduction vector acceptable in step k, $\Delta x(t_k)$ (2.5), by normalizing $\bar{\Delta}x(t_k)$ to $p\delta$, i.e., by setting $\Delta x_i(t_k) = \mu\bar{\Delta}x_i(t_k)$ $(i = 1, \ldots, n)$ with $\mu = p\delta/|\bar{\Delta}x(t_k)|$.

Assumption (A5) implies that the countries agree *a priori* that the norms of the local positive emission reduction vectors, $|\Delta x(t_k)|$, appearing in nondegenerate steps, k, must be proportional to the size of the time step, δ.

Our next assumption, (A6), suggest a rule for the termination of the emission reduction process.

(A6) In a first degenerate step, s, whose degeneracy is identified by the countries through negotiations (see (A4)), the countries terminate the emission reduction process and view $x(t_s)$ as its *outcome*.

Our final assumption, (A7) summarizes the rules for the countries' operation in the emission reduction process.

(A7) In each (nondegenerate) step k preceding the first degenerate step, s, the countries work out a local positive emission reduction vector $\Delta x(t_k)$ (2.5) through negotiations as described in (A4) and (A5) and update the total emission reduction vector using (2.6). If all steps are nondegenerate, then in each step, k, the countries work out a local positive emission reduction vector $\Delta x(t_k)$ (2.5) through negotiations and update the total emission reduction vector using (2.6); in this situation the emission reduction process has no outcome.

3 Outcome of the emission reduction process

Holding a game-theoretic viewpoint, we assume that *a priori* a goal of the countries' community is to bring the full emission reduction vector to a Pareto maximum point for the countries' utilities. A nonnegative emission reduction vector x^* is said to be a *Pareto maximum point* in the emission reduction game if switching from x^* to any nonnegative emission reduction vector $x \neq x^*$ either does not change the countries' utility values, i.e., $w_i(x) = w_i(x^*)$ for all $i = 1, \ldots, n$, or makes at least one country lose in utility, i.e., $w_i(x) < w_i(x^*)$ for some $i \in \{1, \ldots, n\}$. In view of the strict concavity of the utility functions $w_i, \ldots, w_n$ (see (A1) and (A2)), for every positive $z_1, \ldots, z_n$ the maximizer of the sum $z_1w_1(x) + \ldots + z_nw_n(x)$ over all nonnegative emission reduction vectors x is a Pareto maximum point. Note that by (A3) the origin is not a Pareto maximum point. Thanks to the strict concavity of the utility functions (see (A1) and (A2)) a positive emission reduction vector x^* maximizes $z_1w_1(x) + \ldots + z_nw_n(x)$ if and only if

$$z_1\frac{\partial w_1(x^*)}{\partial x_i} + \ldots + z_n\frac{\partial w_n(x^*)}{\partial x_i} = 0 \qquad (3.15)$$

$$(i = 1, \ldots, n)$$

(see, e.g., Germeyer, 1976). Thus, every positive emission reduction vector x^* satisfying (3.15) for some positive $z_1, \ldots, z_n$ is a Pareto maximum point, which can be viewed as a target point in the emission reduction process. We call $z_1, \dots, z_n$ a family of *Pareto multipliers* for the Pareto maximum point x^*.

Our goal in this section is to show that the decentralized boundely rational emission reduction process described in the previous section brings the total emission reduction vector to a small neighborhood of some Pareto maximum point in a finite number of steps.

First, we state that the emission reduction process terminates in some step.

Proposition 3.1 *There is a degenerate step, in which the emission reduction process terminates (see (A6)).*

Proof. Assume, to the contrary, that the emission reduction process never terminates, i.e, all the steps are nondegenerate. By (A5) in each step, k, the local emission reduction vector, $\Delta x(t_k)$, is positive and has the norm $p\delta$; hence, the norms of the total emission reduction vectors, $|x(t_k)|$ (see (2.6)), tend to infinity as $k \to \infty$. Then for each country, i, the total reduction of the pollution load to its territory, $y_i(t_k)$ (2.7), tends to infinity as $k \to \infty$ (here we take into account (2.2)). Therefore, by (2.3)

$$b_i'(y_i(t_k)) \to 0 \quad \text{as} \quad k \to \infty \quad (i = 1, \dots, n). \tag{3.16}$$

By (A1) for each country, i, the cost function c_i, is strictly monotonically increasing and convex, implying that $c_i'(x_i(t_k)) \geq c^0 > 0$ uniformly for all steps k. Combining with (3.16), we find that for every country, i,

$$\lambda_i(t_k) \to \infty \quad \text{as} \quad k \to \infty \quad (i = 1, \dots, n). \tag{3.17}$$

where $\lambda_i(t_k)$ is given by (2.14). For every step, k, let $i_k \in \{1, \dots, n\}$ be such that $\Delta x_{i_k}(t_k) = \max\{\Delta x_1(t_k), \dots, \Delta x_{i_k}(t_k)\}$. In view of (2.8), for every step, k, we have

$$\begin{aligned} \Delta y_{i_k}^0(t_k) - \lambda_{i_k}(t_k)\Delta x_{i_k}(t_k) &= \sum_{j=1,\dots,n,\ j\neq i_k} a_{ji}\Delta x_j(t_k) - \lambda_i(t_k)\Delta x_i(t_k) \\ &\leq [(n-1) - \lambda_i(t_k)]\Delta x_{i_k}(t_k) \end{aligned}$$

By (3.17) the right hand side is negative for all k sufficiently large. Thus, for a large k the local utility growth criterion (2.13) is violated for country i_k; consequently, the local emission reduction vector $\Delta x(t_k)$ is not acceptable in step k. We get a contradiction with our initial assumption and finalize the proof.

As we see from (2.4) and (A3), step 0 is nondegenerate. Therefore, for the first degenerate step, s (see Proposition 3.1), we have $s \geq 1$.

Consider the time interval $[t_{s-1}, t_s]$. For every $t \in [t_{s-1}, t_s]$ we set (see (2.6) and (2.5))

$$x(t) = (x_1(t), \dots, x_n(t)) = x(t_{s-1}) + \frac{t - t_{s-1}}{\delta}\Delta x(t_{s-1}) \tag{3.18}$$

and extend notations (2.7) and (2.14) by setting

$$y_i(t) = \sum_{j=1}^{n} a_{ji}x_j(t), \quad \lambda_i(t) = \frac{c_i'(x_i(t))}{b_i'(y_i(t))} - a_{ii} \tag{3.19}$$

$$(i = 1, \dots, n).$$

For every $t \in [t_{s-1}, t_s]$ let

$$h_i(t,z) = \sum_{j=1,\dots,n,\ j\neq i} a_{ji}z_j - \lambda_i(t)z_i \quad (z=(z_1,\dots,z_n),\ i=1,\dots,n), \tag{3.20}$$

$$H(t) = \{z>0 : |z| = p\delta,\ h_i(t,z)>0\ (i=1,\dots,n)\}; \tag{3.21}$$

here and below $z>0$ marks that a vector z is positive.

The fact that the local emission reduction vector $\Delta x_{s-1}(t_{s-1})$ has the norm $p\delta$ and is acceptable in the nondegenerate step $s-1$, i.e., satisfies the local utility growth criterion for every country in step $s-1$ is equivalent to

$$\Delta x_{s-1}(t_{s-1}) \in H(t_{s-1}) \tag{3.22}$$

(see (A5), (2.13) and (2.8)). Similarly, we see that if $H(t_s)$ is nonempty, then for every $z \in H(t_s)$ the emission reduction vector $\Delta x_s(t_s) = z$ is acceptable in step s; consequently, step s is nondegenerate. Since step s is degenerate, we have

$$H(t_s) = \emptyset. \tag{3.23}$$

Let

$$T = \{t \in [t_{s-1}, t_s] : H(t) \neq \emptyset\}. \tag{3.24}$$

By (3.22) T is nonempty. Denote

$$\tau = \sup T. \tag{3.25}$$

Prior to formulating our main technical statement – Lemma 3.1, we make a few simple observations. In view of the contitnuity of the functions h_i (3.20) the set T is open in $[t_{s-1}, t_s]$. Therefore, if $\tau < t_s$, then $\tau \notin T$, i.e.,

$$H(\tau) = \emptyset; \tag{3.26}$$

note that if $\tau = t_s$, then (3.26) holds by (3.23). By the definition of τ, (3.25), there exist a sequence (τ_m) in $[t_{s-1}, \tau)$ such that $\tau_m \to \tau$ and $H(\tau_m) \neq \emptyset$ $(m = 1, 2, \dots)$. Every sequence (z_m) such that $z_m \in H(\tau_m)$ $(m = 1, 2, \dots)$ is bounded and has a limit point.

Lemma 3.1 *The following statements hold true.*

1) *The emission reduction vector $x^* = x(\tau)$ is a Pareto maximum point.*

2) *Let (τ_m) be a sequence in $[t_{s-1}, \tau)$ such that $\tau_m \to \tau$ and $H(\tau_m) \neq \emptyset$ $(m = 1, 2, \dots)$, $z_m \in H(\tau_m)$ $(m = 1, 2, \dots)$, and $z = (z_1, \dots, z_n)$ be a limit point for the sequence (z_m). Then $z_1, \dots, z_n$ is a family of Pareto multipliers for the Pareto maximum point x^*.*

Proof. Let (τ_m) and (z_m) be the sequences defined above and $z = (z_1, \dots, z_n)$ be a limit point for (z_m). Selecting, without renumeration, an appropriate subsequence, we assume that $z_m \to z$. Taking into account that $z_m > 0$ and $|z_m| = p\delta$ (see (3.21)), we get

$$z_i \geq 0 \quad (i = 1, \dots, n), \tag{3.27}$$

$$|z| = p\delta. \tag{3.28}$$

Since $\tau_m \in T$ and $z_m \in H(\tau_m)$, we have $h_i(\tau_m, z_m) > 0$ $(i = 1, \ldots, n,\ m = 1, 2, \ldots)$. Due to the continuity of h_i $(i = 1, \ldots, n)$ it holds that $h_i(\tau, z) \geq 0$ $(i = 1, \ldots, n)$, or, more specifically (see (3.20)),

$$h_i(\tau, z) = \sum_{j=1,\ldots,n,\ j\neq i} a_{ji} z_j - \lambda_i(\tau) z_i \geq 0 \tag{3.29}$$

$$(i = 1, \ldots, n).$$

Suppose

$$h_{i_0}(\tau, z) = \sum_{j=1,\ldots,n,\ j\neq i_0} a_{ji_0} z_j - \lambda_{i_0}(\tau) z_{i_0} > 0 \tag{3.30}$$

for some $i_0 \in \{1, \ldots, n\}$. Then

$$\sum_{j=1,\ldots,n,\ j\neq i_0} a_{ji_0} z_j - \lambda_{i_0}(\tau)(z_{i_0} + \varepsilon_0) > 0 \tag{3.31}$$

for a sufficiently small $\varepsilon_0 > 0$. Let

$$\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n) = (z_1, \ldots, z_{i_0-1}, z_{i_0} + \varepsilon_0, z_{i_0+1}, \ldots, z_n).$$

Using (3.29), (2.2) and (3.31), we get

$$h_i(\tau, \bar{z}) = \sum_{j=1,\ldots,n,\ j\neq i} a_{ji} \bar{z}_j - \lambda_i(\tau) \bar{z}_i > 0$$

$$(i = 1, \ldots, n).$$

Then

$$h_i(\tau, \bar{z}^*) = \sum_{j=1,\ldots,n,\ j\neq i} a_{ji} \bar{z}_j^* - \lambda_i(\tau) \bar{z}_i^* > 0$$

$$(i = 1, \ldots, n)$$

where

$$\bar{z}^* = (\bar{z}_1^*, \ldots, \bar{z}_n^*) = (\bar{z}_1 + \varepsilon_1, \ldots, \bar{z}_n + \varepsilon_1)$$

with a sufficiently small $\varepsilon_1 > 0$. In view of (3.27) $\bar{z}^* > 0$. For $z^* = p\delta\bar{z}^*/|\bar{z}^*|$ we have $|z^*| = p\delta$ and

$$h_i(\tau, z^*) = \sum_{j=1,\ldots,n,\ j\neq i} a_{ji} z_j^* - \lambda_i(\tau) z_i^* > 0$$

$$(i = 1, \ldots, n).$$

Thus, $z^* \in H(\tau)$. The latter contradicts (3.26). The contradiction shows that (3.30) is not possible for any $i_0 \in \{1, \ldots, n\}$. Hence, in view of (3.29) we get

$$h_i(\tau, z) = \sum_{j=1,\ldots,n,\ j\neq i} a_{ji} z_j - \lambda_i(\tau) z_i = 0 \tag{3.32}$$

$$(i = 1, \ldots, n).$$

As seen from (3.28), there is an $i_* \in \{1, \ldots, n\}$ such that $z_{i_*} > 0$. Then for every $i \in \{1, \ldots, n\}$, $i \neq i_*$,

$$\sum_{j=1,\ldots,n,\ j\neq i} a_{ji} z_j \geq a_{i_* i} z_{i_*} > 0$$

(here we use (2.2)). Now (3.32) shows that $\lambda_i(\tau) > 0$ and $z_i > 0$ for every $i \in \{1, \ldots, n\}$, $i \neq i_*$. Thus, $z > 0$. Multiplying (3.32) by $b_i'(y_i(\tau))$ and using (3.19), we get

$$\sum_{j=1,\ldots,n,\ j\neq i} a_{ji} b_i'(y_i(\tau)) z_j + [a_{ii} b_i'(y_i(\tau)) - c_i'(x_i(\tau))] z_i = 0 \qquad (3.33)$$

$$(i = 1, \ldots, n),$$

or

$$z_1 \frac{\partial w_1(x(\tau))}{\partial x_i} + \ldots + z_n \frac{\partial w_n(x(\tau))}{\partial x_i} = 0$$

$$(i = 1, \ldots, n)$$

(see the form of w_i (2.1)). Thus, the emission reduction vector $x^* = x(\tau)$ is a Pareto maximum point and $z_1, \ldots, z_n$ is a family of Pareto multipliers for x^*. The lemma is proved.

Recall that the emission reduction process terminates in step s (see Proposition 3.1). By (A6) the emission reduction vector $x(t_s)$ is the outcome of the emission reduction process. Our principal statement is the following.

Proposition 3.2 *The outcome of the emission reduction process, $x(t_s)$, lies in the closed $p\delta$-neighborhood of the Pareto maximum point x^* described in Lemma* 3.1.

Proof. By (2.6) and (3.18)

$$x(t_s) - x^* = x(t_s) - x(\tau) = \frac{t_s - \tau}{\delta} \Delta x(t_{s-1}). \qquad (3.34)$$

By (3.22) and (3.21) $|\Delta x(t_{s-1})| = p\delta$ and by (3.25) $0 \leq t_s - \tau \leq \delta$. Hence, the norm of the right hand side in (3.34) is not bigger than $p\delta$. Therefore, $|x(t_s) - x^*| \leq p\delta$. The proposition is proved.

Proposition 3.2 tells us that the discrepancy between a certain Pareto maximum point, x^*, and the output of the emission reduction process, $x(t_s)$, vanishes as the size of the time step in the emission reduction process, δ, goes to zero, or, equivalently, the frequency, in which the countries negotiate on updating their emission reductions, grows infinitely. Let us note that in every nondegenerate step of the emission reduction process, k, the local emission reduction vector, $\Delta x(t_k)$, being a result of the negotiations in step k (see (A7) and (A5)), is not defined uniquely. Therefore, the Pareto maximum point, x^*, that is approached, approximately, in the end of the emission reduction process is not pre-determined and can vary depending on the outcomes of the preceding negotiations. To summarize, we can say that Proposition 3.2 captures a robust but qualitative property of the proposed decentralized boundedly rational emission reduction strategy: in the beginning of the emission reduction process the countries can be sure that the process will bring them close to a solution of the emission reduction game in a finite number of steps; however the countries should also realize that specific features of that solution will be seen after the termination of the process only.

4 Examples

The next example shows that the positivity of the transport coefficients (see (2.2)) is essential for the validity of Proposition 3.2.

Example 4.1 Let the emission reduction process involve two countries, country 1 and country 2 ($n = 2$). Let country 1 pollute itself only ($a_{11} = 1$, $a_{12} = 0$), country 2 pollute itself and country 1 in equal proportions ($a_{21} = a_{22} = 1/2$), and the countries' utility functions be given by

$$w_1(x) = 1 - \frac{1}{x_1 + x_2/2 + 1} - \frac{x_1}{2}, \quad w_2(x) = 1 - \frac{1}{x_2/2 + 1} - \frac{x_2}{4};$$

here, in the right hand sides, the first terms and second terms represent the countries' benefit and cost functions, respectively. One can easily state that (A1) – (A3) are satisfied. We see that in contrast with the earlier assumptions, one of the transport coefficients, a_{12}, is zero. Let us show that Proposition 3.2 is no longer true.

First, we find the Pareto maximum points. We have

$$\frac{\partial w_1(x)}{\partial x_1} = \frac{1}{(x_1 + x_2/2 + 1)^2} - \frac{1}{2}, \quad \frac{\partial w_1(x)}{\partial x_2} = \frac{1}{2(x_1 + x_2/2 + 1)^2}, \tag{4.35}$$

$$\frac{\partial w_2(x)}{\partial x_1} = 0, \quad \frac{\partial w_2(x)}{\partial x_2} = \frac{1}{2(x_2/2 + 1)^2} - \frac{1}{4}, \tag{4.36}$$

We see that for every positive x the gradients $(\partial w_1(x)/\partial, w_1(x)/\partial x_2)$ and $(\partial w_2(x)/\partial, w_2(x)/\partial x_2)$ are linearly independent, implying

$$w_1(x + \varepsilon\Delta x) > w_1(x), \quad w_2(x + \varepsilon\Delta x) > w_2(x) \tag{4.37}$$

for some Δx and for all small $\varepsilon > 0$. Therefore, a Pareto maximum point cannot be positive. For every nonnegative x such that

$$(x_2/2 + 1)^2 < 2 \quad \text{or} \quad x_2 < r = 2(2^{1/2} - 1) \tag{4.38}$$

both $w_1(x)/\partial x_2$ and $w_2(x)/\partial x_2$ are positive; hence, (4.37) holds with $\Delta x = (0, 1)$ and a small $\varepsilon > 0$. Thus, a Pareto maximum point cannot be in the union of the x_1-axis and the part of the x_2-axis, which is located between 0 and r. Take an arbitrary nonnegative x belonging to the rest part of the x_2-axis: $x_1 = 0$ and $x_2 \geq r$ or (see (4.38)) $(x_2/2 + 1)^2 \geq 2$. Hence, by (4.35) and (4.36)

$$\frac{\partial w_1(x)}{\partial x_1} \leq 0, \quad \frac{\partial w_2(x)}{\partial x_2} \leq 0. \tag{4.39}$$

For any nonnegative $\bar{x} \neq x$ we have $\bar{x} = x + \Delta x$ with $\Delta x \neq 0$, $\Delta x_1 \geq 0$. Suppose $\Delta x_2 \neq 0$. Due to (4.39)

$$\frac{\partial w_2(x)}{\partial x_1}\Delta x_1 + \frac{\partial w_2(x)}{\partial x_2}\Delta x_2 \leq 0 \quad \text{if} \quad \Delta x_2 > 0,$$

$$\frac{\partial w_1(x)}{\partial x_1}\Delta x_1 + \frac{\partial w_1(x)}{\partial x_2}\Delta x_2 \leq 0 \quad \text{if} \quad \Delta x_2 < 0.$$

The strict concavity of $w_2(x)$ and $w_1(x)$ in x_2 implies that $w_2(\bar{x}) < w_2(x)$ if $\Delta x_2 > 0$ and $w_1(\bar{x}) < w_1(x)$ if $\Delta x_2 < 0$. Suppose $\Delta x_2 = 0$. Then $\Delta x_1 > 0$ and, in view of (4.39),

$$\frac{\partial w_2(x)}{\partial x_1}\Delta x_1 + \frac{\partial w_2(x)}{\partial x_2}\Delta x_2 = 0, \quad \frac{\partial w_1(x)}{\partial x_1}\Delta x_1 + \frac{\partial w_1(x)}{\partial x_2}\Delta x_2 \le 0.$$

Since $w_2(x)$ is constant in x_1 and $w_1(x)$ is strictly concave in x_1, it holds that $w_2(\bar{x}) = w_2(x)$ and $w_1(\bar{x}) < w_1(x)$. We have either $w_1(\bar{x}) < w_1(x)$, or $w_2(\bar{x}) < w_2(x)$, implying that x is a Pareto maximum point. Thus, all x such that $x_1 = 0$ and $x_2 \ge r$ constitute the set of all Pareto maximum points.

Consider the emission reduction process. The fact that the total emission reduction for country 1, $x_1(t_k)$, grows in each nondegenerate step, k, whereas all the Pareto maximum points, x, have the zero first coordinates, $x_1 = 0$, tells us that the total emission reduction vector, $x(t_k)$, may never approach any Pareto maximum point. To support this intuitive observation, we argue as follows.

Take a step k such that

$$(x_1(t_k) + x_2/2 + 1)^2 < 3/2. \tag{4.40}$$

Using (4.35) and (4.36), we find that for every positive emission reduction vector, $\Delta x(t_k)$, it holds that

$$\frac{\partial w_1(x(t_k))}{\partial x_1}\Delta x_1(t_k) + \frac{\partial w_1(x(t_k))}{\partial x_2}\Delta x_2(t_k) \ge \alpha_{11}\Delta x_1(t_k) + \alpha_{12}\Delta x_2(t_k)$$

where $\alpha_{11} = 2/3 - 1/2 > 0$, $\alpha_{12} = 2/3 > 0$, and

$$\frac{\partial w_2(x(t_k))}{\partial x_1}\Delta x_1(t_k) + \frac{\partial w_2(x(t_k))}{\partial x_2}\Delta x_2(t_k) \ge \alpha_{22}\Delta x_2(t_k)$$

where $\alpha_{22} = 1/3 - 1/4 > 0$. Therefore, every step, k, such that (4.40) holds is nondegenerate and every positive emission reduction vector is acceptable in that step.

For $k = 0$ (4.40) holds since $x(0) = 0$ (see (2.4)). Suppose in every nondegenerate step, k, satisfying (4.40), the countries choose an acceptable local emission vector $\Delta x(t_k)$ such that $\Delta x_1(t_k) = \Delta x_2(t_k) = p\delta$ (we assume that the norm in the two-dimensional space is such that $|\Delta x(t_k)| = \max\{|\Delta x_1(t_k)|, |\Delta x_2(t_k)|\}$). Let k_* be the maximum of all such k. For every $k \le k_*$ we have

$$x_1(t_k) = x_2(t_k) = pk\delta; \tag{4.41}$$

hence, k_* is the maximum of all $k = 0, 1, \ldots$ such that $(3pk\delta/2 + 1)^2 < 3/2$ or $pk\delta < q$ where $q = 2[(3/2)^{1/2} - 1]/3 > 0$. Clearly, $pk_*\delta \ge q - p\delta$, or, in view of (4.41), $x_1(t_{k_*}) \ge q - p\delta$. Let δ be so small that $q - p\delta > q/2$. Since $x_1(t_k)$ grows, $x_1(t_k) > q/2$ in all nondegenerate steps $k \ge k_*$. Thus, the emission reduction process either never terminates or terminates with an $x_1(t_s) > q/2$ in some step $s > k_*$; in the latter case the final emission reduction vector, $x(t_s)$, is at a distance higher than $q/2$ from any Pareto maximum point. The statement of Proposition 3.2 is violated.

As noted in the previous section, the emission reduction process has multiple outcomes. By Proposition 3.2 each of those outcomes approximates a certain Pareto maximum point with accuracy $p\delta$. Let us call a Pareto maximum point *$p\delta$-reachable* if it is approximated by some outcome of the emission reduction process with accuracy $p\delta$. Let us ask ourselves if all the Pareto maximum points are $p\delta$-reachable. The next example shows that there can be a solid gap between the set of all Pareto maximum points and the set of all $p\delta$-reachable ones.

Example 4.2 Let two countries, country 1 and country 2, involved in the emission reduction process ($n = 2$) pollute each other in equal proportions ($a_{ji} = 1/2$, $j, i = 1, 2$), and the countries' utility functions be identical:

$$w_1(x) = 1 - \frac{1}{x_1/2 + x_2/2 + 1} - \frac{x_1}{2}, \quad w_2(x) = 1 - \frac{1}{x_1/2 + x_2/2 + 1} - \frac{x_2}{2};$$

here, in the right hand sides, the first terms and second terms represent the countries' benefit and cost functions, respectively. One can easily state that (A1) – (A3) are satisfied.

We find the Pareto maximum points as nonnegative vectors x satisfying

$$z_1 \frac{\partial w_1(x)}{\partial x_1} + z_2 \frac{\partial w_2(x)}{\partial x_1} = 0, \quad z_1 \frac{\partial w_1(x)}{\partial x_2} + z_2 \frac{\partial w_2(x)}{\partial x_2} = 0$$

with some $z_1, z_2 > 0$. Simple calculations result in the following: the set of all Pareto maximum points consists of all nonnegative x such that

$$x_1/2 + x_2/2 = \beta = 2^{1/2} - 1. \tag{4.42}$$

Geometrically, the latter set is the interval, I, with the end points $x^{(1)} = (2\beta, 0)$ and $x^{(2)} = (0, 2\beta)$. At the end point $x^{(1)}$ the utilities of countries 1 and 2 reach, respectively, their minimum and maximum values, $1-1/(\beta+1)-\beta$ and $1-1/(\beta+1)$, in I; at the end point $x^{(0)} = (0, 2\beta)$ the situation is symmetric. At the middle point of I, $x^{(0)}$, the countries have the same utility value, $1 - 1/(\beta + 1) - \beta/2$. One can view the "middle" Pareto maximum point, $x^{(0)}$, as the "most fair" one and the end points, $x^{(1)}$ and $x^{(2)}$, as the "most unfair" ones. Given a Pareto maximum point, x, the distance from x to the "most unfair" Pareto maximum point closest to x can be treated as "the degree of fairness" of x.

Let us consider the emission reduction process described earlier. Using Proposition 3.2, we find that in every nondegenerate step, k, the total emission reduction vector, $x(t_k)$, lies in the triangle bordered by the x_1-axis, x_2-axis and interval I, in particular,

$$y(t_k) = x_1(t_k)/2 + x_2(t_k)/2 < \beta. \tag{4.43}$$

In the first degenerate step, s, vector $x(t_s)$ constituting the outcome of the emission reduction process lies necessarily beyond the interior of the triangle, implying

$$y(t_s) \geq \beta. \tag{4.44}$$

Take a nondegenerate step k. We have $x(t_{k+1}) = x(t_k) + \Delta x(t_k)$ where $\Delta x(t_k)$ is a positive emission reduction vector acceptable in step k, i.e., satisfying

$$\frac{\partial w_1(x(t_k))}{\partial x_1} \Delta x_1(t_k) + \frac{\partial w_1(x(t_k))}{\partial x_2} \Delta x_2(t_k) > 0,$$

$$\frac{\partial w_2(x(t_k))}{\partial x_1}\Delta x_1(t_k) + \frac{\partial w_2(x(t_k))}{\partial x_2}\Delta x_2(t_k) > 0,$$

or

$$\left(\frac{1}{2(y(t_k)+1)^2} - \frac{1}{2}\right) + \frac{1}{2(y(t_k)+1))^2}\frac{\Delta x_2(t_k)}{\Delta x_1(t_k)} > 0,$$

$$\frac{1}{2(y(t_k)+1)^2} + \left(\frac{1}{2(y(t_k)+1)^2} - \frac{1}{2}\right)\frac{\Delta x_2(t_k)}{\Delta x_1(t_k)} > 0$$

(here we use explicit forms of the partial derivatives). After an elementary transformation, we get

$$\frac{1}{(y(t_k)+1)^2 - 1} > \frac{\Delta x_2(t_k)}{\Delta x_1(t_k)} > (y(t_k)+1)^2 - 1.$$

The latter inequality implies

$$\frac{y(t_{k+1}) - y(t_k)}{\Delta x_1(t_k)} = \frac{1}{2}\left(\frac{\Delta x_2(t_k)}{\Delta x_1(t_k)} + 1\right) > \frac{(y(t_k)+1)^2}{2}. \tag{4.45}$$

We assume δ to be sufficiently small and view (4.45) as a difference approximation to the differential inequality

$$\frac{d\bar{y}(x_1)}{dx_1} > \frac{(\bar{y}(x_1)+1)^2}{2} \tag{4.46}$$

for a function $\bar{y}(x_1)$ at the point $(x_1(t_k), y(x_1(t_k)))$. One can prove that for an arbitrary $\varepsilon >$ and δ sufficiently small, there is a solution to the differential inequality (4.46), $\bar{y}$, defined on $[0,\infty)$, satisfying $\bar{y}(0) = 0$ and such that $|y(t_k) - \bar{y}(x(t_k))| < \varepsilon$ for all nondegenerate steps, k. Clearly, $\bar{y}(x_1) \geq \bar{y}_*(x_1)$ $(x_1 \geq 0)$ where $\bar{y}_*$ is the solution to the differential equation

$$\frac{d\bar{y}_*(x_1)}{dx_1} = \frac{(\bar{y}_*(x_1)+1)^2}{2}, \tag{4.47}$$

defined on $[0,\infty)$ and satisfying $\bar{y}_*(0) = 0$. Therefore, for the last nondegenerate step, $s-1$, it holds that

$$y(t_{s-1}) - \bar{y}_*(x(t_{s-1})) > -\varepsilon. \tag{4.48}$$

By (4.43) with $k = s-1$ and by (4.48) we have $\bar{y}_*(x(t_{s-1})) < \beta + \varepsilon$. Let $\bar{x}_1 > 0$ be such that $\bar{y}_*(x_1) = \beta$. If $\bar{y}_*(x(t_{s-1})) < \beta$, then $x(t_{s-1}) < \bar{x}_1$. If $\bar{y}_*(x(t_{s-1})) \geq \beta$, then $x(t_{s-1}) \geq \bar{x}_1$ and, due to (4.47),

$$x_1(t_{s-1}) - \bar{x}_1 \leq 2\frac{\bar{y}_*(x(t_{s-1})) - \bar{y}_*(\bar{x}_1)}{(\bar{y}_*(\bar{x}_1)+1)^2} \leq 2(\bar{y}_*(x_1(t_{s-1}) - \bar{y}_*(\bar{x}_1))) < 2\varepsilon.$$

Hence, for $x_1(t_s)$, the final emission reduction value for country 1, we have

$$x_1(t_s) \leq x_1(t_{s-1}) + p\delta < \bar{x}_1 + 2\varepsilon + p\delta. \tag{4.49}$$

Let us find $\bar{x}_1$. The integration of the differential equation (4.47) under the initial condition $\bar{y}_*(0) = 0$ yields

$$\bar{y}_*(x_1) = \frac{2}{2 - x_1} - 1.$$

Combining with $\bar{y}_*(\bar{x}_1) = \beta$ and resolving with respect to $\bar{x}_1$, we get

$$\bar{x}_1 = \frac{2\beta}{\beta + 1}.$$

Then by (4.49)

$$x_1(t_s) \leq \frac{2\beta}{\beta + 1} + 2\varepsilon + p\delta. \tag{4.50}$$

Using (4.44) or, equivalently, $x_1(t_s) + x_2(t_s) \geq 2\beta$, we find that

$$x_2(t_s) \geq 2\beta - x_1(t_s) = 2\beta \left(1 - \frac{1}{\beta + 1}\right) - 2\varepsilon - p\delta. \tag{4.51}$$

Let the norm of a vector x in the two-dimensional space be defined as $|x| = \max\{|x_1|, |x_2|\}$. Consider the distance from the outcome vector, $x(t_s)$, to the "most unfair" Pareto maximum point $x^{(1)} = (2\beta, 0)$. From (4.50) and (4.51) we get

$$|x(t_s) - x^{(1)}| \geq 2\beta \left(1 - \frac{1}{\beta + 1}\right) - 2\varepsilon - p\delta.$$

Note that for every $p\delta$-reachable Pareto maximum point, x^*, it holds that $|x(t_s) - x^*| \leq p\delta$. Thus, for every such x^*, we have

$$|x^* - x^{(1)}| \geq |x(t_s) - x^{(1)}| - |x(t_s) - x^*| \geq 2\beta \left(1 - \frac{1}{\beta + 1}\right) - 2\varepsilon - 2p\delta.$$

We see that for an arbitratily small $\gamma > 0$ one can choose ε and δ so small that all Pareto maximum points lying in the $(2\beta[1 - 1/(\beta + 1)] - \gamma)$-neighborhood of the "most unfair" Pareto maximum point $x^{(1)}$ are not $p\delta$-reachable.

A similar argument leads us to a symmetric statement: for an arbitrary $\gamma > 0$ one can choose ε and δ so small that all Pareto maximum points lying in the $(2\beta[1 - 1/(\beta + 1)] - \gamma)$-neighborhood of the "most unfair" Pareto maximum point $x^{(2)} = (2\beta, 0)$ are not $p\delta$-reachable.

Let us note in conclusion that a "converse" statement holds true as well: for an arbitrary $\gamma > 0$ one can choose ε and δ so small that all Pareto maximum points lying beyond the $(2\beta[1 - 1/(\beta + 1)] + \gamma)$-neighborhoods of the "most unfair" Pareto maximum points $x^{(1)}$ and $x^{(2)}$ are $p\delta$-reachable; for brevity, we omit a proof.

5 Negotiation pattern

Here, we discuss a negotiation pattern satisfying assumption (A4), i.e., allowing the countries in each step to either find an acceptable positive emission reduction vector if the step is nondegenerate, or identify the fact that the step is degenerate.

Take an arbitrary step of the emission reduction process, k, which is either initial ($k = 0$) or such that all the preceding steps are nondegenerate and consider negotiations in step k. The goal of the negotiations is to either find a positive emission reduction vector acceptable in that step, or identify that the step is degenerate and terminate the process.

Recall that a positive emission reduction vector $\Delta x(t_k)$ (2.5) is acceptable in step k if for every country, i, its local utility growth criterion (2.13) is satisfied. Substituting (2.8) in (2.13), we represent the set of the countries' local utility growth criteria in step k as a system of inequalities:

$$\lambda_i(t_k)\Delta x_i(t_k) < \sum_{j=1,\dots,n,\ j\neq i} a_{ji}\Delta x_j(t_k) \tag{5.52}$$

$$(i = 1, \dots, n).$$

We see that if $\lambda_i(t_k) \leq 0$, country i satisfies its local utility growth criterion in step k with any $\Delta x_i(t_k) > 0$; we call such a country, i, a *free negotiator* (in step k). Note that the strict inequality, $\lambda_i(t_k) < 0$, or, equivalently, $b_i'(y_i(t_k))a_{ii} - c_i'(x_i(t_k)) > 0$ (see (2.14)), implies that in step k the country's marginal cost is low enough and the country can gain in utility even by slightly reducing its emission solely. The opposite inequality, $\lambda_i(t_k) > 0$, implies that in step k the marginal cost for country i is high enough and a local growth in the country's utility is possible provided other countries reduce emission; we call such a country, i, a *constrained negotiator* (in step k).

Our negotiation pattern suggests that the negotiations in step k go in two phases, phase 1 and phase 2, the latter having two variants, phase 2a and phase 2b.

In phase 1 each country, i, reveals $\lambda_i(t_k)$. Based on that, the countries' community identifies the free negotiators and constrained negotiators. If there are free negotiators, the countries go to phase 2a. Otherwise the countries go to phase 2b.

Phase 2a is organized as follows. Based on some pre-defined rule, one free negotiator, i_*, is selected. The other countries, $i \neq i_*$, propose some $\Delta x_i(t_k) > 0$. The free negotiator i_* responds with a sufficiently large $\Delta x_{i_*}(t_k) > 0$ such that the utility growth criteria (5.52) are satisfied for all $i \neq i_*$; the latter is guaranteed, if, for example,

$$\Delta x_{i_*}(t_k) > \max_{i=1,\dots,n,\ i\neq i_*} \frac{\lambda_i(t_k)\Delta x_i(t_k)}{a_{i_* i}}.$$

For $i = i_*$ (5.52) is satisfied automatically. The vector $\Delta x(t_k)$ (2.5) resulting from the negotiations is acceptable in step k.

Let us give two comments to phase 2a. First, we see that if there exist free negotiators in step k, then step k is nondegenerate. Second, if there are several free negotiators in step k, the proposed simple decision making scheme in phase 2a "discriminates" the selected free negotiator, i_*, which is obliged to compensate for arbitrary choices of all the other negotiators, including the free ones. There are obviously a number of ways to modify the scheme and make it more cooperative; for the sake of brevity, we do not discuss such modifications here.

Phase 2b assuming that there are no free negotiators is organized as follows. In the beginning, the countries represent their local utility growth criteria (5.52) as

$$\Delta x_i(t_k) = \sum_{j=1,\dots,n,\ j\neq i} \frac{\beta_{ji}}{\gamma_i} \Delta x_j(t_k) \tag{5.53}$$

$$(i = 1, \dots, n)$$

where

$$\beta_{ji} = \frac{a_{ji}}{\lambda_i(t_k)} \quad (i = 1, \dots, n)$$

and

$$\gamma_i > 1 \quad (i = 1, \dots, n). \tag{5.54}$$

For each country, i, (5.53) is a formula for its *individual response*, $\Delta x_i(t_k) > 0$, to the proposals of the other countries, $\Delta x_j(t_k) > 0$, $j \neq i$.

Next, the countries switch to negotiations. The negotiations go through an *exploration stage* and a *decision making stage*. In the exploration stage the countries identify if step k is nondegenetate. If step k is degenerate, the countries cancel the decision making stage and terminate the emission reduction process (see (A6)). Otherwise, the countries switch to the decision making stage and find a local emission reduction vector acceptable in step k of the emission reduction process.

In the exploration stage the negotiations proceed in rounds. Round 1 is organized as follows. Country 1 communicates its individual response formula,

$$\Delta x_1(t_k) = \sum_{j=2}^{n} \frac{\beta_{j1}}{\gamma_1} \Delta x_j(t_k), \tag{5.55}$$

to country 2. Country 2 substitutes (5.55) in its individual response formula,

$$\Delta x_2(t_k) = \frac{\beta_{12}}{\gamma_2} \Delta x_1(t_k) + \sum_{j=3}^{n} \frac{\beta_{j2}}{\gamma_2} \Delta x_j(t_k),$$

transforming the latter into

$$\begin{aligned} \Delta x_2(t_k) &= \frac{\beta_{12}}{\gamma_2} \left(\sum_{j=2}^{n} \frac{\beta_{j1}}{\gamma_1} \Delta x_j(t_k) \right) + \sum_{j=3}^{n} \frac{\beta_{j2}}{\gamma_2} \Delta x_j(t_k), \\ &= \frac{\beta_{21}\beta_{12}}{\gamma_1\gamma_2} \Delta x_2(t_k) + \sum_{j=3}^{n} \left(\frac{\beta_{j1}\beta_{12}}{\gamma_1\gamma_2} + \frac{\beta_{j2}}{\gamma_2} \right) \Delta x_j(t_k), \end{aligned}$$

or

$$\Delta x_2(t_k) = \sum_{j=3}^{n} \frac{\beta_{j2}^{(2)}(\gamma_1, \gamma_2)}{\gamma^{(2)}(\gamma_1, \gamma_2)} \Delta x_j(t_k) \tag{5.56}$$

where

$$\beta_{j2}^{(2)}(\gamma_1, \gamma_2) = \frac{\beta_{j1}\beta_{12}}{\gamma_1\gamma_2} + \frac{\beta_{j2}}{\gamma_2} \quad (j = 3, \dots, n), \tag{5.57}$$

$$\gamma^{(2)}(\gamma_1, \gamma_2) = 1 - \frac{\beta_{21}\beta_{12}}{\gamma_1\gamma_2}; \tag{5.58}$$

here an in what follows we omit elementary transformations. The *updated individual response formula* for country 2, (5.56), takes into account the local utility growth criterion for country 1. The requirement that both sides in (5.56) are positive imposes a *positivity constraint* on $\gamma^{(2)}(\gamma_1, \gamma_2)$:

$$\gamma^{(2)}(\gamma_1, \gamma_2) > 0. \tag{5.59}$$

Country 2 communicates its updated individual response formula, (5.56), to country 1, and the latter substitutes (5.56) in its individual response formula (5.55) resulting in

$$\Delta x_1(t_k) = \frac{\beta_2}{\gamma_1}\left(\sum_{j=3}^{n} \frac{\beta_{j2}^{(2)}(\gamma_1, \gamma_2)}{\gamma^{(2)}(\gamma_1, \gamma_2)} \Delta x_j(t_k)\right) + \sum_{j=3}^{n} \frac{\beta_{j1}}{\gamma_1} \Delta x_j(t_k),$$

$$\Delta x_1(t_k) = \sum_{j=3}^{n} \beta_{j1}^{(2)}(\gamma_1, \gamma_2) \Delta x_j(t_k) \tag{5.60}$$

where

$$\beta_{j1}^{(2)}(\gamma_1, \gamma_2) = \frac{\beta_{j2}^{(2)}(\gamma_1, \gamma_2)\beta_{21}}{\gamma_1 \gamma^{(2)}(\gamma_1, \gamma_2)} + \frac{\beta_{j1}}{\gamma_1} \quad (j = 3, \ldots, n). \tag{5.61}$$

Two formulas, (5.60) and (5.56), represent the formula for a collective response of countries 1 and 2 to any proposed local emission reduction values of countries $3, \ldots, n$. The *collective response formula* (5.60), (5.56) and positivity constraint (5.59) constitute the result of round 1.

Round $m - 1$ where $2 \le m < n$ starts with the situation, in which countries $1, \ldots, m$ have generated their *collective response formula* in the form

$$\Delta x_i(t_k) = \sum_{j=m+1}^{n} \beta_{ji}^{(m)}(\gamma_1, \ldots, \gamma_m) \Delta x_j(t_k) \tag{5.62}$$

$$(i = 1, \ldots, m-1),$$

$$\Delta x_m(t_k) = \sum_{j=m+1}^{n} \frac{\beta_{jm}^{(m)}(\gamma_1, \ldots, \gamma_m)}{\gamma^{(m)}(\gamma_1, \ldots, \gamma_m)} \Delta x_j(t_k), \tag{5.63}$$

where $\beta_{ji}^{(m)}(\gamma_1, \ldots, \gamma_m)$ $(j = m+1, \ldots, n)$ are positive automatically, and a set of *positivity constraints*:

$$\gamma^{(i)}(\gamma_1, \ldots, \gamma_i) > 0 \quad (i = 1, \ldots, m). \tag{5.64}$$

Countries $1, \ldots, m$ communicate the collective response formula, (5.62), (5.63), and constraints (5.64) to country $m + 1$.

Let $m+1 < n$. Country $m+1$ substitutes (5.62), (5.63) in its individual response formula

$$\Delta x_{m+1}(t_k) = \sum_{i=1}^{m-1} \frac{\beta_{i\ m+1}}{\gamma_{m+1}} \Delta x_i(t_k) + \frac{\beta_{m\ m+1}}{\gamma_{m+1}} \Delta x_m(t_k) + \sum_{j=m+2}^{n} \frac{\beta_{j\ m+1}}{\gamma_{m+1}} \Delta x_j(t_k).$$

Calculations run as follows:

$$\begin{aligned}
\Delta x_{m+1}(t_k) &= \sum_{i=1}^{m-1} \frac{\beta_{i\ m+1}}{\gamma_{m+1}} \sum_{j=m+1}^{n} \beta_{ji}^{(m)}(\gamma_1, \ldots, \gamma_m)\Delta x_j(t_k) + \\
&\quad \frac{\beta_{m\ m+1}}{\gamma_{m+1}} \sum_{j=m+1}^{n} \frac{\beta_{jm}^{(m)}(\gamma_1, \ldots, \gamma_m)}{\gamma^{(m)}(\gamma_1, \ldots, \gamma_m)} \Delta x_j(t_k) + \\
&\quad \sum_{j=m+2}^{n} \frac{\beta_{j\ m+1}}{\gamma_{m+1}} \Delta x_j(t_k) \\
&= \sum_{i=1}^{m-1} \frac{\beta_{i\ m+1}}{\gamma_{m+1}} \left(\beta_{m+1\ i}^{(m)}(\gamma_1, \ldots, \gamma_m)\Delta x_{m+1}(t_k) + \sum_{j=m+2}^{n} \beta_{ji}^{(m)}(\gamma_1, \ldots, \gamma_m)\Delta x_j(t_k) \right) + \\
&\quad \frac{\beta_{m\ m+1}}{\gamma_{m+1}} \left(\frac{\beta_{m+1\ m}^{(m)}(\gamma_1, \ldots, \gamma_m)}{\gamma^{(m)}(\gamma_1, \ldots, \gamma_m)} \Delta x_{m+1}(t_k) + \sum_{j=m+2}^{n} \frac{\beta_{jm}^{(m)}(\gamma_1, \ldots, \gamma_m)}{\gamma^{(m)}(\gamma_1, \ldots, \gamma_m)} \Delta x_j(t_k) \right) + \\
&\quad \sum_{j=m+2}^{n} \frac{\beta_{j\ m+1}}{\gamma_{m+1}} \Delta x_j(t_k).
\end{aligned}$$

Resolving with respect to $\Delta x_{m+1}(t_k)$, country $m+1$ gets

$$\Delta x_{m+1}(t_k) = \sum_{j=m+2}^{n} \frac{\beta_{j\ m+1}^{(m+1)}(\gamma_1, \ldots, \gamma_{m+1})}{\gamma^{(m+1)}(\gamma_1, \ldots, \gamma_{m+1})} \Delta x_j(t_k) \tag{5.65}$$

where

$$\begin{aligned}
\beta_{j\ m+1}^{(m+1)}(\gamma_1, \ldots, \gamma_{m+1}) &= \sum_{i=1}^{m-1} \frac{\beta_{ji}^{(m)}(\gamma_1, \ldots, \gamma_m)\beta_{i\ m+1}}{\gamma_{m+1}} + \\
&\quad \frac{\beta_{jm}^{(m)}(\gamma_1, \ldots, \gamma_m)\beta_{m\ m+1}}{\gamma^{(m)}(\gamma_1, \ldots, \gamma_m)\gamma_{m+1}} + \frac{\beta_{j\ m+1}}{\gamma_{m+1}}, \qquad (5.66) \\
\gamma^{(m+1)}(\gamma_1, \ldots, \gamma_{m+1}) &= 1 - \sum_{i=1}^{m-1} \frac{\beta_{m+1\ i}^{(m)}(\gamma_1, \ldots, \gamma_m)\beta_{i\ m+1}}{\gamma_{m+1}} - \\
&\quad \frac{\beta_{m+1\ m}^{(m)}(\gamma_1, \ldots, \gamma_m)\beta_{m\ m+1}}{\gamma^{(m)}(\gamma_1, \ldots, \gamma_m)\gamma_{m+1}}. \qquad (5.67)
\end{aligned}$$

The fact that both sides in (5.65) are positive leads to the constraint

$$\gamma^{(m+1)}(\gamma_1, \ldots, \gamma_{m+1}) > 0. \tag{5.68}$$

Equality (5.65) represents an updated individual response formula for country $m+1$, in which the local utility growth criteria for countries $1, \ldots, m$ are taken into account.

Country $m+1$ communicates its updated individual response formula, (5.65), to countries $1, \ldots, m$. Countries $1, \ldots, m$ substitute (5.65) in their collective response formula (5.62), (5.63), transforming the latter into

$$\Delta x_i(t_k) = \sum_{j=m+2}^{n} \beta_{ji}^{(m+1)}(\gamma_1, \ldots, \gamma_m)\Delta x_j(t_k) \tag{5.69}$$

$$(i = 1, \ldots, m)$$

where

$$\beta_{ji}^{(m+1)}(\gamma_1, \ldots, \gamma_{m+1}) = \frac{\beta_{j\ m+1}^{(m+1)}(\gamma_1, \ldots, \gamma_{m+1})\beta_{m+1\ i}^{(m)}(\gamma_1, \ldots, \gamma_m)}{\gamma^{(m+1)}(\gamma_1, \ldots, \gamma_{m+1})} + \beta_{ji}^{(m)}(\gamma_1, \ldots, \gamma_m) \tag{5.70}$$

$$(i = 1, \ldots, m-1),$$

$$\beta_{jm}^{(m+1)}(\gamma_1, \ldots, \gamma_{m+1}) = \frac{\beta_{j\ m+1}^{(m+1)}(\gamma_1, \ldots, \gamma_{m+1})\beta_{m+1\ m}^{(m)}(\gamma_1, \ldots, \gamma_m)}{\gamma^{(m)}(\gamma_1, \ldots, \gamma_m)\gamma^{(m+1)}(\gamma_1, \ldots, \gamma_{m+1})} + \frac{\beta_{jm}^{(m)}(\gamma_1, \ldots, \gamma_m)}{\gamma^{(m)}(\gamma_1, \ldots, \gamma_m)}. \tag{5.71}$$

Equalities (5.69) and (5.65) give a collective response formula for countries $1, \ldots, m+1$. The collective response formula (5.69), (5.65) and positivity constraints (5.64), (5.68) form the result in round $m+1$. Equalities (5.70), (5.71), (5.66), (5.67) show how the collective response formula for countries $1, \ldots, m+1$, (5.69), (5.65), are formed based on the collective response formula for countries $1, \ldots, m$, (5.62), (5.63).

Let $m + 1 = n$. Country n substitutes (5.62), (5.63), where $m = n - 1$, in its individual response formula,

$$\Delta x_n(t_k) = \sum_{i=1}^{n-1} \frac{\beta_{in}}{\gamma_n} \Delta x_i(t_k),$$

and gets in result a simplified analogue of (5.65):

$$\Delta x_n(t_k) = \frac{\varphi^{(n)}(\gamma_1, \ldots, \gamma_{n-1})}{\gamma_n} \Delta x_n(t_k)$$

where

$$\varphi^{(n)}(\gamma_1, \ldots, \gamma_{n-1}) = \sum_{i=1}^{n-2} \beta_{ni}^{(n-1)}(\gamma_1, \ldots, \gamma_{n-1})\beta_{in} + \frac{\beta_{n\ n-1}^{(n-1)}(\gamma_1, \ldots, \gamma_{n-1})\beta_{n-1\ n}}{\gamma^{(n-1)}(\gamma_1, \ldots, \gamma_{n-1})}.$$

The inequality $\Delta x_n(t_k) > 0$ implies $\varphi^{(n)}(\gamma_1, \ldots, \gamma_{n-1}) = \gamma_n$ and, in view of $\gamma_n > 1$ (see (5.54)),

$$\varphi^{(n)}(\gamma_1, \ldots, \gamma_{n-1}) > 1. \tag{5.72}$$

Obviously, (5.72) is a necessary condition for the existence of a positive emission reduction vector $\Delta x(t_k)$ acceptable in step k, i.e., satisfying the countries' utility growth criteria (5.53). Country n communicates the criterion (5.72) to the other countries and finalizes round n.

In the final round of the exploration stage the countries verify if (5.72) is feasible under the constraints imposed on $\gamma_1, \ldots, \gamma_{n-1}$ earlier:

$$\gamma_i > 1, \quad \gamma^{(i)}(\gamma_1, \ldots, \gamma_i) > 0 \quad (i = 1, \ldots, n-1) \tag{5.73}$$

(see (5.54), (5.64)). If the countries find that the system of inequalities (5.72), (5.73) is incompatible, they qualify step k as degenerate, stop the negotiations and terminate the emission reduction process. Otherwise the countries switch to the decision making stage.

Note that in order to tell if the system of inequalities (5.72), (5.73) is compatible, it is sufficient to find $\varphi_*^{(n)} = \sup \varphi^{(n)}(\gamma_1, \ldots, \gamma_{n-1})$ under the constraints (5.73). This constrained optimization problem can be solved numerically using standard optimization techniques; for small n the problem can be treated analytically (for the sake of brevity we do not provide examples). Obviously, the system of inequalities (5.72), (5.73), is compatible if and only if $\varphi_*^{(n)} > 1$.

Let the system of inequalities (5.72), (5.73) be compatible and $\gamma_1, \ldots, \gamma_{n-1}$ satisfy (5.72), (5.73). Consider the decision making stage in the negotiations in phase 2b. The proposed negotiation scheme implies that the compatibility of the system of inequalities (5.72), (5.73) is sufficient for the existence of a positive emission reduction vector, $\Delta x(t_k)$, satisfying the system of the countries' local utility growth criteria (5.53).

In round 1 country n chooses a positive emission reduction value $\Delta x_n(t_k)$ and communicates this value to the other countries. In round 2 countries $1, \ldots, n-1$ compute their emission reduction values, $\Delta x_1(t_k), \ldots, \Delta x_{n-1}(t_k)$, using their collective response formula (5.62), (5.63) designed in round $m = n-2$ of the exploration stage:

$$\Delta x_i(t_k) = \beta_{ni}^{(n-1)}(\gamma_1, \ldots, \gamma_{n-1}) \Delta x_n(t_k) \tag{5.74}$$

$$(i = 1, \ldots, n-2),$$

$$\Delta x_{n-1}(t_k) = \frac{\beta_{n\ n-1}^{(n-1)}(\gamma_1, \ldots, \gamma_{n-1})}{\gamma^{(n-1)}(\gamma_1, \ldots, \gamma_{n-1})} \Delta x_n(t_k). \tag{5.75}$$

A straightforward argument shows that the resulting emission reduction vector, $\Delta x(t_k)$, satisfies the countries' local utility growth criteria (5.53), constituting the desired outcome of the negotiations in step k of the emission reduction process.

In conclusion, we outline a proof of the validity of (5.53). Let us show that $\Delta x(t_k)$ satisfies the collective response formula (5.62), (5.63) designed in round $m = n-3$ of the exploration stage:

$$\Delta x_i(t_k) = \beta_{n-1\ i}^{(n-2)}(\gamma_1, \ldots, \gamma_{n-2}) \Delta x_{n-1}(t_k) + \beta_{ni}^{(n-2)}(\gamma_1, \ldots, \gamma_{n-2}) \Delta x_n(t_k) \tag{5.76}$$

$$(i = 1, \ldots, n-3),$$

$$\Delta x_{n-2}(t_k) = \frac{\beta_{n-1\ n-2}^{(n-2)}(\gamma_1, \ldots, \gamma_{n-2})}{\gamma^{(n-2)}(\gamma_1, \ldots, \gamma_{n-2})} \Delta x_{n-1}(t_k) + \frac{\beta_{n\ n-2}^{(n-2)}(\gamma_1, \ldots, \gamma_{n-2})}{\gamma^{(n-2)}(\gamma_1, \ldots, \gamma_{n-2})} \Delta x_n(t_k). \tag{5.77}$$

By (5.70)

$$\begin{aligned} \beta_{ni}^{(n-1)}(\gamma_1, \ldots, \gamma_{n-1}) \quad = \quad & \frac{\beta_{n\ n-1}^{(n-1)}(\gamma_1, \ldots, \gamma_{n-1}) \beta_{n-1\ i}^{(n-2)}(\gamma_1, \ldots, \gamma_{n-2})}{\gamma^{(n-1)}(\gamma_1, \ldots, \gamma_{n-1})} + \\ & \beta_{ni}^{(n-2)}(\gamma_1, \ldots, \gamma_{n-1}) \end{aligned}$$

$$(i = 1, \ldots, n-3).$$

Substituting in (5.74) for $i = 1, \ldots, n-3$ and using (5.75), we get

$$\begin{aligned} \Delta x_i(t_k) &= \frac{\beta^{(n-1)}_{n\ n-1}(\gamma_1, \ldots, \gamma_{n-1})\beta^{(n-2)}_{n-1\ i}(\gamma_1, \ldots, \gamma_{n-2})}{\gamma^{(n-1)}(\gamma_1, \ldots, \gamma_{n-1})}\Delta x_n(t_k) + \\ &\quad \beta^{(n-2)}_{ni}(\gamma_1, \ldots, \gamma_{n-1})\Delta x_n(t_k) \\ &= \beta^{(n-2)}_{n-1\ i}(\gamma_1, \ldots, \gamma_{n-2})\Delta x_{n-1}(t_k) + \beta^{(n-2)}_{ni}(\gamma_1, \ldots, \gamma_{n-1})\Delta x_n(t_k) \qquad (5.78) \\ &\qquad (i = 1, \ldots, n-3). \end{aligned}$$

We see that (5.76) holds. By (5.71)

$$\begin{aligned} \beta^{(n-1)}_{j\ n-2}(\gamma_1, \ldots, \gamma_{n-1}) &= \frac{\beta^{(n-1)}_{j\ n-1}(\gamma_1, \ldots, \gamma_{n-1})\beta^{(n-2)}_{n-1\ n-2}(\gamma_1, \ldots, \gamma_m)}{\gamma^{(n-2)}(\gamma_1, \ldots, \gamma_{n-2})\gamma^{(n-1)}(\gamma_1, \ldots, \gamma_{n-1})} + \\ &\quad \frac{\beta^{(n-2)}_{j\ n-2}(\gamma_1, \ldots, \gamma_{n-2})}{\gamma^{(n-2)}(\gamma_1, \ldots, \gamma_{n-2})}. \end{aligned}$$

Substituting in (5.74) for $i = n-2$ and using (5.75), we get

$$\begin{aligned} \Delta x_{n-2}(t_k) &= \frac{\beta^{(n-1)}_{n\ n-2}(\gamma_1, \ldots, \gamma_{n-1})\beta^{(n-2)}_{n-1\ n-2}(\gamma_1, \ldots, \gamma_{n-2})}{\gamma^{(n-2)}(\gamma_1, \ldots, \gamma_{n-2})\gamma^{(n-1)}(\gamma_1, \ldots, \gamma_{n-1})}\Delta x_n(t_k) + \\ &\quad \frac{\beta^{(n-2)}_{n\ n-2}(\gamma_1, \ldots, \gamma_{n-1})}{\gamma^{(n-2)}(\gamma_1, \ldots, \gamma_{n-2})}\Delta x_n(t_k) \\ &= \frac{\beta^{(n-2)}_{n-1\ n-2}(\gamma_1, \ldots, \gamma_{n-2})}{\gamma^{(n-2)}(\gamma_1, \ldots, \gamma_{n-2})}\Delta x_{n-1}(t_k) + \frac{\beta^{(n-2)}_{n\ n-2}(\gamma_1, \ldots, \gamma_{n-1})}{\gamma^{(n-2)}(\gamma_1, \ldots, \gamma_{n-2})}\Delta x_n(t_k). \end{aligned}$$

Thus, (5.77) holds.

Similarly, we state, step by step, that $\Delta x(t_k)$ satisfies the collective response formula (5.62), (5.63) for $m = n-3, \ldots, 1$. For $m = 1$ the collective response formula (5.62), (5.63) implies that $\Delta x(t_k)$ satisfies the local utility growth criteria (5.53).

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