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REDUCTION THEOREM AND NORMAL FORMS OF LINEAR SECOND ORDER MIXED TYPE PDE FAMILIES IN THE PLANE*

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ABSTRACT. Normal forms for smooth deformation of germ of characteristic equation of second order partial differential equation being linear with respect to second derivative is found near a point of tangency of characteristic direction with the type change line, when this singular point is nondegenerate and non-resonance.

Keywords: implicit differential equation, mixed type equation, normal form, classification.

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1. Introduction

Consider the second order partial differential equation in the plane

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} = F(x,y,u,u_x,u_y), \tag{1}$$

where x, y are coordinates, a, b, c are smooth functions, and F is some function. The respective *characteristic equation* is defined as

$$a(x,y)dy^2 - 2b(x,y)dxdy + c(x,y)dx^2 = 0. \tag{2}$$

Characteristic directions at the point are the solutions of this equation. At the point there could be two characteristic directions, only one such direction and two imaginary ones if at this point the value of the *discriminant* $D := b^2 - ac$ is positive, zero and negative, respectively.

Net of integral curves of characteristic equation, its local and global behavior play an important role in the theory of partial differential equations (see, for example, [5], [7], [13]). Due to that the problem of getting of local normal forms of characteristic net (or characteristic equation) up to smooth change of coordinates has long history going back to 19-th century [1]. Starting from the beginning of the last century the list of such normal forms includes well known Laplace, wave and Cibrario-Tricomi equations. The respective characteristic equations are [6], [7], [16]

$$dy^2 + dx^2 = 0, \qquad dy^2 - dx^2 = 0 \quad \text{and} \quad dy^2 - xdx^2 = 0. \tag{3}$$

The first and the second normal forms take place near the point of ellipticity and hyperbolicity domain of the initial equation, namely, where the equation (2) has exactly zero and two real solutions $dy : dx$ at the point, respectively.

The third, Cibrario-Tricomi normal form, takes place at a typical point of the *type change line* (or else *discriminant curve*) of the equation, where the discriminant is equal to zero but its

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differential does not and, in addition, the characteristic direction is not tangent to the line at the point. The proof done for this form by F.Tricomi in his treatise [16] had gap and the form was justified correctly by M.Cibrario in [6] a little bit later. This form is the key object in the formulation of well known Tricomi problem and its various modifications.

Complete list of local normal forms of characteristic net for a generic linear second order partial differential equation in the plane was obtained in the end of 20-th century, when smooth normal forms were found near a point of the type change line, at which the characteristic direction is tangent to the line [8], [9], [10], [11]. It was proved that a characteristic equation near a point of such tangency is reduced to the form

$$dy^2 + (kx^2 - y)dx^2 = 0, \tag{4}$$

where k is some real parameter, by multiplication on smooth nonvanishing function and an appropriate selection of new smooth coordinates with the origin at this point, if some standard conditions take place. More precisely, the characteristic direction field could be lifted to the single value field on *equation surface* defined in the space of directions on the plane (with local coordinates $x, y, p, p = dy : dx$) by equation (2). At a point of this surface the value of the lifted direction field is the intersection of the tangent plane to the surface and the contact plane defined as zero of 1-form $dy - pdx$, if these planes are different.

A point of coincidence of these planes are the singular points of the lifted direction field, and ones correspond exactly to the points of tangency of characteristic direction field with the type change line. It is easy to show that near a singular point of the lifted field there exists smooth vector field defining the direction field outside the point. It is clear that the point is also singular one of such vector field. The normal form (4) could be obtained in all cases when there exists such type vector field for which this point is non-degenerate and the field is linearizable near the point. In such situation the parameter k is equal to $\alpha(\alpha+1)^{-2}/4$ for a saddle or a node and $(1+\alpha^2)/16$ for a focus, where *exponent* α is defined as the ratio of the eigenvalue with maximum modulus of the liberalization of the vector field at the point to that with minimum modulus in the fist two cases and as the modulus of the ratio of the imaginary part of the eigenvalue to the real part in the third. In these three cases the parameter is less then zero, greater then zero but less then $1/16$ and greater then $1/16$, respectively. Here the parameter k in the characteristic net equation could be reduced to any one in the respective interval, for example, -1, $1/20$ and 1, respectively [12], [13], [8], [9], by a continuous change of coordinates.

The reduction theorem was one of the key moment in the proofs of the normal forms in [8], [10]. This theorem reduces the problem of normal forms for equation (2) near point of tangency of characteristic field with type change line to the theory of normal forms of pair of the *folding involution* permutating on the equation surface the points with the same coordinates x, y and a vector field on the surface defined the lifted direction field near singular point of the fields.

Here we proof the reduction theorem for the case of families of equations (2), when the equation coefficients smoothly depend on finite dimensional parameter ε. Then using this theorem and well known results we give normal form for smooth deformation of germ of characteristic equation at a point of tangency of characteristic direction with the type change line when the singular point of lifted vector field is nondegenerate and, in addition, the exponent α is irrational in the case of saddle and not natural for a node.

It turns out that normal forms for families near such a points up to multiplication by non-vanishing function and selection of an appropriate coordinates x, y depending the the family's parameter and being of any given in advance order of smoothness are the same as in (4) but with k being already a function on this parameter too. Note that for families of equations (2) the normals forms for Laplace equation, wave equation and Cibrario-Tricomi equation cases are the same.

There are different lists of typical local bifurcations obtained for families of integral curves of equation (2) in the case of one or two dimensional parameter. The review of these results and the respective references are presented in [15].

2. Reduction theorem and its corollaries

Here for a smooth family of the equations (1) with finite dimensional parameter ε we analyze behavior of the respective family of characteristic net near a point P of the type change line, where the differential of discriminant is not zero and the characteristic direction is tangent to the line, and gives some normals forms of the families of characteristic net near the point up to smooth or sufficiently smooth change of coordinates. All our discussions are local near the considered point.

2.1. **Reduction theorem.**

Proposition 2.1. *A smooth family of equations*

$$a(x,y,\varepsilon)dy^2 - 2b(x,y,\varepsilon)dxdy + c(x,y,\varepsilon)dx^2 = 0, \tag{5}$$

with a finite dimensional parameter ε near the point P of discriminant curve, where $D(P) = 0, dD(P) \neq 0$ and the characteristic direction is tangent to the curve, takes the form

$$dy^2 + c(x,y,\varepsilon)dx^2 = 0 \tag{6}$$

with some new smooth function $c, c(O) = 0 = c_x(O) \neq c_y(O)$, after multiplication by some smooth non-vanishing function and an appropriate selection of smooth coordinates with the origin O at this point, which are foliated over the parameter.

This proposition is proved below in the next section.

Near the origin for a given parameter value the folding involution σ of this equation has the form

$$(x,p) \mapsto (x,-p) \tag{7}$$

in coordinates x and $p = dy/dx$ on the equation surface, as it is easy to see. The equation direction field in these coordinates could be calculated by the differentiation of the left hand side of equation $p^2 + c(x,y,\varepsilon) = 0$ and the substitution pdx instead dy. That leads to the equation direction field in the form $(-2p : c_x + pc_y)$ which could be defined on the equation surface by the vector field

$$v := (-2p, c_x + pc_y).$$

The origin is singular point of this vector field, and, in addition, on the line of fixed points of the folding involution (that is on $p = 0$) this field is either zero or has vertical direction, that is the direction of p-axis.

It is easy to check up, that at a point (x,p) of the equation surface the image of the field v under the folding involution is

$$\sigma_* v(x,p) = (2p, -c_x + pc_y).$$

Thus the determinant of the matrix with columns v and $\sigma_* v$ has at a point (x,p) the value $4p^2 c_y$. Due to $c_y(O) \neq 0$ this determinant has exactly the second order zero on the line of fixed points of the folding involution. In particular the fields v and $\sigma_* v$ are collinear only on the line. Accounting that we introduce the following notion of compatibility.

In the plane a vector field and a differentiable involution with a line of fixed points are called *compatible* at a point of the line if near this point the determinant of matrix defined by the field and its image under the involution has second order zero on the line. In the plane a direction field and a differentiable involution with a line of fixed points are called *compatible* at a point of the line if the field could be defined by a vector field being compatible with the involution at this point. Compatibility of germs is defined analogously.

Example. The plane (near the origin, or else germs at the origin of) vector field $(x, \alpha y)$ with $\alpha > 1$ and involution $(x, y) \mapsto ((\alpha+1)x - 2\alpha y)/(\alpha-1), 2x - (\alpha+1)y/(\alpha-1)))$ are compatible.

Two object (functions or germs of functions, maps, etc.) are called *C^r-equivalent along a differentiable (vector) field v* (= *C^r_v-equivalent*) if they could be transformed into each other by C^r-diffeomorphism mapping integral curves of the field into themselves. For a families of objects *C^r_v- equivalence* is C^r-diffeomorphism preserving the natural fibration over the family's parameter ε and mapping the integral curves of the field $(v, \dot{\varepsilon} = 0)$ into themselves; C^r_v- equivalence is strong if, in addition, it preserves the parameter.

Theorem 2.1. *Two germs at the origin of smooth families (v, σ_1) and (v, σ_2) of compatible pairs of direction fields and involutions with the same both finite dimensional parameter and the surface of fixed point of involutions, which passes through the origin, are strongly C^∞_v-equivalent, if for any fixed parameter value being sufficiently closed to zero the field v is transversal to the line of fixed points of involutions almost everywhere.*

We call this statement as the reduction theorem. The theorem is proved in the next Section.

When the discriminant of a characteristic equation is zero but its differential does not this theorem and Proposition 2.1 reduces the problem of getting normal forms of a deformation of this equation near point of tangency of field of characteristic direction with the type change line of finite multiplicity to an analogous problem for the respective deformations of pair of the equation direction field and folding involution.

2.2. **Normal forms of families of mixed type linear PDE in the plane.** Local family of smooth vector fields in $\mathbb{R}^n_x$ with a finite dimensional parameter $\varepsilon \in \mathbb{R}^m$, where n and m are natural number, is the germ at the origin of vector field defined by the equation

$$\dot{x} = v(x, \varepsilon), \qquad \dot{\varepsilon} = 0. \tag{8}$$

The first component of the family, namely, $v, v = v(x, \varepsilon)$, is also called as a *deformation* with a parameter ε of germ of vector field $v(., 0)$ at the origin. Two local families of vector fields are *finite smooth equivalent* if for any finite natural r there are representatives of the respective germs of the families which are C^r-smooth conjugate in their domains, that is there exists C^r-diffeomorphism of the form

$$(x, \varepsilon) \mapsto (X(x, \varepsilon), E(\varepsilon)), \qquad (X(0, 0), E(0)) = (0, 0),$$

which is defined near the origin and conjugates the phase flows of the representatives [14].

A set of complex numbers $\lambda = (\lambda_1, \lambda_2, \dots \lambda_n) \in \mathbb{C}^n$ is *nonresonance* if there is no any relation of the form $\lambda_j = m_1\lambda_1 + \cdots + m_n\lambda_n$ with nonnegative integer m_j, $m_1 + m_2 + \cdots + m_n \geq 2$ [2], [3]. A singular point of differentiable vector field is *nonresonance* if the set of eigenvalues of the linearization of the field at the point is nonresonance.

Theorem 2.2. ([14]) *A deformation of germ of smooth vector field in $\mathbb{R}^n$ at its singular point is finite smooth equivalent to a germ at the origin of a deformation*

$$\dot{z} = A(\varepsilon)z \tag{9}$$

with some matrix function A, if the point is nonresonance.

For $n = 2$ by linear change of coordinate z and an appropriate time dilatation, which depends on the parameter, equation (9) could be reduced to the form

$$\dot{x} = 2y \quad \text{and} \quad \dot{y} = -2k(\varepsilon)x + y, \tag{10}$$

where $z = (x, y)$ and k is a function of the same class of smoothness as the matrix A. Note that $k(0) \neq 0$ due to the origin is nonresonance singular point.

The involution $\sigma : (x, y) \mapsto (x, -y)$ is compatible with the vector field $v, v(x, y) = (2y, k(\varepsilon)x + y)$ of the equation (10) because

$$\begin{vmatrix} v \\ \sigma_* v \end{vmatrix} (x, y) = \begin{vmatrix} 2y & -2k(\varepsilon)x + y \\ -2y & 2k(\varepsilon)x + y \end{vmatrix} = 4y^2.$$

Hence applying now the reduction theorem (like in [8] or [9]) we get the following statement.

Theorem 2.3. *For any given order of smoothness $r \geq 1$ a deformation of germ of characteristic equation (2) at its nonresonance folded singular point takes the form of germ at the origin of deformation*

$$dy^2 + (k(\varepsilon)x^2 - y)dx^2 = 0 \tag{11}$$

with some function k, after multiplication of the equation by nonvanishing C^r-function and an appropriate choice of C^r-coordinates foliated over the parameter with the origin at the point.

Nonresonance folded singular point of the characteristic equation is the point of tangency of characteristic direction field with type change line for which the corresponding singular point of the lifted directon field on the equation surface is nonresonance. For the initial PDE equation that implies the following.

Theorem 2.4. *For any given order of smoothness $r \geq 2$ a deformation of germ of equation (1) at nonresonance folded singular point of its characteristic equation takes the form of germ at the origin of deformation*

$$u_{xx} + (k(\varepsilon)x^2 - y)u_{yy} = F(x, y, u, u_x, u_y) \tag{12}$$

with some function k and new function F, after multiplication of the equation by nonvanishing C^r-function and an appropriate choice of C^r-coordinates foliated over the parameter with the origin at the point.

Here in equation (10) and Theorems 2.3, 2.4 the function k is the same. Note that class of smoothness of this function could be selected as we prefer. But the increasing of this class could lead to reduction of the interval along the parameter axis where the formulas work.

3. Proof of reduction theorem

We use the scheme of the proof proposed in [8] for the case without parameters.

Near the origin smooth local coordinates x, y, ε, which are foliated over the parameter, let us take in such a way that the the family σ_1 of involutions and the surface of the fixed points of involutions take forms $(x, y, \varepsilon) \mapsto (x, -y, \varepsilon)$, and $y = 0$, respectively. It is clear that such selection of coordinates is possible. For example, consider any smooth function $f, f(O) = 0$, with differential at the origin which does not vanish on eigenvectors of derivative $\sigma_{1,*}(O)$ and any its smooth deformation F with the parameter ε. Then appropriate local coordinates could be taken in the form $x = F + \sigma_1^* F$, $y = F - \sigma_1^* F$ with the same coordinate ε along parameter axis.

Denoting $v = (v_1, v_2)$ and accounting the compatibility of v with family σ_1 of involutions we find that the function

$$\begin{vmatrix} v_1(x,y,\varepsilon) & v_2(x,y,\varepsilon) \\ v_1(x,-y,\varepsilon) & -v_2(x,-y,\varepsilon) \end{vmatrix} = -v_1(x,y,\varepsilon)v_2(x,-y,\varepsilon) - v_1(x,-y,\varepsilon)v_2(x,y,\varepsilon)$$

has exactly the second order zero on the surface $y = 0$ of fixed points of involutions. In particular on this surface we have $v_1 v_2 \equiv 0$. Now, accounting that the field $(v, \dot{\varepsilon} = 0)$ is transversal to the surface almost everywhere we get that the last equality implies $v_1(x, 0, \varepsilon) \equiv 0$, and, in addition, the direction of y-axis $(=(0,1,0))$ is eigendirection with eigenvalue -1 of the derivatives σ_{1*}, σ_{2*} at any point of the surface.

Thus this derivatives are the same at any point of the surface, and hence in selected coordinates near the origin the family σ_2 could be written in the form

$$(x, y, \varepsilon) \mapsto (x + y^2 r(x, y, \varepsilon), -y + y^2 s(x, y, \varepsilon), \varepsilon)$$

with some smooth functions r and s. Consequently there exist coordinates

$$\zeta = x + y^2 R(x, y, \varepsilon), \quad \eta = y + y^2 S(x, y, \varepsilon)$$

with some smooth functions R and S and the same ε, in which the family σ_2 of involutions has a form

$$(\zeta, \eta, \varepsilon) \mapsto (\zeta, -\eta, \varepsilon).$$

The rest part of the proof is based on homotopical method proposed by R.Thom [2], [4]. Locally near the origin consider smooth deformation of involutions families σ_1 and σ_2 :

$$\gamma_t : (\zeta_t, \eta_t, \varepsilon) \mapsto (\zeta_t, -\eta_t, \varepsilon),$$

where

$$\zeta_t = x + ty^2 R(x, y, \varepsilon), \quad \eta_t = y + ty^2 S(x, y, \varepsilon).$$

A deformation preserving the natural foliation over the parameter is called as *fibered*. We have $\gamma_0 = \sigma_1$, $\gamma_1 = \sigma_2$. Denote by V_t the respective *infinitesimal deformation field* which is the velocity of motion of the image of a point under a smooth deformation of the involution. It is clear that for considered deformation V_t has zero component along the parameter axis.

Lemma 3.1. *A field V is infinitesimal deformation field of a smooth fibered deformation of family σ if and only if $\sigma_* V = -V$.*

Lemma 3.2. *For a fibered deformation g of the identity with a velocity h the family of involutions $\sigma : (x, y, \varepsilon) \mapsto (x, -y, \varepsilon)$ is deformed with the velocity $h - \sigma_* h$.*

These lemmas are analogous of the respective statements from [8], [9]. Ones can be proofed by direct calculation. Due to that we omit the proofs of these lemmas here.

Due to Lemma 3.2 to prove the theorem it is sufficient locally near the interval $[0, 1]$ of t-axis a deformation velocity to submit in the form

$$V_t = f_t v - (\gamma_t^* f_t)\gamma_{t*} v, \tag{13}$$

where v is our vector field and f_t is a smooth function on t, x, y, ε. Let us show that such presentation really takes place.

The solvability of homological equation (13) with respect to f_t is based on the compatibility both of involutions families σ_1 and σ_2 with the family v. This compatibility immediately implies the compatibility of the families v and γ_t for all $t \in [0, 1]$.

Deformation velocity (the index t is omitted) V has on the curve $y = 0$ (or $\eta = 0$) zero at least of the second order. Hence it could be presented in the form:

$$V = \eta^2 \begin{pmatrix} h(\zeta, \eta, \varepsilon) \\ r(\zeta, \eta, \varepsilon) \end{pmatrix} \tag{14}$$

with some smooth functions h and r.

In addition, due to Lemma 3.1 a deformation velocity has to satisfy the equality $\gamma_* V = -V$. Substituting that to the form (14) of the deformation field we get

$$\begin{pmatrix} \eta^2 h(\zeta, -\eta, \varepsilon) \\ -\eta^2 r(\zeta, -\eta, \varepsilon) \end{pmatrix} = - \begin{pmatrix} \eta^2 h(\zeta, \eta, \varepsilon) \\ \eta^2 r(\zeta, \eta, \varepsilon) \end{pmatrix}.$$

The last equality immediately implies $h(\zeta, -\eta, \varepsilon) = -h(\zeta, \eta, \varepsilon)$ and $r(\zeta, -\eta, \varepsilon) = r(\zeta, \eta, \varepsilon)$. Thus the functions h and r are odd and even, respectively. Hence locally near the origin they could be presented in the form

$$h(\zeta, \eta, \varepsilon) = \eta p(\zeta, \eta^2, \varepsilon) \quad \text{and} \quad r(\zeta, \eta, \varepsilon) = q(\zeta, \eta^2, \varepsilon)$$

with some smooth functions p and q.

Consequently the deformation velocity V could be written near the origin in the form

$$V(\zeta, \eta, \varepsilon) = \eta^3 p(\zeta, \eta^2, \varepsilon)\frac{\partial}{\partial \zeta} + \eta^2 q(\zeta, \eta^2, \varepsilon)\frac{\partial}{\partial \eta} \tag{15}$$

with some smooth functions p and q.

Further due to the reduction theorem conditions the field v is transversal to the surface $\eta = 0$ of fixed points of involutions almost everywhere. Hence on this surface we have $\gamma_* v = -v$, and locally near the origin this field could be written as

$$v(\zeta, \eta, \varepsilon) = \eta l(\zeta, \eta, \varepsilon)\frac{\partial}{\partial \zeta} + m(\zeta, \eta, \varepsilon)\frac{\partial}{\partial \eta} \tag{16}$$

with some smooth functions l and m. Let us submit the function f as the sum of even and odd functions with respect to variable η,

$$f(\zeta, \eta, \varepsilon) = u(\zeta, \eta^2, \varepsilon) + \eta w(\zeta, \eta^2, \varepsilon),$$

where u and w are smooth functions. Substituting this expression for f and expressions (15) and (16) into the terms of the right hand side of equation (13) we obtain:

$$(f_t v)(\zeta, \eta, \varepsilon) = [u(\zeta, \eta^2, \varepsilon) + \eta w(\zeta, \eta^2, \varepsilon)][\eta l(\zeta, \eta, \varepsilon)\frac{\partial}{\partial \zeta} + m(\zeta, \eta, \varepsilon)\frac{\partial}{\partial \eta}], \tag{17}$$

$$(\gamma_t^* f_t)(\zeta, \eta, \varepsilon) = f(\zeta, -\eta, \varepsilon) = u(\zeta, \eta^2, \varepsilon) - \eta w(\zeta, \eta^2, \varepsilon),$$

$$(\gamma_{t*} v)(\zeta, \eta, \varepsilon) = -(\eta l(\zeta, -\eta, \varepsilon)\frac{\partial}{\partial \zeta} + m(\zeta, -\eta, \varepsilon)\frac{\partial}{\partial \eta}),$$

$$((\gamma_t^* f_t)\gamma_{t*} v)(\zeta, \eta, \varepsilon) = -[u(\zeta, \eta^2, \varepsilon) - \eta w(\zeta, \eta^2, \varepsilon)](\eta l(\zeta, -\eta, \varepsilon)\frac{\partial}{\partial \zeta} + m(\zeta, -\eta, \varepsilon)\frac{\partial}{\partial \eta}). \tag{18}$$

Substituting now the expressions (15) (17), (18) (for V, $f_t v$ and $(\gamma_t^* f_t)\gamma_{t*} v$, respectively) into equation (13) we get:

$$\eta^3 p(\zeta, \eta^2, \varepsilon)\frac{\partial}{\partial \zeta} + \eta^2 q(\zeta, \eta^2, \varepsilon)\frac{\partial}{\partial \eta} =$$

$$= [u(\zeta, \eta^2, \varepsilon) + \eta w(\zeta, \eta^2, \varepsilon)][\eta l(\zeta, \eta, \varepsilon)\frac{\partial}{\partial \zeta} + m(\zeta, \eta, \varepsilon)\frac{\partial}{\partial \eta}] +$$

$$+ [u(\zeta, \eta^2, \varepsilon) - \eta w(\zeta, \eta^2, \varepsilon)](\eta l(\zeta, -\eta, \varepsilon)\frac{\partial}{\partial \zeta} + m(\zeta, -\eta, \varepsilon)\frac{\partial}{\partial \eta}) =$$

$$= [u\eta(l(\zeta,\eta,\varepsilon)+l(\zeta,-\eta,\varepsilon))+w\eta^2(l(\zeta,\eta,\varepsilon)-l(\zeta,-\eta,\varepsilon))]\frac{\partial}{\partial\zeta}+$$
$$+[u(m(\zeta,\eta,\varepsilon)+m(\zeta,-\eta,\varepsilon))+w\eta(m(\zeta,\eta,\varepsilon)-m(\zeta,-\eta,\varepsilon))]\frac{\partial}{\partial\eta}.$$

Equating the field components from the left and right parts of the last expression we obtain the following system of linear equations on u and w:

$$\begin{cases} u\eta(l(\zeta,\eta,\varepsilon)+l(\zeta,-\eta,\varepsilon))+w\eta^2(l(\zeta,\eta,\varepsilon)-l(\zeta,-\eta,\varepsilon)) = \eta^3 p(\zeta,\eta^2,\varepsilon) \\ u(m(\zeta,\eta,\varepsilon)+m(\zeta,-\eta,\varepsilon))+w\eta(m(\zeta,\eta,\varepsilon)-m(\zeta,-\eta,\varepsilon)) = \eta^2 q(\zeta,\eta^2,\varepsilon). \end{cases}$$

Dividing here the first equation by η we reduce the system to the form

$$\begin{cases} u(l(\zeta,\eta,\varepsilon)+l(\zeta,-\eta,\varepsilon))+w\eta(l(\zeta,\eta,\varepsilon)-l(\zeta,-\eta,\varepsilon)) = \eta^2 p(\zeta,\eta^2,\varepsilon) \\ u(m(\zeta,\eta,\varepsilon)+m(\zeta,-\eta,\varepsilon))+w\eta(m(\zeta,\eta,\varepsilon)-m(\zeta,-\eta,\varepsilon)) = \eta^2 q(\zeta,\eta^2,\varepsilon). \end{cases} \tag{19}$$

Determinant of the matrix of this system is

$$\begin{vmatrix} l(\zeta,\eta,\varepsilon)+l(\zeta,-\eta,\varepsilon) & \eta(l(\zeta,\eta,\varepsilon)-l(\zeta,-\eta,\varepsilon)) \\ m(\zeta,\eta,\varepsilon)+m(\zeta,-\eta,\varepsilon) & \eta(m(\zeta,\eta,\varepsilon)-m(\zeta,-\eta,\varepsilon)) \end{vmatrix} = \eta^2[L+h(\zeta,\eta,\varepsilon)],$$

where $L = 2[l(0,0,0)m_\eta(0,0,0) - l_\eta(0,0,0)m(0,0,0)]$ and h is a smooth function vanishing at the origin.

Hence system (19) is smoothly resolvable with respect to u and w near the origin if $L \neq 0$ because the right hand side of the system is divided by η^2. But L is not zero due to compatibility condition of families of involutions and fields. Indeed due to compatibility the area of parallelogram defined by values of field v and $\gamma_* v$ has second order zero on the surface $\eta = 0$ of fixed points of involutions. Hence the function

$$\begin{vmatrix} v \\ \gamma_* v \end{vmatrix}(\zeta,\eta,\varepsilon) = \begin{vmatrix} \eta l(\zeta,\eta,\varepsilon) & m(\zeta,\eta,\varepsilon) \\ -\eta l(\zeta,-\eta,\varepsilon) & -m(\zeta,-\eta,\varepsilon) \end{vmatrix} =$$
$$= -\eta[l(\zeta,\eta,\varepsilon)m(\zeta,-\eta,\varepsilon) - l(\zeta,-\eta,\varepsilon)m(\zeta,\eta,\varepsilon)] =$$
$$= 2\eta^2[L+H(\zeta,\eta,\varepsilon)],$$

where H is a smooth function vanishing at the origin, has second order zero on this surface. Consequently, $L \neq 0$, and locally near the origin system (19) is smoothly resolvable with respect to u and w.

Thus homological equation (13) is smoothly resolvable and germs of families of involutions σ_1 and σ_2 at the origin are strongly C^∞_v-equivalent.

The reduction theorem is proved.

3.1. **Proof of Proposition 2.1.** At the first let us select near a point P smooth local coordinates with the origin at this point such that the characteristic direction at the origin is the abscises axis direction. After that one could rewrite the characteristic equation in the form

$$a(x,y,\varepsilon)p^2 - 2b(x,y,\varepsilon)p + c(x,y,\varepsilon) = 0,$$

where $p = dy/dx$ and a, b, c are smooth functions. We have $a(O) \neq 0 = b(O) = c(O)$, $c_y(O) \neq 0$ due to the conditions $D(O) = 0$ and $|D_x(O)| + |D_y(O)| \neq 0$ and also the coordinates selection, where $O = (0,0,0)$. To reduce the coefficient b to zero let us take new coordinate $\tilde{y}$, $y = Y(x,\tilde{y},\varepsilon)$. We have

$$\frac{dy}{dx} = Y_x(x,\tilde{y},\varepsilon) + Y_{\tilde{y}}(x,\tilde{y},\varepsilon)\frac{d\tilde{y}}{dx}.$$

Substituting new variable to the equation we find

$$(Y_x + Y_{\tilde{y}}\frac{d\tilde{y}}{dx})^2 - 2b(x,Y,\varepsilon)(Y_x + Y_{\tilde{y}}\frac{d\tilde{y}}{dx}) + c(x,Y,\varepsilon) = 0,$$

and after simple transformation we arrive to the equation

$$Y_{\tilde{y}}^2(\frac{d\tilde{y}}{dx})^2 + 2Y_{\tilde{y}}[Y_x - b(x,Y,\varepsilon)]\frac{d\tilde{y}}{dx} + c(x,Y,\varepsilon) + Y_x^2 - 2b(x,Y,\varepsilon)Y_x = 0. \quad (20)$$

Thus the second term in this equation is staying with zero coefficient if the expression in square brackets is identically zero, that is

$$Y_x \equiv b(x,Y,\varepsilon).$$

For any given smooth initial conditions on the plane $x = 0$ the last equation has unique and smooth solution. Selecting the solution of this equation for $Y(0,\tilde{y},\varepsilon) = \tilde{y}$ we could write this solution in the form

$$Y(x,\tilde{y},\varepsilon) = \tilde{y} + xB(x,\tilde{y},\varepsilon),$$

where B is some smooth function, due to Hadamard lemma [4]. Now substituting this solution to the equation (20) and dividing the result by $Y_{\tilde{y}}^2$ we arrive to the needed form of the equation with some new function c.

Proposition 2.1 is proved.

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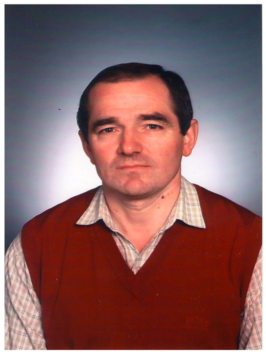

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