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OPTIMIZATION OF STATIONARY SOLUTION OF A MODEL OF SIZE-STRUCTURED POPULATION EXPLOITATION

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Dedicated to Vasilii Vasil'evich Zhikov on the ocassion of his 70th birthday

We establish the global stability of a nontrivial stationary state of the size-structured population dynamics in the case where the growth rate, mortality, and exploitation intensity depend only on the size and certain conditions on the model parameters are imposed. We show that a stationary state maximizing the profit functional of population exploitation, exists and is unique. We also obtain a necessary optimality condition, owing to which this state can be found numerically. Bibliography: 3 *titles.*

1 Introduction

For the dynamics of a size-structured population we prove the existence and uniqueness of a stationary state maximizing the profit on population exploitation under the assumption that the model parameters and exploitation intensity are stationary. More exactly, the population dynamics is described by the equation [1, 2]

$$\frac{\partial x(t,l)}{\partial t} + \frac{\partial [g(l, x(t,\cdot))x(t,l)]}{\partial l} = -[\mu(l, x(t,\cdot)) + u(l, x(t,\cdot))]x(t,l), \tag{1.1}$$

where $x(t,l)$ is the density, $g(l, x(t,\cdot))$ is the growth, and $\mu(l, x(t,\cdot))$ is the mortality of a biomass of size l at time t, whereas $u(l, x(t,\cdot))$ characterizes the exploitation intensity for this population.

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The boundary conditions are interpreted as the reproduction of biomass in natural and industrial ways (for example, reforestation of harvested areas) and are given by the formula

$$x(t,0) = \int_0^L r(l) x^{\beta}(t,l) dl + p(t), \tag{1.2}$$

where L is the size characterizing either the life cycle of biomass or the termination of biomass exploitation, r is the reproduction coefficient, β characterizes the nonlinear dependence of the reproduction property on the biomass density, and p is the density of biomass reproduced in an industrial way.

We look for stationary solutions to the problem (1.1), (1.2) for where it is natural to assume that the growth rate, mortality rate, and exploitation intensity depend only on the biomass size l. Furthermore, we assume that the reproduction coefficient vanishes for $0 \leqslant l < l_0$ and is greater than zero for $l_0 < l \leqslant L$. Here, $l_0 > 0$ is the minimal size of the reproductive biomass. Under these conditions, a stationary solution to the problem (1.1), (1.2) is a solution to the Cauchy problem

$$\frac{d[g(l)x(l)]}{dl} = -[\mu(l) + u(l)]x(l), \quad x(0) = \int_{l_0}^L r(l) x^{\beta}(l) dl + p_0. \tag{1.3}$$

If such a solution exists, then it has the form

$$x(l) = \frac{g(0)x(0)}{g(l)} e^{-\int_0^l m(s)ds}, \qquad m(l) = \frac{\mu(l) + u(l)}{g(l)}. \tag{1.4}$$

Substituting the solution into the initial conditions in the problem (1.3), we obtain an equation with respect to the value $x(0)$. From this equation we see that for $0 < \beta < 1$ there exists a unique value of $x(0)$ providing a positive solution. We note that already for $\beta = 1$ for the existence of a nonnegative solution the following inequality is necessary:

$$\int_{l_0}^L r(l) \frac{g_0}{g(l)} e^{-\int_0^l m(s)ds} dl < 1. \tag{1.5}$$

Below, we will consider only the case $0 < \beta < 1$, but all the results are valid, with minor modifications, in the case $\beta = 1$.

2 The Main Results

In this section, we formulate results about the stability of a stationary state, the existence and uniqueness of such a state with the maximum profit, and the corresponding necessary optimality condition. The proof is given in Section 3.

2.1 Stability of the stationary state

We denote by $x_{\min}$ the minimum of the stationary solution x on the segment $[0, L]$.

Theorem 2.1. *Suppose that $0 < \beta < 1$, $p = p_0 > 0$, $\mu \geqslant 0$ and $u \geqslant 0$ are piecewise continuous functions of l, a function g is positive and differentiable, and*

$$\varepsilon := \inf_{l \in [0,L]} [2\mu(l) + 2u(l) + g'(l)] - g_0 x_{\min}^{2\beta-2} \int_0^L r^2(l)dl > 0. \tag{2.1}$$

Then the solution x_* to the problem (1.1), (1.2), *regarded as a function of l, converges in the L_2-metric as $t \to \infty$ at exponential rate to the stationary state* (1.4); *more exactly, the following estimate holds as $t \to \infty$:*

$$\int_0^L (x_*(t,l) - x(l))^2 dl \leqslant e^{-\varepsilon(t-t_0)} \int_0^L (x_*(t_0,l) - x(l))^2 dl, \tag{2.2}$$

where t_0 is a time moment at which the function x_* is known.

Taking into account the estimate (2.2), we find that, under the assumptions of the theorem, the optimization of population exploitation (for sufficiently large number of its life cycles) is reduced, in essence, to the optimization of its stationary state by choosing the exploitation intensity. For the profit functional we take

$$A(u) = \int_0^L c(l)u(l)x(l)dl - p_0 c_0, \tag{2.3}$$

where $c(l)$ is the cost of biomass of size l, c_0 and p_0 are the cost and density of planting biomass respectively. We assume that the control u satisfies the restriction

$$0 \leqslant u_1 \leqslant u \leqslant u_2, \tag{2.4}$$

is imposed where u_1 and u_2 are piecewise continuous functions, interpreted as minimum maintenance of biomass (for example, thinning on the forest) and the maximal exploitation intensity respectively. Such a measurable control u and the corresponding stationary state are said to be *admissible.*

It is natural to assume that there is a technology restriction on the maximal density of planting biomass, i.e., the function p (or constant p_0) should satisfy the condition

$$0 \leqslant p \leqslant P, \quad \text{where } P > 0 \text{ is a constant.}$$

2.2 Optimization of stationary state

In this subsection, for the above optimization problem we formulate the result about the existence of an optimal stationary state, a necessary optimality condition, and the uniqueness of such a solution under reasonable assumptions on the model parameters.

Theorem 2.2. *Suppose that μ, g, and c are piecewise continuous functions of l on $[0, L]$, and g can converge to zero only as $l \to L$ and $0 < \beta < 1$. Then there exists an admissible stationary state maximizing the profit functional* (2.3).

One of the tools to search an optimal control is a necessary extremum condition. In our case, it can be formulated as follows.

Theorem 2.3. *If, under the assumptions of Theorem 2.2, an admissible control u maximizes the functional (2.3), then for any point $l \in [0, L)$ where this control is the derivative of its integral and $u_1(l) \neq u_2(l)$, the expression*

$$e^{-\int_0^l m(s)ds} c(l) - \int_l^L \frac{u(s)}{g(s)} c(s) e^{-\int_0^s m(\tau)d\tau} ds \tag{2.5}$$

either is nonpositive if $u(l)$ is equal to $u_1(l)$ or is nonnegative if $u(l)$ is equal to $u_2(l)$ or vanishes if $u_2(l)$ lies in $(u_1(l), u_2(l))$.

A similar condition was obtained in [3].

The function (2.5) plays the role of a switching function. However, it is not convenient for handling since its value at the point l depends on the integral over the segment $[l, L]$ which is not known yet at this point if we compute the switching function in the direction from small to large size. However, the expression (2.5) can be written as

$$e^{-\int_0^l m(s)ds} c(l) - \frac{A(u) + p_0 c_0}{x(0)g(0)} + \int_0^l \frac{u(\tau)}{g(\tau)} c(\tau) e^{-\int_0^\tau m(s)ds} d\tau. \tag{2.6}$$

Then, integrating by parts the last term

$$\int_0^l \frac{u(\tau)}{g(\tau)} c(\tau) e^{-\int_0^\tau m(s)ds} d\tau = -\int_0^l c(\tau) e^{-\int_0^\tau \frac{\mu(s)}{g(s)}ds} d\left(e^{-\int_0^\tau \frac{u(s)}{g(s)}ds}\right)$$

$$= -e^{-\int_0^s m(\tau)d\tau} c(s)\Big|_0^l + \int_0^l e^{-\int_0^\tau m(s)ds} \left(c'(\tau) - \frac{\mu(\tau)}{g(\tau)} c(\tau)\right) d\tau$$

$$= c(0) - c(l) e^{-\int_0^l m(s)ds} + \int_0^l e^{-\int_0^\tau m(s)ds} \left(c'(\tau) - \frac{\mu(\tau)}{g(\tau)} c(\tau)\right) d\tau \tag{2.7}$$

and substituting the result into (2.6), we can write the switching function S in the form

$$S(l) := J + \int_0^l e^{-\int_0^\tau m(s)ds} \left(c'(\tau) - \frac{\mu(\tau)}{g(\tau)} c(\tau)\right) d\tau, \tag{2.8}$$

where $J = c(0) - (A(u) + p_0 c_0)/(x(0)g(0))$. For a given value of J the switching function, written as above, can be easily computed in the direction from small to large size.

The value J of the switching function (2.8) at zero is called the *level* of the corresponding stationary solution, whereas $J_{\min}$ and $J_{\max}$ denote the maximal and minimal values of the level so that the switching function is negative if $J < J_{\min}$ and is positive if $J > J_{\max}$ for all admissible controls. According to Theorem 2.3, the optimal stationary state corresponds to the choice of a control for some level $J \in [J_{\min}, J_{\max}]$ which is referred to as *optimal*.

Theorem 2.4. *If, under the assumptions of Theorem 2.2, the function c is differentiable, the function $c' - \mu c/g$ has finitely many zeros on $[0, L)$, and the continuous functions u_1 and u_2, $0 \leqslant u_1 \leqslant u_2$, coincide only at finitely many points, then the optimal level and optimal stationary solution are found in a unique way.*

3 Proof of the Main Results

3.1 Stability. Proof of Theorem 2.1

Lemma 3.1. *The continuous functions x and x_1 on the segment I satisfy the inequality*

$$|x^\beta - x_1^\beta| \leqslant x_{\min}^{\beta-1}|x - x_1| \tag{3.1}$$

where $x > 0$, $x_1 \geqslant 0$, $0 < \beta < 1$, and $x_{\min} = \min\{x(l)|l \in I\}$.

Indeed, in the case $0 < \beta < 1$, if y is nonnegative and real, then $y \leqslant y^\beta$ for $0 \leqslant y \leqslant 1$ and $y \geqslant y^\beta$ for $1 \leqslant y$ and, consequently, $|1 - y^\beta| \leqslant |1 - y|$ in both cases. Hence for $x > 0$ and $x_1 \geqslant 0$

$$|x^\beta - x_1^\beta| = x^\beta \cdot \left|1 - \left(\frac{x_1}{x}\right)^\beta\right| \leqslant x^\beta \cdot \left|1 - \frac{x_1}{x}\right| \leqslant x_{\min}^{\beta-1} \cdot x \cdot \left|1 - \frac{x_1}{x}\right| \leqslant x_{\min}^{\beta-1}|x - x_1|,$$

which is required.

Denote by $\Delta = \Delta(t, l)$ the difference $x(l) - x_1(t, l)$ of two solutions of (1.1), where x is the stationary solution. According to the equation and boundary conditions, the difference satisfies the equalities

$$\frac{\partial \Delta(t,l)}{\partial t} + \frac{\partial [g(l)\Delta(t,l)]}{\partial l} = -[\mu(l) + u(l)]\Delta(t,l), \tag{3.2}$$

$$\Delta(t, 0) = \int_0^L r(l)(x^\beta(l) - x_1^\beta(t,l))dl. \tag{3.3}$$

We have

$$\frac{d}{dt}\int_0^L \Delta^2(t,l)dl = 2\int_0^L \Delta(t,l)\frac{\partial \Delta}{\partial t}(t,l)dl,$$

which, by (3.2), implies

$$\frac{d}{dt}\int_0^L \Delta^2(t,l)dl = -2\int_0^L m(l)\Delta^2(t,l)dl - 2\int_0^L \Delta(t,l)\frac{\partial[g(l)\Delta(t,l)]}{\partial l}dl, \tag{3.4}$$

where $m := \mu + u$. Integrating by parts, we transform the last term on the right-hand side of (3.4) as follows:

$$-2\int_0^L \Delta(t,l)\frac{\partial[g(l)\Delta(t,l)]}{\partial l}dl = -2\Delta^2(t,l)g(l)|_0^L + 2\int_0^L g(l)\Delta(t,l)d(\Delta(t,l))$$

$$= -\Delta^2(t,l)g(l)|_0^L - \int_0^L g'(l)\Delta^2(t,l)dl. \tag{3.5}$$

Substituting (3.5) into (3.4), we find

$$\frac{d}{dt}\int_0^L \Delta^2(t,l)dl = -\int_0^L (2m(l)+g'(l))\Delta^2(t,l)dl - g(L)\Delta^2(t,L) + g(0)\Delta^2(t,0). \tag{3.6}$$

By the Cauchy–Bunyakowsky inequality, (3.3) implies the estimate

$$\Delta^2(t,0) = \left(\int_0^L r(l)\big(x^\beta(l) - x_1^\beta(t,l)\big)dl\right)^2 \leqslant \int_0^L r^2(l)dl \int_0^L \big(x^\beta(l) - x_1^\beta(t,l)\big)^2 dl$$

Using (3.1), we find

$$\Delta^2(t,0) \leqslant x_{\min}^{2\beta-2}\int_0^L r^2(l)dl \int_0^L \Delta^2(t,l)dl \tag{3.7}$$

Substituting (3.7) into (3.6), we obtain

$$\frac{d}{dt}\int_0^L \Delta^2(t,l)dl \leqslant \int_0^L \left[-2m(l) - g'(l) + g_0 x_{\min}^{2\beta-2}\int_0^L r^2(l)dl\right]\Delta^2(t,l)dl - g(L)\Delta^2(t,L)$$

or

$$\frac{d}{dt}\int_0^L \Delta^2(t,l)dl \leqslant \int_0^L \left[-2m(l) - g'(l) + g_0 x_{\min}^{2\beta-2}\int_0^L r^2(l)dl\right]\Delta^2(t,l)dl,$$

since $g(L)\Delta^2(t,L) \geqslant 0$. Using (2.1), we obtain the differential inequality

$$\frac{d}{dt}\int_0^L \Delta^2(t,l)dl \leqslant -\varepsilon\int_0^L \Delta^2(t,l)dl.$$

Solving this inequality, we find

$$\int_0^L \Delta^2(t,l)dl \leqslant e^{-\varepsilon(t-t_0)}\int_0^L \Delta^2(t_0,l)dl. \tag{3.8}$$

Consequently, the solution x_1, regarded as a function of l, converges to the stationary solution x as $t \to \infty$ at exponential rate. Theorem 2.1 is proved. □

3.2 Existence. Proof of Theorem 2.2

Using (1.4), we write the profit functional (2.3) in the form

$$A(u) = x(0)g(0)\int_0^L c(l)e^{-\int_0^l \frac{\mu(s)}{g(s)}ds - \varphi(l)}d\varphi(l) - p_0c_0, \tag{3.9}$$

where

$$\varphi(l) = \int_0^l \frac{u(s)}{g(s)}ds.$$

Lemma 3.2. *The functional* (3.9) *is bounded in the space of admissible controls.*

Indeed, we have

$$|x(0)g(0)\int\limits_0^L c(l)e^{-\int\limits_0^l \frac{\mu(s)}{g(s)}ds-\varphi(l)}d\varphi(l)-p_0c_0|\leqslant x(0)g(0)|\int\limits_0^L c(l)e^{-\varphi(l)}d\varphi(l)|+Pc_0$$
$$\leqslant x(0)g(0)C+Pc_0<\infty,$$

where $C=\sup\limits_{l\in[0,L]}|c|$ is finite since c is piecewise continuous on $[0,L]$, $g(0)$ and P are constants, and $x(0)$ is nonnegative and bounded by assumption. Consequently, the profit functional is bounded on the set of admissible controls, and the least upper bound of the profit functional over this set is attained. We consider a sequence of admissible controls u_n such that $A(u_n)$ converges to this least upper bound as $n\to\infty$. For an admissible control u_n and any $l_1,l_2\in[0,L)$, $l_1\leqslant l_2$ the sequence φ_n satisfies the inequalities

$$\int\limits_{l_1}^{l_2}\frac{u_1(l)}{g(l)}dl\leqslant\varphi_n(l_2)-\varphi_n(l_1)\leqslant\int\limits_{l_1}^{l_2}\frac{u_2(l)}{g(l)}dl \tag{3.10}$$

in view of (2.4). In particular, on the segment $I_k=[0,L-1/k]$ (for a given sufficiently large $k\in\mathbb{N}$ such that $0<L-1/k$), all φ_n satisfy the Lipschitz condition with constant equal to the least upper bound of $u_2(l)/g(l)$ on this segment. Consequently, the set of functions φ_n is bounded and equicontinuous there. Therefore, by the Arzelà–Ascoli theorem, there exists a subsequence $\{\varphi_{n_j,k}\}$ that uniformly converges to $\varphi_{\infty,k}$ on I_k as $j\to\infty$. Passing to the limit in (3.10), we see that the function $\varphi_{\infty,k}$ also satisfies (3.10). Increasing k and taking a uniformly converging subsequence

$$\cdots\supseteq\{\varphi_{n_i,k}\}\supseteq\{\varphi_{n_j,k+1}\}\supseteq,\ldots$$

on the corresponding segments, we conclude that the diagonal subsequence $\{\varphi_{n_k,k}\}$ converges to a limit function φ_∞ on $[0,L)$.

It is clear that, at this limit function, the least upper bound of the values of the functional (2.3) over the set of admissible controls is attained. In particular, on the half-interval $[0,L)$, this function is absolutely continuous, its derivative exists almost everywhere in $[0,L)$, and, by (3.10),

$$\frac{u_1(l)}{g(l)}\leqslant\varphi'(l)\leqslant\frac{u_2(l)}{g(l)}\quad\text{or}\quad u_1(l)\leqslant\varphi'(l)g(l)\leqslant u_2(l)$$

at each point where it exists. Hence, if u is defined at such a point $l\in[0,L)$ by the formula $u(l)=g(l)\varphi'_\infty(l)$ and takes any value in $[u_1,u_2]$ at any other point in $[0,L]$, then u is admissible at any point of $[0,L]$ and provides the least upper bound of the profit functional. Theorem 2.2 is proved. □

3.3 Optimality condition. Proof of Theorem 2.3

It suffices to find the first variation of the functional (2.3). We consider a point $l_0\in[0,L)$, where the optimal admissible control coincides with the derivative of its integral, and a sufficiently small positive number δl such that $[l_0,l_0+\delta l]$ belongs to $[0,L)$. We consider the variation of control $\widetilde{u} = u + h$ such that $\widetilde{u} - u$ is sufficiently small and vanishes outside $[l_0, l_0 + \delta l]$, i.e.,

$$\widetilde{u}(l) - u(l) = \begin{cases} h, & l \in [l_0, l_0 + \delta l], \\ 0, & l \in [0, l_0) \cup (l_0 + \delta l, L]. \end{cases}$$

For the control $\widetilde{u}(l)$ the value of the profit functional $A(\widetilde{u})$ can be represented as

$$x(0)g(0)\Bigg[\int_0^{l_0} \frac{u(l)}{g(l)} c(l) e^{-\int_0^l m(s)ds} dl + \int_{l_0}^{l_0+\delta l} \frac{u(l)+h}{g(l)} c(l) e^{-\int_0^l m(s)ds - h\int_{l_0}^l \frac{1}{g(s)}ds} dl$$
$$+ \int_{l_0+\delta l}^{L} \frac{u(l)}{g(l)} c(l) e^{-\int_0^l m(s)ds - h\int_{l_0}^{\delta l + l_0} \frac{1}{g(s)}ds} dl\Bigg] - p_0 c_0,$$

where m has the form (1.4). We write the last two integrals in the form

$$\int_{l_0}^{l_0+\delta l} \frac{u(l)+h}{g(l)} c(l) e^{-\int_0^l m(s)ds - \int_{l_0}^l \frac{h}{g(s)}ds} dl = \int_{l_0}^{l_0+\delta l} \frac{u(l)}{g(l)} c(l) e^{-\int_0^l m(s)ds} dl + h\cdot\delta l \frac{c(l_0)}{g(l_0)} e^{-\int_0^{l_0} m(s)ds} + \ldots$$

$$\int_{l_0+\delta l}^{L} \frac{u(l)}{g(l)} c(l) e^{-\int_0^l m(s)ds - h\int_{l_0}^{\delta l + l_0} \frac{1}{g(s)}ds} dl = \left(1 - h\int_{l_0}^{\delta l + l_0} \frac{1}{g(s)} ds + \ldots\right) \int_{l_0+\delta l}^{L} \frac{u(l)}{g(l)} \cdot c(l) \cdot e^{-\int_0^l m(s)ds} dl.$$

For the difference $A(\widetilde{u}) - A(u)$ we obtain the expression

$$h\cdot\delta l\cdot \frac{x(0)g(0)}{g(l_0)} \left[e^{-\int_0^{l_0} m(s)ds} c(l_0) - \int_{l_0}^{L} \frac{u(\tau)}{g(\tau)} c(\tau) e^{-\int_0^{\tau} m(s)ds} d\tau \right] + \ldots, \tag{3.11}$$

where dots denote terms of higher order with respect to h and δl. It is easy to see that for small $\delta l > 0$ and $h \neq 0$ the sign of this difference is determined by the signs of h and the expression in the square brackets if the latter differs from zero because the factor $x(0)g(0)/g(l_0)$ is positive. But for the control u maximizing the profit functional and its perturbation $\widetilde{u}$ this difference is nonpositive. Hence the expression in the square brackets either is nonpositive if $u(l_0)$ is equal to $u_1(l_0)$ or is nonnegative if $u(l_0)$ is equal to $u_2(l_0)$ or vanishes if $u(l_0)$ lies in $(u_1(l_0), u_2(l_0))$. Indeed, if $u_1(l_0) \neq u_2(l_0)$, the perturbation h can take any sufficiently small values that are either only nonnegative, or only nonpositive, or arbitrary. Theorem 2.3 is proved. □

3.4 Uniqueness. Proof of Theorem 2.4

We first assume that, in the case of optimal exploitation, the optimal switching function has a single point l_1 at its zero level and its derivative differs from zero at this point. If the level J is perturbed by a sufficiently small ΔJ, the switching point also change, but no new switching points arise. This fact can be easily verified. The corresponding change of the switching point Δl_1 depends on ΔJ and the derivative of the switching function at the point l_1; it is computed by the formula

$$\Delta l_1 = \frac{-\Delta J}{S'} + \ldots, \tag{3.12}$$

where the dots denote terms of higher order of smallness with respect to ΔJ. The corresponding change of the profit functional ΔA is computed by the formula

$$\Delta A = \Delta J \frac{x(0)g(0)\left[u_2(l_1) - u_1(l_1)\right]}{g(l_1)S'} \left(e^{-\psi(l_1)} c(l_1) - \int\limits_{l_1}^{L} \frac{u(l)}{g(l)} c(l) e^{-\psi(l)} dl \right) + \dots$$

or

$$\Delta A = \Delta J \frac{x(0)g(0)\left[u_1(l_1) - u_2(l_1)\right]}{g(l_1)S'} \left(e^{-\psi(l_1)} c(l_1) - \int\limits_{l_1}^{L} \frac{u(l)}{g(l)} c(l) e^{-\psi(l)} dl \right) + \dots,$$

where

$$\psi(l) := \int\limits_{0}^{l} m(s) ds,$$

if the derivative $S'(l_1)$ is positive or negative respectively, or, in both cases, by the formula

$$\Delta A = \Delta J \frac{x(0)g(0)\left[u_2(l_1) - u_1(l_1)\right]}{g(l_1)|S'(l_1)|} \left(e^{-\psi(l_1)} c(l_1) - \int\limits_{l_1}^{L} \frac{u(l)}{g(l)} c(l) e^{-\psi(l)} dl \right) + \dots$$

Transforming the expression on the right-hand side and passing to the limit as $\Delta J \to 0$, we conclude that the profit functional is differentiable at level J and its derivative can be computed at this point by the formula

$$\frac{dA}{dJ}(J) = -\frac{x(0)g(0)\left[u_2(l_1) - u_1(l_1)\right]}{g(l_1)|S'(l_1)|} \left(J + \int\limits_{0}^{L} c(l) \frac{u(l)}{g(l)} e^{-\psi(l)} dl \right).$$

If for a chosen level J the switching function has n zeros and all these zeros are nondegenerate, then a similar argument leads to the derivative

$$\frac{dA}{dJ}(J) = \left(-J - \int\limits_{0}^{L} c(l) \frac{u(l)}{g(l)} e^{-\psi(l)} dl \right) \sum_{i=1}^{n} \frac{x(0)g(0)\left[u_2(l_i) - u_1(l_i)\right]}{g(l_i)|S'(l_i)|}. \tag{3.13}$$

Since the derivative of the switching function is equal to

$$e^{-\int\limits_0^\tau m(s)ds} \left(c' - \frac{\mu c}{g} \right)$$

and the number of zeros on $[0, L]$ is finite, the last sum is positive for almost all levels $J \in [J_{\min}, J_{\max}]$. Hence the maximum of the profit functional is attained at either the endpoints of $[J_{\min}, J_{\max}]$ or the zero J_1 of the expression in parentheses. But, at such a point J_1, the derivative of this expression with respect to J is equal to -1 and, consequently, at any such a point J_1, the derivative (3.13) changes the sign from plus to minus, i.e., this point is a maximum point of the profit functional. Consequently, this functional attains its maximum only at one level in $[J_{\min}, J_{\max}]$. The control corresponding to this level is defined everywhere in a unique way, except for the zeros of the switching function, but the number of such zeros is finite since the number of zeros of its derivative is finite. Theorem 2.4 is proved. □

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