Some Methodological and Empirical Considerations in the Construction of Increment-Decrement Life Tables

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SOME METHODOLOGICAL AND EMPIRICAL CONSIDERATIONS IN THE CONSTRUCTION OF INCREMENT-DECREMENT LIFE TABLES

BY

JACQUES LEDENT

May, 1978

Preface

Interest in human settlement systems and policies has been a critical part of urban-related work at IIASA since its conception. Recently this interest has given rise to a concentrated research effort focusing on migration dynamics and settlement patterns. Four sub-tasks form the core of this research effort:

I. the study of spatial <u>dynamics</u>;

II. the definition and elaboration of a new research area called <u>demometrics</u> and its application to migration analysis and spatial population forecasting;

III. the analysis and design of migration and settlement <u>policy</u>;

IV. a <u>comparative study</u> of national migration and settlement patterns and policies.

This paper, the fifteenth in the spatial population dynamics series, deals with methodological and empirical issues concerning the calculation of those combined life tables that allow entries into, as well as withdrawals from alternative states, namely, increment-decrement life tables. It is especially oriented toward the construction of multiregional life tables: those combined life tables that deal with interregional migration flows as well as mortality.

Related papers in the dynamics series, and other publications of the migration and settlement study, are listed on the back page of this report.

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May 1978

Abstract

The topic of this paper revolves around the calculation of those combined life tables that allow entries as well as withdrawals from alternative states, namely, increment-decrement life tables. The paper provides a complete theoretical presentation of such tables, focusing on the contrasts between the movement and the transition approaches. It also sets forth, for both approaches, life table construction methods based on three alternative methodological variations: the linear and the cubic integration methods, and an interpolative-iterative method. Finally, the paper develops more precise methods for constructing a multiregional life table, for which the generally available death and migration rates are not consistent with either the movement or the transition approaches.

Acknowledgements

In the first place, I wish to express my thanks to Professor Andrei Rogers, who has taught me multiregional mathematical demography. My intellectual debt to him will become clear to the reader as he or she progresses through this paper.

Secondly, I am indebted to Frans Willekens who made helpful comments on an earlier draft.

Thirdly, I benefited greatly from an exchange of correspondence with Robert Schoen.

The burden of editing this paper was borne by Maria Rogers with great skill and good humour. Margaret Leggett typed this difficult paper as well as a previous draft with good cheer.

Although this paper has been entirely written at IIASA, it was initiated when the author was granted generous research time to study increment-decrement life tables during his affiliation with the Division of Economic and Business Research, College of Business and Public Administration, University of Arizona, Tucson.

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Some Methodological and Empirical Considerations in the Construction of Increment-decrement Life Tables

INTRODUCTION

Recently, life tables which can recognize increments (or entrants) as well as decrements (withdrawals) have proved to be of considerable value in various fields of demography. Two approaches to the construction of such combined life tables have emerged: the movement and transition approaches devised by Schoen (1975) and Rogers (1973a, 1973b, 1975a), respectively. These alternatives are not mutually exclusive. On the one hand, they propose different but complementary perspectives on social mobility, and on the other hand, the choice of either approach is mainly determined by the data available.

The purpose of this paper is to develop further the methodological and empirical aspects of both approaches, and to provide a clear understanding of their differences.

Before analyzing the concept of an increment-decrement life table it will be helpful to review briefly the history of life tables. Two of the most commonly used life tables are the single-state life table and the multiple decrement life table.

The single-state life table describes the mortality history of a synthetic group of people who were born at the same moment in a region closed to migration. It is also a model which in probabilistic terms expresses the mortality experience of such a group, called a cohort, as it gradually decreases in size until the death of its last member.

The multiple decrement life table is a more elaborate version of this model, which was originally designed to recognize the existence of different causes of death. Now it is also used as a scheme for analyzing demographic phenomena that can be viewed in cohort terms (marriage, divorce, etc.). However, the multiple decrement model does not permit one to follow persons who have moved from one status category to another and to analyze their subsequent experience.

Such problems may be handled with the help of combined tables which allow for entries into (increments), as well as withdrawals from (decrements) different states. Although "some of the issues involved in the use of combined tables were mentioned by Mertens (1965) and are considered in Jordan (1967) and other actuarial texts" (Schoen and Nelson, 1974)*, it is not until recently that a thorough and systematic discussion of the methodological and empirical problems raised by the construction of such increment-decrement life tables, has appeared in the literature.

The concept of a multiregional life table, an increment-decrement life table applied to the problem of interregional migration, was first developed by Rogers (1973a) who introduced the multiregional counterparts of the single-state life table functions, starting from a given set of age-specific outmigration and death probabilities. As shown in Rogers and Ledent (1974) and Rogers (1975a), these multiregional life table functions can be presented in a matrix format, which makes the general increment-decrement life table appear as a straightforward extension of the single state life table in which matrices replace scalars. In a different application context, Schoen and Nelson (1974) and Schoen (1975) introduced a "life status" table, an increment-decrement life table intended as a framework for a combined analysis of marriage, divorce and mortality.

Although very similar, both of the above efforts presented some significant differences, mainly in the state allocation of

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*Walter Mertens (1965) "Methodological Aspects of the Construction of Nuptiality Tables" Demography, Vol.2. pp.317-348. C.W. Jordan Jr. (1967) Life Contingencies (2nd. ed.) Chicago Society of Actuaries. (These references are mentioned in Schoen and Nelson (1974)).

the initial cohort, in the nature of the observed age-specific data to be introduced, and in the specification of multistate life table functions. First, in the multiregional population system considered by Rogers (1973a, 1975a), the initial cohort may be allocated to several, if not all, states (multiradix system) while, in the life-status system defined by Schoen and Nelson (1974), it is concentrated in one state (single radix system). Second, Rogers (1973a, 1975a) put forward a method of estimating age-specific probabilities from the number of transitions* occurring over the unit time interval to the successive regional groups of survivors at fixed ages of the original cohort. Schoen and Nelson (1974) and Schoen (1975) proposed an alternative method based on the number of movements* made by all the survivors of the original cohort between two fixed ages. Finally, the multistate life table functions specified by Schoen are extensions of the single-state life table functions in which vectors replace scalars, and not matrices as in Rogers. These differences stimulated the recent debate in Demography (Schoen 1975, 1977; Rogers and Ledent 1976, 1977).

Section I of this paper briefly reviews the single-state life table and indicates the elements needed for its extension to the case of an increment-decrement (multistate) life table. It particularly stresses the contrast between the two ways of calculating such a life table referred to as the movement approach (Schoen) and the transition approach (Rogers).

Section II begins with a summarized presentation of the concept of an increment-decrement life table** and its associated functions based on the movement approach. It continues with the empirical problem of calculating such a table, mainly focusing on

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*The distinction between transitions and movements is explained in Section I.

**The concept of increment-decrement life tables can be applied to a large number of fields in which most of the multistate life table functions have a useful interpretation. Besides the problems dealt with by Rogers and Schoen, it has been used for the analysis of working life status (Hoem and Fong, 1976) and in the combined study of nuptiality and birth parity (Oechsli 1972, 1975).

the question of estimating age-specific transition probabilities from observed data on age-specific movement rates.

Section III deals with the alternative perspective, the transition approach. It is necessary only to expose the derivation of the survival probabilities and the life table mortality and mobility rates, since the definitions of the multistate life table functions given in the case of the movement approach apply to the transition approach as well.

Section IV further articulates the contrasts between the movement and the transition approaches.

Finally, since age-specific movement or transition rates needed to construct an increment-decrement life table cannot always be observed as simply as age-specific death rates in the basic life table*, Section V examines alternative ways of correctly originating the calculations of an increment-decrement life table defined in Sections II and III. An empirical evaluation of various methods suggested is provided in the context of interregional human migration (multiregional life table).

The notation used throughout this paper will parallel as much as possible that used by Keyfitz (1968) in dealing with the single-state life table:

- statistics relating to the multistate life table population are denoted by non-capitalized letters, while those referring to the observed population are capitalized, and
- the functional notation f(y) will be used to denote functions of y as a continuous variable, while f_y will be used whenever we mean to denote f for a discrete set of values (y is here in the position of a right subscript).

The following rules will be respected to account for the existence of intercommunicating states:

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*This is so because mortality and mobility rates are not generally pertinent to one of the alternative approaches: mortality data are collected in a way consistent with the movement approach whereas mobility data are generally recorded in terms of transitions (changes of residence) between two points in time rather than in terms of actual moves.

- state-specific values of a statistic f will be denoted by a right superscript specific to the region (f_y^i or $f^i(y)$),
- moves or transitions between two states will be suggested by superscripts located on both sides of the variable concerned: the left superscript will relate to the state of origin, the right one will refer to the state of destination, and
- if reference to the state-of-birth or state-of-presence at any age less than the current age, is necessary, it will be indicated by two subscripts, respectively denoting the relevant region and age: for example, ${}_{iy}l_x^j$ will represent the value of the function l characteristic of those present at age x in state j who were in state i at age y.

A detailed list of all the life table symbols used, along with their interpretation, appears at the end of this paper.

I. THE CONCEPT OF AN INCREMENT-DECREMENT LIFE TABLE

Increment-decrement life tables describe stationary demographic models in which there exists an absorbing state (the state of death) and at least two intercommunicating states (individuals moving freely back and forth). Attached to them are multistate life table functions, expressing facts of mortality and mobility in terms of probabilities. As in the single-state life table, the increment-decrement life tables all originate from age-dependent schedules of mortality and mobility which are here defined state-specifically.

Because mobility is a recurrent event and mortality is not, there exist various ways of defining such forces, two of which have been explored in the past literature. This has resulted in the development of two alternative approaches to constructing increment-decrement life tables, respectively advocated by Rogers (1973a, 1975a) and Schoen (1975).

In order to understand these two approaches one must first look at the methodology used in the single-state life table and then analyze its extension into an increment-decrement life table.

A Review of the Single-state Life Table

The main problem in the single-state life table is to estimate the curve of survivors l(y), at any age y, out of a cohort of l_0 babies born at the same time and going through life together, and submitted to an age dependent mortality schedule $\mu(y)$. This curve is obtained as the integral solution of the basic differential equation (see Keyfitz 1968) expressing the relationship between $\mu(y)$ and l(y):

$$\frac{dl(y)}{dy} = \mu(y)l(y) \quad ; \qquad (1)$$

the integral solution is:

$$l(y) = e^{-\int_0^T \mu(t)dt} l_0 \quad , \qquad (2)$$

which permits one to define the number of survivors l_x, at fixed ages $x = 0,T,2T,\ldots,z$,* by applying a set of age-specific probabilities p_x such that

$$l_{x+T} = p_x l_x \tag{3}$$

in which:

$$p_x = e^{-\int_0^T \mu(x + t)dt} \tag{4}$$

Alternatively, it is possible to think of $l(y)$ as an age distribution of individuals alive at a given time, corresponding to an interpretation of the single-state life table as a stationary population. In this population, the number of persons between exact ages x and x + T is

$$L_x = \int_0^T l(x + t)dt \quad , \tag{5}$$

a quantity which, when the life table represents a cohort, is the number of person-years lived by the cohort between ages x and x + T.

The expected total number of years T_x remaining to the l_x survivors of l_0 may be found by integrating from x to infinity. (The maximum age to which any individual can live is infinite since the last interval is half open):

$$T_x = \int_0^\infty l(x + t)dt \quad . \tag{6}$$

For each of the l_x individuals, the average expectation of life at age x is:

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*Traditionally, all age intervals considered are equal in length (T years) except the last one which is half open: z years and over.

$$e_x = \frac{T_x}{l_x} \quad . \tag{7}$$

Complementary life table functions include survivorship proportions defined as

$$s_x = \frac{L_{x+T}}{L_x} \tag{8}$$

representing the proportion of those in age group x to x + T surviving to age group x + T to x + 2T, and annual age-specific death rates in the synthetically constructed life table stationary population. Since the number of deaths (or decrements to l_x) observed between ages x and x + T is

$$d_x = \int_0^T l(x + t)\mu(x + t)dt = l_x - l_{x+T} \quad , \tag{9}$$

the annual death rate m_x for the age group x to x + T is

$$m_x = \frac{d_x}{L_x} = \frac{l_x - l_{x+T}}{L_x} \quad . \tag{10}$$

Extending the Concept of the Single-state Life Table

By analogy with the single-state case, the first problem in constructing an increment-decrement life table is estimating the state-specific curves of survivors $l^i(y)$, at any age y, out of a cohort of $l_0^{\cdot}$ babies* born at the same time in one or several of the states.**

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*The notation $l_0^{\cdot}$ denotes the size of the initial cohort. Note that $l_0^{\cdot} = \sum_{k=1}^{n} l_0^k$ where l_0^k is the share of the initial cohort allocated to state k.

**The foregoing exposition is quite general and applies to systems with a unique radix.

The basic idea is to start from a set of state-specific mortality schedules as well as a set of schedules of mobility between the intercommunicating states, and then to determine state-specific curves of survivors.

Let $\{l(y)\}$ denote a vector whose typical element $l^i(y)$ is the number of survivors at age y in state i among the members of the initial cohort $l_0^{\cdot}$ whose allocation among states is contained in $\{l_0\}$:

$$\{l(y)\} = \begin{Bmatrix} l^1(y) \\ \\ l^n(y) \end{Bmatrix} ,$$

and let $\{l_x\}$ denote such a vector for predetermined ages $x = 0,T,2T,\ldots,z$, i.e. $\{l_x\} = \{l(x)\}$. The series of the numbers of survivors by state, at those fixed ages, would be generated by a vector extension of (3)

$$\{l_{x+T}\} = \underset{\sim}{p}_x\{l_x\} \tag{11}$$

in which $\underset{\sim}{p}_x$ is a matrix whose $(i-j)^{th}$ element represents the probability ${}^jp_x^i$ that an individual present in state j at age x will move to state i within the next T years.

The estimation of the matrix $\underset{\sim}{p}_x$ is not a simple matter owing to the fact that an individual can make more than one move over a unit time interval. This will be illustrated further with the help of the multistate Lexis diagram first suggested by Rogers (1973a, 1975a) which indicates alternative ways of estimating the transition probabilities contained in $\underset{\sim}{p}_x$.

Alternatively it is possible to think of $\{l(y)\}$ as an allocation vector, by state, of an age distribution of individuals alive at a given time, and thus give the increment-decrement life tables the interpretation of a multistate stationary population. This would then allow for an extension of the single-state L_x and the derivation of the multistate counterparts of the life table functions defined in (6) through (10).

The Multistate Lexis Diagram

The Lexis diagram for a two-state system appears in Figure 1 in which the various moves made by typical individuals over a unit time period are represented. It consists of two separate diagrams, one directly beneath the other, and connects them via the life lines of movers between the two-states. There are five classes of life lines, represented by A, B, C, D, and E respectively.

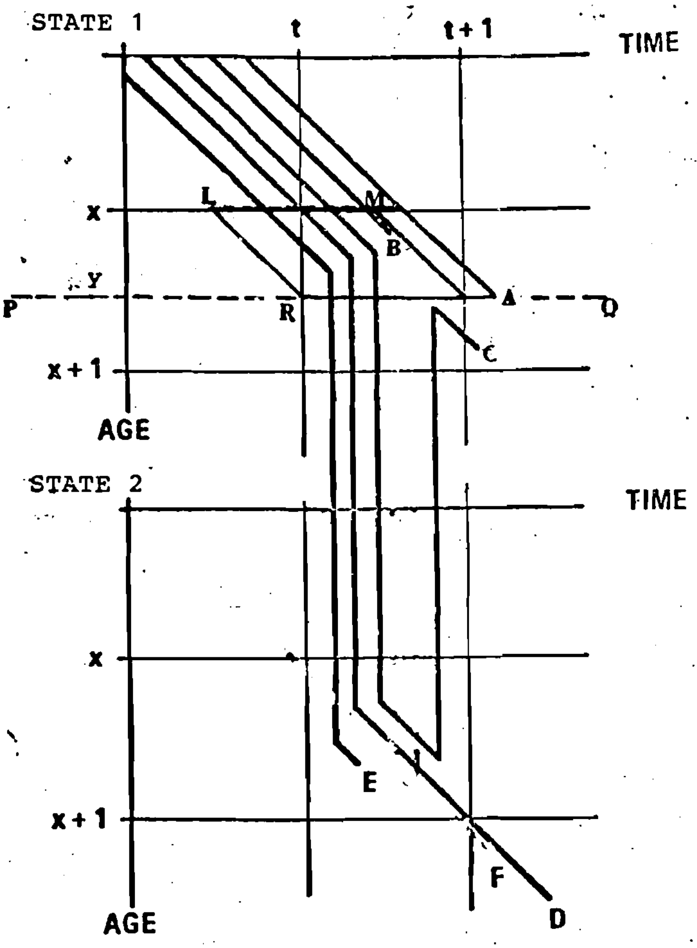

Figure 1: Two-state Lexis diagram

Source: Adapted from Rogers (1975)

Life line A represents the case of an individual surviving in state 1 who does not move out. Life lines B and E relate to individuals in state 1 who die during the unit time interval. In life line B, the death occurs in state 1 while E it takes place in state 2 after the individual concerned has moved from state 1 to state 2. Life line C represents the case of an individual who moves from state 1 to state 2 and returns before the end of the age interval. Finally, life line D refers to an individual in state 1 who moves to state 2, survives the unit time interval and does not return before the end of the interval.

There are other classes of life lines besides the above that consist of more than two moves but these are of a lesser importance. Note that this reasoning can be extended without inconvenience to the n-state case (the focus on a two-state Lexis diagram was adopted for ease of exposition).

Alternative Movement and Transition Approaches Contrasted

As mentioned earlier, two main alternative approaches have been considered to estimate age-specific probabilities such as ${}^{i}p_{x}^{j}$. Their contrast stems from a different emphasis on the life lines described by the multistate Lexis diagram.

Suppose we want to determine the matrix $\underset{\sim}{p}_{kT}$ consisting of the various probabilities of surviving through the age interval (kT, (k + 1)T). As in the single-state life table, the problem is to define a set of forces of mortality and mobility for any specific age $y(kT \leq y \leq (k + 1)T)$ and then to proceed to the age-specific survival probabilities by integration over the whole age interval.

A first possibility consists of defining age-specific forces of mortality and mobility out of a given state i at age y by reference to the group of all individuals present in state i at that age, *no matter what state they were present in at age* $x = kT$. For example, such forces of mobility, for age y, out of state 1 of a two-state system concern all the individuals whose life lines in Figure 1 cross PQ during the period (t, t+1), i.e. between R and S.

A second possibility consists of defining state-specific forces of mobility out of state i by reference to the group of individuals *present in that state at age $x = kT$.* The resulting forces of mobility for age y out of state 1 of a two-state system, relate to the group of individuals whose life lines in Figure 1, not only cross PQ (between R and S) but also cross LM.*

These two alternative definitions express two distinct methods of estimating the age-specific transition probabilities; the movement approach and the transition approach. In the movement approach the focus is on moves viewed as events occurring at *one* given point in time. In the transition approach, the emphasis is on the transitions resulting from the comparison of the states the individuals were in at *two* given points in time, regardless of where the individuals were during the intervening period.

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*The forces of mobility defined here allow an individual to move to another region and come back during the span of time elapsing between the crossing of two lines. This contrasts with an alternative definition of the forces of mobility making no allowances for return moves (Hoem, 1970).

II. THE MOVEMENT APPROACH

This section presents a complete exposition of the methodological and empirical aspects of the construction of increment-decrement life tables based on the movement approach. It includes mathematical developments set in both continuous and discrete terms as well as the applied construction of such tables.

A Theoretical Exposition

In contrast to the single-state case in which one of the main problems is to follow a unique initial set of babies, the multistate case requires following babies born in various states simultaneously.

In the movement approach, this task is carried out by continuously observing all the movements occurring in the system, which does not require focusing on fixed age intervals. For that reason, this approach appears as the more natural way of extending the single-state life table. This will be confirmed later when deriving the multistate life table functions that will appear as straightforward vector or matrix extensions of the single-state life table functions.

Derivation of the Age-Specific Survival Probabilities

Suppose we have an n-state system in which each state i is denoted by the index i (i = 1,...,n). Then, as far as state i is concerned relative to the rest of the system, for an individual aged y at time t, three types of demographic events are possible over the period (t, t + dt):

- survival to age y + dy in state i (dy = dt),
- death before reaching age y + dy in state i, and
- move to one of the other states of the system.

The time interval dt is supposed to be short enough so that multiple transitions, such as move to and death in a state j(j $\neq$ i), are ruled out.

Let ${}^{i}d^{j}(y)$ denote the number of moves from state i to state j made between ages y and y + dy by any person in the system.* On the assumption that no multiple moves can take place in a small interval dy, it appears that these moves are only made by individuals who were members of the group of people surviving in state i at age y,$l^{i}(y)$.

Since the exposure of these individuals to the risk of moving out or dying over the period (t, t + dt) is $l^{i}(y)dy$, the result is that $\frac{{}^{i}d^{j}(y)}{l^{i}(y)\ dy}$ is the corresponding mobility rate from state i to state j (j = 1,...,n, j ≠ i), or death rate in state i (if j = n + 1), attached to age y. Thus, one can define the instantaneous mobility rate (or force of mobility) ${}^{i}\mu^{j}(y)$ as the limiting value of this rate when dy → 0

$$ {}^{i}\mu^{j}(y) = \lim_{dy \to 0} \frac{{}^{i}d^{j}(y)}{l^{i}(y)\ dy} \quad , \qquad \begin{array}{l} \forall i = 1,...,n \\ \forall j = 1,...,n+1 \\ j \neq i \end{array} \tag{12} $$

Once ${}^{i}\mu^{j}(y)$ is available for all j = 1,...,n + 1, the force of retention ${}^{i}\mu^{i}(y)$ is simply obtained from the following equation expressing that the instantaneous process underlying an increment-decrement life table is conservative (Chiang, 1968):

$$ \sum_{j=1}^{n+1} {}^{i}\mu^{j}(y) = 0 \quad , \qquad \forall i = 1,...,n $$

or alternatively,

$$ {}^{i}\mu^{i}(y) = -\ [{}^{i}\mu^{\delta}(y) + \sum_{\substack{j=1 \\ j\neq i}}^{n} {}^{i}\mu^{j}(y)] \quad , \qquad \forall i = 1,...,n \tag{13} $$

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*At this stage, a death in state i is in no way different from a move to another state j of the system: the state of death denoted by δ may be considered as the (n + 1) state of the system. Then, in the following paragraph j = 1,...,n + 1.

As far as the two states i and k = R(i)(i.e.,all states excluding i) are concerned, there exist the six forces of mortality and mobility indicated in Figure 2(a).

(a)

Location at time t / Location at time t + dt	Present in state i	Present in state k
alive in state i	${}^{i}\mu^{i}(y)$	${}^{k}\mu^{i}(y)$
alive in state k	${}^{i}\mu^{k}(y)$	${}^{k}\mu^{k}(y)$
dead	${}^{i}\mu^{\delta}(y)$	${}^{k}\mu^{\delta}(y)$

(b)

Location at time t / Location at time t + dt	Present in state i	Present in state k	
alive in state i	–	${}^{k}d^{i}(y)$	$l^{i}(y + dy)$
alive in state k	${}^{i}d^{k}(y)$	–	$l^{k}(y + dy)$
dead	${}^{i}d^{\delta}(y)$	${}^{k}d^{\delta}(y)$	
	$l^{i}(y)$	$l^{k}(y)$	

Figure 2. Forces of transition and corresponding movements in a two region system.

Clearly the multistate demographic system determined by the above definitions is characterized by state-specific mortality and mobility patterns such that the instantaneous propensity of an individual to make a move only depends on his age and the states of origin and destination for this move. In no way, is this propensity affected by the past mobility history of that individual or the duration of residence in the state out of which the move takes place.

The corresponding movements of the forces of mortality and mobility included in Figure 2(a) are shown in Figure 2(b) permitting us to write the following equation indicating the decrements and increments to the exposed group $l^i(y)$:

$$l^i(y + dy) = l^i(y) - {}^id^\delta(y) - {}^id^k(y) + {}^kd^i(y)$$

$$\forall i = 1,\dots,n \quad .$$

Recalling that k stands for all states excluding i, we can thus rewrite this equation as follows:

$$l^i(y + dy) = l^i(y) - {}^id^\delta(y) - \sum_{\substack{j=1 \\ j\neq i}}^{n} {}^id^j(y) + \sum_{\substack{j=1 \\ j\neq i}}^{n} {}^jd^i(y)$$

$$\forall i = 1,\dots,n \quad . \tag{14}$$

which is precisely the elementary flow equation of Schoen and Land (1976). Substituting (12) into (14) leads to a system of n simultaneous linear differential equations:

$$l^i(y + dy) = l^i(y) - \left[{}^i\mu^\delta(y) + \sum_{\substack{j=1 \\ j\neq i}}^{n} {}^i\mu^j(y)\right] l^i\,(y)dy$$

$$+ \sum_{\substack{j=1 \\ j\neq i}}^{n} {}^j\mu^i(y)\,l^j(y)dy \qquad \forall i = 1,\dots,n \quad ,$$

or, more compactly,

$$\{l(y + dy)\} = \{l(y)\} - \underset{\sim}{\mu}(y)\{l(y)\}dy \tag{15}$$

in which:

$$\underset{\sim}{\mu}(y) = \begin{bmatrix} {}^{1}\mu^{\delta}(y) + [\sum\limits_{\substack{j=1\\ j\neq i}}^{n} {}^{1}\mu^{j}(y)] & -{}^{2}\mu^{1}(y) & & -{}^{n}\mu^{1}(y) \\ -{}^{1}\mu^{2}(y) & {}^{2}\mu^{\delta}(y) + [\sum\limits_{\substack{j=1\\ j\neq i}}^{n} {}^{2}\mu^{j}(y)] & & \\ \vdots & & \ddots & \\ -{}^{1}\mu^{n}(y) & & & {}^{n}\mu^{\delta}(y) + [\sum\limits_{\substack{j=1\\ j\neq i}}^{n} {}^{n}\mu^{j}(y)] \end{bmatrix} \qquad (16$$

or, alternatively, by using (13)

$$\underset{\sim}{\mu}(y) = - \begin{bmatrix} {}^{1}\mu^{1}(y) & {}^{2}\mu^{1}(y) & & {}^{n}\mu^{1}(y) \\ {}^{1}\mu^{2}(y) & {}^{2}\mu^{2}(y) & & \\ & & \ddots & \\ {}^{1}\mu^{n}(y) & & & {}^{n}\mu^{n}(y) \end{bmatrix}$$

The definition of $d\{l(y)\}$

$$d\{l(y)\} = \{l(y + dy)\} - \{l(y)\}$$

leads us to rewrite (15) as:

$$\frac{d\{l(y)\}}{dy} = - \underset{\sim}{\mu}(y)\{l(y)\} \qquad (17)$$

which appears as a straightforward multistate extension of (1).

The system defined by (17) admits n linearly independent solutions $\{l(y)\}_k$ $(k = 1,\ldots,n)$ whose juxtaposition as the columns of a square matrix yields the *integral matrix* of the system (Gantmacher, 1959):

$$\underset{\sim}{l}(y) = [\{l(y)\}_{1,}\ldots,\{l(y)\}_n] \quad .$$

Since every column of $\underset{\sim}{l}(y)$ satisfies (17), the integral matrix $\underset{\sim}{l}(y)$ satisfies the equation:

$$\frac{d\underset{\sim}{l}(y)}{dy} = - \underset{\sim}{\mu}(y)\underset{\sim}{l}(y) \quad . \tag{18}$$

From the theorem on the existence and uniqueness of the solution of a system of linear differential equations, it follows that $\underset{\sim}{l}(y)$ is *uniquely* determined when the value of $\underset{\sim}{l}(y)$ for some initial value $y = 0$ is known, say $\underset{\sim}{l}(0)$ or $\underset{\sim 0}{l}$ (Gantmacher, 1959):

$$\underset{\sim}{l}(y) = {}_0\underset{\sim}{\Omega}(y) \; \underset{\sim 0}{l} \tag{19}$$

in which the matrix ${}_0\underset{\sim}{\Omega}(y)$, uniquely defined as the normalized solution of (18) in that it becomes the unit matrix for $y = 0$, is called the matricant (Gantmacher, 1959).

Note that ${}_0\underset{\sim}{\Omega}(y)$ cannot be simply expressed as a function of the $\underset{\sim}{\mu}(y)$'s as its counterpart in the basic life table was in (2). However, as indicated in Schoen and Land (1976) and Krishnamoorthy (1977), it can be determined by using the infinitesimal calculus of Volterra. (Gantmacher, 1959). Such a determination takes advantage of the following property displayed by the matricant:

$${}_{x_1}\underset{\sim}{\Omega}(x_3) = {}_{x_2}\underset{\sim}{\Omega}(x_3)\,{}_{x_1}\underset{\sim}{\Omega}(x_2) \quad . \tag{20}$$

If we divide the basic interval $(0 = y_0, y = y_n)$ into n parts by introducing intermediate points $y_1, y_2, \ldots, y_{n-1}$ and set

$\Delta y_k = y_k - y_{k-1}$ $(k = 1,\ldots,n)$, then we have from (20)

$${}_{0}\underset{\sim}{\Omega}(y) = {}_{y_{n-1}}\underset{\sim}{\Omega}(y)\,{}_{y_{n-2}}\underset{\sim}{\Omega}(y_{n-1})\cdots{}_{y_1}\underset{\sim}{\Omega}(y_2)\,{}_{0}\underset{\sim}{\Omega}(y_1) \quad .$$

If the intervals Δy_k are small, we can calculate ${}_{y_{k-1}}\underset{\sim}{\Omega}(y_k)$ by taking $\underset{\sim}{\mu}(t) \sim \underset{\sim}{\mu}(T_k)$, a constant matrix, such that T_k is an intermediate point in the interval (y_{k-1}, y_k). We have:

$${}_{y_{k-1}}\underset{\sim}{\Omega}(y_k) = e^{-\underset{\sim}{\mu}(T_k)\Delta y_k} + (**)$$

in which the symbol (**) denotes the sum of terms of order two or greater. Since

$$e^{-\underset{\sim}{\mu}(T_k)\Delta y_k} = \underset{\sim}{I} - \underset{\sim}{\mu}(T_k)\Delta y_k + (**)$$

we can then rewrite ${}_{0}\underset{\sim}{\Omega}(y)$ as:

$$\begin{aligned}{}_{0}\underset{\sim}{\Omega}(y) = {} & [\underset{\sim}{I} - \underset{\sim}{\mu}(T_n)\Delta y_n][\underset{\sim}{I} - \underset{\sim}{\mu}(T_{n-1})\Delta y_{n-1}]\cdots \\ & [\underset{\sim}{I} - \underset{\sim}{\mu}(T_1)\Delta y_1] + (**) \quad . \end{aligned} \tag{21}$$

Having derived an integral matrix solution of (17), we now face the difficulty of interpreting it. What is the meaning of $\underset{\sim}{l}(y)$ with regard to the problem on hand?

First let us say that $\underset{\sim}{l}(y)$ is a matrix containing n vectors, each one of them representing an independent solution of (17). With reference to the "initial" values y = 0, it is clear that n independent solutions can be obtained by separately generating the subsequent evolution of the state-specific groups of the initial cohort $l_0^{\cdot}$. Thus $\underset{\sim}{l}_0$ is a diagonal matrix which denotes the state-specific allocation of the initial cohort: its typical diagonal element is l_0^i . Furthermore, $\underset{\sim}{l}(y)$ is a square matrix whose i^{th} column is a vector representing the state specific allocation of the survivors of l_0^i at age y (in the remainder of the paper it will be denoted by ${}_{0}\underset{\sim}{l}(y)$).

Since the columns of ${}_0\underset{\sim}{1}(y)$ are n linearly independent solutions, their sum is also a solution of (17). Then $\{1(y)\}$ is given by:

$$\{1(y)\} = {}_0\underset{\sim}{\Omega}(y)\{1_0\}$$

in which $\{1_0\}$ is the allocation vector of the initial cohort $1_0^{\bullet}$. Clearly, the matrix ${}_0\underset{\sim}{\Omega}(y)$ defines a set of survival probabilities: its $(i,j)^{th}$ element represents the probability for a person born in state j to survive at age y in state i.

From the property (20) of the matricant, it can be concluded that the probability ${}^{i}p_x^{j}$ that an individual present at age x in state i will survive in state j, T years later, is the $(j,i)^{th}$ element of the matrix $\underset{\sim}{p}_x = {}_x\underset{\sim}{\Omega}(x + T)$. Hence:

$$\underset{\sim}{p}_x = {}_0\underset{\sim}{\Omega}(x + T)\,{}_0\underset{\sim}{\Omega}(x)^{-1} \qquad (22)$$

An expression of $\underset{\sim}{p}_x$ can be derived from the expressions of ${}_0\underset{\sim}{\Omega}(x + T)$ and ${}_0\underset{\sim}{\Omega}(x)$ obtained by use of the infinitesimal calculus of Volterra:

$$\underset{\sim}{p}_x = \prod_{k=m}^{1} [\underset{\sim}{I} - \underset{\sim}{\mu}(x + \theta_k)\Delta y_k] + (**) \qquad (23)$$

where $x + y_1$, $x + y_2, \ldots, x + y_{m-1}$, are $(m - 1)$ intermediate points dividing the interval $(x, x + T)$ into m parts containing respectively the intermediate points $x + \theta_1, x + \theta_2, \ldots, x + \theta_{m-1}$.*

---

*Note that the application of the infinitesimal calculus of Volterra, leads us to write

$$e^{-\int_0^T \underset{\sim}{\mu}(y + t)dt} = \underset{\sim}{I} - \sum_{k=1}^{n} \underset{\sim}{\mu}(x + \theta_k)\Delta y_k + (**)$$

. Since (23) can be rewritten as $\underset{\sim}{p}_x = \underset{\sim}{I} - \sum_{k=1}^{n} \underset{\sim}{\mu}(x + \theta_k)\Delta y_k + (**)$, one may conclude that $e^{-\int_0^T \underset{\sim}{\mu}(y + t)dt}$ is a good approximation of $\underset{\sim}{p}_x$: the discrepancy represents terms of at least the second order.

Also, note that it is possible to define a matrix $\underset{\sim}{q}_x$ of the probabilities of dying within the next T years analogous to the q_x of the single-state life table. Let ${}^{i}q_x^{j}$ denote the probability for a person present in state i at age x to die within the next T years in state j. Then the number of deaths occurring in state j between ages x and x + T for the member of l_x^i is equal to $l_x^i \cdot {}^{i}q_x^{j}$ as well as to $\int_0^T {}^{j}\mu^{\delta}(x + t)\,{}_{ix}l^{j}(x + t)dt$ in which ${}_{ix}l^{j}(x + t)$ denotes the members of l^{i}_{x} surviving to age x + 1 in state j. Therefore,

$$\underset{\sim}{q}_x = [\int_0^T \underset{\sim}{\overset{\delta}{\mu}}(x + t)\,{}_{x}\underset{\sim}{l}(x + t)dt]\,{}_{x}\underset{\sim}{l}_x^{-1} \quad ,$$

$$\forall x = 0,T,\ldots,Z - T$$

in which $\underset{\sim}{\overset{\delta}{\mu}}(y)$ is a diagonal matrix of instantaneous death rates, ${}_{x}\underset{\sim}{l}(y)$ a matrix whose (i,j)th element is ${}_{jx}l^{i}(y)$ and ${}_{x}\underset{\sim}{l}_x$ a diagonal matrix whose ith element is l_x^i.* Finally, substituting (24) into that last expression leads to:

$$\underset{\sim}{q}_x = \int_0^T \underset{\sim}{\overset{\delta}{\mu}}(x + t)\,{}_{x}\underset{\sim}{\Omega}(x + t)dt \qquad (25)$$

or alternatively,

$$\underset{\sim}{q}_x = [\int_0^T \underset{\sim}{\overset{\delta}{\mu}}(x + t)\,{}_{0}\underset{\sim}{\Omega}(x + t)dt]\,{}_{0}\underset{\sim}{\Omega}(x)^{-1}$$

a precise evaluation of which could also be obtained by use of the infinitesimal calculus of Volterra.

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*The notation ${}_{x}\underset{\sim}{l}(y)$ generalizes the above notation ${}_{0}\underset{\sim}{l}(y)$ by describing the state changes in the system with reference to the state of the system at any age y (0 < y < x) rather than with reference to the state-of-birth only. Note that (19) can then be generalized into

$${}_{x}\underset{\sim}{l}(y) = {}_{x}\underset{\sim}{\Omega}(y)\,\underset{\sim}{l}_x \qquad (24)$$

The relevance of Markov processes to the interpretation of increment-decrement life tables has not gone unnoticed (Rogers, 1973a, 1975a; Schoen, 1975; Schoen and Land, 1976; Krishnamoorthy, 1977). It is, in fact, simple to establish that the matrices of probabilities $\underset{\sim}{p}_x$ determine a Markov transition probability model* characterizing the multistate stationary population defined by $\{l(y)\}$:

- the matrix $\underset{\sim}{p}_x$ is such that its elements are conditional upon occupancy of a specific state at age x and are independent of the history of previous moves or the duration of residence in the state (this follows from the property (20) of the matricant), and
- the elements of $\underset{\sim}{p}_x$ satisfy, as indicated by Schoen and Land (1976), the three standard conditions specified in Cox and Miller (1965):
  a) $0 \leq {}^{i}p_x^{j}$
  b) $0 \leq \sum_{j=1}^{n} {}^{i}p_x^{j} \leq 1$
  c) transitivity property defined in (20).

Indeed, the Markov process interpretation is simply due to the nature of the instantaneous pattern of mortality and mobility defined by (12). All individuals present at a fixed age in a given region have identical propensities to move out, independent of the past mobility history of each individual.

To summarize, the mortality and mobility process underlying an increment-decrement life table, characterized by the existence of a unique survival probability function ${}_0\underset{\sim}{\Omega}(y)$, leads to an age-specific distribution $\{l(y)\}$ that represents a linear combination of n independent age distributions, respectively generated by each of the state-specific groups of the initial cohort.

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*The word transition must be understood in its common meaning in stochastic processes. To avoid any confusion, the transition probability matrix $\underset{\sim}{p}_x$ will be referred to as the matrix of survival probabilities.

There are as many linearly independent distributions as non-empty states in the initial cohort.

Consequently, in the multiradix case (more than one state, possibly n states, are initially non-empty), the age-specific distribution $\{l(y)\}$ depends on the state allocation of the initial cohort. However, in the single radix case (all individuals born in a unique state), the age-specific distribution $\{l(y)\}$ is uniquely defined.

This distinction is extremely important since

- as we will see later, the multiradix case causes additional problems with respect to the single radix case in the discrete formulation of the model underlying the construction of an increment-decrement life table, and
- the use of matrix algebra for the derivation of the multistate functions is more suitable for the multiradix case than for the single radix case.*

The Multistate Life Table Functions

Two different generalizations of the single-state life table functions are possible and have given rise to a subject of controversy between Schoen and Rogers/Ledent.

The first generalization, introduced by Schoen, consists of multistate life table functions which are attached to the state-specific age distributions $l^{i}(y)$ considered in their entirety.

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*This especially applies to life table functions containing the inverse of ${}_0\underset{\sim}{l}_x$. Clearly, if at least one state of the system is initially empty, ${}_0\underset{\sim}{l}_x$ is not invertible. (It contains at least a zero column and its determinant is thus equal to zero.) However, the formulas containing such a term ${}_0\underset{\sim}{l}_x$ will remain valid if one reduces the scope of the matrices involved: ${}_0\underset{\sim}{l}_x$ (or more generally any matrix to be inverted) will be reduced to a r x r matrix (in which r is the number of states initially empty), while the other matrices will be reduced to s x r matrices (in which s is not necessarily equal to r: $r \leq s \leq n$).

Schoen and Nelson (1974) define:

$$L_x^i = \int_0^T l^i(x + t)dt \quad , \qquad \forall i = 1,\dots,n \qquad (26)$$

as a function which, like the L_x variable in the single-state life table, has a dual meaning. It represents first the number of people alive in state i of the increment-decrement life table between ages x and x + T, and second, the number of person-years lived by the initial life table cohort $l_0^\cdot$ in state i between those ages. (26) can be rewritten in a vector format as:

$$\{L_x\} = \int_0^T \{l(x + t)\}dt \quad .$$

We can define $\{T(x)\}$, the state-specific allocation vector of the number of people alive in the life table aged x and over, as:

$$\{T_x\} = \int_0^\infty \{l(x + t)\}dt \quad .$$

With the idea of extending the definition (7) of expectations of life at exact ages, Schoen and Land (1976) define the mean duration of stay in a given state after age x for all survivors in the system at age x as,

$$(ae)_x^i = \frac{T_x^i}{l_x^\cdot} \quad . \qquad \forall i = 1,\dots,n$$

This is a statistic that we would like to further qualify by state of presence at age x. However, this is not straightforward since the person-years lived included in the quantities T_x^i involve members of l_x^i as well as members of all the groups l_x^j $(j = 1,\dots,n,\ j \neq i)$. We need to have recourse to variables such as ${}_{ix}e_x^j$ denoting the number of years that a member of l_x^i can expect to spend in region j before his death. We then have the following equation linking l, e and T functions.

$$\sum_{j=1}^{n} {}_{jx}e_x^i \; l_x^j = T_x^i \qquad \forall i = 1,...,n \quad .$$

or more compactly,

$${}_{x}\underset{\sim}{e}_{x}\{l_x\} = \{T_x\} \tag{27}$$

in which the $(i,j)^{th}$ element of ${}_{x}\underset{\sim}{e}_{x}$ is ${}_{jx}e_x^i$.

This vector equation (27) is clearly insufficient to draw ${}_{x}\underset{\sim}{e}_{x}$ from the availability of $\{l_x\}$. However, it suggests that the generation of n linearly independent $\{l(y)\}$ distributions, would allow for a derivation of ${}_{x}\underset{\sim}{e}_{x}$. Let $\{l_x\}_1$ denote the age-distribution relating to the first increment-decrement life table generated and $\{T_x\}_1$ the corresponding number of person-years lived over age x. Then, it is possible to write

$${}_{x}\underset{\sim}{e}_{x} \; \underset{\sim}{l}_{x} = \underset{\sim}{T}_{x}$$

in which

$$\underset{\sim}{l}_{x} = [\{l_x\}_1,...,\{l_x\}_n] \text{ and } \underset{\sim}{T}_{x} = [\{T_x\}_1,...,\{T_x\}_n] \quad ,$$

which leads to:

$${}_{x}\underset{\sim}{e}_{x} = \underset{\sim}{T}_{x} \; \underset{\sim}{l}_{x}^{-1} \tag{28}$$

In fact, the generation of n linearly independent increment-decrement life tables is not necessary to obtain ${}_{x}\underset{\sim}{e}_{x}$. Let us recall that the differential equation (17) underlying an increment-decrement life table admits n linearly independent solutions corresponding to n initial cohorts, each of which has a radix concentrated in a different state. Then, it suffices to attach an additional subscript referring to the state of birth to define multistate life table functions leading to the derivation of $_{x}\underset{\sim}{e}_{x}$ (Rogers 1973a, 1975a).

The second generalization of the single-state life table functions thus starts with the definition of $_{j0}L^{i}_{x}$. It represents the number of people born in j and alive in state i of the life table between ages x and x + T, which is also the number of person-years lived in state i between those ages by the members of the initial cohort born in state j as:

$$_{j0}L^{i}_{x} = \int_{0}^{T} {}_{j0}l^{i}(x + t)dt \qquad \forall i,j = 1,\ldots,n \ ,$$

which can be written more compactly as:

$$_{0}\underset{\sim}{L}_{x} = \int_{0}^{T} {}_{0}\underset{\sim}{l}(x + t)dt \ . \qquad (29)$$

The total number of person-years lived in state i in prospect for the group born in j may be taken as

$$_{j0}T^{i}_{x} = \int_{0}^{\infty} {}_{j0}l^{i}(x + t)dt \qquad \forall i,j = 1,\ldots,n \ ,$$

or, more compactly:

$$_{0}\underset{\sim}{T}_{x} = \int_{0}^{\infty} {}_{0}\underset{\sim}{l}(x + t)dt \ . \qquad (30)$$

The superiority of this matrix generalization of the single life table L_x is evident in that, unlike the vector generalization (Schoen), it permits a direct derivation of $_{x}\underset{\sim}{e}_{x}$ from (28) rewritten as:

$$_{x}\underset{\sim}{e}_{x} = {}_{0}\underset{\sim}{T}_{x} \, {}_{0}\underset{\sim}{l}_{x}^{-1} \qquad (31)$$

Note that on substituting (30) into that last equation and replacing ${}_0\underset{\sim}{l}(x+t)\,{}_0\underset{\sim}{l}_x^{-1}$ by ${}_x\underset{\sim}{\Omega}(x+t)$ yields

$$ {}_x\underset{\sim}{e}_x = \int_0^\infty {}_x\underset{\sim}{\Omega}(x+t)dt \qquad (32)$$

an expression that indicates the independence of ${}_x\underset{\sim}{e}_x$ vis-a-vis the state allocation of the initial cohort. Rogers (1975b) also develops the notion of a net migraproduction matrix as an alternative measure of mobility. Specified in a discrete setting, the latter expresses mobility in terms of the number of expected moves out of each state of the system beyond some given exact ages 0,T,2T,...,z. Below, we re-examine this concept using a continuous specification. Let ${}_{ix}n_x^j$ be the number of moves that an individual present at age x in region i can expect to make out of state j before his death, then $\sum_k {}_{i0}l_x^k\ {}_{kx}n_x^j$ is the total number of moves that the members of l_0^i can expect to make out of state j beyond age x.

Alternatively, this number can be obtained by applying the total mobility rate $\sum_{\substack{k=1\\k\neq j}}^{n} {}^j\mu^k(x+t)$ to ${}_{i0}l^j(x+t)$ for the $t \geq 0$, and summing them:

$$\sum_{\substack{k=1\\k\neq j}}^{n} {}_{i0}l_x^k\ {}_{kx}n_x^j = \int_0^\infty \Big[\sum_{\substack{k=1\\k\neq j}}^{n} {}^j\mu^k(x+t)\Big]{}_{i0}l^j(x+t)dt$$

$$\forall i,j = 1,\ldots,n$$

which can be expressed more compactly as:

$$ {}_x\underset{\sim}{n}_x\ {}_0\underset{\sim}{l}_x = \int_0^\infty \overset{mt}{\underset{\sim}{\mu}}(x+t)\,{}_0\underset{\sim}{l}(x+t)dt$$

in which ${}_x\underset{\sim}{n}_x$ is a net migraproduction matrix whose (i,j)th element is ${}_{jx}n_x^i$ and $\overset{mt}{\underset{\sim}{\mu}}(x+t)$ a diagonal matrix whose ith diagonal element is $[\sum_{\substack{k=1\\k\neq i}}^{n} {}^{i}\mu^{k}(x+t)]$. Consequently

$$ {}_{x}\underset{\sim}{n}_{x} = [\int_{0}^{\infty} \underset{\sim}{\overset{mt}{\mu}}(x+t)\,{}_{0}\underset{\sim}{l}(x+t)dt]\,{}_{0}\underset{\sim}{l}_{x}^{-1} \qquad (33)$$

On substituting ${}_{x}\underset{\sim}{\Omega}(x+t)$ for ${}_{0}\underset{\sim}{l}(x+t)\,{}_{o}\underset{\sim}{l}(x)^{-1}$ yields,

$$ {}_{x}\underset{\sim}{n}_{x} = \int_{0}^{\infty} \underset{\sim}{\overset{mt}{\mu}}(x+t)\,{}_{x}\underset{\sim}{\Omega}(x+t)dt \quad , \qquad (34)$$

an expression that also shows the independence of ${}_{x}\underset{\sim}{n}_{x}$ vis-a-vis the state allocation of the initial cohort.

Another consequence of the matrix notation is the possibility of extending the definitions (29) and (30) by relating the multistate functions to the states of presence at any age y rather than to the state-of-birth. For example, ${}_{y}\underset{\sim}{L}_{x}$ denotes a matrix whose typical element ${}_{jy}L_{x}^{i} = \int_{0}^{T} {}_{jy}l^{i}(x+t)dt$ is the number of people present at age y in state j $(0 \leq y \leq x)$ and alive in state i between ages x and x + T. In a similar way, ${}_{y}\underset{\sim}{T}_{x}$ denotes a matrix whose typical element ${}_{jy}T_{x}^{i} = \int_{0}^{\infty} {}_{jy}l^{i}(x+t)dt$ is the total number of years that a person present at age y in state j can expect to live in state i beyond age x.

It can immediately be established that the following relationships extending (31) and (33) hold:

$$ {}_{x}\underset{\sim}{e}_{x} = {}_{y}\underset{\sim}{T}_{x}\,{}_{y}\underset{\sim}{L}_{x}^{-1} \quad , \qquad \forall y\ 0 \leq y \leq x$$

$$ {}_{x}\underset{\sim}{n}_{x} = [\int_{0}^{\infty} \underset{\sim}{\overset{mt}{\mu}}(x+t)\,{}_{y}\underset{\sim}{l}(x+t)dt]\,{}_{y}\underset{\sim}{l}_{x}^{-1} \quad , \qquad \forall y\ 0 \leq y \leq x$$

Note that this generalization of the multistate life table functions, focusing on the states of presence at any age rather than on states of birth, is very useful. As mentioned earlier, in the case of a system with some initially empty states, the knowledge of ${}_{0}\underset{\sim}{l}_{x}$ and ${}_{0}\underset{\sim}{L}_{x}$ only permits the calculations of expectations of life or migraproduction rates at any age relating to the initially non-empty states. Fortunately, the knowledge of ${}_{x}\underset{\sim}{l}_{x}$ and ${}_{x}\underset{\sim}{L}_{x}$ and the use of the just derived formulas permit deriving those statistics relating to all states which are initially empty but non-empty at age x.

It is also possible to extend the two alternative measures of mobility (expectations of life and migraproduction rates) by defining them with reference to the state of presence at age y $(0 \le y \le x)$. This leads to a matrix of expectations of life ${}_{y}\underset{\sim}{e}_{x}$ by place of presence at age y defined as

$${}_{y}\underset{\sim}{e}_{x} = {}_{y}\underset{\sim}{T}_{x}\;{}_{y}\overset{mt}{\underset{\sim}{l}}_{x}{}^{-1} \qquad \forall y \;\; 0 \le y \le x \;\; .$$

in which ${}_{y}\overset{mt}{\underset{\sim}{l}}_{x}$ is a diagonal matrix whose typical element is $\sum_{k=1}^{n} {}_{iy}l_{x}^{k}$. In a similar way, one may define a matrix of migraproduction rates ${}_{y}\underset{\sim}{n}_{x}$ by place of presence at age y as

$${}_{y}\underset{\sim}{n}_{x} = \left[\int_{0}^{\infty} \overset{mt}{\underset{\sim}{\mu}}(x+t)\,{}_{y}\underset{\sim}{l}(x+t)dt\right]{}_{y}\overset{t}{\underset{\sim}{l}}_{x}{}^{-1} \;\; , \qquad \forall y \;\; 0 \le y \le x \;\; .$$

Note that, if y is zero, the above definitions reduce to those of expectations of life and migraproduction rates by place-of-birth put forward by Rogers (1975a).*

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*All types of expectations of life and migraproduction rates are independent of the state allocation of the initial cohort. We can establish the following relationships between the multistate functions just defined:

$${}_{0}\underset{\sim}{e}_{x} = {}_{x}\underset{\sim}{e}_{x}\;{}_{0}\underset{\sim}{\Omega}(x) \quad , \quad {}_{0}\underset{\sim}{n}_{x} = {}_{x}\underset{\sim}{n}_{x}\;{}_{0}\underset{\sim}{\Omega}(x)$$

and

$${}_{0}\underset{\sim}{e}_{x}\;{}_{0}\underset{\sim}{n}_{x}{}^{-1} = {}_{x}\underset{\sim}{e}_{x}\;{}_{x}\underset{\sim}{n}_{x}{}^{-1}$$

Age-specific Mortality/Mobility Rates and Survivorship Proportions

The extension of the age-specific death rate m_x of the single-state life table is straightforward in the present version of the multistate life table. The age-specific movement rate ${}^{i}m_x^{j}$, the discrete counterpart of ${}^{i}\mu^{j}(y)$, is defined as the ratio of the number of moves ${}^{i}d_x^{j}$ from i to j between ages x and x + T to the exposed population L_x^i:

$$ {}^{i}m_x^{j} = \frac{{}^{i}d_x^{j}}{L_x^i} \qquad \begin{matrix} \forall i = 1,\ldots,n \\ \forall j = 1,\ldots,n+1 \\ j \neq i \end{matrix} \quad . \tag{35} $$

From the definition (12) of the instantaneous rate ${}^{i}\mu_x^{j}(y)$, it follows that the number of movements ${}^{i}d_x^{j}$ is equal to $\int_0^T {}^{i}\mu^{j}(x+t)\,l^i(x+t)dt$. Then recalling the definition of L_x^i and substituting into the above definition yields:

$$ {}^{i}m_x^{j} = \frac{\int_0^T {}^{i}\mu^{j}(x+t)\,l^i(x+t)dt}{\int_0^T l^i(x+t)dt} \qquad \begin{matrix} \forall i = 1,\ldots,n \\ \forall j = 1,\ldots,n+1 \\ j \neq i \end{matrix} \quad . \tag{36} $$

It is clear that the above definition of the age-specific rates involves the consideration of all persons (whatever their state of birth) alive in the system between ages x and x + T. Consequently, the value of ${}^{i}m_x^{j}$ is affected by the state allocation of the initial cohort as indicated by this equivalent specification of (35):*

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* This specification of ${}^{i}m_x^{j}$ also shows that unlike the instantaneous mortality and mobility rates which are independent of each other, the discrete mortality and mobility rates are not independent within and between regions.

$$ {}^{i}m_{x}^{j} = \frac{\int_{0}^{T} {}^{i}\mu^{j}(x+t) \sum_{k=1}^{n} {}_{k0}l^{i}(x+t)\, dt}{\sum_{k=1}^{n} \int_{0}^{T} {}_{k0}l^{i}(x+t)dt} \qquad \forall i,j = 1,\ldots,n \quad j \neq i \quad . $$

A further consequence of this dependence of the age-specific mortality/mobility rates on $\{l_0\}$ is the impossibility of drawing the age-specific movement rates from the life table functions, as can be done in the single-state case. The discrete equivalent to the elementary flow equation (14) can be written as:

$$ l_{x+T}^{i} = l_{x}^{i} - {}^{i}d_{x}^{\delta} - \sum_{\substack{j=1 \\ j\neq i}}^{n} {}^{i}d_{x}^{j} + \sum_{\substack{j=1 \\ j\neq i}}^{n} {}^{j}d_{x}^{i} \qquad \forall i = 1,\ldots,n \tag{37} $$

Substituting the definition equations (35) then leads to:

$$ l_{x+T}^{i} = l_{x}^{i} - [{}^{i}m_{x}^{\delta} + \sum_{\substack{j=1 \\ j\neq i}}^{n} {}^{i}m_{x}^{j}]L_{x}^{i} + \sum_{\substack{j=1 \\ j\neq i}}^{n} {}^{j}m_{x}^{i}\, L_{x}^{j} \qquad \forall i = 1,\ldots,n \quad , $$

which can be rewritten as:

$$ \{l_{x+T}\} = \{l_{x}\} - \underset{\sim}{m}_{x}\{L_{x}\} \tag{38} $$

in which $\underset{\sim}{m}_{x}$ is the discrete counterpart of (16), i.e.,

$$\underset{\sim}{m}_x = \left[\begin{array}{cccc} {}^1m^\delta_x + [\sum_{\substack{j=1\\ j\neq 1}}^{n} {}^im^j_x] & -{}^2m^1_x & \cdots & {}^nm^1_x \\ -{}^1m^2_x & {}^2m^\delta_x + [\sum_{\substack{j=1\\ j\neq 2}}^{n} {}^2m^j_x] & & \\ & & \ddots & \\ -{}^1m^n_x & & & {}^nm^\delta_x + \left[\sum_{\substack{j=1\\ j\neq n}}^{n} {}^nm^j_x\right] \end{array}\right] \qquad (39)$$

Clearly, the vector equation (38) is insufficient to draw $\underset{\sim}{m}_x$ from the availability of $\{1_x\}$, $\{1_{x+T}\}$ and $\{L_x\}$. Therefore, it is rather tempting to generalize (38) and write it in a matrix format as

$${}_0\underset{\sim}{l}_{x+T} = {}_0\underset{\sim}{l}_x - \underset{\sim}{m}_x \, {}_0\underset{\sim}{L}_x$$

However, this relationship does not hold since $\underset{\sim}{m}_x$ is not a constant matrix. This result is not surprising since the differential equation (17) admits n linearly independent solutions, corresponding to the groups of survivors in each initial radix, and suggests the constancy of age-specific mortality and death rates by place of birth.

Let ${}_{k0}^{\;\;i}m^j_x$ denote the mobility rate from state i to state j between ages x and x + T for those born in state k. Its expression is easily obtained from (36) by substituting ${}_{k0}l^i(x + t)$ for $l^i(x + t)$:

$${}_{k0}^{\;\;i}m^j_x = \frac{\int_0^T {}^i\mu^j(x + t)\,{}_{k0}l^i(x + t)dt}{\int_0^T {}_{k0}l^i(x + t)dt} \qquad \begin{array}{l} \forall i = 1,\ldots,n \\ \forall j = 1,\ldots,n + 1 \\ \;\; j \neq i \end{array} \qquad (40)$$

Observing that ${}_{k0}l^i(x + t) = {}^k_0\Omega^i(x + t)\,l^k_0$ leads to the equivalent expression

$$ {}^{i}_{k0}m^j_x = \frac{\int_0^T {}^i\mu^j(x + t)\,{}^k_0\Omega^i(x + t)dt}{\int_0^T {}^k_0\Omega^i(x + t)dt} \qquad \begin{array}{l} \forall i = 1,\dots,n \\ \forall j = 1,\dots,n + 1 \\ j \neq i \end{array} $$

which shows the constancy of age-specific mortality and mobility rates by place-of-birth.

Clearly ${}^im^j_x$ denoted by ${}^{i}_{.0}m^j_x$ to be consistent with the notations just adopted is such that:

$$ {}^{i}_{.0}m^j_x = \frac{\sum_{k=1}^{n} {}^{i}_{k0}m^j_x \; {}_{k0}L^i_x}{\sum_{k=1}^{n} {}_{k0}L^i_x} \qquad \begin{array}{l} \forall i = 1,\dots,n \\ \forall j = 1,\dots,n + 1 \\ j \neq i \end{array} \qquad (41) $$

To summarize, the existence of a predetermined pattern of mortality/mobility, defined in continuous terms by assumption (12), does not lead to the constancy of age-specific mortality and mobility rates but to the constancy of such rates further indexed by place of birth. Indeed, in the single-radix case, the age-specific mortality and mobility rates do not bear any ambiguity since there exists a single state-of-birth.

We could also define age-specific mortality and mobility rates by reference to states of presence at anyage y $(0 \leq y \leq x)$ rather than to states of birth. In fact, this generalizes (40) to:

$$ {}^{i}_{ky}m^j_x = \frac{\int_0^T {}^i\mu^j(x + t)\,{}_{ky}l^i(x + t)dt}{\int_0^T {}_{ky}l^i(x + t)dt} \qquad \begin{array}{l} \forall i = 1,\dots,n \\ \forall j = 1,\dots,n \\ j \neq i \\ \forall k = 1,\dots,n \\ \forall y \; 0 \leq y \leq x \end{array} $$

and (41) to:

$$
{}^{i}_{\cdot y}m^{j}_{x} = \frac{\sum_{k=1}^{n} {}^{i}_{ky}m^{j}_{x} \; {}_{ky}L_{x}}{\sum_{k=1}^{n} {}_{ky}L_{x}} \qquad \begin{array}{c} \forall i,j = 1,\ldots,n \\ j \neq i \\ \forall y \;\; 0 \leq y \leq x \end{array}
$$

Note the dependence of these rates on the state allocation of the initial cohort.

Another life table function that one would like to extend to the multiregional case is the survivorship probability s_x denoting the proportion of individuals aged x to x + T who survive to be x + T to x + 2T, T years later.

For example, we define the proportion ${}^{i}s^{j}_{x}$ of individuals present in state i between ages x and x + T who move to state j and survive to be included in that state's x + T to x + 2T years old population T years later, then

$$
L^{i}_{x+T} = \sum_{k=1}^{n} {}^{i}s^{k}_{x} \; L^{k}_{x} \qquad \forall i = 1,\ldots,n
$$

which can be written more compactly as:

$$
\{L_{x+T}\} = \underset{\sim}{s}_{x}\{L_{x}\} \tag{42}
$$

in which $\underset{\sim}{s}_x$ is a matrix whose $(i,j)^{th}$ element is ${}^{j}s^{i}_{x}$.

Again (42), a vector equation, is insufficient to draw $\underset{\sim}{s}_x$ from the availability of the multistate stationary population $\{L_x\}$. Furthermore, it suggests that the survivorship proportions depend on the state-specific allocation of the initial cohort. Then, as is the case of the age-specific mortality and mobility rates, it is necessary to characterize the survivorship by a third index relating to the state of birth. Let ${}^{i}_{k0}s^{j}_{x}$ denote the proportion of ${}_{k0}L^{k}_{x}$ who move to state j within a T-year period.

Then:

$$ {}_{k0}^{\ i}s_x^j = \frac{{}_k\alpha_i\ {}_{k0}L_{x+T}^j}{{}_{k0}L_x^i} \qquad \forall i,j,k = 1,\ldots,n $$

in which the numerator ${}_k\alpha_i\ {}_{k0}L_{x+T}^j$ represents the fraction of the total number of years lived in state j between ages x + T and x + 2T by the k-born individuals who were also living in stage i between ages x and x + T. Recalling the notation ${}_{y1}^{\ i}\Omega^j(y_2)$, it follows that

$$ {}_k\alpha_i\ {}_{k0}L_x^j + T = \int_0^T {}_{x+t}^{\ \ i}\Omega^j(x + t + T)\,{}_{k0}l^i(x + t)dt \quad , $$

and that:

$$ {}_{k0}^{\ i}s_x^j = \frac{\int_0^T {}_{x+t}^{\ \ i}\Omega^j(x + t + T)\,{}_{k0}l^i(x + t)dt}{\int_0^T {}_{k0}l^i(x + t)dt} \qquad (43) $$

$$ \forall i,j,k = 1,\ldots,n $$

Note the independence of ${}_{k0}^{\ i}s_x^j$ vis-a-vis the state allocation of the initial radix that affects the survivorship proportion:

$$ {}^is_x^j = \frac{\sum_{k=1}^{n} {}_{0k}^{\ i}s_x^j\ {}_{k0}L_x^i}{\sum_{k=1}^{n} {}_{k0}L_x^i} \qquad \forall i,j = 1,\ldots,n \ . \qquad (44) $$

Because of the definition of the discrete model of multiregional population growth, this dependence of the survivorship proportions on $\{l_0\}$ is unfortunate. Rogers/Ledent (1974) have thus defined approximate survivorship proportions based on the desirable property of independence vis-a-vis of $\{l_0\}$. Under this assumption, (42) holds for each age distribution $\{l_x\}$ generated by the total allocations of the initial cohort to a unique state. Therefore the unvarying matrix $\underset{\sim}{s}_x$ thus defined can be obtained from:

$$\underset{\sim}{s}_x = {}_0\underset{\sim}{L}_{x+T}\ {}_0\underset{\sim}{L}_x^{-1} \quad . \tag{45}$$

It is possible to extend the multistate functions (43) and (44) by relating them to the states of presence at any age y ($0 \le y \le x$) rather than to states of birth. For example, ${}_{ky}^{i}s_x^{j}$ could be obtained from (43) in which ${}_{ky}l^i(x + t)$ is substituted for ${}_{k0}l^i(x + t)$.*

Continuous and Discrete Aspects of an Increment-Decrement Life Table

In the above presentation of standard increment-decrement life tables, the key element lies in the definition (12) of the forces of mortality and mobility contained in $\underset{\sim}{\mu}(y)$ such that an individual's instantaneous propensities to move (or die) are independent of his past mobility history. This gives a Markov process interpretation to such tables and guarantees the independence vis-a-vis the initial radix of multistate life table functions characteristic of an exact age. Formulas (21), (23), (32) and (34) show that such functions as ${}_0\underset{\sim}{\Omega}(x)$, $\underset{\sim}{p}_x$, ${}_x\underset{\sim}{e}_x$ and ${}_x\underset{\sim}{n}_x$ only depend on the curves $\underset{\sim}{\mu}(y)$ and are, in no circumstances, affected by the state allocation of the initial cohort.

In contrast to these continuous age life table functions, the functions relating to discrete age intervals depend not only on the curve of instantaneous forces of mortality and transition but also on

---

*Note the independence of $\underset{\sim}{s}_x$ as defined by (45) vis-a-vis the choice of the state of presence at age y ($0 \le y \le x$)

$$\underset{\sim}{s}_x = {}_y\underset{\sim}{L}_{x+T}\ {}_y\underset{\sim}{L}_x^{-1} \quad , \qquad \forall y \ 0 \le y \le x \quad .$$

state/age distribution of the resulting stationary population. Since the latter is determined by the same curves of instantaneous forces and by the state allocation of the initial cohort, as shown by (19), it follows that $\underset{\sim}{m}_x$ and the matrix of true survivorship proportions $\underset{\sim}{s}_x$ are affected by the state allocation of $l_0^{\cdot}$. Nevertheless, the pattern of mortality and mobility is such that constant mortality/mobility rates and survivorship proportions can be found in each of the multistate stationary populations originating from each state-specific group of the initial cohort.

The assumption of (12), defining the instantaneous mortality and mobility pattern, leads to constant age-specific mortality and mobility rates for each of the multistate stationary populations generated from the n independent solutions of (17). Note that, although the forces of mortality and mobility depend only on the states of origin and destination, the age-specific mortality and mobility rates "by state of birth" depend on all states in the models as suggested by (40). Consequently, for a given x, the matrices ${}_{k0}\underset{\sim}{m}_x$ for all k = 1,...,n are not independent. The importance of this finding will be made clear later.

Multistate Life Table Functions in Terms of the Life Table Mortality and Mobility Rates

The above exposition of increment-decrement life tables suggests that a point of choice in proceeding from the life table age-specific mortality and mobility rates is the integration of $\{l(y)\}$ and ${}_0\underset{\sim}{l}(y)$ over successive intervals (x, x + T). As in the single-state case, this problem can be illustrated further, even without supposing any explicit method for deriving $\{L_x\}$. This requires the consideration of a matrix $\underset{\sim}{a}_x$ of mean durations of transfers. It is the multistate analog of the average number of years a_x lived in the interval (x, x + T) by those of the single-state life table who died in that interval.

The Matrix of Mean Durations of Transfers over a Time Period

In order to understand the matrix of mean durations of tranfers over a time period, it is sufficient to focus on the subsequent evolution of the group of people between ages x and x + T present at age x in state i, (x = 0,T,...,z-2T).

Let:

- ${}_{ix}z_j$ denote the number of times that a member of l^i_x enters state j
- $x + {}^{E}_{ix}t^{l}_{j}$ and $x + {}^{W}_{ix}t^{l}_{j}$ denote the age at which this individual respectively enters and leaves a state j for the l^{th} time.

Assimilating the reaching of age x + T in state j to a withdrawal from state j, leads us to determine the time spent between ages x and x + T in state j by a member of l^i_x as $\sum_{l=1}^{z_j} [{}^{W}_{ix}t^{l}_{j} - {}^{E}_{ix}t^{l}_{j}]$ in which z_j is the number of entries into state j.*

Letting the variable ${}^{j}_{ix}d^{k}(y)$ denote the number of moves from state j to state k made by the members of l^1_x between ages y and y + dy and insisting on the fact that reaching age x + T in a certain state is equivalent to withdrawing from this state at a time x + T, we can then write the total number of person-years lived between ages x and x + T in state j by all the members of l^i_x as:

$$ {}_{ix}L^j_x = T\ {}_{ix}l^j_{x+T} + \int_{y=x}^{y=x+T} y \left[\sum_{\substack{k=1\\k\neq j}}^{n+1} {}^{j}_{ix}d^{k}(y)\right] - \int_{y=x}^{y=x+T} y \left[\sum_{\substack{k=1\\k\neq j}}^{n} {}^{k}_{ix}d^{j}(y)\right] $$

$$ \forall i,j = 1,\dots,n \quad . $$

It follows that:

$$ L^j_x = Tl^j_{x+t} + \int_{y=x}^{y=x+T} y \left[\sum_{\substack{k=1\\k\neq j}}^{n+1} {}^{j}_{.x}d^{k}(y)\right] - \int_{y=x}^{y=x+T} y \left[\sum_{\substack{k=1\\k\neq j}}^{n} {}^{k}_{.x}d^{j}(y)\right] \qquad (46) $$

---

*Note that ${}^{E}_{ix}t^{1}_{i} = o$ and ${}^{W}_{ix}t^{z_k}_{k} = T$ if k is the state of presence at age x + T.

Let ${}^{i}a_{x}^{j}$ denote the average time elapsed between age x and the age y at which a move is made from state j to state k $(x \leq y \leq x + T)$. By definition:

$$\int_{y=x}^{y=x+T} y \; {}^{j}_{.x}d^{k}(y) = {}^{j}a_{x}^{k} \; {}^{j}d_{x}^{k} \qquad \forall j,k = 1,\ldots,n \qquad (47)$$

in which ${}^{j}d_{x}^{k}$ is the total number of moves made by all members of the system between ages x and x + T from state j to state k. Substituting (47) into (46) we have:

$$L_{x}^{j} = T \; l_{x+T}^{j} + \sum_{\substack{k=1 \\ k\neq j}}^{n+1} {}^{j}a_{x}^{k} \; {}^{j}d_{x}^{k} - \sum_{\substack{k=1 \\ k\neq j}}^{n} {}^{k}a_{x}^{j} \; {}^{k}d_{x}^{j} \quad ,$$

$$\forall j = 1,\ldots,n \quad .$$

Recalling the definition of the age-specific mortality and mobility rates (35), we then obtain:

$$L_{x}^{j} = T \; l_{x+T}^{j} + \sum_{\substack{k=1 \\ k\neq j}}^{n+1} {}^{j}a_{x}^{k} \; {}^{j}m_{x}^{k} \; L_{x}^{j} - \sum_{\substack{k=1 \\ k\neq j}}^{n} {}^{k}a_{x}^{j} \; {}^{k}m_{x}^{j} \; L_{x}^{k}$$

$$\forall j = 1,\ldots,n$$

or, more compactly, using a matrix notation:

$$\{L_{x}\} = T \; \{l_{x+T}\} + \overset{0}{\underset{\sim}{a}}_{x} \{L_{x}\} \qquad (48)$$

in which:

$$\underset{\sim}{\overset{0}{a}}_x = \begin{bmatrix} {}^1a_x^\delta\ {}^1m_x^\delta + \left[\sum_{\substack{j=1\\ j\neq i}}^{n} {}^1a_x^j\ {}^1m_x^j\right] & -\ {}^2a_x^1\ {}^2m_x^1 & \cdots\cdots & -\ {}^na_x^1\ {}^nm_x^1 \\ -\ {}^1a_x^2\ {}^1m_x^2 & {}^2a_x^\delta\ {}^2m_x^\delta + \left[\sum_{\substack{j=1\\ j\neq 2}}^{n} {}^2a_x^j\ {}^2m_x^j\right] & & \\ & & \ddots & \\ -\ {}^1a_x^n\ {}^1m_x^n & & & {}^na_x^\delta\ {}^nm_x^\delta + \left[\sum_{\substack{j=1\\ j\neq i}}^{n} {}^na_x^j\ {}^nm_x^j\right] \end{bmatrix}$$

Note the likeness of $\underset{\sim}{\overset{0}{a}}_x$ with $\underset{\sim}{m}_x$: the $(i,j)^{th}$ element of $\underset{\sim}{\overset{0}{a}}_x$ is obtained from the corresponding element of $\underset{\sim}{m}_x$ by multiplying it by a coefficient characteristic of both the i and j^{th} states. Thus,

$$\{L_x\} = T(\underset{\sim}{I} - \underset{\sim}{\overset{0}{a}}_x)^{-1}\{l_{x+T}\} \quad . \qquad (49)$$

Substituting into (38) leads to an expression linking $\{l_{x+t}\}$ with $\{l_x\}$

$$\{l_{x+T}\} = \left[\underset{\sim}{I} + T\ \underset{\sim}{m}_x[\underset{\sim}{I} - \underset{\sim}{\overset{0}{a}}_x]^{-1}\right]^{-1}\{l_x\} \quad . \qquad (50)$$

Comparing this last expression with the definition (11) of $\{l_{x+t}\}$

$$\{l_{x+t}\} = \underset{\sim}{p}_x\{l_x\}$$

provides the desired expression of the age-specific matrix of probabilities

$$\underset{\sim}{p}_x = \left[\underset{\sim}{I} + T\,\underset{\sim}{m}_x[\underset{\sim}{I} - \overset{0}{\underset{\sim}{a}}_x]^{-1}\right]^{-1\;*} \qquad (51)$$

Since $\underset{\sim}{I} = (\underset{\sim}{I} - \overset{0}{\underset{\sim}{a}}_x)(\underset{\sim}{I} - \overset{0}{\underset{\sim}{a}}_x)^{-1}$, (51) can be successfully rewritten as:

$$\underset{\sim}{p}_x = (\underset{\sim}{I} - \overset{0}{\underset{\sim}{a}}_x)(\underset{\sim}{I} - \overset{0}{\underset{\sim}{a}}_x)^{-1} + T\,\underset{\sim}{m}_x[\underset{\sim}{I} - \overset{0}{\underset{\sim}{a}}_x]^{-1}$$

$$= \left[(\underset{\sim}{I} - \overset{0}{\underset{\sim}{a}}_x + T\,\underset{\sim}{m}_x)(\underset{\sim}{I} - \overset{0}{\underset{\sim}{a}}_x)^{-1}\right]^{-1}$$

and finally,

$$\underset{\sim}{p}_x = [\underset{\sim}{I} - \overset{0}{\underset{\sim}{a}}_x][\underset{\sim}{I} - \overset{0}{\underset{\sim}{a}}_x + T\,\underset{\sim}{m}_x]^{-1}$$

a relationship from which we can draw $\overset{0}{\underset{\sim}{a}}_x$ in terms of $\underset{\sim}{p}_x$ and $\underset{\sim}{m}_x$

$$\overset{0}{\underset{\sim}{a}}_x = \underset{\sim}{I} - T(\underset{\sim}{I} - \underset{\sim}{p}_x)^{-1}\underset{\sim}{p}_x\,\underset{\sim}{m}_x \quad .\;*$$

Although $\underset{\sim}{p}_x$ is independent of the state allocation of the initial cohort, $\overset{0}{\underset{\sim}{a}}_x$ depends on it since $\underset{\sim}{m}_x$ in (51) generally varies with this initial allocation.**

Moreover, substituting (50) into (49) yields:

$$\{L_x\} = T[\underset{\sim}{I} - \overset{0}{\underset{\sim}{a}}_x + T\,\underset{\sim}{m}_x]^{-1}\{l_x\} \qquad (52)$$

---

*In the case of the last age group (48) reduces to $\{L_z\} = \overset{0}{\underset{\sim}{a}}_z\{L_z\}$ or using (38) $\underset{\sim}{m}_z^{-1}\{l_z\} = \underset{\sim}{m}_z^{-1}\,\overset{0}{\underset{\sim}{a}}_z\{l_z\}$, an equality that indicates the dependence of the a-coefficients on the state allocation of the initial vector $\{l_0\}$.

**As for the other interval-related multistate life table functions, $\overset{0}{\underset{\sim}{a}}_x$ depends on the state allocation of the initial cohort whereas for each of the n independent multistate stationary populations, there exists a constant matrix of mean durations of transfers ${}_{k0}\overset{0}{a}_x$.

an expression that will later allow the derivation of $\underset{\sim}{q}_x$. Since those who die in state j between ages x and x + T were members at age x of any cohort l_x^k, the corresponding number of deaths can be written either $\sum_{k=1}^{n} {}^k q_x^j \, l_x^k$ or ${}^j m_x^\delta \, L_x^j$ so that we have in matrix form $\underset{\sim}{q}_x \{l_x\} = \underset{\sim}{\overset{\delta}{m}}_x \{L_x\}$ in which $\underset{\sim}{\overset{\delta}{m}}_x$ is a diagonal matrix of death rates. Substituting the expression of $\{L_x\}$ into this last formula then leads to:

$$\underset{\sim}{T}_x \{l_x\} = T \, \underset{\sim}{\overset{\delta}{m}}_x [\underset{\sim}{I} - \underset{\sim}{\overset{0}{a}}_x + T \, \underset{\sim}{m}_x]^{-1} \{l_x\}$$

and finally, because of the independence of $\underset{\sim}{q}_x$ from $\{l_x\}$, we have

$$\underset{\sim}{q}_x = T \, \underset{\sim}{\overset{\delta}{m}}_x [\underset{\sim}{I} - \underset{\sim}{\overset{0}{a}}_x + T \, \underset{\sim}{m}_x]^{-1} \quad . \qquad (53)$$

The Case of a Uniform Distribution of All Moves

Before looking at the case of a uniform distribution of moves, let us consider the case in which all moves out of a region are similarly distributed. Then:

$${}^i a_x^1 = {}^i a_x^2 \cdots = {}^i a_x^n = {}^i a_x^\delta = {}^i a_x$$

which permits us to express $\underset{\sim}{\overset{0}{a}}_x$ as the product of two matrices:

$$\underset{\sim}{\overset{0}{a}}_x = \underset{\sim}{m}_x \underset{\sim}{a}_x$$

in which $\underset{\sim}{a}_x$ is a diagonal matrix whose typical element is ${}^i a_x$. In such circumstances, (48) becomes

$$\{L_x\} = T\{l_{x+t}\} + \underset{\sim}{m}_x \underset{\sim}{a}_x \{L_x\} \quad . \qquad (54)$$

After substitution of (38)

$$\{L_x\} = T\{l_{x+t}\} + \underset{\sim}{m}_x \underset{\sim}{a}_x \underset{\sim}{m}_x^{-1} [\{l_x\} - \{l_{x+t}\}] \quad ,$$

becomes a formula generalizing the single-state identity

$$L_x = T\ l_{x+t} + a_x(l_x - l_{x+t}) \quad .$$

The single-state function a_x is extended as the more complex function $\underset{\sim}{m}_x \underset{\sim}{a}_x \underset{\sim}{m}_x^{-1}$; the latter however, reduces to $\underset{\sim}{a}_x$ if all moves in the system are uniformly distributed. Substituting the expression of $\overset{0}{\underset{\sim}{a}}_x$ into (51) and (53) yields:

$$\underset{\sim}{p}_x = [\underset{\sim}{I} - \underset{\sim}{m}_x \underset{\sim}{a}_x][\underset{\sim}{I} + \underset{\sim}{m}_x(T\ \underset{\sim}{I} - \underset{\sim}{a}_x)]^{-1} \tag{55}$$

and

$$\underset{\sim}{q}_x = T\ \overset{\delta}{\underset{\sim}{m}}_x\ [\underset{\sim}{I} + \underset{\sim}{m}_x(T\ \underset{\sim}{I} - \underset{\sim}{a}_x)]^{-1} \quad . \tag{56}$$

Note that (55) corrects the formula given in Rogers/Ledent (1976) in which the two expressions between brackets were inverted.

Furthermore, if all moves out of each region are uniformly distributed for each closed interval, i.e.,

$$\underset{\sim}{a}_x = \frac{T}{2}\ \underset{\sim}{I} \qquad \text{for all } x \neq z \quad , \tag{57}$$

we obtain by substitution into (55)

$$\underset{\sim}{p}_x = [\underset{\sim}{I} - \frac{T}{2}\ \underset{\sim}{m}_x][\underset{\sim}{I} + \frac{T}{2}\ \underset{\sim}{m}_x]^{-1} \quad ,^{*} \tag{58}$$

---

*$[\underset{\sim}{I} - \frac{T}{2}\ \underset{\sim}{m}_x][\underset{\sim}{I} + \frac{T}{2}\ \underset{\sim}{m}_x] = [\underset{\sim}{I} + \frac{T}{2}\ \underset{\sim}{m}_x][\underset{\sim}{I} - \frac{T}{2}\ \underset{\sim}{m}_x]$, $\underset{\sim}{p}_x$ can be rewritten

as: $\underset{\sim}{p}_x = [\underset{\sim}{I} + \frac{T}{2}\ \underset{\sim}{m}_x]^{-1}[\underset{\sim}{I} - \frac{T}{2}\ \underset{\sim}{m}_x]$ (58a).

This alternate expression of $\underset{\sim}{p}_x$ is found in Rogers/Ledent (1976).

and by substitution into (56)

$$\underset{\sim}{q}_x = T \, \underset{\sim}{\overset{\delta}{m}}_x [\underset{\sim}{I} + \frac{T}{2} \underset{\sim}{m}_x]^{-1} \quad . \tag{59}$$

Conversely from (58) we can draw an expression of $\underset{\sim}{m}_x$ in terms of $\underset{\sim}{p}_x$

$$\underset{\sim}{m}_x = \frac{2}{T} \, [\underset{\sim}{I} + \underset{\sim}{p}_x]^{-1} [\underset{\sim}{I} - \underset{\sim}{p}_x]$$

an equation which indicates that $\underset{\sim}{m}_x$ is uniquely defined in terms of $\underset{\sim}{p}_x$ and is thus independent of the state allocation of the initial vector.

Consequently, assuming a uniform distribution of moves, we find constant age-specific mortality and mobility rates by place-of-birth for any choice of the state allocation of the initial cohort, i.e.,

$$^{i}m^{j}_{x} = \cdots = {}_{k0}^{\;i}m^{j}_{x} = \cdots = {}_{n0}^{\;i}m^{j}_{x} = {}^{i}m^{j}_{x} \quad .$$

Then, (38) can be generalized as:

$${}_{0}\underset{\sim}{l}_x - {}_{0}\underset{\sim}{l}_{x+T} = \underset{\sim}{m}_x \, {}_{0}\underset{\sim}{L}_x \quad , \tag{60}$$

an equation from which we can draw

$$\underset{\sim}{m}_x = ({}_{0}\underset{\sim}{l}_x - {}_{0}\underset{\sim}{l}_{x+T}) \, {}_{0}\underset{\sim}{L}_x^{-1} \quad ,^{*} \tag{61}$$

As in the single-state case, the assumption of uniformly distributed moves leads to the derivation of survival probabilities that are identical to those obtained by supposing a linear integration over $\{l_x\}$. This result can be demonstrated directly by comparing

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*(61) also holds if the multistate life table rates relate to the state of presence at age y $(0 \leq y \leq x)$ rather than to state of birth:

${}_{ky}^{\;i}m^{j}_{x} = {}^{i}m^{j}_{x}$ = constant independent of y $(0 \leq y \leq x)$ and k(= 1,...,n).

$\{L_x\}$ and $\frac{T}{2}[\{l_x\} + \{l_{x+T}\}]$. Assuming (57) yields

$$\{l_{x+T}\} = \underset{\sim}{p}_x\{l_x\} = (\underset{\sim}{I} - \frac{T}{2}\,\underset{\sim}{m}_x)(\underset{\sim}{I} + \frac{T}{2}\,\underset{\sim}{m}_x)^{-1}\{l_x\} \quad ,$$

and

$$\frac{T}{2}[\{l_x\} + \{l_{x+T}\}] = \frac{T}{2}[\underset{\sim}{I} + (\underset{\sim}{I} - \frac{T}{2}\,\underset{\sim}{m}_x)(\underset{\sim}{I} + \frac{T}{2}\,\underset{\sim}{m}_x)]^{-1}\{l_x\} \ .$$

Since $\underset{\sim}{I}$ can be decomposed into $(\underset{\sim}{I} + \frac{T}{2}\,\underset{\sim}{m}_x)(\underset{\sim}{I} + \frac{T}{2}\,\underset{\sim}{m}_x)^{-1}$, this eventually leads to:

$$\frac{T}{2}[\{l_x\} + \{l_{x+T}\}] = T[\underset{\sim}{I} + \frac{T}{2}\,\underset{\sim}{m}_x]^{-1}\{l_x\} \quad . \tag{62}$$

Setting $\underset{\sim}{a}_x = \frac{T}{2}\,\underset{\sim}{I}$ in (52) gives

$$\{L_x\} = T[\underset{\sim}{I} + \frac{T}{2}\,\underset{\sim}{m}_x]^{-1}\{l_x\} \quad , \tag{63}$$

and consequently we obtain after comparing (62) and (63)

$$\{L_x\} = \frac{T}{2}[\{l_x\} + \{l_{x+T}\}] \quad . \tag{64}$$

Conversely, if one assumes that $\{L_x\}$ is given by a linear integration such as (64) for any choice of the state allocation of the initial matrix, one finds by comparing (38) with (64) that:

$$\underset{\sim}{I} - \underset{\sim}{p}_x = \frac{T}{2}\,\underset{\sim}{m}_x[\underset{\sim}{I} + \underset{\sim}{p}_x]$$

i.e.,

$$\underset{\sim}{p}_x = [\underset{\sim}{I} + \frac{T}{2}\,\underset{\sim}{m}_x]^{-1}[\underset{\sim}{I} - \frac{T}{2}\,\underset{\sim}{m}_x]$$

Further comparison with (51) leads to

$$\underset{\sim}{a}_x = \frac{T}{2}\,\underset{\sim}{m}_x$$

showing that all movements are uniformly distributed in each age group.

In other words, as in the single-state case, the assumption of a uniform distribution of movements is equivalent to the linear derivation of the person-years in the stationary population.

This equivalence, shown here by reference to the vectorial age distributions, also applies to the matrical age distributions Since $\underset{\sim}{m}_x$ is independent of the initial radix, the matrix extension of (64) holds, giving

$$ {}_0\underset{\sim}{L}_x = \frac{T}{2}[{}_0\underset{\sim}{l}_x + {}_0\underset{\sim}{l}_{x+T}] \quad , \qquad (65)$$

a relationship which permits us to obtain the values of Rogers' multistate life table functions such as ${}_0\underset{\sim}{T}_x, {}_x\underset{\sim}{e}_x$ from the knowledge of $\underset{\sim}{m}_x$.*

Carrying the linear integration on $\{\underset{\sim}{l}_x\}$ is equivalent to performing it not only on ${}_0\underset{\sim}{L}_x$ but also on ${}_y\underset{\sim}{L}_x$ (for all y, $0 \leq y \leq x$). This finding permits us to compute some of the multistate functions relating to initially empty states by using the generalized expressions of multistate life table functions relating to states of presence at age y rather than to states of birth.**

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*In addition, the property of independence displayed by $\underset{\sim}{m}_x$ allows rewriting (33) in a discrete form as:

$${}_x\underset{\sim}{n}_x = {}^{mt}\underset{\sim}{m}_x\ {}_0\underset{\sim}{L}_x\ {}_0\underset{\sim}{l}_x^{-1} \quad ,$$

in which ${}^{mt}\underset{\sim}{m}_x$ is a diagonal matrix whose i^{th} element is $[\sum_{\substack{k=1\\k\neq i}}^{n} {}^{i}m^{k}_{x}]$, thus making it possible to express ${}_x\underset{\sim}{n}_x$ in terms of the life table rates.

**This observation is important as the non equivalence of the cubic integration of ${}_0\underset{\sim}{L}_x$ and ${}_y\underset{\sim}{L}_x$ will suggest later.

In the case of the terminal age interval which is half open, a different treatment is used:

- $\underset{\sim}{p}_z$ is set to zero since everybody eventually dies, and
- since the length of the interval is infinite, ${}_0\underset{\sim}{L}_z$ cannot be obtained by linear integration.

For this age group, we assume the independence of the life table mortality and mobility rates vis-a-vis the state allocation of the initial cohort, a property equivalent to the linear integration hypothesis in the case of the closed age intervals. Thus (60) in which ${}_0\underset{\sim}{l}_{z+T} = 0$ holds, leads to

$${}_0\underset{\sim}{L}_z = {}_0\underset{\sim}{T}_z = \underset{\sim}{m}_z^{-1}\ {}_0\underset{\sim}{l}_z \ .^*$$

Thus,

$${}_x\underset{\sim}{e}_z = {}_0\underset{\sim}{T}_z\ {}_0\underset{\sim}{l}_z^{-1} = {}_0\underset{\sim}{L}_z\ {}_0\underset{\sim}{l}_z^{-1} = \underset{\sim}{m}_z^{-1} \quad . \qquad (66)$$

Note that $\underset{\sim}{m}_z$ being independent of the state allocation of the intial vector,

$$\overset{0}{\underset{\sim}{a}}_z = \underset{\sim}{I} \quad ,$$

is an equality giving:

$${}^i a_z^d = \frac{1}{{}^i m_z^\delta} \quad \text{and} \quad {}^i a_z^j = 0 \qquad (\forall j \neq i) .$$

In other words, the assumption made about $\underset{\sim}{m}_z$ is equivalent to supposing that in the last interval, all moves (except deaths) out of a region, take place instantaneously at exact age z.

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*This formula is the matrix expression of the various scalar formulas derived by Schoen in the appendix of his 1975 article. Also, note that $\underset{\sim}{m}_z$ is not a diagonal matrix: non-zero mobility rates are here allowed, unlike in Rogers (1973a).

The derivation of the true survivorship probabilities by state of birth as defined by (43) requires a further assumption concerning the integration of the numerator. We can use a linear integration method which would be consistent with the method of integration used for deriving $\{L_x\}$, then:

$$ {}_{k0}^{\,i}s_x^j = \frac{{}_{k0}l_x^i \; {}^{i}p_x^j + {}_{k0}l_{x+T}^i \; {}^{i}p_{x+T}^j}{{}_{k0}l_x^i + {}_{k0}l_{x+T}^i} \qquad \forall i,j = 1,...,n \quad . $$

By contrast, because of the linear integration assumption for deriving $\{L_x\}$, the approximate survivorship probabilities as defined by (45) can be simply expressed in terms of the age-specific mortality and mobility rates.

From (65) rewritten as

$$ {}_0\underset{\sim}{L}_x = \frac{T}{2}(\underset{\sim}{I} + \underset{\sim}{p}_x)\,{}_0\underset{\sim}{l}_x $$

it follows that:

$$ \underset{\sim}{s}_x = (\underset{\sim}{I} + \underset{\sim}{p}_{x+T})\,{}_0\underset{\sim}{l}_{x+t} \; {}_0\underset{\sim}{l}_x^{\;-1}(\underset{\sim}{I} + \underset{\sim}{p}_x) $$

$$ = (\underset{\sim}{I} + \underset{\sim}{p}_{x+t})\underset{\sim}{p}_{x+T}(\underset{\sim}{I} + \underset{\sim}{p}_x) $$

and eventually, after substitution of (55) for the age-specific probabilities:

$$ \underset{\sim}{s}_x = (\underset{\sim}{I} + \frac{T}{2}\,\underset{\sim}{m}_{x+T})^{-1}(\underset{\sim}{I} - \frac{T}{2}\,\underset{\sim}{m}_x) \qquad (67) $$

The comparison of (67) with (58) suggests that $\underset{\sim}{s}_x$ is simply obtained from the formula giving $\underset{\sim}{p}_x$ by replacing the age-specific matrix $\underset{\sim}{m}_x$ within the first brackets with the similar matrix $\underset{\sim}{m}_{x+T}$ corresponding to the next age group.

Of course, (67) is valid only for x = 0,T,...,z-2T whereas $\underset{\sim}{s}_{z-T}$ is given by:

$$\underset{\sim}{s}_{z-T} = \frac{1}{T}\,\underset{\sim}{m}_{z}^{-1}[\underset{\sim}{I} - \frac{T}{2}\,\underset{\sim}{m}_{z-T}] \qquad (68)$$

obtained by combining (45), (55), (65), and (66).

Another statistic needed in Section III is the matrix which gives the regional allocation of survivors at time t + T among those born between times t and t + T.

If a child is born in state i at time t_1 $(0 < t_1 < T$ since we can suppose t = 0 without imposing any further restriction), the possibility he or she will live to the end of the interval (age $T - t_1$) in state j is $\dfrac{{}_{i0}l^{j}_{x-t_1}}{l^{i}_{0}}$. Summing this through the T-year interval of time and age, with births uniformly distributed in time within the T years, gives the proportion of survivors in state j among children born throughout the interval in state i:

$$\frac{1}{T}\,\frac{\int_0^T {}_{i0}l^{j}(y)\,dy}{{}_{i0}l^{i}_{0}} = \frac{1}{T}\,\frac{{}_{i0}L^{j}_{0}}{{}_{i0}l^{i}_{0}} \qquad \forall i,j = 1,\ldots,n \ . \qquad (69)$$

Then we have,

$${}^{i}\underset{\sim}{s}^{j}_{-T} = \frac{1}{T}\,{}_{0}\underset{\sim}{L}_{0}\,\underset{\sim}{l}_{0}^{-1} \ ,$$

and since

$${}_{0}\underset{\sim}{L}_{0}\,\underset{\sim}{l}_{0}^{-1} = \frac{T}{2}[\underset{\sim}{I} + \underset{\sim}{p}_{0}] = \frac{T}{2}[\underset{\sim}{I} + \frac{T}{2}\,\underset{\sim}{m}_{0}]^{-1} \ ,$$

we obtain:

$$\underset{\sim}{s}_{-T} = [\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_0]^{-1} \quad .$$

The similarity mentioned above between the formulas giving the $\underset{\sim}{p}$- and $\underset{\sim}{s}$- matrices is further illustrated by Table 1 below.

Table 1. Comparison of the survival probabilities $\underset{\sim}{p}_x$ and the approximate survivorship proportions $\underset{\sim}{s}_x$.

$\underset{\sim}{p}_x$	$\underset{\sim}{s}_x$
	$= [\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_o]^{-1}$ for $x = -T$
$= [\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x]^{-1} [\underset{\sim}{I} - \frac{T}{2}\underset{\sim}{m}_x]$ for $x = 0,T,2T,\ldots,z-T$	$= [\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_{x+T}]^{-1} [I - \frac{T}{2}\underset{\sim}{m}_x]$ for $x = 0,T,2T,\ldots,z-2T$
	$= \frac{2}{T}\underset{\sim}{m}_z^{-1} [\underset{\sim}{I} - \frac{T}{2}\underset{\sim}{m}_{z-T}]$ for $x = z-T$
$= \underset{\sim}{0}$ for $x = z$	

Applied Calculation of an Increment-decrement Life Table Based on the Movement Approach

The above exposition of increment-decrement life tables suggests that their applied calculation requires first a linkage of the life table age-specific rates with observed data, and second, the availability of a method of integration for deriving $\{L_x\}$ and ${}_0\underset{\sim}{L}_x$.

Linkage with Data on Observed Population

By analogy with the single-state life table, the linkage of life table rates with observed data is performed by positing some relationship between the mortality and mobility patterns of the observed and synthetic (that of the increment-decrement life table) populations.

As presented above, increment-decrement life tables are based on the predetermined knowledge of mortality and mobility patterns defined by continuous curves of such forces. Ideally, one should carry out the linkage with the observed population system by assuming identical curves of mortality and mobility forces in both the synthetic and observed populations. However, the difficulties encountered in implementing such an assumption when calculating an applied life table, make it necessary to link observed and life table patterns of mortality and mobility at a discrete level.*

Then, as in the single-state case, we are left with relating life table age-specific mortality and mobility rates to observed data. But, is it possible to implement a linkage analogous to the one of the single-state life tables in which a simple equality of the age-specific mortality rates of both the life table and observed populations is generally posited?

Earlier, we pointed out that the assumptions underlying movement increment-decrement life tables led to n elementary multistate stationary populations, characterized by constant age-specific rates. In addition the consolidated stationary population displayed age-specific rates varying with the state allocation of the initial cohort. Consequently, the most efficient strategy would be to estimate age-specific mortality and mobility rates by

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*The generalization of two single-state methods assuming identical curves of mortality and mobility forces are possible:

(1) a method iterating to the "data" analogous to the method proposed by Keyfitz (1968, Chapter 1), and

(2) a method extending that of Keyfitz and Frauenthal (1975). Although no attempt to evaluate and compare the validity of these two methods was undertaken, it can be said that the former alternative is feasible, whereas the latter, studied by Krishnamboodiri (1977) is likely to lead to highly inaccurate results. The rationale for this a priori judgement is that the curves of instantaneous mobility forces encountered in multistate models are not as nicely shaped as the curves of instantaneous forces of mortality in the single-state life table.

state of birth for the actual population and to equate them to their life table counterparts. Unfortunately, for most choices of the integration method for deriving $\{L_x\}$, this would yield age-specific survival probabilities different for each one of the n elementary stationary populations since the age-specific rates of these populations are not independent.

Under these conditions, the only practical way to proceed is to reduce the generality of the increment-decrement life table by further assuming that all types of moves out of each state are evenly distributed and that the typical distribution is independent of the state allocation of the initial cohort. This is equivalent to imposing identical life table age-specific rates in each elementary stationary population.*

On imposing the above restriction, the calculation of movement increment-decrement life tables is greatly simplified since:

- the equality of the life table and observed rates of mortality and mobility no longer raises a problem, and
- matrix generalizations of vector equations such as (38), and (54), now hold.

From there, the applied calculation of multistate life table functions still requires a method of integration for deriving $\{L_x\}$. The most common way to proceed is to assume a uniform distribution of these moves over time (linear integration). The columns of increment-decrement life tables directly follow from the application of formulas that pertain to the linear case in which the matrices of age-specific life table rates are set equal to their observed counterparts.

Two of the most popular alternatives to the linear integration method are, in the single-state case, a cubic integration method and an interpolative-iterative procedure. Can these methods be extended to the standard approach of the multistate case?

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*Indeed, no such restriction has to be imposed in the single radix case.

Case of a Cubic Integration Method for Deriving $\{L_x\}$

Schoen and Nelson (1974) have proposed to perform the integration of $\{L_x\}$ from a third-degree curve through values $\{l_{x-T}\}$, $\{l_x\}$, $\{l_{x+T}\}$ and $\{l_{x+2T}\}$.

$$\{L_x\} = \frac{13T}{24}\,[\{l_x\} + \{l_{x+T}\}] - \frac{T}{24}\,[\{l_{x-T}\} + \{l_{x+T}\}] \quad . \; (70)^*$$

In the first step, they compute initial values of the l- vectors using the linear integration method. Plugging these estimates into (70), they obtain new estimates of $\{L_x\}$, which lead to new estimates of the l- vectors by using (38). These new estimates of $\{l_x\}$ lead to improved estimates of $\{L_x\}$. The procedure is repeated until convergence of the l-estimates.

As such, the integration method proposed by Schoen and Nelson raised some important problems. On the one hand, Schoen and Nelson do not indicate what is the appropriate state allocation of the initial cohort necessary to begin the iterative procedure. The reason is that their focus on marriage and divorce analysis causes everybody to be born in the same state (the state of being single), so that their system has a unique multistate stationary population that can be characterized by vectors only, instead of matrices as in the multiradix case.

Is their method applicable to the multiradix case? The answer to this question follows from our previous development on the linkage between life table and observed populations: if one is willing to assume the validity of (70) for any choice of $\{l_0\}$ (i.e., to fix the constancy of the life table rates that are assumed equal to their discrete counterparts), then the cubic integration method applies to the multiradix case as well, thus validating the matrix generalization of (70).

On the other hand, Schoen and Nelson indicate how to find $\{l_x\}$, $\{L_x\}$, $\{T_x\}$ and $\{ae_x\}$ but give no hint of how to find $\underset{\sim}{p}_x$, ${}_x\underset{\sim}{e}_x$ (and ${}_0\underset{\sim}{e}_x$), ${}_x\underset{\sim}{n}_x$ (or ${}_0\underset{\sim}{n}_x$). These can, however, be found as

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*This general formula is not valid for the first, next to the last, and last age groups.

follows. In theory, the availability of $\{l_x\}$, $\{L_x\}$ and therefore that of ${}_{0}\underset{\sim}{l}_{x}$ and ${}_{0}\underset{\sim}{L}_{x}$ allows for a direct calculation of ${}_{x}\underset{\sim}{e}_{x}$ $({}_{0}\underset{\sim}{e}_{x})$, ${}_{x}\underset{\sim}{n}_{x}$ $({}_{0}\underset{\sim}{n}_{x})$, by using the formulas that express these functions in terms of $\underset{\sim}{m}_{x}$, ${}_{0}\underset{\sim}{l}_{x}$ and ${}_{0}\underset{\sim}{L}_{x}$. The age-specific survival probabilities could be obtained from.

$$\underset{\sim}{p}_{x} = {}_{0}\underset{\sim}{l}_{x+T} \; {}_{0}\underset{\sim}{l}_{x}^{-1} \quad .$$

However, these calculations can be performed only if ${}_{0}\underset{\sim}{l}_{x}$ is invertible, which is not the case if a whole column of ${}_{0}\underset{\sim}{l}_{x}$ consists of zeros, i.e., if at least one state is initially empty. As indicated above, one could then reduce the $\underset{\sim}{l}$ matrices to invertible r by r matrices (where r is the number of states that initially are not empty) and apply to them the above formulas. Unfortunately, this would yield only the requested multistate functions of the states that are initially not empty.*

To summarize, as proposed by Schoen and Nelson, the cubic integration method for deriving $\{L_x\}$ is feasible (any choice of $\{l_0\}$ will lead to the correct multistate stationary populations). However the estimates of all multistate life table functions can be obtained only when no state is initially empty.

An Interpolative-Iterative Procedure

An interpolative-iterative procedure for calculating a more accurate single-state life table by presenting a finite approximation of the continuous-time process underlying such a table, was developed by Keyfitz (1968, Chapter 11). The application of this method to the multistate case was first suggested by

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*It naturally comes to mind that one could calculate the multistate functions attached to all states by estimating those functions related to exact age or age group x from the formulas that express these functions in terms of $\underset{\sim}{m}_{x}$, ${}_{x}\underset{\sim}{l}_{x}$ and ${}_{x}\underset{\sim}{L}_{x}$ (rather than ${}_{0}\underset{\sim}{l}_{x}$ and ${}_{0}\underset{\sim}{L}_{x}$). Indeed this requires the knowledge of ${}_{x}\underset{\sim}{l}_{x}$ and ${}_{x}\underset{\sim}{L}_{x}$ that could perhaps be computed with the means of the method used to calculate ${}_{0}\underset{\sim}{l}_{x}$ and ${}_{0}\underset{\sim}{L}_{x}$. However, in contrast to the linear case, the values of ${}_{x}\underset{\sim}{L}_{x}$ thus obtained would not be consistent with those of ${}_{0}\underset{\sim}{L}_{x}$.

Oechsli (1972, 1975) in a study of the parity and nuptiality problem and later used by Ledent/Rogers (1972) in the context of interregional migration.

Fundamentally, the calculation procedure in the multistate case is based on the graduation of the mortality and mobility curves to small intervals,(possibly using a linear *interpolation* between "pivotal" values except for the first age group),* and the process of *iterating* to the data (Keyfitz 1968).

Suppose now that each age group (x, x + T) is divided into equal h-year segments (whose number amounts to T/h) and that for each one of these age groups are available:

- a matrix $\underset{\sim}{m}_x$ of movement rates relating to the whole interval, and
- initial estimates of the matrices ${}_h\underset{\sim}{m}_y$ of mortality and mobility rates characteristic of each k-year period (y, y + h) and obtained by an approximate interpolative method (Ledent/Rogers 1972).

Thus, it is possible to obtain the evolution (between ages x and x + T) of the survivors of the initial cohort from

$$ {}_0\underset{\sim}{l}_{y+h} = {}_h\underset{\sim}{p}_y \; {}_y\underset{\sim}{l}_y \qquad \forall y \text{ such that } x \leq y \leq x + T - h $$

in which ${}_h\underset{\sim}{p}_x$ could be given by (58).

However, here we take advantage of the fact that h is small enough to forbid multiple movements, so that ${}_h\underset{\sim}{p}_y$ is given by**

$$ {}_h\underset{\sim}{P}_y = \left[\underset{\sim}{I} - \frac{h}{2}\left({}_h\overset{\delta}{\underset{\sim}{m}}_y + {}_h\overset{mt}{\underset{\sim}{m}}_y - {}_h\overset{m}{\underset{\sim}{m}}_y\right)\right]\left[\underset{\sim}{I} + \frac{h}{2}\left({}_n\overset{\delta}{\underset{\sim}{m}}_y + {}_h\overset{mt}{\underset{\sim}{m}}_y\right)\right]^{-1} \qquad (71) $$

---

*In the case of the first age-group an interpolation of the mortality curve can be obtained by supposing that l(y) is an hyperbola (Keyfitz 1968).

**The demonstration is analogous to the one underlying the derivation of $\underset{\sim}{p}_x$ in the transition approach (see Section III).

in which ${}_{h}\underset{\sim}{\overset{\delta}{m}}$ is a diagonal matrix whose typical element is ${}^{i}_{h}m^{\delta}$, ${}_{h}\underset{\sim}{\overset{mt}{m}}$ is a diagonal matrix whose typical element is $\sum_{k\neq i} {}^{i}_{h}m^{k}$ and ${}_{h}\underset{\sim}{\overset{m}{m}}$ a matrix whose diagonal is zero and the off diagonal elements are age specific migration rates.*

The smallness of the age interval (y, y + h) also makes possible the use of the following linear integration:

$$ {}^{h}_{0}\underset{\sim}{L}_{y} = \frac{h}{2}[{}_{0}\underset{\sim}{l}_{y} + {}_{0}\underset{\sim}{l}_{y+h}] \qquad \forall y \text{ such that } x < y < x + T - h \quad . $$

This leads to an estimate of ${}_{0}\underset{\sim}{L}_{x}$ obtained from:

$$ {}_{0}\underset{\sim}{L}_{x} = \sum_{y=x}^{x+T-h} {}^{h}_{0}\underset{\sim}{L}_{y} \quad . $$

Assuming independence of the movement rates from the initial radix (as for the linear and cubic integration), we can then obtain estimates of their values in the synthetic population just constructed from

$$ \underset{\sim}{m}^{*}_{x} = ({}_{0}\underset{\sim}{l}_{x} - {}_{0}\underset{\sim}{l}_{x+T})\ {}_{0}\underset{\sim}{L}_{x}^{-1} \tag{72} $$

In general m^{*}_{x} will not coincide with the available estimates of m_{x}. We will obtain improved estimates of ${}_{h}m_{x}$ from:

$$ {}^{i-\delta}_{h}m_{y} = {}^{i}_{h}m^{\delta}_{y}\ \frac{{}^{i}m^{\delta}_{x}}{{}^{i}m^{\delta *}_{x}} \qquad \forall y \text{ such that } x \leq y \leq x + T - h $$

and

$$ {}^{i-j}_{h}m_{y} = {}^{i}_{h}m^{j}_{y}\ \frac{{}^{i}m^{j}_{x}}{{}^{i}m^{j *}_{x}} \qquad \begin{array}{l} \forall y \text{ such that } x \leq y \leq x + T - h \\ \forall j \neq i \end{array} $$

---

*Note that ${}_{h}\underset{\sim}{\overset{\delta}{m}}_{y} + {}_{h}\underset{\sim}{\overset{mt}{m}}_{y} = {}_{h}\underset{\sim}{m}_{y} + {}_{h}\underset{\sim}{\overset{m}{m}}_{y}$.

and consequently revised estimates of ${}_h\underset{\sim}{p}_y$ by plugging the new estimates of ${}_h\underset{\sim}{m}_x$ into (71). This then allows us to compute new estimates of ${}_0\underset{\sim}{l}_y$, ${}_0^h\underset{\sim}{L}_y$ ($\forall y$ such that $x \leq y \leq x + T - h$) that leads to new values of $\underset{\sim}{m}^*_x$ and so forth. The process is continued until convergence of the estimates of the transition rates in the synthetic population to those contained in $\underset{\sim}{m}_x$.

Note that the procedure just outlined generalizes Oechsli's methodology to the multiradix case.*

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*Oechsli's methodology was defined for a single radix system: in this case, the above matrices reduce to vectors and (72) to:

$$ {}^i m_x^j = \frac{{}^i d_x^j}{L_x^i} \qquad \begin{array}{l} \forall i = 1,\ldots,n \\ \forall j = 1,\ldots,n \\ j \neq i \end{array} $$

in which ${}^i d_x^j = \sum\limits_{y=x}^{x+T-h} {}_h^i p_y^j \; l_y^i$

III. THE TRANSITION APPROACH

The columns of an increment-decrement life table can be derived from the prior knowledge of the survival probabilities $\underset{\sim}{p}_x$ (rather than from that of the mortality and mobility rates contained in $\underset{\sim}{m}_x$).

A priori, such survival probabilities can be determined by simply comparing the individual's state of presence at the start and end of each time interval. The advantages of such a procedure are: first, the ability to deal separately with individuals present in the system,since the age-specific transition probabilities out of each region can be separately obtained, and second, the ability to limit the data requirements because observation of all the moves made by individuals within each age interval is not necessary.

The purpose of this section is to discuss how such survival probabilities can be obtained. This question is subsequently examined in both continuous and discrete settings.

A Continuous-time Exposition of the Transition Approach

As a consequence of the focus on transitions (changes in the states of presence between two fixed ages), the present approach introduces an additional time dimension so that the mortality and mobility patterns can only be studied as a continuous time process within each predetermined age interval (again assumed to be T years long except for the last age interval).

Derivation of the Age-specific Survival Probabilities

As indicated above, it is possible to study separately the subsequent evolution of each state-specific group of individuals surviving at a given age x. Here it is sufficient to examine such an evolution over a T-year span, concerning the groups of individuals l_x^i present at age x in state i.

The survivors of the cohort l_x^i at age y $(x \leq y \leq x + T)$ can be present in either state of the system. Let ${}_{ix}l^k(y)$ be the total number of survivors in either state at age y. The corresponding individuals in each state are subject to the three types of demographic events described in the standard approach. In particular, during a short time interval dy = dt, assumed to be small enough to rule out the possibility of multiple moves, the members of ${}_{ix}l^k(y)$ generate deaths, denoted as ${}^{k}_{ix}d^{\delta}(y)$ as well as movements to the rest of the system, denoted as

$$\sum_{\substack{k=1 \\ l \neq k}}^{n} {}^{k}_{ix}d^{l}(y) \quad .$$

Unlike the standard approach in which deaths occur according to state-specific mortality patterns, the transition approach recognizes an identical mortality pattern for all the survivors at any age y $(x \leq y \leq x + T)$ of each of the state-specific groups l^i_x.

Observe that the exposure of the cohort ${}_{ix}l^k(y)$ over the period (t, t + dt) to the risk of death is ${}_{ix}l^k(y)dy$. Thus the age-specific death rate of the numbers of the cohort l^i_x surviving to age y in region k is between ages y and y + dy,

$$\frac{{}^{k}_{ix}d^{\delta}(y)}{{}_{ix}l^{k}(y)dy} \quad .$$

The existence of a unique mortality pattern for all survivors of the cohort l^i_x then leads to the following series of equalities:

$$\frac{{}^{1}_{ix}d^{\delta}(y)}{{}_{ix}l^{1}(y)} = \frac{{}^{2}_{ix}d^{\delta}(y)}{{}_{ix}l^{2}(y)} = \ldots = \frac{{}^{i}_{ix}d^{\delta}(y)}{{}_{ix}l^{i}(y)} \ldots = \frac{{}^{n}_{ix}d^{\delta}(y)}{{}_{ix}l^{n}(y)} \quad . \quad (73)$$

Each term of (73) is also equal to the ratio of the sum of all numerators to the sum of all denominators, i.e.,

$$\frac{\sum_{k=1}^{n} {}^{k}_{ix}d^{\delta}(y)}{\sum_{k=1}^{n} {}_{ix}l^{k}(y)} \quad .$$

The numerator of this ratio is the total number of deaths occurring between ages y and y + dy to the survivors of l_x^i, i.e., ${}_{ix}l^{\cdot}(y) - {}_{ix}l^{\cdot}(y + dy)$, a quantity that we denote by ${}^i(ad)_x^{\delta}(y)$. We can rewrite the age-specific death rate as:

$$\frac{{}^i(ad)_x(y)}{{}_{ix}l^{\cdot}(y)dy} = \frac{{}_{ix}l^{\cdot}(y) - {}_{ix}l^{\cdot}(y + dy)}{{}_{ix}l^{\cdot}(y)dy} .$$

We then can define the instantaneous death rates or forces of mortality,* attached to the survivors of cohort l_x^i, as the limiting value of the above rate when dy → 0

$$ {}^i\hat{\mu}_x^{\delta}(y) = \lim_{dy \to 0} \frac{{}^i(ad)_x(y)}{{}_{ix}l^{\cdot}(y)dy} = \lim_{dy \to 0} \frac{{}_{ix}l^{\cdot}(y) - {}_{ix}l^{\cdot}(y + dy)}{{}_{ix}l^{\cdot}(y)dy}$$

$$\forall i = 1,\ldots,n \quad . \quad (74)$$

We observe that the net change, between ages y to y + dy in the number of individuals, members of l_x^i, who are present in state j is simply ${}_{ix}l^j(y + dy) - {}_{ix}l^j(y)$, a quantity that we denote by ${}^i(ad)_x^j(y)$. We can define the "apparent" instantaneous rate of mobility from state i to state j, attached to the cohort l_x^i, as the limiting value of this rate when dy → 0.

$$ {}^i\hat{\mu}_x^j(y) = \lim_{dy \to 0} \frac{{}^i(ad)_x^j(y)}{{}_{ix}l^i(y)dy} = \lim_{dy \to 0} \frac{{}_{ix}l_x^j(y + dy) - {}_{ix}l_x^j(y)}{{}_{ix}l^i(y)dy}$$

$$\forall i,j = 1,\ldots,n \quad (75)$$
$$j \neq i$$

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*The symbol used here to denote transition forces of mortality and mobility is identical to the one used to denote the movement forces of mobility and mortality, but a caret is here added to indicate the difference in the origin of these forces. A subscript is also added to make clear which age interval these forces relate to.

Once ${}^{i}\hat{\mu}_{x}^{\delta}(y)$ and ${}^{i}\hat{\mu}_{x}^{j}(y)$ (for all $j = 1,\dots,n, j \neq i$) are defined, the force of retention ${}^{i}\hat{\mu}_{x}^{i}(y)$ is simply obtained from the following equation expressing that the instantaneous process of this approach is conservative (Chiang 1968):

$$
{}^{i}\hat{\mu}_{x}^{i}(y) + {}^{i}\hat{\mu}_{x}^{\delta}(y) + \sum_{\substack{j=1 \\ j \neq i}}^{n} {}^{i}\hat{\mu}_{x}^{j}(y) = 0 \quad ,
$$

or alternatively:

$$
{}^{i}\hat{\mu}_{x}^{i}(y) = - [{}^{i}\hat{\mu}^{\delta}(y) + \sum_{\substack{j=1 \\ j \neq i}}^{n} {}^{i}\hat{\mu}_{x}^{j}(y)] \qquad i = 1,\dots,n. \quad (76)
$$

The specification of the transition model immediately follows from the above definitions. First we have the following equation indicating the decrements to the group ${}_{ix}l^{\cdot}(y)$ between ages y and y + dy:

$$
{}_{ix}l^{\cdot}(y + dy) = {}_{ix}l^{\cdot}(y) - {}^{i}(ad)_{x}^{\delta}(y) \qquad \forall i = 1,\dots,n \quad (77)
$$

Second, we have the (n - 1) equations indicating the changes experienced between ages y and y + dy by the group ${}_{ix}l^{j}(y)$

$$
{}_{ix}l^{j}(y + dy) = {}_{ix}l^{j}(y) + {}^{i}(ad)_{x}^{j}(y)^{*} \qquad \forall i,j = 1,\dots,n \quad j \neq i . \quad (78)
$$

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*Note that ${}^{i}(ad)_{x}^{j}(y)$ is a number of additional transitions from state i to state j between ages y and y + dy, representing a compounded effect of moves out of state j$[\sum_{k=1}^{n} {}_{ix}^{j}d^{k}(y)]$ and into state j$[\sum_{k=1}^{n} {}_{ix}^{k}d^{j}(y)]$ as well as deaths occurring in state j$[{}_{ix}^{j}d^{\delta}(y)]$.

By subtracting the (n - 1) equations contained in (78) from (77) we obtain the equation showing the decrements and increments to the exposed group ${}_{ix}l^{i}$

$$ {}_{ix}l^{i}(y + dy) = {}_{ix}l^{i}(y) - {}^{i}(ad)^{\delta}_{x}(y) - \sum_{\substack{j=1 \\ j\neq i}}^{n} {}^{i}(ad)^{j}_{x}(y) \qquad \forall i = 1,\ldots,n \qquad (79) $$

Substituting (74) and (75) into the equations (78) and (79) leads to

$$ {}_{ix}l^{i}(y + dy) = {}_{ix}l^{i}(y) - {}^{i}\hat{\mu}^{\delta}_{x}(y)\,{}_{ix}l^{\cdot}(y)dy - \left[\sum_{\substack{j=1 \\ j\neq i}}^{n} {}^{i}\hat{\mu}^{j}_{x}(y)\right]{}_{ix}l^{i}(y)dy \qquad \forall i = 1,\ldots,n $$

$$ {}_{ix}l^{j}(y + dy) = {}_{ix}l^{i}(y) + {}^{i}\hat{\mu}^{j}_{x}(y)\,{}_{ix}l^{i}(y)dy \qquad \forall \underset{j\neq i}{i,j} = 1,\ldots,n $$

or more compactly, in matrix format,

$$ \{{}_{ix}l(y + dy)\} = \{{}_{ix}l(y)\} - {}^{i}\underset{\sim}{\hat{\mu}}_{x}(y)\{{}_{ix}l(y)\}dy \qquad (80) $$

where:

$$ {}^{i}\underset{\sim}{\hat{\mu}}_{x}(y) = \left[\begin{array}{ccc} 0 & -\,{}^{i}\hat{\mu}^{1}_{x}(y) & 0 \\ {}^{i}\hat{\mu}^{\delta}_{x}(y)\ldots & {}^{i}\hat{\mu}^{\delta}_{x}(y) + \sum_{\substack{j=1 \\ j\neq i}}^{n} {}^{i}\hat{\mu}^{j}_{x}(y) & \ldots{}^{i}\hat{\mu}^{\delta}_{x}(y) \\ 0 & \underbrace{-\,{}^{i}\hat{\mu}^{n}_{x}(y)}_{\text{ith column}} & 0 \end{array}\right] \begin{array}{l} \\ \left.\vphantom{\sum_{j=1}^{n}}\right\}\text{ith row} \\ \\ \end{array} \qquad (81) $$

The matrix ${}^{i}\hat{\underset{\sim}{\mu}}_x(y)$ is the sum of two matrices: the first one contains mortality elements (in the i^{th} row) and the second one consists of migration elements (in the i^{th} column).

Since, by definition, $\{{}_{ix}l(y + dy)\} - \{{}_{ix}l(y)\} = d\{{}_{ix}l(y)\}$, we may rewrite (80) as:

$$\frac{d\{{}_{ix}l(y)\}}{dy} = - {}^{i}\hat{\underset{\sim}{\mu}}_x(y)\ \{{}_{ix}l(y)\} \qquad (82)$$

An integral matrix of this system

$${}_{ix}l(y) = [\{{}_{ix}l(y)\}_i,\ldots,\{{}_{ix}l(y)\}_n]$$

is such that each column verifies equation (82). From the theorem on the existence and uniqueness of the solution of such a system, it follows that ${}_{ix}\underset{\sim}{l}(y)$ is determined when the value of ${}_{ix}\underset{\sim}{l}(y)$ for some initial value $y = x$ is known, say ${}_{ix}\underset{\sim}{l}(x)$:

$${}_{ix}\underset{\sim}{l}(y) = {}^{i}\hat{\Omega}_x(y)\,{}_{ix}l(x) \qquad \forall i = 1,\ldots,n$$

in which the matrix ${}^{i}\hat{\Omega}_x(y)$, uniquely defined as the normalized solution of (82), is not a simple expression of the ${}^{i}\hat{\underset{\sim}{\mu}}_x$'s. Again, ${}^{i}\hat{\underset{\sim}{\Omega}}_x(y)$ could be determined by using the infinitesimal calculus of Volterra. However, this determination is not necessary as shown below. Since ${}_{ix}\underset{\sim}{l}(x)$ is a zero matrix except for the $(i,i)^{th}$ element, (82) has a unique solution

$$\{{}_{ix}l(y)\} = {}^{i}\underset{\sim}{\Pi}_x(y)\{{}_{ix}l(x)\} \qquad \forall i = 1,\ldots,n \qquad (83)$$

in which ${}^{i}\underset{\sim}{\Pi}_x(y)$ is a matrix whose elements are zero except for the i^{th} column denoted by $\{{}^{i}\Pi_x(y)\}$ and identical to the i^{th} column of the matrix ${}^{i}\hat{\Omega}_x(y)$. These non-zero elements will be determined later, using simple calculus.

Because of the following identity:

$$\{l(y)\} = \sum_{i=1}^{n} \{_{ix}l(y)\} \quad ,$$

we have:

$$\{l(y)\} = \sum_{i=1}^{n} \{_{ix}l(y)\} = \sum_{i=1}^{n} {}^{i}\underset{\sim}{\Pi}_{x}(y)\{_{ix}l(x)\} \quad .$$

(83) can also be rewritten in matrix form as

$$_{x}\underset{\sim}{l}(y) = \underset{\sim}{\Pi}_{x}(y)\,_{x}\underset{\sim}{l}_{x}$$

$$\Pi_{x}(y) = [\{^{1}\Pi_{x}(y)\}, \cdots \{^{j}\Pi_{x}(y)\}, \cdots \{^{n}\Pi_{x}(y)\}]$$

and $_{x}\underset{\sim}{l}_{x}$ is a diagonal matrix allocating the survivors of $l_{0}^{\cdot}$ at age x.

Clearly, the probability ${}^{i\hat{}j}\mu_{x}$ that an individual present at age x in state i will survive in state j, T years later, is the $(j,i)^{th}$ element of the matrix $\underset{\sim}{\hat{p}}_{x} = \underset{\sim}{\Pi}_{x}(x + T)$.

In contrast to the movement case, we can express the elements of $\hat{p}_{x}$ as functions of the forces of mortality and mobility using simple calculus. Equation (77) can be rewritten as:

$$_{ix}l^{\cdot}(y + dy) = {}_{ix}l^{\cdot}(y) - {}^{i\hat{}\delta}\mu_{x}(y)\ {}_{ix}l^{\cdot}(y)dy \qquad \forall i = 1,\ldots,n$$

whose integral solution is:

$$_{ix}l^{i}(y) = e^{-\int_{0}^{y} {}^{i\hat{}\delta}\mu_{x}(x+t)dt}\ l_{x}^{i} \qquad \forall i = 1,\ldots,n \quad (84)$$

It follows that the probability ${}^{i}\hat{p}^{\cdot}_{x}$ for a member of l^{i}_{x} to survive in any state of the system at age x + T is

$$ {}^{i}\hat{p}^{\cdot}_{x} = e^{-\int_{0}^{T} {}^{i}\hat{\mu}^{\delta}_{x}(x+t)\,dt} \qquad \forall i = 1,\dots,n \quad (85)$$

while the corresponding probability of dying in any state before reaching age x + T is:

$$ {}^{i}\hat{p}^{\delta}_{x} = 1 - e^{-\int_{0}^{T} {}^{i}\hat{\mu}^{\delta}_{x}(x+t)\,dt} \quad . \qquad \forall i = 1,\dots,n \quad (86)$$

Substituting (84) into (78) leads to the following differential equation, permitting us to determine ${}_{ix}l^{i}(y)$

$$ \frac{d\,{}_{ix}l^{i}(y + dy)}{dy} = - {}^{i}\hat{\mu}^{\delta}_{x}(y)\; e^{-\int_{0}^{y-x} {}^{i}\hat{\mu}^{\delta}_{x}(x+t)\,dt}\; l^{i}_{x} - \Big[\sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i}\hat{\mu}^{j}_{x}(y)\Big]\,{}_{ix}l^{i}(y)$$

$$\forall i = 1,\dots,n \quad (87)$$

whose solution clearly has the general specification:

$$ {}_{ix}l^{i}(y) = e^{-\int_{0}^{y-x} \sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i}\hat{\mu}^{j}_{x}(x+t)\,dt}\; {}^{i}A(y) \quad .$$

Substituting this general solution into (87) yields

$$ e^{-\int_{0}^{y-x} \sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i}\hat{\mu}^{j}_{x}(x+t)\,dt}\; \frac{d\,{}^{i}A(y)}{dy} = - {}^{i}\hat{\mu}^{\delta}_{x}(y)\; e^{-\int_{0}^{y-x} {}^{i}\hat{\mu}^{\delta}_{x}(x+t)\,dt}\; l^{i}_{x} \quad .$$

Thus, we have

$${}^{i}A(y) = - \int_{0}^{y-x} {}^{i\hat{}\delta}\mu_{x}(x+\theta)\; e^{-\int_{0}^{\theta} \left[{}^{i\hat{}\delta}\mu(x+t) - \sum\limits_{\substack{j=1\\ j\neq i}}^{n} {}^{i\hat{}\delta}\mu(x+t)\right] dt}\, d\theta$$

and finally

$${}_{ix}l^{i}(y) = \left[e^{\int_{o}^{y-x} \sum\limits_{\substack{j=1\\ j\neq i}}^{n} {}^{i\hat{}j}\mu(x+t)\,dt} \int_{0}^{y-x} (- {}^{i\hat{}\delta}\mu_{x}(x+\theta) \; e^{-\int_{0}^{\theta} [{}^{i\hat{}\delta}\mu(x+t) - \sum\limits_{\substack{j=1\\ j\neq i}}^{n} {}^{i\hat{}j}\mu(x+t)]dt}\, d\theta \right] l_{x}^{i}$$

$$\forall i = 1,\ldots,n \qquad (88)$$

The result is that the probability ${}^{i\hat{}i}p_{x}$ for a member of l_{x}^{i} to survive in state i of the system at age x + T is the expression between brackets in (88) in which y - x would be replaced by T.

Next, the probabilities ${}^{i\hat{}j}p_{x}$ for ($j\neq i$) could be obtained by substituting (88) into equation (78) and solving the ensuing differential equation. Their expression, involving the use of several integral signs, is not reported here since it does not add any special insights into the transition process.*

Age-specific Mortality and Mobility Rates

The age-specific transition rate ${}^{i}\hat{m}_{x}^{\delta}$, the discrete counterpart of ${}^{i}\hat{\mu}_{x}^{\delta}(y)$, is defined as the ratio of the number of deaths occurring to the members of l_{x}^{i} between ages x and x + T to the exposed population,

$$ {}^{i}\hat{m}_{x}^{\delta} = \frac{{}^{i}(ad)_{x}^{\delta}}{{}_{ix}L_{x}^{\bullet}} \quad , \qquad \forall i = 1,\ldots,n \quad . \tag{89}$$

From the definition (74) of the instantaneous rate ${}^{i}\hat{\mu}_{x}^{\delta}(y)$, it follows that the number of transitions ${}^{i}(ad)_{x}^{j}$ is equal to $\int_{0}^{T} {}^{i}\hat{\mu}_{x}^{\delta}(x+t)\,{}_{ix}l^{\bullet}(x + t)dt$. Then, recalling the definition of ${}_{ix}L_{x}^{\bullet}$

$$ {}^{i}\hat{p}_{x}^{j} = \int_{o}^{T} {}^{i}\hat{\mu}_{x}^{j}(x + u)\; e^{-\int_{o}^{u}\left[{}^{i}\hat{\mu}_{x}^{\delta}(x+\theta) + \sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i}\hat{\mu}_{x}^{j}(x+\theta)\right] d\theta} du \tag{90}$$

$$ {}^{i}\hat{p}_{x}^{i} = e^{-\int_{o}^{T}\left[{}^{i}\hat{\mu}_{x}^{\delta}(x+u) + \sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i}\hat{\mu}_{x}^{j}(x+u)\right] du} \tag{91}$$

$$ {}^{i}\hat{p}_{x}^{\delta} = \int_{o}^{T} {}^{i}\hat{\mu}_{x}^{\delta}(x + u)\; e^{-\int_{o}^{u}\left[{}^{i}\hat{\mu}_{x}^{\delta}(x+\theta) + \sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i}\hat{\mu}_{x}^{j}(x+\theta)\right] d\theta} du \tag{92}$$

---

*Note that the age-specific survival probabilities obtained here differ from those obtained by Hoem (1970) and Ledent (1972) who have assumed a more restrictive hypothesis (no more than one movement allowed over each time interval).

and substituting into the above definition yields

$$ {}^{i\hat{}\delta}m_x = \frac{\int_0^T {}^{i\hat{}\delta}\mu(x+t)\,{}_{ix}l^{\cdot}(x+t)\,dt}{\int_0^T {}_{ix}l^{\cdot}(x+t)\,dt} \qquad \forall i = 1,\ldots,n \ . \qquad (93) $$

In a similar way, we can define the age-specific transition rate ${}^{i\hat{}j}m_x$, the discrete counterpart of ${}^{i\hat{}j}\mu_x(y)$, as the following ratio:

$$ {}^{i\hat{}j}m_x = \frac{{}^{i}(ad)^{j}_{x}}{{}_{ix}L^{i}_{x}} \qquad \forall i,j = 1,\ldots,n \quad j \neq i \qquad (94) $$

that, using continuous functions, can be rewritten as:

$$ {}^{i\hat{}j}m_x = \frac{\int_0^T {}^{i\hat{}j}\mu(x+t)\,{}_{ix}l^{i}(x+t)\,dt}{\int_0^T {}_{ix}l^{i}(x+t)\,dt} \qquad \forall i,j = 1,\ldots,n \quad j \neq i \ . \qquad (95) $$

It is clear that, unlike the age-specific mortality and mobility rates of the movement approach, the rates just defined do not depend on the state allocation of the initial cohort: this is merely a consequence of the independent evolution of the survivors of the various groups l^i_x over the next T years.

A corollary of this property is that it is possible to derive the age-specific migration rates from the knowledge of the l and L functions.

The discrete equivalents to the elementary flow equations (78) and (79) can be rewritten as:

$$ {}_{ix}l^{j}_{x+T} = {}^{i}(ad)^{j}_{x} \qquad \forall i,j = 1,\ldots,n \quad j \neq i \ , \qquad (96) $$

and

$$ {}_{ix}l^{i}_{x+T} = {}_{ix}l^{i}_{x} - {}^{i}(ad)^{\delta}_{x} - \sum_{i=1}^{n} {}^{i}(ad)^{i}_{x} \quad , \quad \forall i = 1,\ldots,n \ . \tag{97} $$

Substituting the definition equations (89) and (94) leads to:

$$ {}_{ix}l^{j}_{x+T} = {}^{i}\hat{m}^{j}_{x}\ {}_{ix}L^{i}_{x} \qquad \forall i,j = 1,\ldots,n \quad j \neq i \tag{98} $$

$$ {}_{ix}l^{i}_{x+T} = {}_{ix}l^{i}_{x} - {}^{i}\hat{m}^{\delta}_{x}\ {}_{ix}L^{\cdot}_{x} - \sum_{j=1}^{n} {}^{i}\hat{m}^{j}_{x}\ {}_{ix}L^{i}_{x} \qquad \forall i = 1,\ldots,n \tag{99} $$

or, more compactly,

$$ \{{}_{ix}l_{x+T}\} = \{{}_{ix}l_{x}\} - {}^{i}\hat{\underset{\sim}{m}}_{x}\ \{{}_{ix}L_{x}\} \qquad \forall i = 1,\ldots,n, \tag{100} $$

in which ${}^{i}\hat{\underset{\sim}{m}}_{x}$ is the discrete counterpart of (81)

$$ {}^{i}\hat{\underset{\sim}{m}}_{x} = \begin{array}{c} \left[\begin{array}{ccc} 0 & {}^{i}\hat{m}^{j}_{x} & 0 \\ {}^{i}\hat{m}^{\delta}_{x} \cdots {}^{i}\hat{m}^{\delta}_{x} & {}^{i}\hat{m}^{j}_{x} + \left[\sum\limits_{\substack{j=1 \\ j\neq i}}^{n} {}^{i}\hat{m}^{j}_{x}\right] & {}^{i}\hat{m}^{\delta}_{x} \cdots {}^{i}\hat{m}^{\delta}_{x} \\ 0 & {}^{i}\hat{m}^{n}_{x} & 0 \end{array}\right] \begin{array}{c} \\ \left.\right\} i^{th} \text{ row} \\ \\ \end{array} \\ \underbrace{\qquad}_{\substack{i^{th} \\ \text{column}}} \end{array} \tag{101} $$

The expression of ${}^{i}\hat{\underset{\sim}{m}}^{\delta}_{x}$ and ${}^{i}\hat{\underset{\sim}{m}}^{j}_{x}$ (j≠i) in terms of the multistate functions is easily obtained from (98) and (99):

$$ {}^{i}\hat{m}^{\delta}_{x} = \frac{{}_{ix}l^{i}_{x} - {}_{ix}l^{\cdot}_{x+T}}{{}_{ix}L^{\cdot}_{x}} \qquad \forall i = 1,\ldots,n \ , $$

$$ {}^{i}\hat{m}^{j}_{x} = \frac{{}_{ix}l^{j}_{x+T}}{{}_{ix}L^{i}_{x}} \qquad \forall i,j = 1,\ldots,n \ . \quad j \neq i $$

In the case of the last age group, z years and over, ${}^{i}(ad)^{j}_{z} = 0(\forall j)$ follows from the fact that everybody ultimately dies and, consequently, ${}^{i}\hat{m}^{j}_{z} = 0 \ (\forall j \neq i)$ and ${}^{i}\hat{m}^{\delta}_{x} = \dfrac{{}_{iz}l^{i}_{z}}{{}_{iz}l^{\cdot}_{z}}$, (Rogers, 1973a).*

Applied Calculation of an Increment-decrement Life Table Based on the Transition Approach

The main idea in constructing such a table is to derive an expression of the age-specific survival probabilities in terms of the mortality and mobility rates and then to provide applied estimates of these probabilities by assuming identical rates in both the life table and observed populations (especially because there is here, in contrast to the movement approach, no problem in implementing such an assumption).

However, in opposition to the movement approach, the present approach does not permit deriving a generalized expression of the age-specific survival probabilities. Specific formulas must therefore be established for each particular choice of the integration method for calculating $\{{}_{ix}L_{x}\}$.**

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*The matrix of expectations of life ${}_{z}\underset{\sim}{e}_{z}$ is thus given by the same formula as in the standard approach, but the non-diagonal elements are here zeros since ${}^{i}\hat{m}^{j}_{z} = 0 \ (\forall j \neq i)$.

**There is no simple way of defining variables similar to the ${}^{i}a^{j}_{x}$ variables of the movement approach when the system contains more than two regions.

The Linear Case

Suppose that $\{{}_{ix}L_x\}$ is determined by the following linear approximation:

$$\{{}_{ix}L_x\} = \frac{T}{2}[\{{}_{ix}l_x\} + \{{}_{ix}l_{x+T}\}] \quad , \quad \forall i = 1,\dots,n \quad .$$

Thus:

$$\{{}_{ix}l_{x+t}\} = [\underset{\sim}{I} + \frac{T}{2}\,{}^{i\wedge}\underset{\sim}{m}_x]^{-1}[\underset{\sim}{I} - \frac{T}{2}\,{}^{i\wedge}\underset{\sim}{m}_x]\{{}_{ix}l_x\} \quad \forall i = 1,\dots,n$$

and the age-specific probabilities ${}^{i\wedge j}p_x$ (for $j = 1,\dots,n$) are contained in the i^{th} column of ${}^{i\wedge}\underset{\sim}{p}_x$ defined by:

$${}^{i\wedge}\underset{\sim}{p}_x = [\underset{\sim}{I} + \frac{T}{2}\,{}^{i\wedge}\underset{\sim}{m}_x]^{-1}[\underset{\sim}{I} - \frac{T}{2}\,{}^{i\wedge}\underset{\sim}{m}_x] \quad \forall i = 1,\dots,n \quad . \qquad (102)$$

Note the likeness of (102) with the corresponding formula (58a) obtained in the movement approach.

Since ${}^{i\wedge}\underset{\sim}{m}_x$ contains many zero entries, it is possible to compute the inverse of $[\underset{\sim}{I} + \frac{T}{2}\,{}^{i\wedge}\underset{\sim}{m}_x]^{-1}$ and therefore ${}^{i\wedge}\underset{\sim}{p}_x$. However the calculations are rather tedious. Fortunately, the simplicity of the scalar formulas (98) and (99) permit a direct derivation of the elements of the i^{th} column of ${}^{i\wedge}\underset{\sim}{p}_x$ in which we are interested.

Introducing the linear hypothesis in (98) and (99) yields

$$\begin{aligned} {}_{ix}l^i_{x+T} = {} & l^i_x - \frac{T}{2}\,{}^{i\wedge\delta}m_x[l^i_x + {}_{ix}l^i_{x+t}] - \frac{T}{2}\,{}^{i\wedge\delta}m_x\left[\sum_{\substack{j=1\\ j\neq i}} {}_{ix}l^j_{x+t}\right] \\ & - \frac{T}{2}\,[\sum_{\substack{j=1\\ j\neq i}} {}^{i\wedge j}m_x]\,[l^i_x + {}_{ix}l^i_{x+t}] \qquad \forall i = 1,\dots,n \end{aligned}$$

and

$${}_{ix}l^j_{x+T} = \frac{T}{2}\,{}^{i\wedge j}m_x\,[l^i_x + {}_{ix}l^i_{x+t}] \qquad \forall i,\underset{j\,\neq\, i}{j} = 1,\dots,n$$

Substituting (98) for ${}_{ix}l^{j}_{x+T}$ leads to a relationship linking ${}_{ix}l^{i}_{x+T}$ to l^{i}_{x}

$$\left(1 + \frac{T}{2}\,{}^{i\hat{}\delta}m_x + \frac{T^2}{4}\,{}^{i\hat{}\delta}m_x\left[\sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i\hat{}j}m_x\right] + \frac{T}{2}\sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i\hat{}j}m_x\right)\,{}_{ix}l^{i}_{x+T}$$

$$= \left(1 - \frac{T}{2}\,{}^{i\hat{}\delta}m_x - \frac{T^2}{4}\,{}^{i\hat{}\delta}m_x\left[\sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i\hat{}j}m_x\right] - \frac{T}{2}\sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i\hat{}j}m_x\right)\,l^{i}_{x}$$

Then, we have the probability of remaining in state i at age x + T as:

$${}^{i\hat{}i}p_x = \frac{1 - \frac{T}{2}\left[{}^{i\hat{}\delta}m_x + \sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i\hat{}j}m_x\right] - \frac{T^2}{4}\,{}^{i\hat{}\delta}m_x\left[\sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i\hat{}j}m_x\right]}{\left(1 + \frac{T}{2}\,{}^{i\hat{}\delta}m_x\right)\left(1 + \frac{T}{2}\sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i\hat{}j}m_x\right)} \qquad \forall i = 1,\ldots,n$$

or

$${}^{i\hat{}i}p_x = 1 - \frac{T\left[{}^{i\hat{}\delta}m_x + \sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i\hat{}j}m_x + \frac{T}{2}\,{}^{i\hat{}\delta}m_x\left[\sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i\hat{}j}m_x\right]\right]}{\left(1 + \frac{T}{2}\,{}^{i\hat{}\delta}m_x\right)\left(1 + \frac{T}{2}\sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i\hat{}j}m_x\right)} \qquad \forall i = 1,\ldots,n \qquad (103)$$

It follows that the probability of being in state j at age x + T is:

$${}^{i\hat{}j}p_x = \frac{T\,{}^{i\hat{}j}m_x}{\left(1 + \frac{T}{2}\,{}^{i\hat{}\delta}m_x\right)\left(1 + \frac{T}{2}\sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i\hat{}j}m_x\right)} \qquad \forall i,j = 1,\ldots,n \qquad (104)$$

and the probability of dying, in either state, between ages x and x + T, is

$$
{}^{i\wedge\delta}p_x = 1 - \sum_{j=1}^{n} {}^{i\wedge j}p_x = \frac{T\left[{}^{i\wedge\delta}m_x + \frac{T}{2}\,{}^{i\wedge\delta}m_x\left[\sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i\wedge j}m_x\right]\right]}{(1+\frac{T}{2}\,{}^{i\wedge\delta}m_x)(1+\frac{T}{2}\sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i\wedge j}m_x)}, \qquad \forall i = 1,\ldots,n
$$

which reduces to:

$$
{}^{i\wedge\delta}p_x = \frac{T\,{}^{i\wedge\delta}m_x}{1+\frac{T}{2}\,{}^{i\wedge\delta}m_x} \qquad \forall i = 1,\ldots,n \qquad (105)
$$

$$
{}^{i\wedge i}p_x = \frac{1-\frac{T}{2}[{}^{i\wedge\delta}m_x + \sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i\wedge j}m_x]}{1+\frac{T}{2}[{}^{i\wedge\delta}m_x + \sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i\wedge j}m_x]} \qquad \forall i = 1,\ldots,n \qquad (106)
$$

$$
{}^{i\wedge j}p_x = \frac{T\,{}^{ij}m_x}{1+\frac{T}{2}[{}^{i\wedge\delta}m_x + \sum_{\substack{j=1\\ j\neq i}}^{n} {}^{ij}m_x]} \qquad \forall i,j = 1,\ldots,n \quad j \neq i \qquad (107)
$$

$$
{}^{i\wedge\delta}p_x = \frac{T\,{}^{i\wedge\delta}m_x}{1+\frac{T}{2}[{}^{i\wedge\delta}m_x + \sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i\wedge j}m_x]} \qquad \forall i = 1,\ldots,n \qquad (108)
$$

which clearly constitute the discrete counterparts of (90) through (92).

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* Note that if we assume with Rogers (1975a) that people cannot make more than one move over a T-year period, the terms containing products of two rates drop out in (103) through (105) and yield the formulas obtained by Rogers (1975a).

The above scalar transition probabilities ${}^{i}\hat{p}^{j}_{x}$ can be collected into a matrix $\hat{\underset{\sim}{p}}_x$ similar to $\underset{\sim}{p}_x$:

$$\hat{\underset{\sim}{p}}_x = [\underset{\sim}{I} - \frac{T}{2}\,\underset{\sim}{u}_x][\underset{\sim}{I} + \frac{T}{2}\,\underset{\sim}{v}_x]^{-1} \qquad (109)$$

in which:

$$\underset{\sim}{u}_x = \left[\begin{array}{ccc} {}^{1}\hat{m}^{\delta}_{x} + (1 + \frac{T}{2}\,{}^{1}\hat{m}^{\delta}_{x}) \sum\limits_{\substack{j=1\\ j\neq i}}^{n} {}^{1}\hat{m}^{j}_{x} & -2\,{}^{2}\hat{m}^{1}_{x} & -2\,{}^{n}\hat{m}^{1}_{x} \\ -2\,{}^{1}\hat{m}^{2}_{x} & {}^{2}\hat{m}^{\delta}_{x} + (1 + \frac{T}{2}\,{}^{2}\hat{m}^{\delta}_{x}) \sum\limits_{\substack{j=1\\ j\neq i}}^{n} {}^{2}\hat{m}^{j}_{x} & \\ \vdots & & \ddots \\ -2\,{}^{1}\hat{m}^{n} & & \end{array}\right]$$

(110)

and $\underset{\sim}{v}_x$ a diagonal matrix whose diagonal is identical to that of $\underset{\sim}{u}_x$. Let $\hat{\underset{\sim}{m}}_x$ be a matrix of transition rates similar to the matrix $\underset{\sim}{m}_x$ of movement rates, previously defined,

$\overset{\delta}{\hat{m}}_x$ a diagonal matrix whose typical element is the transition mortality rate ${}^{i}\hat{m}^{\delta}_{x}$ and,

$\overset{mt}{\hat{\underset{\sim}{m}}}_x$ a diagonal matrix whose typical element is the total mobility rate out of state i, $\sum\limits_{\substack{j=1\\ j\neq i}}^{n} {}^{i}\hat{m}^{j}_{x}$.

The matrix $\overset{m}{\hat{\underset{\sim}{m}}}_x$ has zero diagonal elements and the (i,j)th off-diagonal element equal to ${}^{j}\hat{m}^{i}_{x}$.

We can rewrite (109) as:

$$\hat{\underset{\sim}{p}}_x = [\underset{\sim}{I} - \frac{T}{2}(\hat{\underset{\sim}{m}}_x - \overset{m}{\hat{\underset{\sim}{m}}}_x + \frac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_x \overset{mt}{\hat{\underset{\sim}{m}}}_x)][\underset{\sim}{I} + \frac{T}{2}(\hat{\underset{\sim}{m}}_x + \overset{m}{\hat{\underset{\sim}{m}}}_x + \frac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}\,\overset{mt}{\hat{\underset{\sim}{m}}}_x)]^{-1}, \text{(111)}^{*}$$

or, alternatively,

$$\hat{\underset{\sim}{p}}_x = \underset{\sim}{I} - T\,\hat{\underset{\sim}{m}}_x\,(\underset{\sim}{I} + \frac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_x)^{-1}(\underset{\sim}{I} + \frac{T}{2}\overset{mt}{\hat{\underset{\sim}{m}}}_x)^{-1} \quad . \quad **$$

A simple expression of $\{L_x\}$ follows after substituting (111) into $\{L_x\} = \frac{T}{2}[\underset{\sim}{I} + \hat{\underset{\sim}{p}}_x]\{l_x\}$:

$$\{L_x\} = T[\underset{\sim}{I} + \frac{T}{2}\overset{m}{\hat{\underset{\sim}{m}}}_x][\underset{\sim}{I} + \frac{T}{2}(\underset{\sim}{m} + \overset{m}{\hat{\underset{\sim}{m}}}_x + \frac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_x \overset{mt}{\hat{\underset{\sim}{m}}}_x)]^{-1}\{l_x\} \text{ .(112)}$$

Finally, we can derive an expression of $\hat{\underset{\sim}{s}}_x$ in terms of the age-specific cohort rates:

$$\hat{\underset{\sim}{s}}_x = [\underset{\sim}{I} + \frac{T}{2}\overset{m}{\hat{\underset{\sim}{m}}}_{x+T}][\underset{\sim}{I} + \frac{T}{2}(\hat{\underset{\sim}{m}}_{x+T} + \overset{m}{\hat{\underset{\sim}{m}}}_{x+T} + \frac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_{x+T}\overset{mt}{\hat{\underset{\sim}{m}}}_{x+T})]^{-1}$$

$$[\underset{\sim}{I} - \frac{T}{2}(\hat{\underset{\sim}{m}}_x - \overset{m}{\hat{\underset{\sim}{m}}}_x + \frac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_x \overset{mt}{\hat{\underset{\sim}{m}}}_x)][\underset{\sim}{I} + \frac{T}{2}\overset{m}{\hat{\underset{\sim}{m}}}_x]^{-1} , \qquad (113)$$

Compare (112) with (63) and (113) with (67).

---

*Compare (111) with (58). Note that the second quantity between brackets is a diagonal matrix that can be rewritten as

$$(\underset{\sim}{I} + \frac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_x)(\underset{\sim}{I} + \frac{T}{2}\overset{mt}{\hat{\underset{\sim}{m}}}_x) .$$

**In the case considered by Rogers (1973a, 1975a), $\hat{\underset{\sim}{p}}_x$ reduces to:

$$\hat{\underset{\sim}{p}}_x = \underset{\sim}{I} - T\,\hat{\underset{\sim}{m}}_x[\underset{\sim}{I} + \frac{T}{2}(\overset{\delta}{\hat{\underset{\sim}{m}}}_x - \overset{mt}{\hat{\underset{\sim}{m}}}_x)]^{-1} \quad .$$

Substituting observed life table rates into (109) provides estimates of the age-specific survival probabilities from which all other multistate functions can then be derived. In the case of the last age group, (100) becomes:

$$\{_{iz}l_z\} = {}^{i}\hat{\underset{\sim}{m}}_z\{_{iz}L_z\}$$

in which ${}^{i}\hat{\underset{\sim}{m}}_z$ is a zero matrix except for the i^{th} row whose elements are all equal to ${}^{i}\hat{m}_z^{\delta}$ (since ${}^{i}\hat{m}_z^{j} = 0 \quad \forall j \neq i$).

This last vector equality actually reduces to a unique scalar equation

$$l_z^i = {}^{i}\hat{m}_z^{\delta} \ {}_{iz}L_z^{\cdot} = {}^{i}\hat{m}_z^{\delta} \ (\sum_{j=i}^{n} {}_{iz}L_z^j) \quad ,$$

which is insufficient to determine ${}_{iz}L_z^j$ for all j = 1,...,n. Thus, the general assumptions embodied in this transition approach do not permit us to determine, from the availability of the transition mortality rates, the various numbers of person-years lived in each state.*

Consequently, only the movement approach allows for an exact calculation of the multistate life table functions of the last group.

Alternative Methods for Deriving $\{L_x\}$

In opposition to the movement approach, the transition approach cannot use the cubic integration method which requires the simultaneous consideration of different age groups. However, an interpolative-iterative is possible. Such a method adapting the general procedure developed by Oechsli (1972, 1975) to the

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*Because he supposes that no more than one move is made within each age interval (including the last one), Rogers (1973a, 1975a) has

$${}_{ix}L_z^j = 0 \ (\forall j \neq i) \text{ and } l_z^i = {}^{i}\hat{m}_z^{\delta} \ {}_{iz}L_z^i \quad .$$

transition approach was set forth in Ledent (1972) and Ledent/Rogers (1972). Actually, it does not differ much from the method proposed in the movement approach (see Section II).

The main difference is that the multistate stationary population is further broken down into groups characterized by the state of presence at the beginning of the period and that the separate consideration of these groups makes it possible to "do away" with the radix problem.

The method used in the movement approach remains valid here, with vectors replacing matrices*, but the age-specific life table rates for the consolidated intervals are now obtained from:

$${}^{i}\hat{m}^{\delta *}_{x} = \frac{l^{i}_{x} - {}_{ix}l^{\cdot}_{x+T}}{{}_{ix}L^{\cdot}_{x}}$$

and

$${}^{i}\hat{m}^{j *}_{x} = \frac{{}_{ix}l^{j}_{x+T}}{{}_{ix}L^{i}_{x}} \qquad \forall j \neq i$$

instead of (72).

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*The use of the interpolative-iterative methodology in the transition approach is equivalent to its use in the movement approach for a system with a single radix.

IV. MOVEMENT APPROACH VERSUS TRANSITION APPROACH: A FINAL THEORETICAL ASSESSMENT

The purpose of this section is to compare the respective merits of the two alternative approaches to the construction of increment-decrement life tables, and thus to shed some light on the controversy that has been going on between Schoen (1975, 1977) and Rogers/Ledent (1976, 1977).

Nature of the Two Approaches Contrasted

In both approaches the discrete age distribution $\{l_x\}$ is obtained by the application of a series of transition matrices to an initial cohort $\{l_0\}$. However, these matrices are estimated differently owing to the distinct focus of both approaches. To be more specific, when estimating the probability of an individual moving out of a state i over a fixed period of time, (1) the movement approach takes into account all of the moves made by the individual over that one period of time (whether state i is involved or not) while, (2) the transition approach compares the individual's state of presence at the beginning and end of that period, i.e., at two given points in time.

Consider a group of individuals present at age x in state i. The transition approach focuses on the net balance of moves from state i to state j made by the members of this group between ages x and x + T. On the other hand, the movement approach follows all the moves made by these individuals over the same T-year period, thus explicitly considering all gross flows of moves between each pair of states k,j (= 1,...,n). The information needed in the transition approach is somewhat less than in the movement approach,* and may be considered as a "reduced form" of the movement approach.

Further insights can be made by comparing the continuous patterns of mortality and mobility that underline each approach.

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*From an applied point of view, the information sought in the transition approach is also easier to collect, which explains why migration data are generally available in terms of transitions rather than moves.

Indeed, the instantaneous mortality and mobility rates ${}^{i}\hat{\mu}^{j}_{x}(y)$ of the transition approach are not identical to their analogs of the movement approach, as can be seen from the respective definitions of these forces.

For example, rewriting the definition of the instantaneous mortality rate in the transition approach:

$$ {}^{i}\hat{\mu}^{\delta}_{x}(y) = \lim_{dy \to 0} \frac{{}^{i}(ad)^{\delta}_{x}(y)}{{}_{ix}l^{\cdot}(y)dy} \qquad \forall i = 1,...,n \qquad (74) $$

and observing that:

$$ {}^{i}(ad)_{x}(y) = \sum_{j=i}^{n} {}_{ix}^{\,j}d^{\delta}(y) = {}_{ix}^{\,\cdot}d^{\delta}(y) $$

we have:

$$ {}^{i}\hat{\mu}^{\delta}_{x}(y) = \lim_{dy \to 0} \frac{{}_{ix}^{\,\cdot}d^{\delta}(y)}{{}_{ix}l^{\cdot}(y)dy} \qquad \forall i = 1,...,n $$

This is clearly different from the definition (12) of the instantaneous death rate in the movement approach that can be rewritten as:

$$ {}^{i}\mu^{\delta}(y) = \lim_{dy \to 0} \frac{{}^{i}_{.x}d^{\delta}(y)}{{}_{.x}l^{i}(y)dy} \quad * \qquad \forall i = 1,...,n \qquad (114) $$

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* Note that we may obtain ${}^{i}\hat{\mu}^{\delta}_{x}(y)$ from ${}^{i}\mu^{\delta}(y)$ by simply exchanging the index i with the dot, representing the whole set of states in the system.

The instantaneous mobility rate of the transition approach is defined by:

$$ {}^{i}\hat{\mu}^{j}_{x}(y) = \lim_{dy \to 0} \frac{{}^{i}(ad)^{j}_{x}(y)}{{}_{ix}l^{\cdot}(y)dy} \qquad \forall i,j = 1,\dots,n \quad j \neq i \tag{75} $$

in which

$$ {}^{i}(ad)^{j}_{x}(y) = \sum_{\substack{k=1\\k\neq j}}^{n} {}^{k}_{ix}d^{j}(y) - \sum_{\substack{k=1\\k\neq j}}^{n+1} {}^{j}_{ix}d^{k}(y) $$

$$ = {}^{\cdot}_{ix}d^{j}(y) - {}^{j}_{ix}d^{\cdot}(y) - {}^{j}_{ix}d^{\delta}(y) \tag{115} $$

Thus, we have:

$$ {}^{i}\hat{\mu}^{j}_{x}(y) = \lim_{dy \to 0} \frac{{}^{\cdot}_{ix}d^{j}(y) - {}^{j}_{ix}d^{\cdot}(y) - {}^{j}_{ix}d^{\delta}(y)}{{}_{ix}l^{i}(y)dy} \qquad \forall i,j = 1,\dots,n \quad j \neq i \tag{116} $$

while the instantaneous mobility rate in the movement approach is:

$$ {}^{i}\mu^{j}(y) = \lim_{dy \to 0} \frac{{}^{i}_{.x}d^{j}(y)}{{}_{.x}l^{i}(y)dy} \qquad \forall i,j = 1,\dots,n \quad j \neq i \quad . $$

Aside from the non-equality of the movement and transition instantaneous rates, we note that the relationship between mortality and mobility patterns is of a different nature in each approach.

In the movement approach the instantaneous rates ${}^{i}\mu^{j}(y)$ for $j = 1,\dots,n$ $(j \neq i)$ are clearly independent. This is merely a consequence of the assumption that no more than two events can take place in a small interval of time, so that the continuous patterns of mortality and mobility, characteristic of each region are unrelated.

In the transition approach the instantaneous death and migration rates at age y are dependent on the choice of the exact age x immediately below y in the series of fixed ages from which discrete life tables are constructed. Moreover, the instantaneous death rates are not attached to the state of presence in which the deaths actually occur at age y but to the state of presence at the earlier age x. Also, note that the mobility patterns is a composite of pure mobility [because of ${}_{ix}^{\cdot}d^{j}(y) - {}_{ix}^{j}d^{\cdot}(y)$ in (116)] and mortality [because of ${}_{ix}^{j}d^{\delta}(y)$]. Therefore, in the transition approach, unlike the movement approach, the mobility pattern is clearly affected by mortality.

Consolidated Flow Equations and Multistate Functions Contrasted

The contrast between moves on the one hand and transitions on the other hand is further substantiated by comparing the consolidated flow equations.

The integration of (14) yields the consolidated flow equation of the movement approach (Schoen and Nelson, 1974; Schoen, 1975)

$$l_{x+T}^{i} = l_{x}^{i} - {}^{i}d_{x}^{\delta} - \sum_{\substack{j=1 \\ j \neq i}}^{n} {}^{j}d_{x}^{i} \qquad \forall i = 1,\ldots,n \quad , \qquad (117)$$

in which ${}^{i}d_{x}^{j}$ is the total number of moves from state i to state j between ages x and x + T: (n is the number of decrements to l_{x}^{i} due to mobility).

$${}^{i}d_{x}^{j} = \int_{0}^{T} {}^{i}\mu^{j}(x + t) \; l^{i}(x + t)dt \qquad \forall i,j = 1,\ldots,n \atop j \neq i \qquad ,$$

or, in discrete form,

$$ {}^{i}d_{x}^{j} = {}^{i}m_{x}^{j}\ L_{x}^{i} \qquad \forall i,j = 1,\dots,n \quad j \neq i \quad . \qquad (118)$$

The integration of (78) and (79) yields:

$$ {}_{ix}l_{x+T}^{i} = {}_{ix}l_{x}^{i} - {}^{i}(ad)_{x}^{j} - \sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i}(ad)_{x}^{j} \qquad \forall i = 1,\dots,n \qquad (119)$$

$$ {}_{jx}l_{x+T}^{i} = {}^{j}(ad)_{x}^{i} \qquad \forall i,j = 1,\dots,n \quad j \neq i \quad . \qquad (120)$$

Adding (119) and the (n - 1) equations composing (120) leads us to the consolidated flow equation of the transition approach (Rogers 1973a, 1975a):

$$ l_{x+T}^{i} = l_{x}^{i} - {}^{i}(ad)_{x}^{\delta} - \sum_{\substack{j=1\\ j\neq i}}^{n} {}^{i}(ad)_{x}^{j} + \sum_{\substack{j=1\\ j\neq i}}^{n} {}^{j}(ad)_{x}^{i} \qquad (121)$$

$$\forall i,j = 1,\dots,n \quad ,$$

in which ${}^{i}(ad)_{x}^{j}$ is the number of net moves (transitions) from state i to state j between ages x and x + T

$$ {}^{i}(ad)_{x}^{j} = \int_{0}^{T} {}^{i}\hat{\mu}_{x}^{j}(x+t)\ {}_{ix}l^{i}(x+t)dt \qquad \forall i,j = 1,\dots,n \quad j \neq i \quad .$$

and ${}^{i}(ad)_{x}^{\delta}$ the number of deaths occuring between ages x and x + T to those present in state i at age x

$$ {}^{i}(ad)_{x}^{\delta} = \int_{0}^{T} {}^{i}\hat{\mu}_{x}^{\delta}(x+t)\ {}_{ix}l^{\cdot}(x+t)dt \qquad \forall i = 1,\dots,n \quad .$$

In discrete form, the <u>net</u> decrements to l^{i}_{x} are respectively:

$$^{i}(ad)^{j}_{x} = {}^{i}\hat{m}^{j}_{x}\ {}_{ix}L^{i}_{x} \qquad \forall i,j = 1,\dots,n \quad j \neq i \qquad ,$$

and

$$^{i}(ad)^{\delta}_{x} = {}^{i}\hat{m}^{\delta}_{x}\ {}_{ix}L^{\cdot}_{x} \qquad \forall i = 1,\dots,n \quad .$$

An important aspect of the comparison between the two alternative approaches is that (121) of the transition approach can be broken down into n separate equations [contained in (119) and (120)] while (117) of the movement approach cannot.

The substitution of the net decrements into the flow equations of the transition approach lead to n^2 scalar equations, summarized as

$$\{{}_{ix}l_{x}\} = \{{}_{ix}l_{x+T}\} = {}^{i}\hat{\underset{\sim}{m}}_{x}\{{}_{ix}L_{x}\} \qquad \forall i = 1,\dots,n \qquad (122)$$

while substitution of gross decrements into the flow equation of the movement approach yields only n scalar equations, summarized as:

$$\{l_{x}\} - \{l_{x+T}\} = \underset{\sim}{m}_{x}\{L_{x}\} \quad . \qquad (123)$$

Consequently, from the knowledge of multistate functions, (122), unlike (123), allows for the derivation of life table rates, which permits the elimination of the radix problem without imposing any further assumptions. In the movement approach however, a further assumption (independence of the life table rates from the state allocation of the initial cohort) must be introduced. Actually it is equivalent to suppose that (117) holds for each group ${}_{jy}l^{i}_{x}$ rather than for $l^{i}_{x} = {}_{\cdot y}l^{i}_{x}$ alone.

Table 2: A Tabular Comparison of the Movement and Transition Approaches

MOVEMENT APPROACH

ow equation

$$ {}_{iy}l^{j}_{x+T} = {}_{iy}l^{j}_{x} - {}_{iy}^{\;i}d^{\delta}_{x} - \sum_{\substack{j=1\\ j\neq i}}^{n} {}_{iy}^{\;i}d^{j}_{x} + \sum_{\substack{j=1\\ j\neq i}} {}_{iy}^{\;j}d^{i}_{x} $$

Age specific mortality and mobility rates

$$ {}^{i}m^{j}_{x} = \frac{{}_{1y}^{\;i}d^{j}_{x}}{{}_{1y}L^{i}_{x}} = \cdots\cdots = \frac{{}_{ny}^{\;i}d^{j}_{x}}{{}_{ny}L^{i}_{x}} \qquad \begin{array}{l} \forall i = 1,\ldots,n \\ \forall j = 1,\ldots,n+1 \\ j\neq i \end{array} $$

Matrix of age-specific rates

$$ \underset{\sim}{m}_{x} = \begin{bmatrix} {}^{1}m^{\delta}_{x} + \left[\sum\limits_{\substack{j=1\\ j\neq 1}}^{n} {}^{1}m^{j}_{x}\right] & -\,{}^{2}m^{1}_{x} & -\,{}^{n}m^{1}_{x} \\ -\,{}^{1}m^{2}_{x} & {}^{2}m^{\delta}_{x} + \left[\sum\limits_{\substack{j=1\\ j\neq 2}}^{n} {}^{2}m^{j}_{x}\right] & \\ \vdots & & \ddots \\ -\,{}^{1}m^{n}_{x} & & {}^{n}m^{\delta}_{x} + \left[\sum\limits_{\substack{j=1\\ j\neq n}}^{n} {}^{n}m^{j}_{x}\right] \end{bmatrix} $$

Model in compact form

$$ {}_{y}\underset{\sim}{l}_{x} - {}_{y}\underset{\sim}{l}_{x+T} = \underset{\sim}{m}_{x}\; {}_{y}\underset{\sim}{L}_{x} $$

LINEAR INTEGRATION

Survival probabilities

$$ \underset{\sim}{p}_{x} = [\underset{\sim}{I} - \frac{T}{2}\underset{\sim}{m}_{x}][\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_{x}]^{-1} $$

Person-years lived

$$ {}_{y}\underset{\sim}{L}_{x} = T[\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_{x}]^{-1}\; {}_{y}\underset{\sim}{l}_{x} $$

Survivorship proportions

$$ \underset{\sim}{s}_{x} = [\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_{x+T}]^{-1}[\underset{\sim}{I} - \frac{T}{2}\underset{\sim}{m}_{x}] $$

Table 2. (continued)

$$\left.\begin{array}{ll} {}_{ix}l^{i}_{x+T} = {}_{ix}l^{i}_{x} - {}^{i}(ad)^{\delta}_{x} - \sum\limits_{\substack{j=1\\ j\neq i}}^{n} {}^{i}(ad)^{j}_{x} & \forall i = 1,\ldots,n \\ {}_{ix}l^{j}_{x+T} = {}^{i}(ad)^{j}_{x} & \forall i,j = 1,\ldots,n \\ & j\neq i \end{array}\right\}$$

Flow equation

$${}^{i}\hat{m}^{\delta}_{x} = \frac{{}^{i}(ad)^{\delta}_{x}}{{}_{ix}L^{\cdot}_{x}} \quad \forall i = 1,\ldots,n \;; \quad {}^{i}\hat{m}^{j}_{x} = \frac{{}^{i}(ad)^{j}_{x}}{{}_{ix}L^{i}_{x}} \quad \forall i,j = 1,\ldots,n \;\; j\neq i$$

Age specific mortality and mobility rates

$${}^{i}\hat{\underset{\sim}{m}}_{x} = \begin{bmatrix} 0 & -{}^{i}\hat{m}^{1}_{x} & 0 \\ {}^{i}\hat{m}^{\delta}_{x} \;.... & {}^{i}\hat{m}^{\delta}_{x} + \left[\sum\limits_{\substack{j=1\\ j\neq i}}^{n} {}^{i}\hat{m}^{j}_{x}\right] & ....\; {}^{i}\hat{m}^{\delta}_{x} \\ 0 & -{}^{i}\hat{m}^{n}_{x} & 0 \end{bmatrix}$$

Matrix of age-specific rates

$$\{{}_{ix}l_{x}\} - \{{}_{ix}l_{x+T}\} = {}^{i}\hat{\underset{\sim}{m}}_{x}\{{}_{ix}L_{x}\}$$

Model in compact form

LINEAR INTEGRATION

$$\hat{\underset{\sim}{p}}_{x} = [\underset{\sim}{I} - \tfrac{T}{2}(\hat{\underset{\sim}{m}}_{x} - \overset{m}{\hat{\underset{\sim}{m}}}_{x} + \tfrac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_{x}\overset{mt}{\hat{\underset{\sim}{m}}}_{x})][\underset{\sim}{I} + \tfrac{T}{2}(\hat{\underset{\sim}{m}}_{x} + \overset{m}{\hat{\underset{\sim}{m}}}_{x} + \tfrac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_{x}\overset{mt}{\hat{\underset{\sim}{m}}}_{x})]^{-1}$$

Survival probabilities

$${}_{y}\underset{\sim}{L}_{x} = T[\underset{\sim}{I} + \tfrac{T}{2}\overset{m}{\hat{\underset{\sim}{m}}}_{x+T}][\underset{\sim}{I} + \tfrac{T}{2}(\hat{\underset{\sim}{m}}_{x} + \hat{\underset{\sim}{m}}_{x} + \tfrac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_{x}\overset{mt}{\hat{\underset{\sim}{m}}}_{x})]^{-1}$$

Person-years lived

$$\hat{\underset{\sim}{s}}_{x} = [\underset{\sim}{I} + \tfrac{T}{2}\hat{\underset{\sim}{m}}_{x+T}][\underset{\sim}{I} + \tfrac{T}{2}(\hat{\underset{\sim}{m}}_{x} - \overset{m}{\hat{\underset{\sim}{m}}}_{x} + \tfrac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_{x+T}\overset{mt}{\hat{\underset{\sim}{m}}}_{x+T})]^{-1}$$

$$[\underset{\sim}{I} - \tfrac{T}{2}(\hat{\underset{\sim}{m}}_{x} - \overset{m}{\hat{\underset{\sim}{m}}}_{x} + \underset{\sim}{m}_{x} + \tfrac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_{x}\overset{mt}{\hat{\underset{\sim}{m}}}_{x})][\underset{\sim}{I} + \tfrac{T}{2}\overset{m}{\hat{\underset{\sim}{m}}}_{x}]^{-1}$$

Survivorship proportions

$$ {}_{jy}l^{i}_{x+T} = {}_{jy}l^{i}_{x} - {}_{jy}^{\ i}d^{\delta}_{x} - \sum_{\substack{k=1\\k\neq i}}^{n} {}_{jy}^{\ i}d^{k}_{x} + \sum_{\substack{k=1\\k\neq i}}^{n} {}_{jy}^{\ k}d^{i}_{x} \qquad (124) $$

$$ \forall i,j = 1,\ldots,n \quad . $$

Thus we have n^2 scalar equations that can be summarized in vector format as

$$ \{{}_{iy}l_{x}\} = \{{}_{iy}l_{x+T}\} = \underset{\sim}{m}_{x}\{{}_{iy}L_{x}\} \qquad \begin{array}{l} \forall i = 1,\ldots,n \\ \forall y \ 0 \leq y \leq x \end{array} \qquad (125) $$

or, in matrix format

$$ {}_{y}\underset{\sim}{l}_{x} - {}_{y}\underset{\sim}{l}_{x+T} = \underset{\sim}{m}_{x} \ {}_{y}\underset{\sim}{L}_{x} \qquad \forall y \ 0 \leq y \leq x \quad , $$

which permits the estimation of all elements of $\underset{\sim}{m}_{x}$ from the knowledge of the multistate life table functions.

The contrast of the two approaches is continued in Table 2, which shows the flow and orientation equations as well as the expressions of some multistate life table functions (in the linear case).*

Relationship between Movement and Transition Rates (Linear Case)

Expression of Movement Rates in Terms of Transition Rates

Formulas expressing movement rates in terms of transition rates can be obtained by equating the age-specific probabilities

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*It is interesting to note that the formulas corresponding to the transition approach collapse into those of the movement approach by simply setting $\overset{m}{\hat{\underset{\sim}{m}}}_{x}$ and $\overset{mt}{\hat{\underset{\sim}{m}}}_{x}$ equal to a zero matrix.

$\underset{\sim}{p}_x$ and $\hat{\underset{\sim}{p}}_x$ derived in both approaches.*

From (58) we can draw $\underset{\sim}{m}_x$:

$$\underset{\sim}{m}_x = \frac{2}{T}\,[\underset{\sim}{I} - \underset{\sim}{p}_x]\,[\underset{\sim}{I} + \underset{\sim}{p}_x]^{-1} \quad .$$

By equating $\underset{\sim}{p}_x$ to $\hat{\underset{\sim}{p}}_x$ and substituting (103) for $\hat{\underset{\sim}{p}}_x$, we have

$$\underset{\sim}{m}_x = \underset{\sim}{w}_x\,[\underset{\sim}{I} + \frac{T}{2}\,\underset{\sim}{v}_x]^{-1}[\underset{\sim}{I} + \frac{T}{2}\,\underset{\sim}{v}_x]\,[\underset{\sim}{I} + \frac{T}{2}\,\overset{m}{\hat{\underset{\sim}{m}}}_x]^{-1} \quad ,$$

in which:

$$\underset{\sim}{w}_x = \begin{bmatrix} {}^{i\delta}\hat{m}_x + (1 + \frac{T}{2}\,{}^{i\delta}\hat{m}_x)\sum\limits_{\substack{j=1\\ j\neq i}}^{n} {}^{1j}\hat{m}_x & -\,{}^{21}\hat{m}_x & & -\,{}^{n1}\hat{m}_x \\ -\,{}^{12}\hat{m}_x & {}^{2\delta}\hat{m}_x + (1 + \frac{T}{2}\,{}^{2\delta}\hat{m}_x)\sum\limits_{\substack{j=1\\ j\neq i}}^{n} {}^{2j}\hat{m}_x & & \\ & & \ddots & \\ -\,{}^{1n}\hat{m}_x & & & \end{bmatrix}$$

---

*The rationale for equating these probabilities lies in the equivalence of the linear integration methods used in both the transition and movement approaches. This equivalence can immediately be established from the observation that $\{L_x\} = \frac{T}{2}\,[\{l_x\} + \{l_{x+T}\}]$ is identical to $\{{}_{jx}L_x\} = \frac{T}{2}\,[\{{}_{jx}l_x\} + {}_{jx}\{l_{x+T}\}]$ if one supposes independence of $\{l_x\}$ with respect to the state allocation of the initial cohort.

(i.e., $\underset{\sim}{w}_x$ is the same as $\underset{\sim}{u}_x$ defined in (110) except for the fact that the off-diagonal elements are half of those of $\underset{\sim}{u}_x$) and $\underset{\sim}{\overset{m}{\hat{m}}}_x$ is the same as in Section III.

Since $(\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{v}_x)^{-1}(\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{v}_x) = \underset{\sim}{I}$, the above equation reduces to:

$$\underset{\sim}{m}_x = \underset{\sim}{w}_x(\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{\overset{m}{\hat{m}}}_x)^{-1} \quad .$$

Observing that $\underset{\sim}{w}_x = \underset{\sim}{\hat{m}}_x + \frac{T}{2}\overset{\delta}{\hat{m}}_x \overset{mt}{\hat{m}}_x$, we also have:

$$\underset{\sim}{m}_x = (\underset{\sim}{\hat{m}}_x + \frac{T}{2}\underset{\sim}{\overset{\delta}{\hat{m}}}_x \underset{\sim}{\overset{mt}{\hat{m}}}_x)(\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{\overset{m}{\hat{m}}}_x)^{-1} \quad , \qquad (126)^*$$

from which we can draw an explicit relationship linking standard and cohort death rates by premultiplying by a row vector of ones $\{i\}'$. Noting that:

$$\{i\}' \underset{\sim}{m}_x = \{{}^{1}\overset{\delta}{m}_x, \ldots, {}^{n}\overset{\delta}{m}_x\}'$$

$$\{i\}'(\underset{\sim}{\hat{m}}_x + \frac{T}{2}\overset{\delta}{\hat{m}}_x \overset{mt}{\hat{m}}_x) = \{{}^{1}\overset{\delta}{\hat{m}}_x(1 + \frac{T}{2}\sum_{\substack{j=1\\j\neq i}}^{n} {}^{1}\hat{m}^{j}_x), \ldots, {}^{n}\overset{\delta}{\hat{m}}_x(1 + \frac{T}{2}\sum_{\substack{j=1\\j\neq i}}^{n} {}^{n}\hat{m}^{j}_x)\}' \quad ,$$

---

*Note that, in the case examined by Rogers (1975), $\underset{\sim}{w}_x$ reduces to $\underset{\sim}{\hat{m}}_x$ so that the relationship between movement and transition rates is simply:

$$\underset{\sim}{m}_x = \underset{\sim}{\hat{m}}_x[\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{\overset{m}{\hat{m}}}_x]^{-1} \quad .$$

we finally obtain after transposing:

$$\{m_x^\delta\} = (\underset{\sim}{I} + \frac{T}{2}\overset{m\prime}{\hat{\underset{\sim}{m}}}_x)^{-1}(I + \frac{T}{2}\overset{mt}{\hat{\underset{\sim}{m}}}_x)\{\hat{m}_x^\delta\} \quad , \qquad (127)^*$$

in which $\overset{m}{\hat{m}}{}'_x$ is the transpose of $\overset{m}{\hat{m}}_x$.

In the case of a two-state linear system, it is obvious from (126) that

$${}^1m_x^2 = {}^1\hat{m}_x^2 \frac{(1 + \frac{T}{2}\,{}^2\hat{m}_x^\delta)(1 + \frac{T}{2}\,{}^2\hat{m}_x^1)}{1 - \frac{T^2}{4}\,{}^1\hat{m}_x^2\,{}^2\hat{m}_x^1}$$

and

$${}^1m_x^\delta = \frac{{}^1\hat{m}_x^\delta(1 + \frac{T}{2}\,{}^1\hat{m}_x^2) - \frac{T}{2}\,{}^1\hat{m}_x^2\,{}^2\hat{m}_x^\delta(1 + \frac{T}{2}\,{}^2\hat{m}_x^1)}{1 - \frac{T^2}{4}\,{}^1\hat{m}_x^2\,{}^2\hat{m}_x^1} \quad .$$

Subtracting $\hat{\underset{\sim}{m}}_x$ from both sides of (126) leads to an estimate of the difference between movement and transition rates:

$$\underset{\sim}{m}_x - \hat{\underset{\sim}{m}}_x = [\hat{\underset{\sim}{m}}_x + \frac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_x\overset{mt}{\hat{\underset{\sim}{m}}}_x)(\underset{\sim}{I} + \frac{T}{2}\overset{m}{\hat{\underset{\sim}{m}}}_x)^{-1} - \hat{\underset{\sim}{m}}_x \quad ,$$

$$= [\hat{\underset{\sim}{m}}_x + \frac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_x\overset{mt}{\hat{\underset{\sim}{m}}}_x - \hat{\underset{\sim}{m}}_x(\underset{\sim}{I} + \frac{T}{2}\overset{m}{\hat{\underset{\sim}{m}}}_x)][\underset{\sim}{I} + \frac{T}{2}\overset{m}{\hat{\underset{\sim}{m}}}_x]^{-1} \quad ,$$

---

*In the case examined by Rogers (1975a), the relationship between standard and transition death rates is simply:

$$\{m_x^\delta\} = [\underset{\sim}{I} + \frac{T}{2}\overset{m}{\hat{\underset{\sim}{m}}}{}'_x]^{-1}\{\hat{m}_x^\delta\} \quad .$$

which reduces to:

$$\underset{\sim}{m}_x - \hat{\underset{\sim}{m}}_x = \frac{T}{2}[\hat{\underset{\sim}{m}}^{\delta}_x \hat{\underset{\sim}{m}}^{mt}_x - \hat{\underset{\sim}{m}}_x \hat{\underset{\sim}{m}}^{m}_x][\underset{\sim}{I} + \frac{T}{2}\hat{\underset{\sim}{m}}^{m}]^{-1} \quad , \qquad (128)$$

a relationship indicating that the difference between corresponding movement and transition rates is likely to be small since each scalar element of $\underset{\sim}{m}_x - \hat{\underset{\sim}{m}}_x$ contains terms that consist of products of at least two rates. For example, in the case of a two-region system, we can establish that:

$$\frac{{}^1m^2_x - {}^1\hat{m}^2_x}{{}^1m^2_x} = \frac{T}{2}\,\frac{{}^2\hat{m}^\delta_x + {}^2\hat{m}^1_x[1 + \frac{T}{2}({}^1\hat{m}^2_x + {}^2\hat{m}^\delta_x)]}{1 - \frac{T^2}{4}\,{}^1\hat{m}^2_x\,{}^2\hat{m}^1_x}$$

$$\simeq \frac{T}{2}[{}^2\hat{m}^\delta_x + {}^2\hat{m}^1_x] \qquad (129)$$

$$\frac{{}^1m^\delta_x - {}^1\hat{m}^\delta_x}{{}^1\hat{m}^\delta_x} = \frac{T}{2}\,{}^1\hat{m}^2_x\,\frac{({}^1\hat{m}^\delta_x - {}^2\hat{m}_x)(1 + \frac{T}{2}\,{}^2\hat{m}^1_x)}{1 - \frac{T^2}{4}\,{}^1\hat{m}^2_x\,{}^2\hat{m}^1_x}$$

$$\simeq \frac{T}{2}\,{}^1\hat{m}^2_x[{}^1\hat{m}^\delta_x - {}^2\hat{m}^\delta_x] \qquad (130)$$

Three important contrasts between the two approaches should be noted:

1. The relative difference between movement and transition rates is approximately a linear function of the length of age intervals.
2. The relative discrepancy between movement and transition rates of mobility is largely influenced by the level of mortality in higher age groups.

3. The relative discrepancy between movement and transition rates of mortality is generally trifling, as suggested by (130) whose right-hand side contains the product of two rates. As expected, (130) also shows that the larger the relative discrepancy between the movement rates of mortality in each region, the larger the relative discrepancy between movement and transition rates of mortality.

In the case of the last age group, there is no possibility to express movement rates in terms of transition death rates (mobility rates are zero by definition).

Expression of Transition Rates in Terms of Movement Rates

Alternatively, formulas expressing transition rates in terms of movement rates can be derived from the following relationship (obtained by comparing (122) and (125) in which y is set equal to x):

$$\underset{\sim}{m}_x\{{}_{jx}L_x\} = {}^j\underset{\sim}{\hat{m}}_x\{{}_{jx}L_x\} \qquad \forall j = 1,\ldots,n \quad .$$

Substituting (63) yields:

$$\underset{\sim}{m}_x(\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x)^{-1}\{{}_{jx}l_x\} = {}^j\underset{\sim}{\hat{m}}_x(\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x)^{-1}\{{}_{jx}l_x\} \quad . \qquad (131)$$

Since all components of $\{{}_{jx}l_x\}$ are zeros except for the j^{th} one, (131) means that the j^{th} columns of $[\underset{\sim}{m}_x(\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x)^{-1}]$ and $[{}^j\underset{\sim}{\hat{m}}_x(\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x)^{-1}]$ are equal.

Let $\{{}^jh\}$ denote the vector $(\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x)^{-1}\{j\}$ in which $\{j\}$ is a column vector of zeros, except for the j^{th} component being equal to one. Then recalling the definition of ${}^j\underset{\sim}{\hat{m}}_x$ and observing that ${}^j\underset{\sim}{\hat{m}}_x\{j\}$ is a vector whose j^{th} component is

${}^{j}\hat{m}_x^{\delta}[\sum_{k=1}^{n} {}^{j}h_k] + {}^{j}h_j[\sum_{\substack{k=1\\k\neq j}}^{n} {}^{j}\hat{m}_x^{k}]$ and any l^{th} component

$(l \neq j)$ is ${}^{j}h_j \; {}^{j}\hat{m}_x^{l}$, we have:

$$[{}^{j}m_x^{\delta} + \sum_{\substack{k=i\\k\neq j}}^{n} {}^{j}m_x^{k}] \; {}^{j}h_j - \sum_{\substack{k=1\\k\neq j}} {}^{j}m^{k} \; {}^{j}h_k = {}^{j}\hat{m}_x^{\delta}[\sum_{k=1}^{n} {}^{j}h_l] + {}^{j}h_j[\sum_{\substack{k=1\\k\neq j}}^{n} {}^{j}\hat{m}^{k}]$$

$$\forall j = 1,\ldots,n$$

and:

$$[{}^{l}m_x^{\delta} + \sum_{\substack{k=1\\k\neq l}}^{n} {}^{l}m_x^{k}] \, {}^{j}h_l - \sum_{\substack{k=1\\k\neq l}} {}^{k}m^{l} \; {}^{j}h_k = - \, {}^{j}h_j \; {}^{j}\hat{m}_x^{l} \qquad \forall j,l = 1,\ldots,n$$

Therefore:

$${}^{j}\hat{m}_x^{l} = \frac{\sum_{\substack{k=1\\k\neq}}^{n} {}^{k}m^{l} \; {}^{j}h_k - [{}^{l}m_x^{\delta} + \sum_{\substack{k=1\\k\neq}}^{n} {}^{l}m^{k}] \, {}^{j}h_l}{{}^{j}h_j} \qquad \begin{matrix}\forall j,l = 1,\ldots,n\\ l \neq j\end{matrix}$$

and:

$${}^{j}\hat{m}_x^{\delta} = \frac{\sum_{=1}^{n} {}^{l}m_x^{\delta} \; {}^{j}h_l}{{}^{j}h_j} \qquad \forall j = 1,\ldots,n \quad .$$

Similarly in the case of the last age group:

$${}^{j}m_z = \frac{1}{\{j\}' \; \underset{\sim}{m}_z'^{-1}\{i\}}$$

in which $\{j\}'$ and $\underset{\sim}{m}_z'^{-1}$ are the transposes of $\{j\}$ and $\underset{\sim}{m}_z^{-1}$ respectively.

Assessment of the Discrepancy Between the Alternative Approaches (Linear Case)

Suppose that we put the same set of rates into both formulas (58) and (111), expressing the age-specific probabilities in the movement and transition approaches respectively. What would be the difference between the two types of probabilities thus obtained?

Let $\Delta\underset{\sim}{p}_x$ denote the quantity obtained by subtracting the transition formula from the movement formula:

$$\Delta\underset{\sim}{p}_x = [\underset{\sim}{I} - \frac{T}{2}\underset{\sim}{m}_x][\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x]^{-1} - [\underset{\sim}{I} + \frac{T}{2}(\underset{\sim}{m}_x - \overset{m}{\underset{\sim}{m}}_x + \frac{T}{2}\overset{\delta}{\underset{\sim}{m}}_x \overset{mt}{\underset{\sim}{m}}_x)]$$

$$[\underset{\sim}{I} + \frac{T}{2}(\underset{\sim}{m}_x + \overset{m}{\underset{\sim}{m}}_x + \frac{T}{2}\overset{\delta}{\underset{\sim}{m}}_x \overset{mt}{\underset{\sim}{m}}_x)]^{-1}$$

Using the property that $[\underset{\sim}{I} - \frac{T}{2}\underset{\sim}{m}_x][I + \frac{T}{2}\underset{\sim}{m}_x]^{-1} = [\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x]^{-1}$ $[I - \frac{T}{2}\underset{\sim}{m}_x]$, we can rearrange Δp_x as:

$$\Delta\underset{\sim}{p}_x = [\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x]^{-1}[(\underset{\sim}{I} - \frac{T}{2}\underset{\sim}{m}_x)(\underset{\sim}{I} + \frac{T}{2}(\underset{\sim}{m}_x + \overset{m}{\underset{\sim}{m}}_x + \frac{T}{2}\overset{\delta}{\underset{\sim}{m}}_x \overset{mt}{m}_x))$$

$$(\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x)(\underset{\sim}{I} - \frac{T}{2}(\underset{\sim}{m}_x - \overset{m}{\underset{\sim}{m}}_x + \frac{T}{2}\overset{\delta}{\underset{\sim}{m}}_x \overset{mt}{\underset{\sim}{m}}_x))]$$

$$[\underset{\sim}{I} + \frac{T}{2}(\underset{\sim}{m}_x + \overset{m}{\underset{\sim}{m}}_x + \frac{T}{2}\overset{\delta}{\underset{\sim}{m}}_x \overset{mt}{\underset{\sim}{m}}_x)]^{-1}$$

and finally obtain:

$$\Delta\underset{\sim}{p}_x = \frac{T^2}{2}[\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x]^{-1}[\overset{\delta}{\underset{\sim}{m}}_x \overset{mt}{m}_x - \underset{\sim}{m}_x \overset{m}{\underset{\sim}{m}}_x][\underset{\sim}{I} + \frac{T}{2}(\underset{\sim}{m}_x + \overset{m}{\underset{\sim}{m}}_x + \frac{T}{2}\overset{\delta}{\underset{\sim}{m}}_x \overset{mt}{\underset{\sim}{m}}_x)]^{-1}$$

Now, suppose that we put the two alternative sets of rates in the same formula, say (58) normally valid in the movement case.

Using movement rates leads to the true transition probabilities:

$$\underset{\sim}{p}_x = (\underset{\sim}{I} - \frac{T}{2}\underset{\sim}{m}_x)(\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x)^{-1} \tag{58}$$

while using transition rates yields the approximate transition probabilities $\bar{p}_x$ such that:

$$\bar{p}_x = (\underset{\sim}{I} - \frac{T}{2}\hat{\underset{\sim}{m}}_x)(\underset{\sim}{I} + \frac{T}{2}\hat{\underset{\sim}{m}}_x)^{-1} \quad .$$

Since we can permute the two matrices in (58), we can write the difference between the exact and approximate probability matrices as:

$$\underset{\sim}{p}_x - \bar{\underset{\sim}{p}}_x = (\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x)^{-1}[(\underset{\sim}{I} - \frac{T}{2}\underset{\sim}{m}_x)(\underset{\sim}{I} + \frac{T}{2}\hat{\underset{\sim}{m}}_x) - (\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x)(\underset{\sim}{I} - \frac{T}{2}\hat{\underset{\sim}{m}}_x)]$$

$$(\underset{\sim}{I} + \frac{T}{2}\hat{\underset{\sim}{m}}_x)^{-1}$$

or:

$$\underset{\sim}{p}_x - \bar{\underset{\sim}{p}}_x = T[\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x]^{-1}[\hat{\underset{\sim}{m}}_x - \underset{\sim}{m}_x][\underset{\sim}{I} + \frac{T}{2}\hat{\underset{\sim}{m}}_x]^{-1}$$

$$= \frac{T^2}{2}[\underset{\sim}{I} + \frac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_x]^{-1}[\underset{\sim}{I} + \frac{T}{2}\overset{mt}{\hat{\underset{\sim}{m}}}_x]^{-1}[\underset{\sim}{m}_x\,\overset{m}{\hat{\underset{\sim}{m}}}_x - \overset{\delta}{\hat{\underset{\sim}{m}}}_x\,\hat{\underset{\sim}{m}}_x][\underset{\sim}{I} + \frac{T}{2}\overset{m}{\hat{\underset{\sim}{m}}}_x]^{-1}$$

$$[\underset{\sim}{I} + \frac{T}{2}\hat{\underset{\sim}{m}}_x]^{-1}$$

Alternatively we can calculate the difference between the two probability matrices that can be obtained if (111) is used instead of (58).

The true probability matrix is given by

$$\hat{\underset{\sim}{p}}_x = \underset{\sim}{I} - T\,\hat{\underset{\sim}{m}}_x[\underset{\sim}{I} + \frac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_x]^{-1}[\underset{\sim}{I} + \frac{T}{2}\overset{mt}{\hat{\underset{\sim}{m}}}_x]^{-1}$$

while the approximate probability matrix is derived from:

$$\bar{\hat{\underset{\sim}{p}}}_x = \underset{\sim}{I} - T\,\underset{\sim}{m}_x[\underset{\sim}{I} + \frac{T}{2}\overset{\delta}{\underset{\sim}{m}}_x]^{-1}[\underset{\sim}{I} + \frac{T}{2}\overset{mt}{\underset{\sim}{m}}_x]^{-1} \quad .$$

The result is that:

$$\hat{\underset{\sim}{p}}_x - \bar{\hat{\underset{\sim}{p}}}_x = T[\underset{\sim}{m}_x(\underset{\sim}{I} + \frac{T}{2}\overset{\delta}{\underset{\sim}{m}}_x)^{-1}(\underset{\sim}{I} + \frac{T}{2}\overset{mt}{\underset{\sim}{m}}_x)^{-1} - \hat{\underset{\sim}{m}}_x(\underset{\sim}{I} + \frac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_x)^{-1}(\underset{\sim}{I} + \frac{T}{2}\overset{mt}{\hat{\underset{\sim}{m}}}_x)^{-1}] \quad .$$

Finally:

$$ {}^{i}\hat{p}^{j}_x - {}^{i}\bar{\hat{p}}^{j}_x = T\left[\frac{{}^{i}\hat{m}^{j}_x}{(1 + \frac{T}{2}\,{}^{i}\hat{m}^{\delta}_x)(1 + \frac{T}{2}\sum\limits_{\substack{k=1\\k\neq i}}^{n} {}^{i}\hat{m}^{k}_x)} - \frac{{}^{i}m^{j}_x}{(1 + \frac{T}{2}\,{}^{i}m^{\delta}_x)(1 + \frac{T}{2}\sum\limits_{\substack{k=1\\k\neq i}}^{n} {}^{i}m^{k}_x)}\right]$$

$$\forall i,j = 1,\ldots,n$$

Since $\hat{\underset{\sim}{m}}^{\delta}_x$, $\underset{\sim}{m}^{mt}_x$, $\overset{\delta}{\hat{\underset{\sim}{m}}}_x$, $\overset{mt}{\hat{\underset{\sim}{m}}}_x$ are diagonal matrices, we have:

$$\frac{{}^{i}\bar{\hat{p}}^{j}_x}{{}^{i}\hat{p}^{j}_x} = \frac{{}^{i}m^{j}_x(1 + \frac{T}{2}\,{}^{i}\hat{m}^{\delta}_x)(1 + \frac{T}{2}\sum\limits_{k=1}^{n} {}^{i}\hat{m}^{k}_x)}{{}^{i}\hat{m}^{j}_x(1 + \frac{T}{2}\,{}^{i}m^{\delta}_x)(1 + \frac{T}{2}\sum\limits_{k=1}^{n} {}^{i}m^{k}_x)} \quad , \quad \forall j \neq i \qquad (132)$$

and

$$ {}^{i}\hat{p}^{i}_x - {}^{i}\bar{\hat{p}}^{i}_x = T\left[\frac{{}^{i}m^{\delta}_x + \sum\limits_{\substack{k=1\\k\neq i}}^{n} {}^{i}m^{k}_x}{(1 + \frac{T}{2}\,{}^{i}m^{\delta}_x)(1 + \frac{T}{2}\sum\limits_{\substack{k=1\\k\neq i}}^{n} {}^{i}m^{k}_x)} - \frac{{}^{i}m^{\delta}_x + \sum\limits_{\substack{k=1\\k\neq i}}^{n} {}^{i}m^{k}_x}{(1 + \frac{T}{2}\,{}^{i}\hat{m}^{\delta}_x)(1 + \frac{T}{2}\sum\limits_{\substack{k=1\\k=i}}^{n} {}^{i}\hat{m}^{k}_x)}\right] \qquad (133)$$

The equations (132) and (133) indicate that the larger the difference between movement and transition rates, the larger the discrepancy between the true and approximate transition probabilities.

Also (132) indicates that, all else being equal,

${}^{i}\hat{m}^{\delta}_{x} > {}^{i}m^{\delta}_{x}$ leads to ${}^{i}\bar{\hat{p}}^{j}_{x} > {}^{i}\hat{p}^{j}_{x}$

${}^{i}\hat{m}^{k}_{x} > {}^{i}m^{k}_{x}$ $(k \neq j$ or $i)$ also leads to ${}^{i}\bar{\hat{p}}^{j}_{x} > {}^{i}\hat{p}^{j}_{x}$

${}^{i}\hat{m}^{j}_{x} > {}^{i}m^{j}_{x}$ however, leads to ${}^{i}\bar{\hat{p}}^{j}_{x} < {}^{i}\hat{p}^{j}_{x}$.

Dividing both sides of (133) by ${}^{i}\hat{p}^{i}_{x}$ and further rearranging the ensuing relationship, yields

$$\frac{{}^{i}\bar{\hat{p}}^{i} - {}^{i}\hat{p}^{i}}{{}^{i}\hat{p}^{i}} = \frac{({}^{i}\hat{m}^{\delta}_{x} - {}^{i}m^{\delta}_{x})(1 - \frac{T^2}{4} \sum_{\substack{k=1\\k\neq i}}^{n} {}^{i}m^{k}_{x} \sum_{\substack{k=1\\k\neq i}}^{n} {}^{i}\hat{m}^{k}_{x}) - (\sum_{\substack{k=1\\k\neq i}}^{n} {}^{i}\hat{m}^{k}_{x} - \sum_{\substack{k=1\\k\neq i}}^{n} {}^{i}m^{k}_{x})(1 - \frac{T^2}{4} {}^{i}m^{\delta}_{x} \, {}^{i}\hat{m}^{\delta}_{x})}{({}^{i}\hat{m}^{\delta} + \sum_{\substack{k=1\\k\neq i}}^{n} {}^{i}m^{k}_{x})(1 + \frac{T}{2} {}^{i}m^{\delta}_{x})(1 + \frac{T}{2} \sum_{\substack{k=1\\k\neq i}}^{n} {}^{i}m^{k}_{x})}$$

Clearly,

$$\frac{{}^{i}\bar{\hat{p}}^{i} - {}^{i}\hat{p}^{i}}{{}^{i}\hat{p}^{i}} \approx \frac{1 - \dfrac{{}^{i}m^{\delta}_{x} + \sum_{\substack{k=1\\k\neq i}}^{n} {}^{i}m^{k}_{x}}{{}^{i}\hat{m}^{\delta}_{x} + \sum_{\substack{k=1\\k\neq i}}^{n} {}^{i}m^{k}_{x}}}{1 + \frac{T}{2}({}^{i}m^{\delta}_{x} + \sum_{\substack{k=1\\k\neq i}}^{n} {}^{i}m^{k}_{x})} ,$$

which, in the case of a two-region system (in which regions are denoted by i and j), reduces to:

$$\frac{{}^{i}\bar{\hat{p}}^{i} - {}^{i}\hat{p}^{i}}{{}^{i}\hat{p}^{i}} \approx - \frac{T}{2} \, {}^{i}m^{j}_{x} \, \frac{{}^{i}\hat{m}^{\delta}_{x} + {}^{j}\hat{m}^{i}_{x}}{{}^{i}\hat{m}^{\delta}_{x} + {}^{i}\hat{m}^{j}_{x}} \, \frac{1}{1 + \frac{T}{2}\,({}^{i}\hat{m}^{\delta}_{x} + {}^{i}\hat{m}^{j}_{x})} .$$

V. CALCULATION OF A MULTIREGIONAL LIFE TABLE: THE INCREMENT - DECREMENT LIFE TABLE APPLIED TO THE PROBLEM OF INTERREGIONAL MIGRATION

There are two alternative methods of calculating an increment-decrement (multiregional) life table (Rogers 1975a):

- the Option 1 method simply consists of setting life table age-specific rates of mortality and mobility equal to their observed counterparts, and
- the Option 2 method calculates a multiregional life table in which survivorship (or migration) proportions are equal to their observed counterparts.

Calculation of a Multiregional Life Table (Option 1)

A prerequisite to the use of either the movement of the transition approach, as defined in Sections II and III respectively, is clearly the measurement of the observed mortality and mobility rates. Unfortunately, the mortality and mobility data commonly available do not permit a measurement of age-specific rates consistent with either approach: vital statistics data allow for the estimation of mortality rates according to the movement approach whereas population census data permit us to estimate mobility rates compatible with the transition approach.

The Measurement of Age-Specific Mortality and Migration Rates

Defining the age-specific mortality and migration rates in observed multiregional systems does not raise any problem because their definitions are direct analogs of the corresponding life table rates' definitions.

In the movement approach, the observed analogs of the definition (35) of life table rates are simply:

A. in the case of mortality,

$$ {}^{i}M_x^{\delta} = \frac{{}^{i}D_x^{\delta}}{K_x^{i}} \qquad \forall i = 1,\ldots,n \quad , \qquad (134)$$

in which ${}^{i}D_{x}^{\delta}$ is the observed number of deaths occurring in region i over a T-year period, to people aged x to x + T (at time of death) and K_{x}^{i} the average population exposed to the risk of death in region i over the T-year period, and

B. in the case of migration,

$$ {}^{i}M_{x}^{j} = \frac{{}^{i}D_{x}^{j}}{K_{x}^{i}} \qquad \forall i,j = 1,\dots,n \quad j \neq i \quad , \tag{135}$$

in which ${}^{i}D_{x}^{j}$ represents the total number of moves from region i to region j made over a T-year period by individuals aged x to x + T (at the time of move).

In the transition approach, the observed age-specific death rates are defined, for each regional cohort, by the analog of (89).

$$ {}^{i}\hat{M}_{x}^{\delta} = \frac{{}^{i}(AD)_{x}^{\delta}}{{}_{ix}K_{x}^{\cdot}} \tag{136}$$

in which ${}^{i}(AD)_{x}^{\delta}$ is the observed number of deaths occurring (in either region) and over a T-year period to people aged x to x + T (at time of death) <u>but present in region i at age x</u> and ${}_{ix}K_{x}^{\cdot}$ the average population exposed to the risk of death over a T year period.

In the same manner, age-specific migration rates related to each regional cohort are given by the analog of (94)

$$ {}^{i}M_{x}^{j} = \frac{{}^{i}(AD)_{x}^{j}}{{}_{ix}K_{x}^{i}} \tag{137}$$

in which ${}^{i}(AD)_{x}^{j}$ is the number of transitions made over a T year period between regions i and j by people aged x in region i at the beginning of the observation period, and ${}_{ix}K_{x}^{i}$ the average population exposed to the risk of migrating.

Since vital statistics are generally collected by place of occurrence, estimates of age-specific mortality rates, by region, consistent with the movement approach can be easily measured by application of (134): ${}^{i}D_{x}^{\delta}$ is directly provided by vital statistics data and K_{x}^{i} can be approximated by the mid-period population of each age group. In contrast to this, since no link is generally made, between reporting deaths, the region of death occurrence, and the region of presence at any earlier age, no age-specific mortality rates consistent with the transition approach can be simply measured.

Very few countries have compulsory registration, that makes it possible to evaluate the total number of moves between pairs of regions over a given period. In most instances, migration rates consistent with the movement approach cannot be measured. Fortunately, a population census generally proves data on place-to-place migration in terms of reported changes of residence from a fixed prior date (i.e., viewed as transitions rather than moves) and thus constitutes a data source consistent with the transition approach.

Unfortunately, typical migration figures released by most censuses do not correspond exactly to the numerator of (137). Generally, census data reports the number of people ${}^{i}K_{x}^{j}$ (aged x to x + T at the end of the observed period) present in region i at the beginning of the period and in region j at the end of the period. Therefore, we must approximate to determine ${}^{i}(AD)_{x}^{j}$, the numerator of (137):

$$ {}^{i}(AD)_{x}^{j} = \frac{{}^{i}K_{x+T}^{j} + {}^{i}K_{x}^{j}}{2} \quad . \qquad (138)^{*} $$

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*If $x = 0$, then ${}^{i}K_{-T}^{j}$ denotes the number of babies ${}^{i}B^{j}$ born in region i over the T-year period who were present in region j at the end of the period.

We also need to estimate the denominator of (137) which is not K_x^i but ${}_{ix}K_x^i$. This quantity can be calculated by a linear approximation

$$ {}_{ix}K_x^i = \frac{{}_{ix}K_x^i(t) + {}_{i,x-T}K_x^i(t + T)}{2} \quad , \tag{139} $$

in which ${}_{i,y}K_x^i(u)$ represents the number of people aged y to y + T in region i at time t and present in the same region u - t years later. ${}_{i,x}K_x^i(t)$ is nothing more than the population aged x to x + T in region i at the beginning of the period, whereas, ${}_{i,x-T}K_x^i(t + T)$ is immediately obtained as K_x^i . Also, note the existence of further complications for the last age group that are not reported here.

To summarize, measures of age-specific mortality and migration rates consistent with either approach generally cannot be obtained. Most common data only permit us to derive mortality and migration rates compatible with the movement and transition approaches respectively. Fortunately, this does not hamper the applied calculation of a multiregional life table since an alternative mixed approach based on the availability of movement death rates and transition migration rates is possible.

Illustration of Linear and Interpolative-Iterative Variations

The construction of a multiregional life table from the type of data generally available can be performed using either the linear integration method for deriving $\{L_x\}$ or an interpolative-iteration method.

If the linear integration method is retained the relationship (125) linking the death rates of the movement and transition approaches can be reformulated as

$$\{\hat{m}^{\delta}\} = (\underset{\sim}{I} + \frac{T}{2}\overset{mt}{\underset{\sim}{\hat{m}}}_x)^{-1}(\underset{\sim}{I} + \frac{T}{2}\overset{m}{\underset{\sim}{\hat{m}}}_x)\{m_x^{\delta}\} \qquad (140)$$

which provides a simple expression of the death rates (of the transition approach) in terms of the life table (movement) death rates and (transition) migration rates.

The age-specific survival probabilities of a multiregional life table can be expressed in terms of life table (movement) death rates and (transition) migration rates by substituting (140) into the formula (111) of the transition approach.

The result is, assuming equality of life table and observed rates, that the age-specific survival probabilities can be obtained from:

$$\underset{\sim}{\hat{p}}_x = [\underset{\sim}{I} - \frac{T}{2}(\underset{\sim}{\hat{M}}_x - \underset{\sim}{\hat{M}}_x + \frac{T}{2}\overset{\delta}{\underset{\sim}{\hat{M}}}_x \overset{mt}{\underset{\sim}{\hat{M}}}_x)][(\underset{\sim}{I} + \frac{T}{2}\overset{\delta}{\underset{\sim}{\hat{M}}}_x)(\underset{\sim}{I} + \frac{T}{2}\overset{mt}{\underset{\sim}{\hat{M}}}_x)]^{-1}$$

where the diagonal of $\overset{\delta}{\underset{\sim}{\hat{M}}}_x$ is identical to the vector

$$\{\overset{\delta}{\hat{M}}_x\} = (\underset{\sim}{I} + \frac{T}{2}\overset{mt}{\underset{\sim}{\hat{M}}}_x)^{-1}(\underset{\sim}{I} + \frac{T}{2}\overset{m}{\hat{M}}_x)\{M_x^{\delta}\}$$

in which $\{\overset{\delta}{M}_x\}$ is a vector of observed (movement) death rates and $\underset{\sim}{\hat{M}}_x$, $\overset{\delta}{\underset{\sim}{\hat{M}}}_x$, $\overset{m}{\underset{\sim}{\hat{M}}}_x$ and $\overset{mt}{\underset{\sim}{\hat{M}}}_x$ are the observed counterparts of $\hat{m}_x$, $\overset{\delta}{\hat{m}}_x$, $\overset{m}{\hat{m}}_x$, and $\overset{mt}{\hat{m}}_x$. Thus, initiated by the estimation of the age-specific survival probabilities, the calculation of the other multistate life table functions is completed as indicated in Sections II and III.

Such a calculation is illustrated using mortality and migration data for the four region system of the U.S. female population (period of observation 1965-1970).* Age-specific (movement) mortality rates for the regions of this system have been measured by the application of (134) to available data (see the second column of Movement Rates in Table 6, which provides estimates of such mortality rates relating to the third region of the system: South) while age-specific (transition) migration rates have been measured by application of (137) through (139) (the three columns of Transition Rates in Table 6 providing estimates of such migration rates out of the third region).

The complete set of probabilities of dying and outmigrating concerning the South region is given by Survival Probabilities in Table 3. For instance, a twenty-five year old woman living in the South has a probability of dying with the next five years equal to 0.00470. Moreover, her probability of still living in the South region five years later is equal to 0.92226, while the probabilities of migrating to the North East, North Central and West regions are respectively equal to 0.01975, 0.02946 and 0.02383. The two alternative mobility statistics to which the above transition probabilities lead, expectations of life and net migraproduction rates, are set out in Table 4. It appears that a woman born in the South has a life expectancy of 74.30 years, of which 52.16 can be expected to be spent in the South, 5.73 in the North East, 8.71 in the North Central and 7.71 in the West. Alternatively, such a woman is expected to make an average of 0.72 moves out of a U.S. Census region, including 0.52 out of the South region.

Exact survivorship proportions by place-of-birth for those residing in the South at age x are displayed in Table 5. The probability for a woman aged 25 to 30 in the South to survive

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*This system is composed of four regions which are precisely the four regions of the United States considered by the U.S. Census Bureau: North East, North Central, South and West.

five years later in the same region is equal to 0.93033 if she was born in the South. But, for a woman born in other regions this probability increases to 0.93121 if she was born in the West. Then, the survivorship proportions for women aged 25 to 30 in the South, independently of their place of residence, stands somewhere in between 0.93033 and 0.93121. However, the corresponding approximate survivorship proportions calculated from $s_x = {}_0\underset{\sim}{L}_{x+5}\ {}_0\underset{\sim}{L}_x^{-1}$ is only equal to 0.93017 (as indicated in Table 4), which provides an order of magnitude of the approximation made by using the aforementioned formula.

Taking advantage of the formulas linking movement and transition rates (see Section IV), we have calculated the (transition) death rates and the (movement) migration rates compatible with the input rates. From the figures in Table 6 we find the following two discrepancies:

- transition (mortality) rates are, as expected, only slightly different from their movement counterparts: slightly smaller in the young age groups (0.00531 versus 0.00533 for the first age group in the South region), they become much smaller in the middle age groups and then slightly higher in the old age groups (0.14948 versus 0.14944 for the last age group).
- The discrepancy between movement and transition rates of migration is larger than in the case of mortality. Although, movement rates are always higher than transition rates, the discrepancy is relatively small when mortality has little influence (up to 50 years old) - the migration rate from South to West in age group 20 to 25 is equal to 0.520 (movement rate) versus 0.515 (transition rate)* - and tends to augment sharply with age: the movement rate for the last age group is almost fifty percent higher than the corresponding transition rate. Indeed, these results were more or less expected since movement rates of migration, unlike their transition counterparts, are only slightly influenced by mortality.

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*The discrepancy increases with the intensity of migration.

Table 3. Multiregional life table based on movement death rates and transition rates of migration, linear case, United States, four region system (1965-1970), females, age specific survival probabilities and approximate survivorship proportions (South Region).

SURVIVAL PROBABILITIES

x	${}^{3}p_x^{\delta}$	${}^{3}p_x^{1}$	${}^{3}p_x^{2}$	${}^{3}p_x^{3}$	${}^{3}p_x^{4}$
0	0.02622	0.02270	0.03443	0.88657	0.03008
5	0.00211	0.01248	0.02117	0.94616	0.01808
10	0.00173	0.01045	0.01718	0.95619	0.01445
15	0.00319	0.01607	0.02615	0.93587	0.01872
20	0.00383	0.02214	0.03413	0.91521	0.02468
25	0.00470	0.01975	0.02946	0.92226	0.02383
30	0.00692	0.01378	0.02117	0.93896	0.01917
35	0.01117	0.01037	0.01559	0.94817	0.01471
40	0.01640	0.00717	0.01099	0.95482	0.01061
45	0.02345	0.00513	0.00791	0.95595	0.00756
50	0.03221	0.00408	0.00629	0.95171	0.00564
55	0.04644	0.00337	0.00559	0.94018	0.00442
60	0.06444	0.00326	0.00524	0.92323	0.00383
65	0.09974	0.00348	0.00509	0.88810	0.00359
70	0.14800	0.00378	0.00550	0.83928	0.00344
75	0.23463	0.00368	0.00541	0.75305	0.00323
80	0.35188	0.00307	0.00452	0.63783	0.00270
85	1.00000	0.00000	0.00000	0.00000	0.00000

APPROXIMATE SURVIVORSHIP PROPORTIONS

x	${}^{3}s_x^{\cdot}$	${}^{3}s_x^{1}$	${}^{3}s_x^{2}$	${}^{3}s_x^{3}$	${}^{3}s_x^{4}$
-5	0.98689	0.01155	0.01722	0.94329	0.01504
0	0.98563	0.01803	0.02856	0.91414	0.02490
5	0.99808	0.01149	0.01925	0.95097	0.01637
10	0.99754	0.01316	0.02150	0.94634	0.01655
15	0.99649	0.01899	0.02990	0.92601	0.02159
20	0.99574	0.02106	0.03197	0.91852	0.02418
25	0.99420	0.01690	0.02556	0.93017	0.02158
30	0.99097	0.01212	0.01848	0.94333	0.01704
35	0.98623	0.00881	0.01335	0.95134	0.01273
40	0.98011	0.00616	0.00948	0.95534	0.00913
45	0.97218	0.00460	0.00710	0.95387	0.00661
50	0.96076	0.00371	0.00592	0.94609	0.00503
55	0.94477	0.00329	0.00538	0.93197	0.00412
60	0.91850	0.00333	0.00512	0.90636	0.00368
65	0.87737	0.00357	0.00521	0.86512	0.00347
70	0.81212	0.00363	0.00531	0.79993	0.00326
75	0.71454	0.00328	0.00482	0.70355	0.00288
80	1.07570	0.00937	0.01415	1.04282	0.00936

Table 4. Multiregional life table based on movement death rates and transition rates of migration, linear case, United States, four region system (1965-1970), females, age specific expectations of life and migraproduction rates by place-of-birth (South Region)

EXPECTATIONS OF LIFE

x	${}^{3\cdot}e_x$	${}^{3\,1}e_x$	${}^{3\,2}e_x$	${}^{3\,3}e_x$	${}^{3\,4}e_x$
0	74.30710	5.73413	8.70722	52.15824	7.70751
5	71.24020	5.83021	8.85322	48.71898	7.83779
10	66.38396	5.69821	8.64856	44.37607	7.66112
15	61.49184	5.51497	8.35940	40.20349	7.41399
20	56.67720	5.28791	7.99872	36.27846	7.11210
25	51.87977	4.99381	7.53658	32.61716	6.73222
30	47.10232	4.63072	6.97934	29.21747	6.27480
35	42.39338	4.22262	6.36365	26.04045	5.76665
40	37.81362	3.79430	5.72373	23.06072	5.23487
45	33.37009	3.35753	5.07655	20.24329	4.69272
50	29.07399	2.92161	4.43435	17.56780	4.15024
55	24.93456	2.49299	3.80530	15.02198	3.61429
60	21.00023	2.08130	3.20208	12.62112	3.09573
65	17.26464	1.69078	2.62912	10.34898	2.59577
70	13.88216	1.33923	2.11185	8.29395	2.13713
75	10.88301	1.02894	1.65250	6.47726	1.72431
80	8.49644	0.78092	1.28346	5.03595	1.39611
85	6.78769	0.59916	1.01174	4.00578	1.17101

NET MIGRAPRODUCTION RATES

x	${}^{3\cdot}n_x$	${}^{3\,1}n_x$	${}^{3\,2}n_x$	${}^{3\,3}n_x$	${}^{3\,4}n_x$
0	0.72318	0.04896	0.08735	0.51633	0.07054
5	0.62875	0.04821	0.08583	0.42599	0.06872
10	0.57472	0.04698	0.08349	0.37815	0.06610
15	0.53106	0.04552	0.08084	0.34122	0.06348
20	0.46737	0.04253	0.07572	0.28952	0.05960
25	0.38124	0.03753	0.06662	0.22421	0.05286
30	0.30292	0.03237	0.05687	0.16947	0.04421
35	0.24401	0.02786	0.04863	0.13143	0.03608
40	0.19961	0.02413	0.04181	0.10410	0.02957
45	0.16810	0.02129	0.03659	0.08537	0.02484
50	0.14493	0.01899	0.03233	0.07222	0.02138
55	0.12589	0.01683	0.02839	0.06210	0.01856
60	0.10761	0.01418	0.02387	0.05365	0.01590
65	0.08647	0.01091	0.01862	0.04581	0.01312
70	0.07000	0.00780	0.01378	0.03796	0.01046
75	0.05342	0.00559	0.01016	0.02957	0.00810
80	0.03807	0.00389	0.00718	0.02108	0.00593
85	0.02449	0.00244	0.00458	0.01352	0.00395

Table 5. Multiregional life table based on movement death rates and transition rates of migration, linear case, United States, four region system (1965-1970), females, exact survivorship proportions by place-of-birth and place-of-residence at age x (South Region).

FOR THOSE BORN IN THE NORTH EAST

x	${}^{3}_{10}s^{\cdot}_{x}$	${}^{3}_{10}s^{1}_{x}$	${}^{3}_{10}s^{2}_{x}$	${}^{3}_{10}s^{3}_{x}$	${}^{3}_{10}s^{4}_{x}$
-5	0.00000	0.00000	0.00000	0.00000	0.00000
0	0.99789	0.01248	0.02117	0.94616	0.01808
5	0.99812	0.01124	0.01874	0.95227	0.01587
10	0.99744	0.01364	0.02227	0.94467	0.01687
15	0.99644	0.01955	0.03073	0.92403	0.02214
20	0.99568	0.02080	0.03152	0.91916	0.02420
25	0.99411	0.01655	0.02502	0.93121	0.02133
30	0.99085	0.01199	0.01825	0.94378	0.01683
35	0.98613	0.00872	0.01322	0.95160	0.01259
40	0.98001	0.00613	0.00942	0.95539	0.00906
45	0.97239	0.00460	0.00709	0.95381	0.00659
50	0.96062	0.00572	0.00594	0.94591	0.00503
55	0.94449	0.00332	0.00541	0.93164	0.00412
60	0.91783	0.00357	0.00516	0.90559	0.00371
65	0.87664	0.00363	0.00529	0.86421	0.00351
70	0.81141	0.00373	0.00546	0.79888	0.00334
75	0.71397	0.00341	0.00502	0.70254	0.00300

FOR THOSE BORN IN THE NORTH CENTRAL

x	${}^{3}_{20}s^{\cdot}_{x}$	${}^{3}_{20}s^{1}_{x}$	${}^{3}_{20}s^{2}_{x}$	${}^{3}_{20}s^{3}_{x}$	${}^{3}_{20}s^{4}_{x}$
-5	0.00000	0.00000	0.00000	0.00000	0.0000
0	0.99789	0.01248	0.02117	0.94616	0.01808
5	0.99812	0.01127	0.01879	0.95214	0.01591
10	0.99746	0.01357	0.02217	0.94490	0.01682
15	0.99645	0.01945	0.03061	0.92435	0.02204
20	0.99569	0.02081	0.03155	0.91912	0.02421
25	0.99411	0.01655	0.02502	0.93121	0.02133
30	0.99085	0.01199	0.01825	0.94379	0.01683
35	0.98614	0.00872	0.01322	0.95160	0.01259
40	0.98001	0.00613	0.00943	0.95539	0.00906
45	0.97210	0.00460	0.00710	0.95381	0.00660
50	0.96062	0.00372	0.00594.	0.94593	0.00503
55	0.94455	0.00332	0.00541	0.93169	0.00412
60	0.91798	0.00337	0.00516	0.90574	0.00371
65	0.87682	0.00363	0.00529	0.86440	0.00352
70	0.81159	0.00373	0.00546	0.79906	0.00334
75	0.71410	0.00341	0.00502	0.70267	0.00300

Table 5. (Continued)

FOR THOSE BORN IN THE SOUTH

x	${}^{3}_{30}s^{\cdot}_{x}$	${}^{3}_{30}s^{1}_{x}$	${}^{3}_{30}s^{2}_{x}$	${}^{3}_{30}s^{3}_{x}$	${}^{3}_{30}s^{4}_{x}$
-5	0.98689	0.01135	0.01722	0.94329	0.01504
0	0.98511	0.01790	0.02820	0.91458	0.02444
5	0.99807	0.01149	0.01923	0.95104	0.01631
10	0.99756	0.01320	0.02158	0.94624	0.01654
15	0.99650	0.01901	0.03002	0.92585	0.02161
20	0.99575	0.02099	0.03189	0.91860	0.02427
25	0.99423	0.01686	0.02546	0.93033	0.02158
30	0.99101	0.01212	0.01845	0.94345	0.01699
35	0.98627	0.00880	0.01334	0.95143	0.01270
40	0.98014	0.00617	0.00948	0.95537	0.00911
45	0.97222	0.00461	0.00712	0.95387	0.00662
50	0.96079	0.00373	0.00595	0.94606	0.00504
55	0.94479	0.00332	0.00542	0.93192	0.00413
60	0.91851	0.00337	0.00516	0.90626	0.00371
65	0.87743	0.00362	0.00528	0.86500	0.00352
70	0.81230	0.00373	0.00546	0.79976	0.00334
75	0.71482	0.00342	0.00503	0.70338	0.00300

FOR THOSE BORN IN THE WEST

x	${}^{3}_{40}s^{\cdot}_{x}$	${}^{3}_{40}s^{1}_{x}$	${}^{3}_{40}s^{2}_{x}$	${}^{3}_{40}s^{3}_{x}$	${}^{3}_{40}s^{4}_{x}$
-5	0.00000	0.00000	0.00000	0.00000	0.00000
0	0.99789	0.01248	0.02117	0.94616	0.01808
5	0.99812	0.01125	0.01876	0.95222	0.01588
10	0.99746	0.01356	0.02215	0.94494	0.01681
15	0.99646	0.01937	0.03050	0.92463	0.02196
20	0.99570	0.02084	0.03160	0.91903	0.02422
25	0.99411	0.01655	0.02503	0.93119	0.02134
30	0.99085	0.01199	0.01824	0.94380	0.01682
35	0.98614	0.00872	0.01322	0.95160	0.01259
40	0.98002	0.00614	0.00943	0.95539	0.00906
45	0.97212	0.00460	0.00710	0.95382	0.00660
50	0.96066	0.00373	0.00594	0.94596	0.00503
55	0.94463	0.00332	0.00541	0.93177	0.00413
60	0.91818	0.00337	0.00516	0.90593	0.00371
65	0.87704	0.00362	0.00528	0.86462	0.00352
70	0.81179	0.00373	0.00546	0.79926	0.00334
75	0.71426	0.00342	0.00502	0.70282	0.00300

Table 6. Multiregional life table, linear case, United States, four region system (1965-1970), females, consistent age-specific movement and transition rates (South Region)

MOVEMENT RATES

x	${}^{3\delta}m_x$	${}^{31}m_x$	${}^{32}m_x$	${}^{33}m_x$	${}^{34}m_x$
0	0.00533	0.00491	0.00755	0.00000	0.00669
5	0.00042	0.00259	0.00441	0.00000	0.00379
10	0.00035	0.00216	0.00356	0.00000	0.00299
15	0.00064	0.00338	0.00552	0.00000	0.00390
20	0.00077	0.00472	0.00735	0.00000	0.00520
25	0.00095	0.00417	0.00628	0.00000	0.00505
30	0.00140	0.00288	0.00445	0.00000	0.00404
35	0.00226	0.00216	0.00326	0.00000	0.00308
40	0.00331	0.00149	0.00229	0.00000	0.00221
45	0.00475	0.00107	0.00165	0.00000	0.00157
50	0.00656	0.00086	0.00132	0.00000	0.00118
55	0.00951	0.00072	0.00119	0.00000	0.00093
60	0.01332	0.00071	0.00114	0.00000	0.00082
65	0.02100	0.00079	0.00115	0.00000	0.00079
70	0.03196	0.00091	0.00131	0.00000	0.00080
75	0.05314	0.00098	0.00142	0.00000	0.00083
80	0.08538	0.00094	0.00136	0.00000	0.00079
85	0.14944	0.00102	0.00148	0.00000	0.00085

TRANSITION RATES

x	${}^{3\delta}\hat{m}_x$	${}^{31}\hat{m}_x$	${}^{32}\hat{m}_x$	${}^{33}\hat{m}_x$	${}^{34}\hat{m}_x$
0	0.00531	0.00481	0.00730	0.00000	0.00638
5	0.00042	0.00256	0.00435	0.00000	0.00372
10	0.00035	0.00214	0.00351	0.00000	0.00295
15	0.00064	0.00332	0.00540	0.00000	0.00387
20	0.00077	0.00462	0.00713	0.00000	0.00515
25	0.00094	0.00411	0.00613	0.00000	0.00496
30	0.00139	0.00284	0.00437	0.00000	0.00395
35	0.00225	0.00213	0.00320	0.00000	0.00302
40	0.00331	0.00147	0.00225	0.00000	0.00217
45	0.00475	0.00105	0.00162	0.00000	0.00155
50	0.00656	0.00084	0.00129	0.00000	0.00116
55	0.00951	0.00070	0.00115	0.00000	0.00091
60	0.01332	0.00068	0.00109	0.00000	0.00080
65	0.02099	0.00074	0.00108	0.00000	0.00076
70	0.03197	0.00082	0.00120	0.00000	0.00075
75	0.05316	0.00084	0.00123	0.00000	0.00074
80	0.08540	0.00075	0.00110	0.00000	0.00066
85	0.14948	0.00072	0.00105	0.00000	0.00063

Alternatively, a multiregional life table can be calculated by using an iterative-interpolative method similar to those developed earlier in both the movement and transition cases. In order to do this, the relevant elements of the movement and transition approaches must be combined so that the resulting iterative process relies on successive estimates of life table rates converging to the predetermined ones: movement death rates and transition migration rates. Application to our four-region system of the U.S. female population was performed by assuming small age intervals equal to 0.2 years (i.e. 1/25th of the normal age interval).

The sets of expectations of life and migraproduction rates for a woman born in the South region are displayed in Table 8. A comparison of the values of these multistate life table functions with those obtained in the linear case (Tables 3 and 4) indicates no dramatic change in the life table statistics so that the gains expected from the use of the interpolative-iterative methodology appear largely outweighed by the extra resources necessary to perform the iterative calculation of the age-specific survival probabilities. Although the calculations of these probabilities do not require, for any of the eighteen age groups considered, more than four iterations to obtain convergence,* the time required by a computer to perform these calculations is much greater than in the linear case.

Finally, the main advantage of the interpolative-iterative method is to permit the calculation of a mean duration of transfer, the estimates of which for the South Region are displayed in Table 7. It appears that, for all groups except the first,** the values of mean durations of transfers do not differ much from 5/2 (the value they take in the linear case). Note that mean duration of moves are consistently less than this value except for the age group 20-25.

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*The iteration process was stopped when the highest absolute value of the discrepancies between the life table and observed rates was narrowed down to less than 10^{-6}.

**As mentioned in Section II, the linear approximation was not used for the first age group which accounts for the higher infant mortality in the first year of life.

Table 7. Multiregional life table based on movement death rates and transition rates of migration, iterative-interpolative case, United States, four region system (1965-1970), females, mean duration of transfers and survival probabilities (South Region).

MEAN DURATIONS OF TRANSFERS

x	$^{3}a_{x}^{\delta}$	$^{3}a_{x}^{1}$	$^{3}a_{x}^{2}$	$^{3}a_{x}^{3}$	$^{3}a_{x}^{4}$
0	0.47452	2.39242	2.38043	0.00000	2.37207
5	2.32169	2.27972	2.32438	0.00000	2.31332
10	2.47299	2.27642	2.26770	0.00000	2.32227
15	2.55205	2.48824	2.50038	0.00000	2.46759
20	2.51086	2.56261	2.54639	0.00000	2.54618
25	2.51145	2.47885	2.45582	0.00000	2.48025
30	2.52601	2.39065	2.39942	0.00000	2.42991
35	2.53572	2.43486	2.42074	0.00000	2.43046
40	2.52691	2.40256	2.40545	0.00000	2.41510
45	2.52252	2.39614	2.39690	0.00000	2.40616
50	2.51602	2.42114	2.40756	0.00000	2.41142
55	2.51440	2.39071	2.42137	0.00000	2.40374
60	2.50340	2.39876	2.41172	0.00000	2.40749
65	2.49777	2.39145	2.36842	0.00000	2.40258
70	2.47199	2.36792	2.37221	0.00000	2.36643
75	2.45075	2.27400	2.28901	0.00000	2.29873
80	2.36369	1.99190	2.00797	0.00000	2.04733
85	0.69177	0.00000	0.00000	0.00000	0.00000

SURVIVAL PROBABILITIES

x	$^{3}p_{x}^{\delta}$	$^{3}p_{x}^{1}$	$^{3}p_{x}^{2}$	$^{3}p_{x}^{3}$	$^{3}p_{x}^{4}$
0	0.02594	0.02245	0.03401	0.88791	0.02971
5	0.00211	0.01245	0.02113	0.94626	0.01804
10	0.00173	0.01043	0.01715	0.95627	0.01442
15	0.00349	0.01606	0.02615	0.93588	0.01872
20	0.00384	0.02215	0.03416	0.91515	0.02470
25	0.00470	0.01974	0.02945	0.92229	0.02382
30	0.00692	0.01376	0.02116	0.93901	0.01915
35	0.01117	0.01037	0.01558	0.94819	0.01470
40	0.01640	0.00717	0.01099	0.95484	0.01060
45	0.02345	0.00512	0.00791	0.95595	0.00756
50	0.03228	0.00408	0.00629	0.95171	0.00564
55	0.04645	0.00337	0.00559	0.94017	0.00442
60	0.06444	0.00326	0.00523	0.92325	0.00383
65	0.09974	0.00348	0.00508	0.88811	0.00359
70	0.14788	0.00377	0.00549	0.83942	0.00344
75	0.23392	0.00366	0.00539	0.75381	0.00322
80	0.34855	0.00304	0.00447	0.64127	0.00267
85	1.00000	0.00000	0.00000	0.00000	0.00000

Table 8. Multiregional life table based on movement death rates and transition rates of migration, iterative-Interpolative case, United States, four region system (1965-1970), females, expectations of life and net migraproduction rates by place-of-birth (South Region)

EXPECTATIONS OF LIFE

x	$^{3}e_{x}^{\cdot}$	$^{3}e_{x}^{1}$	$^{3}e_{x}^{2}$	$^{3}e_{x}^{3}$	$^{3}e_{x}^{4}$
0	74.32567	5.72116	8.68842	52.22654	7.68955
5	71.23811	5.81594	8.83245	48.77168	7.81803
10	66.38189	5.68537	8.62997	44.42331	7.64323
15	61.48966	5.50357	8.34297	40.24518	7.39794
20	56.67507	5.27789	7.98435	36.31502	7.09781
25	51.87767	4.98499	7.52399	32.64922	6.71947
30	47.10025	4.62295	6.96826	29.24563	6.26341
35	42.39131	4.21583	6.35395	26.06501	5.75652
40	37.81161	3.78843	5.71531	23.08195	5.22591
45	33.36812	3.35254	5.06931	20.26142	4.68484
50	29.07209	2.91744	4.42823	17.58305	4.14337
55	24.93274	2.48963	3.80024	15.03451	3.60836
60	20.99853	2.07868	3.19800	12.63116	3.09069
65	17.26290	1.68886	2.62594	10.35658	2.59153
70	13.88018	1.33794	2.10945	8.29920	2.13358
75	10.87867	1.02801	1.65049	6.47909	1.72108
80	8.47967	0.77923	1.28023	5.02857	1.39165
85	6.78740	0.60014	1.01122	4.00895	1.16709

NET MIGRAPRODUCTION RATES

x	$^{3}n_{x}^{\cdot}$	$^{3}n_{x}^{1}$	$^{3}n_{x}^{2}$	$^{3}n_{x}^{3}$	$^{3}n_{x}^{4}$
0	0.72426	0.04893	0.08730	0.51759	0.07044
5	0.62969	0.04819	0.08580	0.42707	0.06864
10	0.57560	0.04697	0.08347	0.37910	0.06605
15	0.53190	0.04552	0.08083	0.34210	0.06345
20	0.46821	0.04254	0.07575	0.29033	0.05959
25	0.38211	0.03757	0.06669	0.22496	0.05290
30	0.30377	0.03242	0.05696	0.17014	0.04425
35	0.24483	0.02792	0.04873	0.13204	0.03614
40	0.20043	0.02420	0.04192	0.10466	0.02964
45	0.16890	0.02137	0.03671	0.08590	0.02492
50	0.14572	0.01907	0.03246	0.07273	0.02146
55	0.12666	0.01691	0.02852	0.06259	0.01865
60	0.10838	0.01426	0.02401	0.05413	0.01598
65	0.08923	0.01100	0.01876	0.04627	0.01321
70	0.07074	0.00789	0.01392	0.03839	0.01055
75	0.05407	0.00566	0.01028	0.02995	0.00818
80	0.03852	0.00394	0.00726	0.02134	0.00598
85	0.02449	0.00244	0.00457	0.01353	0.00394

Table 9. Multiregional life table based on the movement approach, linear case, United States, four region system (1965-1970), females, age-specific survival probabilities and expectation of life by place-of-birth (South Region).

SURVIVAL PROBABILITIES

x	${}^{3}p_{x}^{\delta}$	${}^{3}p_{x}^{1}$	${}^{3}p_{x}^{2}$	${}^{3}p_{x}^{3}$	${}^{3}p_{x}^{4}$
0	0.02622	0.02229	0.03338	0.88939	0.02873
5	0.00211	0.01238	0.02089	0.94688	0.01774
10	0.00173	0.01036	0.01698	0.95667	0.01426
15	0.00319	0.01579	0.02561	0.93683	0.01858
20	0.00384	0.02170	0.03318	0.91680	0.02448
25	0.00470	0.01948	0.02879	0.92363	0.02340
30	0.00692	0.01362	0.02081	0.93989	0.01876
35	0.01117	0.01025	0.01534	0.94881	0.01444
40	0.01640	0.00708	0.01081	0.95526	0.01044
45	0.02345	0.00504	0.00777	0.95629	0.00745
50	0.03228	0.00399	0.00615	0.95204	0.00554
55	0.04644	0.00326	0.00541	0.94056	0.00432
60	0.06444	0.00311	0.00500	0.92372	0.00371
65	0.09974	0.00326	0.00478	0.88879	0.00344
70	0.14800	0.00343	0.00503	0.84033	0.00321
75	0.23462	0.00317	0.00472	0.75461	0.00288
80	0.35186	0.00247	0.00368	0.63973	0.00225
85	1.00000	0.00000	0.00000	0.00000	0.00000

EXPECTATIONS OF LIFE

x	${}^{3}e_{x}^{\cdot}$	${}^{3}e_{x}^{1}$	${}^{3}e_{x}^{2}$	${}^{3}e_{x}^{3}$	${}^{3}e_{x}^{4}$
0	74.29087	5.67333	8.56229	52.48186	7.57339
5	71.22379	5.76887	8.70714	49.04425	7.70353
10	66.36758	5.63900	8.50802	44.68658	7.53397
15	61.47551	5.45813	8.22509	40.49767	7.29462
20	56.66088	5.23384	7.87136	36.55523	7.00046
25	51.86352	4.94335	7.41797	32.87400	6.62820
30	47.08619	4.58447	6.87103	29.45201	6.17868
35	42.37746	4.18063	6.26596	26.25195	5.67892
40	37.79800	3.75641	5.63635	23.24961	5.15563
45	33.35469	3.32358	4.99902	20.41033	4.62176
50	29.05876	2.89141	4.36620	17.71391	4.08723
55	24.91922	2.46637	3.74601	15.14799	3.55884
60	20.98454	2.05796	3.15095	12.72825	3.04737
65	17.24808	1.67031	2.58530	10.43845	2.55401
70	13.86414	1.32097	2.07400	8.36803	2.10114
75	10.86168	1.01215	1.61912	6.53742	1.69299
80	8.46804	0.76460	1.25238	5.08359	1.36747
85	6.74323	0.58163	0.97953	4.04025	1.14180

Let us recall that these mean durations of transfers are derived from the matrix $\overset{0}{\underset{\sim}{a}}_x$ defined in Section II. However, although the matrix was found to be equal to $\underset{\sim}{I} - T(\underset{\sim}{I} - \underset{\sim}{p}_x)^{-1}\underset{\sim}{p}_x \, \underset{\sim}{m}_x$, this last expression cannot be used to estimate $\overset{0}{\underset{\sim}{a}}_x$ because a computer does not provide a precise estimation of the difference between the matrices $\underset{\sim}{I}$ and $T(\underset{\sim}{I} - \underset{\sim}{p}_x)^{-1}\underset{\sim}{p}_x \, \underset{\sim}{m}_x$. Therefore, the matrix $\overset{0}{a}_x$ was obtained from (49) rewritten in matrix form as:

$$\overset{0}{\underset{\sim}{a}}_x = (\underset{\sim}{L}_x - T\underset{\sim}{l}_{x+t})\underset{\sim}{L}_x^{-1} \quad .$$

Numerical Assessment of the Discrepancy Between Movement, Transition, and Mixed Approaches

In order to assess numerically the discrepancy between the alternative approaches to multiregional life table construction, we have applied the formulas (linear case) of both the movement and transition approaches using the age-specific rates previously used as input data. Table 9 shows the survival probabilities and expectations of life (by place-of-birth) of U.S. females in the South region. This was obtained by constructing the multiregional life table of the four region system of the female population from the movement approach with transition migration rates substituted for movement migration rates.

As expected, the probabilities of dying obtained with such a method are almost identical to those acquired earlier using the correct mixed method. In contrast to this, the outmigration probabilities appear to be much smaller than when correctly estimated, the discrepancy becoming larger in the older age groups (compare the bottom parts of Tables 3 and 9). Thus, the use of the movement approach when only transition rates are available, has little consequence on the total expectations of life but may modify the estimates of their regional shares. For example, a woman born in the South has a life expectancy of 74.29 years (versus 74.31) which is allocated among the regions as follows: North East 5.67 years (versus 5.73), North Central 8.56 years (versus 8.70), South 52.48 (versus 52.15) and West 7.57 years (versus 7.70).

On the other hand, constructing the multiregional life table of the same population system from the transition approach with movement mortality rates substituted for the transition mortality rates leads to survival probabilities and expectation of life (top parts of Tables 10 and 11) which only differ slightly from their correct values (see Tables 1 and 2).*

To summarize, the type of mortality and migration data commonly available calls for a third approach, the *mixed approach*, to the construction of a multiregional life table. Based on movement mortality rates and transition migration rates, it is in fact, a slightly modified variant of the transition approach in which movement mortality rates are used as inputs rather than transition mortality rates. It turns out that, since the discrepancy between movement and transition mortality rates is small, the numerical values of the multistate life table function obtained with the mixed appraoch do not significantly differ from those obtained with the transition approach. In contrast to this, the use of the movement approach rather than the use of the mixed approach would yield more inaccurate results.

Calculation of a Multiregional Life Table (Option 2)

So far, the calculation of multiregional life tables has been performed by simply setting life table age-specific rates equal to their observed counterparts. Rogers (1975) has developed an alternative, generalizing the census survival method

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*Note that the transition approach, as developed in Section III, does not rule out, as in Rogers' transition approach, the occurrence of a migration followed by a death within the same unit interval. Tables 10 and 11 - the bottom parts of which show what the age-specific survival probabilities and expectations of life would be using Rogers' transition approach - indicate the necessity of using the revised approach developed here rather than Rogers' approach. For instance, ruling out the possibility for an individual to die before the end of the time period in which he has moved from one region to another, contributes to increasing the expectations of life by a large amount: in the case of the South region, life expectancy at birth then increases from 74.29 to 74.52 years.

Table 10. Multiregional life table based on the transition approach, linear case, United States, four region system (1965-1970), females, age-specific survival probabilities from revised and Rogers' definitions (South Region).

FROM REVISED DEFINITION OF SURVIVAL PROBABILITIES

x	$^{3\hat{}\delta}p_x$	$^{3\hat{}1}p_x$	$^{3\hat{}2}p_x$	$^{3\hat{}3}p_x$	$^{3\hat{}4}p_x$
0	0.02631	0.02270	0.03443	0.88648	0.03008
5	0.00212	0.01248	0.02117	0.94616	0.01808
10	0.00174	0.01045	0.01718	0.95618	0.01445
15	0.00320	0.01607	0.02615	0.93586	0.01872
20	0.00386	0.02214	0.03413	0.91519	0.02468
25	0.00473	0.01974	0.02946	0.92223	0.02383
30	0.00696	0.01378	0.02117	0.93892	0.01917
35	0.01121	0.01037	0.01559	0.94812	0.01470
40	0.01644	0.00717	0.01099	0.95479	0.01061
45	0.02349	0.00513	0.00791	0.95591	0.00756
50	0.03229	0.00408	0.00629	0.95169	0.00564
55	0.04646	0.00337	0.00559	0.94016	0.00442
60	0.06445	0.00326	0.00524	0.92322	0.00383
65	0.09975	0.00348	0.00509	0.88809	0.00359
70	0.14796	0.00378	0.00550	0.83933	0.00344
75	0.23455	0.00368	0.00541	0.75314	0.00323
80	0.35181	0.00307	0.00452	0.63790	0.00270
85	1.00000	0.00000	0.00000	0.00000	0.00000

FROM ROGERS' DEFINITION OF SURVIVAL PROBABILITIES

x	$^{3\hat{}\delta}p_x$	$^{3\hat{}1}p_x$	$^{3\hat{}2}p_x$	$^{3\hat{}3}p_x$	$^{3\hat{}4}p_x$
0	0.02516	0.02271	0.03445	0.88758	0.03009
5	0.00207	0.01248	0.02117	0.94621	0.01808
10	0.00170	0.01045	0.01718	0.95622	0.01445
15	0.00310	0.01607	0.02615	0.93596	0.01872
20	0.00370	0.02214	0.03414	0.91534	0.02468
25	0.00456	0.01975	0.02946	0.92240	0.02383
30	0.00677	0.01378	0.02118	0.93911	0.01917
35	0.01099	0.01037	0.01559	0.94834	0.01471
40	0.01620	0.00717	0.01099	0.95502	0.01061
45	0.02320	0.00513	0.00791	0.95615	0.00757
50	0.03204	0.00408	0.00630	0.95194	0.00564
55	0.04615	0.00337	0.00559	0.94047	0.00442
60	0.06405	0.00326	0.00524	0.92362	0.00383
65	0.09914	0.00349	0.00509	0.88870	0.00359
70	0.14700	0.00378	0.00550	0.84026	0.00344
75	0.23310	0.00368	0.00541	0.75457	0.00323
80	0.34999	0.00308	0.00452	0.63970	0.00270
85	1.00000	0.00000	0.00000	0.00000	0.00000

Table 11. Multiregional life table based on the transition approach, linear case, United States, four region system (1965-1970), females, expectations of life at birth from revised and Rogers' definitions of survival probabilities, (South Region).

FROM REVISED DEFINITION OF SURVIVAL PROBABILITIES

x	$^{3}e_x^{:}$	$^{3}e_x^{1}$	$^{3}e_x^{2}$	$^{3}e_x^{3}$	$^{3}e_x^{4}$
0	74.29462	5.73260	8.70589	52.14762	7.70851
5	71.23465	5.82921	8.65273	48.71311	7.83959
10	66.37887	5.69724	8.64812	44.37053	7.66297
15	61.48711	5.51403	8.35900	40.19821	7.41587
20	56.67290	5.28700	7.99837	36.27350	7.11403
25	51.87630	4.99295	7.53633	32.61277	6.73425
30	47.09976	4.62992	6.97920	29.21371	6.27693
35	42.39171	4.22188	6.36359	26.03736	5.76888
40	37.81280	3.79358	5.72373	23.05831	5.23717
45	33.36982	3.35680	5.07654	20.24145	4.69504
50	29.07412	2.92083	4.43428	17.56647	4.15254
55	24.93489	2.49214	3.80513	15.02109	3.61653
60	21.00072	2.08035	3.20178	12.62067	3.09792
65	17.26531	1.68974	2.62867	10.34900	2.59790
70	13.88313	1.33814	2.11127	8.29451	2.13922
75	10.88406	1.02781	1.65176	6.47819	1.72629
80	8.49739	0.77977	1.28256	5.03707	1.39799
85	6.78874	0.59800	1.01065	4.00716	1.17294

FROM ROGERS' DEFINITION OF SURVIVAL PROBABILITIES

x	$^{3}e_x^{:}$	$^{3}e_x^{1}$	$^{3}e_x^{2}$	$^{3}e_x^{3}$	$^{3}e_x^{4}$
0	74.51743	5.75425	8.74080	52.28649	7.73589
5	71.37640	5.84453	8.87807	48.79539	7.85840
10	66.51716	5.71234	8.67314	44.45019	7.68148
15	61.62332	5.52902	8.38385	40.27623	7.43423
20	56.80588	5.30159	8.02260	36.34786	7.13183
25	51.99948	5.00688	7.55956	32.68190	6.75114
30	47.21518	4.64318	7.00139	29.27772	6.29288
35	42.49966	4.23450	6.38480	26.09645	5.78390
40	37.91285	3.80555	5.74391	23.11221	5.25118
45	33.46284	3.36821	5.09582	20.29067	4.70814
50	29.16121	2.93182	4.45286	17.61167	4.16486
55	25.01662	2.50279	3.82311	15.06259	3.62813
60	21.07654	2.09060	3.21902	12.65822	3.10870
65	17.33392	1.69936	2.64485	10.38207	2.60764
70	13.94361	1.34684	2.12603	8.32297	2.14776
75	10.93722	1.03550	1.66505	6.50291	1.73376
80	8.54422	0.78652	1.29460	5.05858	1.40452
85	6.83311	0.60428	1.02232	4.02742	1.17910

of the basic life table, in which the calculation is based on setting life table age-specific survivorship proportions equal to their observed counterparts.*

Generalities

In Section II, we have defined approximate survivorship proportions as

$$\underset{\sim}{s}_x = {}_0\underset{\sim}{L}_{x+T} \; {}_0\underset{\sim}{L}_x^{-1} \tag{45}$$

which can be rewritten in the linear integration variant as:

$$\underset{\sim}{s}_x = [\underset{\sim}{I} + \underset{\sim}{p}_{x+T}] \; \underset{\sim}{p}_x \; [\underset{\sim}{I} + \underset{\sim}{p}_x]^{-1} \qquad \forall x = 0,T,\ldots,z - 2T \quad . \tag{141}$$

This relationship indicates that $\underset{\sim}{p}_{x+T}$ can be derived if $\underset{\sim}{s}_x$ and $\underset{\sim}{p}_x$ are known and suggests that, if $\underset{\sim}{p}_0$ is available, the series of matrices $\underset{\sim}{p}_x$ (for x = T,...,z - T) can be obtained from the knowledge of the survivorship matrices for x = 0,...,z - 2T.

Since the following relationship holds between $\underset{\sim}{s}_{-T}$ and $\underset{\sim}{p}_0$

$$\underset{\sim}{s}_{-T} = \frac{1}{2}[\underset{\sim}{I} + \underset{\sim}{p}_0] \quad , \tag{142}$$

we can then derive $\underset{\sim}{p}_0$ from (142) in which $\underset{\sim}{s}_{-T}$ is set equal to the observed $\underset{\sim}{S}_{-T}$

$$\underset{\sim}{p}_0 = 2 \; \underset{\sim}{S}_{-T} - \underset{\sim}{I} \quad .$$

Then an estimate of $\underset{\sim}{p}_T$ can be obtained from the knowledge of $\underset{\sim}{s}_0$

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*The Option 2 method yields a unique set of age-specific transition probabilities. Mortality and migration rates consistent with both approaches (movement and transition rates) could then be estimated from the relationships expressing life table rates in terms of survival probabilities.

(set equal to $\underset{\sim}{S}_o$) using (141) rewritten as:

$$\underset{\sim}{p}_T = \underset{\sim}{p}_o^{-1}[\underset{\sim}{I} + \underset{\sim}{p}_o]\underset{\sim}{S}_o^{-1} - \underset{\sim}{I}$$

and so forth.

More generally, $\underset{\sim}{p}_x$ can be obtained from the observed $\underset{\sim}{S}_{x-T}$ and the just calculated $\underset{\sim}{p}_{x-T}$ by using

$$\underset{\sim}{p}_x = \underset{\sim}{p}_{x-T}^{-1}[\underset{\sim}{I} + \underset{\sim}{p}_{x-T}]\ \underset{\sim}{S}_{x-T}^{-1} - \underset{\sim}{I}$$

For the last age group, (141) is to be replaced by

$$\underset{\sim}{s}_{Z-T} = \frac{2}{T}\ \underset{\sim}{m}_Z^{-1}\ \underset{\sim}{p}_{Z-T}[\underset{\sim}{I} + \underset{\sim}{p}_{x+T}]^{-1}$$

so that an estimate of $\underset{\sim}{m}_z$ can be obtained from:

$$\underset{\sim}{m}_z = \frac{2}{T}\ \underset{\sim}{p}_{z-T}[\underset{\sim}{I} + \underset{\sim}{p}_{z-T}]^{-1}\ \underset{\sim}{S}_{z-T}$$

The availability of the series of age-specific survival probabilities (and the age-specific rates of the last age group) then allows for the complete calculation of a multiregional life table.*

This method of estimating the age-specific probabilities is initiated with the first age group (from an observed value of the survivorship proportions relating to the babies born in the

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*Note that, since there exists a simple relationship between mortality and mobility rates of the movement approach and survivorship proportions $\underset{\sim}{s}_x = (I + \frac{T}{2}\ \underset{\sim}{m}_{x+T})^{-1}(\underset{\sim}{I} - \frac{T}{2}\ \underset{\sim}{m}_x)$, the procedure described above can be used to directly obtain movement rates, thus bypassing the intermediate calculation of the survival probabilities.

period considered) while Rogers' (1975) calculations proceed from the last age group (from a value of $\underset{\sim}{M}_z$ that could not be observed and which had to be assumed).

The life table construction method just described can be used when the information available consists of either lifetime migration data for two consecutive censuses, or current migration and mortality data.

Calculation from Lifetime Migration Data

Suppose that the information available consists of lifetime migration data for two consecutive censuses, taken in years t and t + T.

Typically, the figures available for both census years describe the regional allocation of survivors by T-year age groups according to their place of birth. This permits the construction of age-specific $\underset{\sim}{K}_x^y$ whose $(i,j)^{th}$ element denotes the number of persons born in region j and aged x to x + T in region i at the time of the census (y = t and t + T).

Rogers and Von Rabenau (1971) have shown that the availability of such data allows for a simple measurement of the observed matrix of survivorship proportions:

$$\underset{\sim}{S}_x = \underset{\sim}{K}_{x+T}^{t+T} \underset{\sim}{K}_x^{t^{-1}} \qquad \forall x = 0,\ldots,z - T \quad .$$

In a similar way, the matrix of survivorship proportions relating to those born during the observation period can be measured from:

$$\underset{\sim}{S}_{-T} = \underset{\sim}{K}_0^{t+T} \underset{\sim}{B} \quad ,$$

in which B is a diagonal matrix whose typical element is the number of births that occurred in region i between years t and t + T.

Thus, lifetime migration data from two consecutive sources permits the measurement of the series of matrices of observed survivorship proportions allowing for the utilization of the Option 2 method.

Calculation from Current Mortality Rates and Migration Proportions

In this alternative case, the information regarding both mortality and mobility patterns is supposed to be identical to that used in Option 1. The mortality data are again converted into age-specific mortality rates consistent with the movement approach but the migration data are now used to measure survivorship proportions rather than migration rates. The problem is then one of estimating the mobility proportions that would prevail in absence of mortality over the observation period and then using the Option 2 method to obtain estimates of the migration rates.*

Typically, if z years and over represents the last age group considered, $\frac{z}{T} + 2$ matrices (for $x = 0, T, \ldots, z + T$) describing the transition flows (changes of residence) over the T-year period preceding the census are needed. Let $\underset{\sim}{AD}_x$ denote the matrix of age-specific transitions relating to age groups x to x + T** in which the flows of stayers (people present in the same region at the beginning and end of the observation period) are included in the diagonal.

Disregarding mortality, the fraction of those present in region j between ages x and x + T among the group of people present in region i, T years later is:

$$ {}_i{}^{m}S_x^j = \frac{{}^{i}AD^{j}_{x+T}}{\sum\limits_{j=1}^{n} {}^{i}AD^{j}_{x+T}} \qquad \begin{array}{l} \forall i,j = 1,\ldots,n \\ \forall x = 0,\ldots,z - 2T \end{array} \quad , $$

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*Because age-specific mortality and migration are not independent, the migration rates estimated here are slightly different from those that would be obtained if mortality was accounted for.

**${}^{i}AD^{j}_{x}$ denotes the number of people aged x - T to x in region i at the beginning of the period and present in region j, T years later.

while the corresponding fraction of those born between t and t + T in region i and present in region j at time t + T is:

$$ {}_{i}^{m}S^{j}_{-T} = \frac{{}^{i}AD^{j}_{0}}{\sum\limits_{j=1}^{n} {}^{i}AD^{j}_{0}} \qquad \forall i,j = 1,\ldots,n \quad . $$

In the case x = z - T, the numerator of the fraction of persons surviving contains two terms in order to be consistent with the treatment of the last age group in Section II

$$ {}_{i}^{m}S^{j}_{z-T} = \frac{{}^{i}AD^{j}_{z} + {}^{i}AD^{j}_{x+T}}{\sum\limits_{j=1}^{n} {}^{i}AD^{j}_{z}} \qquad \forall i,j = 1,\ldots,n \quad . $$

Having measured the observed mobility proportions, we then derive the movement rates of migration compatible with these observed mobility proportions.

Since:

$$ \underset{\sim}{s}_{-T} = \frac{1}{2}[\underset{\sim}{I} + \underset{\sim}{p}_{0}] = [\underset{\sim}{I} + \frac{T}{2}\,\underset{\sim}{m}_{0}]^{-1} \quad , $$

we obtain an estimate of the migration rates for the first age group from

$$ \underset{\sim}{m}_{0} = \frac{2}{T}[\underset{\sim}{s}_{-T}^{-1} - \underset{\sim}{I}] $$

in which $\overset{m}{\underset{\sim}{S}}_{-T} = ({}_{j}^{m}S^{i}_{-T})$ is substituted from $\underset{\sim}{s}_{-T}$. Then the migration rates for the second age group can be obtained from (67), rewritten as:

$$ \underset{\sim}{m}_{T} = \frac{2}{T}[(\underset{\sim}{I} - \frac{T}{2}\,\underset{\sim}{M}_{0})\ \underset{\sim}{s}_{0}^{-1} - \underset{\sim}{I}] $$

in which $\overset{m}{\underset{\sim}{S}}_0$ and the estimate of $\underset{\sim}{M}_0$ just derived are substituted for $\underset{\sim}{s}_0$ and $\underset{\sim}{m}_0$ and so forth.

To the matrices of these migration rates are then added the corresponding diagonal matrices of mortality rates which yields the matrices of rates $\underset{\sim}{M}_x$ needed to perform the calculation of a multiregional life table according to the Option 1 method.

Numerical Application

A numerical application of this method was performed from current mortality rates and mobility proportions for the four region system of the U.S. female population, previously considered (the period of observation was again 1965-1970). Unfortunately, the results turned out to be different from our expectations, since we obtained negative outmigration rates and, consequently, negative survival probabilities for some age groups. Nevertheless, we calculated the number of person-years lived in each age group and found acceptable results except in the case of the last two age groups where we obtained negative migration rates. We then calculated the expectations of life and approximate survivorship proportions shown in Table 12.

The question is then why the Option 2 method, unlike the normal construction method starting from observed rates, produces such unfortunate results. The answer is two-fold. *First*, the time process of the two methods is exactly reversed. On the one hand, the Option 1 method, based on mortality and migration figures observed in a given point in time,* calculates multistate functions from the assumption that these mortality and migration rates, and thus the resulting survival probabilities, remain constant over time. Indeed, the survivorship proportions to which this method lead are different from those which would be observed over the data collection period. On the other hand, Option

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*Although the migration data can be collected on a five-year period, the resulting migration rates are no more than averages characterizing the middle year of the data collection period.

Table 12. Multiregional life table, Option 2, linear case, United States, four region system (1965-1970), females, expectations of life and survivorship proportions (South Region).

EXPECTATIONS OF LIFE

x	${}^{3}e_{x}^{\cdot}$	${}^{3}e_{x}^{1}$	${}^{3}e_{x}^{2}$	${}^{3}e_{x}^{3}$	${}^{3}e_{x}^{4}$
0	73.91586	5.75522	8.66145	51.89149	7.60769
5	70.83619	5.83468	8.78951	48.49638	7.71561
10	65.97917	5.70759	8.59206	44.13222	7.54730
15	61.08656	5.53420	8.32031	39.92007	7.31198
20	56.27090	5.32505	7.98844	35.92502	7.03239
25	51.47200	5.03157	7.52498	32.26117	6.65428
30	46.69234	4.65340	6.95195	28.89957	6.18742
35	41.98043	4.23549	6.32474	25.74633	5.67387
40	37.39595	3.79845	5.67557	22.78340	5.13854
45	32.94567	3.35474	5.02175	19.97362	4.59556
50	28.63963	2.91175	4.37161	17.30617	4.05010
55	24.48614	2.47661	3.73550	14.76117	3.51287
60	20.53036	2.05598	3.12092	12.36491	2.98855
65	16.76285	1.65899	2.53828	10.08077	2.48481
70	13.32540	1.29699	2.00358	8.01507	2.00975
75	10.22880	0.97598	1.52183	6.15315	1.57785
80	7.61957	0.69666	1.09161	4.66203	1.16927
85	5.40088	0.40666	0.70978	3.43048	0.85396

APPROXIMATE SURVIVORSHIP PROPORTIONS

x	${}^{3}s_{x}^{\cdot}$	${}^{3}s_{x}^{1}$	${}^{3}s_{x}^{2}$	${}^{3}s_{x}^{3}$	${}^{3}s_{x}^{4}$
-5	0.98691	0.01466	0.02042	0.93300	0.01882
0	0.98563	0.01377	0.02390	0.92863	0.01933
5	0.99808	0.01023	0.01678	0.95564	0.01542
10	0.99754	0.01100	0.01813	0.95457	0.01385
15	0.99654	0.02323	0.03756	0.90980	0.02591
20	0.99574	0.02499	0.03636	0.90655	0.02784
25	0.99420	0.01553	0.02427	0.93295	0.02145
30	0.99097	0.01176	0.01770	0.94494	0.01657
35	0.98623	0.00800	0.01201	0.95477	0.01145
40	0.98011	0.00620	0.00978	0.95452	0.00962
45	0.97218	0.00414	0.00618	0.95620	0.00565
50	0.96077	0.00436	0.00702	0.94308	0.00630
55	0.94476	0.00234	0.00415	0.93575	0.00253
60	0.91851	0.00505	0.00788	0.89887	0.00672
65	0.87738	0.00195	0.00243	0.87250	0.00050
70	0.81206	0.00741	0.01194	0.78295	0.00975
75	0.71460	-0.00051	-0.00159	0.72040	-0.00371
80	0.86323	-0.01267	-0.02360	0.93171	-0.03222

2 starts from the observation of survivorship proportions and attempts to determine the constant (mortality and) migration rates or survival probabilities that would lead to such proportions. Unfortunately, the survivorship proportions are observed for a time period, say five years as in the above numerical illustration, during which the age-specific migration rates are not necessarily constant and may fluctuate greatly. Second, the nature of the Option 2 method does not permit us to estimate mortality/mobility rates and survival probabilities separately for each group. Since equation (141) relates statistics of two consecutive age groups, estimation errors made on a given age group are passed on the the next.

In brief, since migration is a more volatile phenomenon than mortality, (i.e., age-specific outmigration rates, unlike age-specific mortality rates, may present large fluctuations over a short period of time), the Option 2 method does not appear to be as useful a method for constructing a multiregional life table as for constructing a single-region life table.

Evaluation of the Alternative Variants in Multiregional Life Table Construction

As just seen, the choice of the Option 2 method as a way of constructing a multiregional life table must be avoided whenever possible: a multiregional life table is best constructed when using the Option 1 method based on the equalization of life table rates with their observed counterparts.

Moreover, because of the type of mortality and mobility data available, the mixed approach (a combination of the movement and transition approaches) must preferably be chosen among the variations of the Option 1 method. However, the use of the transition approach yields acceptable results in view of the slight discrepancy existing between corresponding movement and transition death rates. That statement would not be true if the movement approach was used instead. In other words, in contrast to the analysis of life status (Schoen and Nelson, 1974; Schoen, 1975)* the study of interregional migration generally requires the choice of the mixed approach which is closely related to the transition approach.

It is clear that the most feasible integration methods to derive $\{L_x\}$ are the linear and cubic integration methods. However, in contrast to the linear method that can be easily used whatever the approach chosen (movement, transition, mixed), the cubic method can only be used in the case of the movement approach. Since movement migration data are sometimes available, we have used this integration method to calculate a multiregional life table of the four region system for the U.S. female population in which the values of the observed transition rates were substituted for those of the movement rates. The age-specific survival probabilities thus obtained (Table 13) were then directly comparable with the ones similarly obtained when using a linear integration method (Table 9).

The result is that: a) the cubic integration method does not yield radically different estimates, b) the discrepancy between the linear and cubic estimates mostly affects the retention probabilities and the probabilities of dying, and c) this discrepancy tends to be higher for older ages (see age group 75 to 80).** The mean durations of transfers implied by the choice of the cubic integration method appear in the bottom part of Table 13.

The discrepancies between the linear and cubic integration methods on the one hand, and the linear and the interpolative-iterative methods on the other hand point in opposite directions. Whereas the interpolative-iterative method yields higher retention probabilities and smaller probabilities of dying than the linear integration method (as suggested by the comparison of the survival

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*The type of data available for the problem studied by Schoen makes the use of the movement approach preferable.

**The estimates of the survival probabilities for age groups 5 to 10 and 80 to 85 were identical in both Tables 9 and 13 since the linear integration method was substituted for the cubic integration method.

probabilities in Tables 3 and 7), the cubic integration method yields smaller retention probabilities and higher probabilities of dying. Also, the interpolative-iterative method yields (see Table 7) a_x^δ coefficients slightly higher than 2.5 (except for the first age group), while the cubic integration method leads to a_x^δ coefficients **much** higher than 2.5 (see Table 13).

If the interpolative-iterative method is assumed to be more accurate than any other method, it then appears that the linear integration method yields estimates of the multistate life table functions which are better than those of the cubic integration method. Then, even if its use is made possible by the type of data available, the cubic integration method will not be prefered to the linear integration method. Moreover, since the interpolative-iterative method yields estimates of the multistate life table functions only slightly different from those obtained in the linear case, the linear integration method would generally be prefered because of the larger computer time required for the interpolative-iterative method.

Finally, the mixed approach of the Option 1 method based on a linear integration over $\{l_y\}$ for deriving $\{L_x\}$ appears as the best variant in calculating a multiregional life table.*

Migration Rates and the Calculation of a Multiregional Life Table

Clearly, the accuracy of the columns of a multiregional life table calculated by the Option 1 method depends on the precision of observed mortality and mobility rates' measurement.

Impact of Alternative Measures of Transition Migration Rates

Whereas the measurement of movement rates as proposed by (134) and (135) does not raise any particular problem (straightforward extension of the single region case), the measurement of transition rates suggested in (136) and (137) raises some difficulties because the numerators and denominators of these definitions

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*The present conclusion is indeed limited to the case of a demographic system for which available data are movement mobility data and transition migration data. However, it can be extended to the case of any demographic system, as we will see later.

Table 13. Multiregional life table based on the movement approach, cubic case, United States, four region system (1965–1970), females, survival probabilities and mean durations of transfers (South Region).

SURVIVAL PROBABILITIES

x	${}^{3}p_x^{\delta}$	${}^{3}p_x^{1}$	${}^{3}p_x^{2}$	${}^{3}p_x^{3}$	${}^{3}p_x^{4}$
0	0.02622	0.02234	0.03349	0.88910	0.02884
5	0.00211	0.01238	0.02089	0.94688	0.01774
10	0.00173	0.01036	0.01699	0.95667	0.01426
15	0.00319	0.01582	0.02566	0.93672	0.01861
20	0.00384	0.02170	0.03316	0.91681	0.02448
25	0.00470	0.01943	0.02869	0.92382	0.02336
30	0.00692	0.01358	0.02074	0.94004	0.01871
35	0.01117	0.01023	0.01530	0.94889	0.01441
40	0.01641	0.00707	0.01080	0.95531	0.01042
45	0.02347	0.00504	0.00776	0.95630	0.00744
50	0.03230	0.00399	0.00615	0.95203	0.00554
55	0.04652	0.00326	0.00541	0.94051	0.00432
60	0.06456	0.00311	0.00500	0.92360	0.00372
65	0.10001	0.00326	0.00478	0.88852	0.00344
70	0.14855	0.00342	0.00502	0.83979	0.00321
75	0.23555	0.00313	0.00466	0.75379	0.00286
80	0.35186	0.00247	0.00368	0.63973	0.00225
85	1.00000	0.00000	0.00000	0.00000	0.00000

MEAN DURATIONS OF TRANSFERS

x	${}^{3}a_x^{\delta}$	${}^{3}a_x^{1}$	${}^{3}a_x^{2}$	${}^{3}a_x^{3}$	${}^{3}a_x^{4}$
0	2.51620	2.59865	2.60911	0.00000	2.60465
5	2.49986	2.50000	2.50000	0.00000	2.50000
10	2.62432	2.53131	2.51537	0.00000	2.46887
15	2.63360	2.61387	2.59328	0.00000	2.58048
20	2.57625	2.49532	2.47575	0.00000	2.50532
25	2.62855	2.36395	2.35279	0.00000	2.40650
30	2.68628	2.31195	2.31137	0.00000	2.35240
35	2.66887	2.33183	2.32342	0.00000	2.33639
40	2.64744	2.31271	2.31669	0.00000	2.32218
45	2.63958	2.33883	2.33875	0.00000	2.32780
50	2.63436	2.37042	2.38114	0.00000	2.34749
55	2.62593	2.39591	2.40888	0.00000	2.37073
60	2.64128	2.43464	2.41154	0.00000	2.39758
65	2.63475	2.42979	2.41150	0.00000	2.39780
70	2.62554	2.36789	2.37599	0.00000	2.35855
75	2.56507	2.26004	2.27092	0.00000	2.27750
80	2.50000	2.50000	2.50000	0.00000	2.50002
85	6.63177	0.00000	0.00000	0.00000	0.00000

must be approximated. As a matter of fact, Rogers (1975a) proposes approximations of these quantities different from those put down earlier in this paper. He simply measures the rate of migration from region i to region j for age group x, x + T as the ratio of the number of changes of residence (from i to j made during the observation period by those aged x to x + T at the end of the interval) to the average population in region i, i.e.,

$$ {}^{i}\hat{m}_{x}^{j} = \frac{{}^{i}K_{x}^{j}(t + T)}{\frac{T}{2}[K_{x}^{i}(t) + K_{x}^{i}(t + T)]} \quad . \tag{143}$$

Clearly, this contrasts with our measurement of ${}^{i}\hat{m}_{x}^{j}$ defined as

$$ {}^{i}\hat{m}_{x}^{j} = \frac{\frac{1}{2}[{}^{i}K_{x-T}^{j}(t + T) + {}^{i}K_{x}^{j}(t + T)]}{\frac{T}{2}[K_{x}^{i}(t) + \sum_{j=1}^{n} {}^{j}K_{x}^{i}(t + T)]} \quad , \tag{144}$$

a relationship obtained by combining (137) through (139).

The alternative measures of migration rates out of the South Region in our four region system of the U.S. calculated from (143) and (144) appear in Table 14 which shows that transition migration rates as measured by (143) are generally higher than when measured by (144).

Since the denominators of ${}^{i}\hat{m}_{x}^{j}$ in both (143) and (144) take similar values, the discrepancy between our measurement method and that of Rogers originates for the larger part from the different values taken by the numerators of (143) and (144). Indeed, the numerator of (143) concerns migrants who were aged x to x + T at the end of the observation period, while the numerator of (144) represents an approximation of the number of migrations from region i to region j relating to the people aged x to x + T in the middle year of the observation period. Clearly, our measurement method (144) is more legitimate than that of Rogers (143) which does not properly estimate the age disaggregation of migration flows over the observation period.

Table 14. United States, four region system (1965-1970), females, transition migration rates from Ledent's and Rogers' definitions contrasted.

FROM LEDENT'S DEFINITION OF MIGRATION RATES

x	$\overset{3\ \delta}{m_x}$	$\overset{3\hat{}1}{m_x}$	$\overset{3\hat{}2}{m_x}$	$\overset{3\hat{}3}{m_x}$	$\overset{3\hat{}4}{m_x}$
0	0.00533	0.00481	0.00730	0.00000	0.00638
5	0.00042	0.00256	0.00435	0.00000	0.00372
10	0.00035	0.00214	0.00351	0.00000	0.00295
15	0.00064	0.00332	0.00540	0.00000	0.00387
20	0.00077	0.00462	0.00713	0.00000	0.00515
25	0.00095	0.00411	0.00613	0.00000	0.00496
30	0.00140	0.00284	0.00437	0.00000	0.00395
35	0.00226	0.00213	0.00320	0.00000	0.00302
40	0.00331	0.00147	0.00225	0.00000	0.00217
45	0.00475	0.00105	0.00162	0.00000	0.00155
50	0.00656	0.00084	0.00129	0.00000	0.00116
55	0.00951	0.00070	0.00115	0.00000	0.00091
60	0.01332	0.00068	0.00109	0.00000	0.00080
65	0.02100	0.00074	0.00108	0.00000	0.00076
70	0.03196	0.00082	0.00120	0.00000	0.00075
75	0.05314	0.00084	0.00123	0.00000	0.00074
80	0.08538	0.00075	0.00110	0.00000	0.00066
85	0.14944	0.00072	0.00105	0.00000	0.00063

FROM ROGERS' DEFINITION OF MIGRATION RATES

x	$\overset{3\hat{}\delta}{m_x}$	$\overset{3\hat{}1}{m_x}$	$\overset{3\hat{}2}{m_x}$	$\overset{3\hat{}3}{m_x}$	$\overset{3\hat{}4}{m_x}$
0	0.00533	0.00324	0.00564	0.00000	0.00455
5	0.00042	0.00218	0.00357	0.00000	0.00328
10	0.00035	0.00209	0.00344	0.00000	0.00263
15	0.00064	0.00424	0.00686	0.00000	0.00474
20	0.00077	0.00407	0.00593	0.00000	0.00454
25	0.00095	0.00274	0.00427	0.00000	0.00378
30	0.00140	0.00231	0.00347	0.00000	0.00324
35	0.00226	0.00177	0.00267	0.00000	0.00255
40	0.00331	0.00120	0.00188	0.00000	0.00184
45	0.00475	0.00086	0.00129	0.00000	0.00119
50	0.00656	0.00070	0.00112	0.00000	0.00096
55	0.00951	0.00061	0.00106	0.00000	0.00075
60	0.01332	0.00065	0.00096	0.00000	0.00073
65	0.02100	0.00068	0.00098	0.00000	0.00064
70	0.03196	0.00072	0.00106	0.00000	0.00063
75	0.05314	0.00062	0.00092	0.00000	0.00055
80	0.08538	0.00045	0.00066	0.00000	0.00039
85	0.14944	0.00000	0.00000	0.00000	0.00000

The impact of the measurement of migration rates on age-specific transition probabilities can easily be obtained by analytically comparing the matrices $\underset{\sim}{p}_x$ and $\underset{\sim}{p}'_x$ corresponding to the alternative measurement of migration rates.

Let $\underset{\sim}{m}_x$ and $\underset{\sim}{m}'_x$ be the matrices of movement rates consistent with the two alternatives. Since

$$\underset{\sim}{p}_x - \underset{\sim}{p}'_x = [\underset{\sim}{I} + \underset{\sim}{p}_x] - [\underset{\sim}{I} + \underset{\sim}{p}'_x] \quad ,$$

in which

$$[\underset{\sim}{I} + \underset{\sim}{p}_x] = 2[\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x]^{-1}$$

$$[\underset{\sim}{I} + \underset{\sim}{p}'_x] = 2[\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}'_x]^{-1}$$

it follows that:

$$\underset{\sim}{p}_x - \underset{\sim}{p}'_x = 2[(\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x)^{-1} - (\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}'_x)^{-1}]$$

Replacing $\underset{\sim}{m}_x$ and $\underset{\sim}{m}'_x$ by their expressions in terms of transition rates such as

$$\underset{\sim}{m}_x = (\overset{\delta}{\hat{\underset{\sim}{m}}}_x + \overset{mt}{\hat{\underset{\sim}{m}}}_x - \overset{m}{\hat{\underset{\sim}{m}}}_x + \frac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_x\overset{mt}{\hat{\underset{\sim}{m}}}_x)(\underset{\sim}{I} + \frac{T}{2}\overset{m}{\hat{\underset{\sim}{m}}}_x)^{-1}$$

in which $\overset{\delta}{\hat{\underset{\sim}{m}}}_x$ is given by

$$\{\hat{m}^{\delta}_x\} = [\underset{\sim}{I} + \frac{T}{2}\overset{mt}{\hat{\underset{\sim}{m}}}_x]^{-1}[\underset{\sim}{I} + \frac{T}{2}\overset{m}{\hat{\underset{\sim}{m}}}_x]\{\underset{\sim}{m}^{\delta}_x\}$$

we then have

$$\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x = (\underset{\sim}{I} + \frac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_x)(\underset{\sim}{I} + \frac{T}{2}\overset{mt}{\hat{\underset{\sim}{m}}}_x)(\underset{\sim}{I} + \frac{T}{2}\overset{m}{\hat{m}}_x)^{-1}$$

and:

$$\underset{\sim}{p}_x - \underset{\sim}{p}'_x = 2\Big[\Big(\underset{\sim}{I} + \frac{T}{2}\overset{m}{\hat{\underset{\sim}{m}}}_x\Big)\Big(\underset{\sim}{I} + \frac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}_x\Big)^{-1}\Big(\underset{\sim}{I} + \frac{T}{2}\overset{mt}{\hat{\underset{\sim}{m}}}_x\Big)^{-1}$$

$$- \Big(\underset{\sim}{I} + \frac{T}{2}\overset{m}{\hat{\underset{\sim}{m}}}{}'_x\Big)\Big(\underset{\sim}{I} + \frac{T}{2}\overset{\delta}{\hat{\underset{\sim}{m}}}{}'_x\Big)^{-1}\Big(\underset{\sim}{I} + \frac{T}{2}\overset{mt}{\hat{\underset{\sim}{m}}}{}'_x\Big)^{-1}\Big] .$$

Since $\overset{\delta}{\hat{\underset{\sim}{m}}}_x$ and $\overset{mt}{\hat{\underset{\sim}{m}}}_x$ are diagonal matrices, we can write:

$${}^ip^i_x - {}^ip^{i\prime}_x = 2\left[\frac{1}{\Big(1 + \frac{T}{2}\,{}^i\hat{m}^\delta_x\Big)\Big(1 + \frac{T}{2}\sum\limits_{\substack{j=1\\ j\neq i}}^{n}{}^i\hat{m}^j_x\Big)} - \frac{1}{\Big(1 + \frac{T}{2}\,{}^i\hat{m}^{\delta\prime}_x\Big)\Big(1 + \frac{T}{2}\sum\limits_{\substack{j=1\\ j\neq i}}^{n}{}^i\hat{m}^{j\prime}_x\Big)}\right]$$

$$\forall i = 1,\ldots,n \quad . \qquad (145)$$

and:

$${}^ip^j_x - {}^ip^{j\prime}_x = T\left[\frac{{}^i\hat{m}^j_x}{\Big(1 + \frac{T}{2}\,{}^i\hat{m}^\delta_x\Big)\Big(1 + \frac{T}{2}\sum\limits_{\substack{j=1\\ j\neq i}}^{n}{}^i\hat{m}^j_x\Big)} - \frac{{}^i\hat{m}^{j\prime}_x}{\Big(1 + \frac{T}{2}\,{}^i\hat{m}^{\delta\prime}_x\Big)\Big(1 + \frac{T}{2}\sum\limits_{\substack{j=1\\ j\neq i}}^{n}{}^i\hat{m}^{j\prime}_x\Big)}\right]$$

$$\forall i,j = 1,\ldots,n \quad . \qquad (146)$$

Because of the impact of the measurement of migration rates $j \neq i$ on the values of the transition death rates:

- ${}^ip^i_x - {}^ip^{i\prime}_x$ is approximately proportional to $\sum\limits_j {}^i\hat{m}^{j\prime}_x - \sum\limits_j {}^i\hat{m}^j_x$

  (the higher the total outmigration rate of a given region the smaller the retention probability),
- ${}^ip^j_x - {}^ip^{j\prime}_x$ is more or less proportional to ${}^i\hat{m}^j_x - {}^i\hat{m}^{j\prime}_x$

  (the higher the rate of outmigration to a given region the higher the probability of moving to this region), and

- the magnitude of a variation Δm in the measurement of migration rates on the values of the probabilities of out-migrating is roughly $T\Delta m$ for an age group with small mortality and migration rates.

We also have:

$$ {}^{i}q_{x}^{\delta} - {}^{i}q_{x}^{\delta'} = 2\left[\frac{1}{1 + \frac{T}{2}\,{}^{i}\hat{m}_{x}^{\delta}} - \frac{1}{1 + \frac{T}{2}\,{}^{i}\hat{m}_{x}^{\delta'}} \right] , \qquad \forall i = 1,\ldots,n \quad . \tag{147} $$

which suggests that the probabilities of dying are only slightly affected by the measurement of migration rates.

The impact of the measurement of migration rates on other life table functions must be determined not analytically but numerically. We have calculated the multiregional life table functions - based on the mixed approach and a linear integration for deriving $\{L_x\}$ - which uses Rogers' definition of migration rates (see Tables 15 and 16). Their comparison with the life table functions obtained with our definition of migration rates reveals that, as it could be expected, probabilities of dying, total survivorship proportions, and total expectations of life appearing in the second columns of Tables 15 and 16 are clearly similar to those obtained with our definition of migration rates (see Tables 3 and 4). The only exception to this concerns the last age group for which Rogers posits zero migration rates.

Survival probabilities and approximate survivorship proportions are clearly sensitive to the measurement of migration rates: for example, the probabilities for a woman aged 35 in the South region to be in one of the four regions of the system five years later, North East, North Central, South, and West, taken in that order, are respectively 0.00866, 0.01303, 0.95470 and 0.01244 (versus 0.01037, 0.01559, 0.94817 and 0.01471 with our definition). These important differences then result in large discrepancies relating to the expectations of life (by place-of-birth) and especially to the migraproduction rates. The average number of moves out of the four U.S. census regions made by a woman born in the South region decreases from 0.72 (with our definition) to 0.59 (with Rogers' definition).

In brief, the columns of a multiregional life table are very sensitive to the proper measurement of migration rates as illustrated by the above comparison of two multiregional life tables constructed with alternative definitions of the migration rates. Moreover, it was shown that the discrepancy resulting from these two alternative definitions was much higher than the discrepancies implied by the theoretical points of choice, i.e., the choice between the transition and movement approaches or the choice between the linear method of integration and the interpolative-iterative method.

Influence of the Length of the Observation Period

Again, since the measurement of observed movement rates is a straightforward extension of the measurement of observed rates in the single-region case, the length of the time interval over which actual movement rates are observed does not raise any particular difficulty. However, as put forward at the beginning of this section, the correct measurement of transition migration rates for use in an increment-decrement life table implicitly requires that the length of the typical age group be equal to the duration of the period over which migration is recorded. Thus, the numerical applications concerning our U.S. four-region system were carried out from migration data relating to five-year age groups observed on a five-year period (1965-70).

It is clear that if the migration data had been available for a one-year period only, the resulting migration rates would have led to an overestimation of the probabilities of moving from one region to another. Intuitively, this problem stems from the well known fact that annual migration rates are higher than $1/n^{th}$ time n-year migration rates, owing to the peculiarities caused by multiple moves and especially return migration. (See Table 17 comparing typical one-year and five-year migration rates for British regions.)

Table 15. Multiregional life table based on movement death rates and transition rates of migration (Rogers' definition), linear case, United States, four region system (1965-1970), females, age-specific survival probabilities and approximate survivorship proportions (South Region).

SURVIVAL PROBABILITIES

x	${}_3p_x^{\delta}$	${}_3p_x^{1}$	${}_3p_x^{2}$	${}_3p_x^{3}$	${}_3p_x^{4}$
0	0.02625	0.01545	0.02691	0.90968	0.02170
5	0.00211	0.01063	0.01742	0.95383	0.01601
10	0.00173	0.01022	0.01686	0.95832	0.01286
15	0.00318	0.02038	0.03295	0.92075	0.02275
20	0.00384	0.01962	0.02854	0.92616	0.02184
25	0.00471	0.01329	0.02076	0.94287	0.01837
30	0.00693	0.01125	0.01691	0.94911	0.01580
35	0.01117	0.00866	0.01303	0.95470	0.01244
40	0.01641	0.00586	0.00921	0.95950	0.00902
45	0.02346	0.00421	0.00632	0.96017	0.00584
50	0.03228	0.00342	0.00546	0.95414	0.00470
55	0.04644	0.00297	0.00515	0.94177	0.00366
60	0.06445	0.00315	0.00464	0.92427	0.00350
65	0.09974	0.00320	0.00462	0.88943	0.00300
70	0.14800	0.00331	0.00487	0.84091	0.00291
75	0.23461	0.00274	0.00403	0.75622	0.00240
80	0.35185	0.00184	0.00270	0.64200	0.00161
85	1.00000	0.00000	0.00000	0.00000	0.00000

APPROXIMATE SURVIVORSHIP PROPORTIONS

x	${}_3s_x^{\cdot}$	${}_3s_x^{1}$	${}_3s_x^{2}$	${}_3s_x^{3}$	${}_3s_x^{4}$
-5	0.98688	0.00773	0.01346	0.95484	0.01085
0	0.98563	0.01325	0.02264	0.93055	0.01920
5	0.99808	0.01042	0.01716	0.95599	0.01452
10	0.99754	0.01514	0.02460	0.94009	0.01771
15	0.99649	0.02010	0.03089	0.92326	0.02224
20	0.99573	0.01662	0.02488	0.93411	0.02013
25	0.99419	0.01229	0.01890	0.94586	0.01714
30	0.99097	0.00999	0.01502	0.95178	0.01418
35	0.98623	0.00729	0.01116	0.95700	0.01077
40	0.98010	0.00504	0.00779	0.95981	0.00746
45	0.97218	0.00381	0.00588	0.95722	0.00527
50	0.96476	0.00318	0.00528	0.94812	0.00418
55	0.94476	0.00304	0.00487	0.93328	0.00357
60	0.91850	0.00315	0.00460	0.90752	0.00323
65	0.87737	0.00321	0.00468	0.86657	0.00292
70	0.81214	0.00296	0.00436	0.80222	0.00261
75	0.71456	0.00225	0.00331	0.70793	0.00197
80	1.05242	0.00168	0.00253	1.04654	0.00167

Table 16. Multiregional life table based on movement death rates and transition rates of migration (Rogers' definition), linear case, United States, four region system (1965-1970), females, age-specific expectations of life and net migraproduction rates by place-of-birth (South Region).

EXPECTATIONS OF LIFE

x	${}^{3}e_{x}^{\cdot}$	${}^{3}e_{x}^{1}$	${}^{3}e_{x}^{2}$	${}^{3}e_{x}^{3}$	${}^{3}e_{x}^{4}$
0	74.24352	5.04406	7.85896	54.43270	6.90780
5	71.17735	5.14034	8.00170	50.99703	7.03828
10	66.32155	5.04735	7.84109	46.52755	6.90556
15	61.42994	4.90611	7.60244	42.21471	6.70667
20	56.61558	4.70923	7.27459	38.18924	6.44252
25	51.81844	4.43703	6.83706	34.45030	6.09405
30	47.24178	4.10804	6.32308	30.93023	5.68042
35	42.53460	3.74599	5.76493	27.59895	5.22474
40	37.75743	3.36647	5.18628	24.45873	4.74595
45	33.31634	2.97958	4.60084	21.48034	4.25558
50	29.02276	2.59369	4.02010	18.64484	3.76413
55	24.88413	2.21437	3.45124	15.94049	3.27802
60	20.94991	1.85072	2.90587	13.38641	2.80691
65	17.21156	1.50553	2.38722	10.96773	2.35108
70	13.82451	1.19247	1.91579	8.78491	1.93134
75	10.81265	0.91196	1.49114	6.85822	1.55133
80	8.39987	0.68349	1.14357	5.32782	1.24500
85	6.63450	0.50960	0.07681	4.22221	1.02588

NET MIGRAPRODUCTION RATES

x	${}^{3}n_{x}^{\cdot}$	${}^{3}n_{x}^{1}$	${}^{3}n_{x}^{2}$	${}^{3}n_{x}^{3}$	${}^{3}n_{x}^{4}$
0	0.59437	0.03608	0.06511	0.44331	0.04986
5	0.52615	0.03567	0.06422	0.37735	0.04892
10	0.48030	0.03488	0.06256	0.33562	0.04724
15	0.43914	0.03373	0.06049	0.29945	0.04546
20	0.35882	0.03036	0.05451	0.23246	0.04149
25	0.28441	0.02646	0.04719	0.17521	0.03555
30	0.22834	0.02309	0.04072	0.13507	0.02947
35	0.18090	0.01972	0.03456	0.10290	0.02371
40	0.14387	0.01681	0.02925	0.07883	0.01899
45	0.11723	0.01453	0.02584	0.06229	0.01537
50	0.09852	0.01274	0.02165	0.05124	0.01288
55	0.08202	0.01086	0.01826	0.04214	0.01076
60	0.06494	0.00829	0.01393	0.03423	0.00849
65	0.04775	0.00545	0.00942	0.02660	0.00628
70	0.03281	0.00329	0.00605	0.01919	0.00428
75	0.01947	0.00191	0.00352	0.01145	0.00258
80	0.00801	0.00076	0.00142	0.00477	0.00107
85	0.00000	0.00000	0.00000	0.00000	0.00000

The impact of using migration rates relating to a period whose length is not equal to that of the typical age group can be assessed by comparing the age-specific transition probabilities obtained in the case of observation periods having the correct and incorrect lengths.

Table 17. Comparison between one-year and five-year rates.*

Region	One-year rates	Five-year rates
North	0.1101	0.3385
Yorkshire/Humberside	0.1061	0.3382
North West	0.1056	0.3233
East Midlands	0.1080	0.3254
West Midlands	0.1074	0.3302
East Anglia	0.1280	0.3519
South East	0.1269	0.3584
South West	0.1266	0.3602
Wales	0.0938	0.2913
Scotland	0.1187	0.3504

This can be performed by applying the formulas (145) through (147) in which the annual transition migration rates for the observation period having the correct length are contained in $\overset{m}{\hat{m}}_{\underset{\sim}{x}}$ and the rates corresponding to the alternative observation period are contained in $\overset{m}{\hat{m}}'_{x}$.

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*The one-year and five-year migration rates shown in this table are aggregate migration rates for the British Regions observed in 1970-71 and 1966-71 respectively. They are drawn from Ph. Rees, *The Measurement of Migration From Census Data and Other Sources*, Environment and Planning A, 9, 1977, 247-272.

Since $\overset{m}{\hat{\underset{\sim}{m}}}_x$ is generally much less than $\overset{m}{\hat{\underset{\sim}{m}}}'_x$ ($\overset{m}{\hat{\underset{\sim}{m}}}_x \approx .65\ \overset{m}{\hat{\underset{\sim}{m}}}'_x$ in the case of the British regions shown in Table 17), it follows that the use of annual migration data instead of five-year migration data in a model in which population is broken down into five-year age groups leads to inaccurate estimates of the multiregional life table functions. This is illustrated by the comparison of Tables 18 and 19 displaying four multiregional life table functions (survival probabilities, approximate survivorship proportions, expectations of life and migraproduction rates) obtained by multiplying all transition rates of our U.S. four-region example by 1/0.65 - with those of Tables 3 and 4.

For example, the life expectancy of a woman born in the South slightly increased form 74.31 to 74.39 years while the times of this life expectancy spent in other regions increase dramatically: 5.73 to 7.50 (North East), 8.71 to 11.2 (North Central) and 7.70 to 10.09 (West). The higher mobility is also reflected by the total migraproduction rate for a woman born in the South which jumps from .72 to 1.13.

Clearly, the difficulties relating to the measurement of migration rates (more specifically number of moves or transitions, length of the period of observation) have an impact on the calculation of multiregional life tables that is much larger than those created by methodological aspects. In the future, improved methods for calculating multiregional life tables should not focus so much on extending theoretical grounds (developed in this paper) but rather on proposing better methods of measuring migration rates from data commonly available.

Comparison of the Actual and Modeled Migration Processes

One of the strengths of the single-state life table is that its underlying mortality process replicates the actual mortality process.* The reason for this is that the propensity to die at

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*The discrepancy between actual and modeled mortality processes results from the more or less regular age composition of the observed population (owing to variations in the fertility pattern and, at a lesser degree, in the mortality pattern over time).

Table 18. Multiregional life table based on movement death rates and transition rates of migration, linear case, hypothetical four region system, age-specific survival probabilities and approximate survivorship proportions (South Region).

SURVIVAL PROBABILITIES

x	$^{3}p_x^{\delta}$	$^{3}p_x^{1}$	$^{3}p_x^{2}$	$^{3}p_x^{3}$	$^{3}p_x^{4}$
0	0.02617	0.03412	0.05175	0.84277	0.04520
5	0.00211	0.01893	0.03212	0.91941	0.02743
10	0.00173	0.01590	0.02614	0.93426	0.02198
15	0.00318	0.02432	0.03958	0.90456	0.02833
20	0.00382	0.03333	0.05139	0.87430	0.03716
25	0.00468	0.02979	0.04445	0.88512	0.03596
30	0.00690	0.02089	0.03211	0.91104	0.02906
35	0.01114	0.01578	0.02372	0.92697	0.02238
40	0.01638	0.01095	0.01678	0.93969	0.01620
45	0.02344	0.00784	0.01210	0.94505	0.01157
50	0.03226	0.00625	0.00964	0.94320	0.00864
55	0.04643	0.00517	0.00857	0.93306	0.00678
60	0.06444	0.00500	0.00803	0.91666	0.00587
65	0.09973	0.00534	0.00780	0.88163	0.00550
70	0.14803	0.00579	0.00843	0.83249	0.00527
75	0.23468	0.00564	0.00829	0.74645	0.00495
80	0.35191	0.00471	0.00693	0.63231	0.00414
85	1.00000	0.00000	0.00000	0.00000	0.00000

APPROXIMATE SURVIVORSHIP PROPORTIONS

x	$^{3}s_x^{\cdot}$	$^{3}s_x^{1}$	$^{3}s_x^{2}$	$^{3}s_x^{3}$	$^{3}s_x^{4}$
-5	0.98692	0.01706	0.02587	0.92139	0.02260
0	0.98564	0.02740	0.04342	0.87680	0.03803
5	0.99808	0.01746	0.02930	0.92637	0.02495
10	0.99755	0.01988	0.03247	0.92012	0.02508
15	0.99650	0.02857	0.04494	0.89050	0.03250
20	0.99576	0.03184	0.04832	0.87921	0.03639
25	0.99423	0.02565	0.03884	0.89704	0.03270
30	0.99100	0.01846	0.02815	0.91843	0.02597
35	0.98626	0.01346	0.02040	0.93293	0.01946
40	0.98013	0.00943	0.01450	0.94222	0.01398
45	0.97220	0.00704	0.01088	0.94414	0.01014
50	0.96017	0.00569	0.00907	0.93830	0.00772
55	0.94477	0.00505	0.00825	0.92517	0.00631
60	0.91650	0.00511	0.00785	0.89989	0.00565
65	0.87735	0.00548	0.00799	0.85856	0.00532
70	0.81208	0.00556	0.00815	0.79338	0.00499
75	0.71450	0.00503	0.00740	0.69765	0.00442
80	1.08874	0.01446	0.02187	1.03786	0.01455

Table 19. Multiregional life table based on movement death rates and transition rates of migration, linear case, hypothetical four region system, age-specific expectations of life and net migraproduction rates (South Region).

EXPECTATIONS OF LIFE

x	${}^{3}e_{x}^{\cdot}$	${}^{3}e_{x}^{1}$	${}^{3}e_{x}^{2}$	${}^{3}e_{x}^{3}$	${}^{3}e_{x}^{4}$
0	74.39092	7.50300	11.18121	45.61314	10.09358
5	71.32264	7.61702	11.34881	42.10804	10.24876
10	66.46544	7.41864	11.04158	38.02067	9.98454
15	61.57200	7.14946	10.61825	34.17978	9.62450
20	56.75594	6.82261	10.10113	30.64068	9.19152
25	51.95599	6.41103	9.45900	27.42826	8.65770
30	47.17459	5.91486	8.70741	24.52196	8.03036
35	42.45909	5.36644	7.89459	21.84830	7.34976
40	37.87025	4.79791	7.06258	19.35917	6.65060
45	33.41787	4.22386	6.23129	17.01551	5.94721
50	29.11219	3.65515	5.41401	14.79391	5.24912
55	24.96702	3.10033	4.62078	12.68216	4.56376
60	21.02714	2.57181	3.86701	10.68524	3.90309
65	17.29071	2.07715	3.16057	8.78423	3.26876
70	13.90597	1.63787	2.53065	7.04960	2.68785
75	10.91506	1.25579	1.97870	5.51257	2.16800
80	8.54647	0.95332	1.53949	4.29640	1.75725
85	6.87105	0.73436	1.22092	3.43655	1.47922

NET MIGRAPRODUCTION RATES

x	${}^{3}n_{x}^{\cdot}$	${}^{3}n_{x}^{1}$	${}^{3}n_{x}^{2}$	${}^{3}n_{x}^{3}$	${}^{3}n_{x}^{4}$
0	1.13394	0.09972	0.17555	0.71335	0.14533
5	0.98693	0.09797	0.17200	0.57587	0.14109
10	0.90310	0.09516	0.16661	0.50622	0.13511
15	0.83545	0.09186	0.16066	0.45369	0.12925
20	0.73689	0.08526	0.14940	0.38152	0.12070
25	0.60314	0.07451	0.12991	0.29258	0.10615
30	0.48104	0.06365	0.10961	0.21997	0.08781
35	0.38865	0.05435	0.09286	0.17039	0.07106
40	0.31898	0.04676	0.07924	0.13508	0.05790
45	0.26951	0.04104	0.06898	0.11101	0.04848
50	0.23292	0.03645	0.06068	0.09416	0.04164
55	0.20252	0.03217	0.05307	0.08118	0.03610
60	0.17261	0.02698	0.04444	0.07030	0.03089
65	0.14077	0.02065	0.03452	0.06013	0.02547
70	0.11035	0.01470	0.02548	0.04988	0.02030
75	0.08390	0.01051	0.01879	0.03889	0.01571
80	0.05989	0.00731	0.01330	0.02777	0.01151
85	0.03868	0.00460	0.00851	0.01788	0.00770

any age y in an observed population is roughly the same for all individuals of each age cohorts as hypothesized in the life table (this propensity to die only depends on the value of y).

Does the multiregional life table provide a similar duplication of the actual mobility process between regions of the system? First, note that the single-region assumption of homogeneous age cohorts cannot be extended to the multiregional case because the propensities to migrate vary among individuals in a very sensitive manner. Some individuals ("chronic" movers) have a tendency to move repeatedly. In actual populations, members of a group of individuals present at the same age in a given region have differential propensities to move, largely dependent on their past migratory history; the larger the number of moves made in the past, the higher their propensity to move.

The question is one of determining how unrealistic the assumptions contained in the multiregional life table are. Whatever the focus chosen (movement or transition approach), the multiregional life table (or more generally the increment-decrement life table) is in fact concerned with transitions between predetermined ages rather than with moves: in essence, it looks at net balances of migrations rather than at gross migration flows.*

The multiregional life table thus describes a (Markovian) <u>transition</u> scheme in which the consolidation of moves into transitions occurs within the model (movement approach) rather than outside of the model (transition approach).

The multiregional life table must be judged on its ability to replicate consolidated moves (transitions) rather than gross flows (moves). Let us summarize the two main alternatives (movement and transition approaches).

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*The multiregional life table functions generally relate to age-cohorts independently of the region of presence at any earlier age and only require the knowledge of consolidated moves (an exception to this is the case of migraproduction rates).

First, the movement approach permits us to calculate all moves accurately but not transitions because of the non-validity of the Markovian assumption in the real world (multiplying by x movement rates would result in an approximate multiplication by x of transition rates).* Therefore, its use is to be avoided when calculating a multiregional life table.

Second, the transition approach makes it possible to calculate accurately observed transitions if the transition migrations are appropriately chosen (i.e., if the length of the typical age group is equal to the length of the observation period).

The homogeneous and Markovian assumptions underlying the multiregional life table seriously limit the ability of the multiregional life table to replicate the observed migration process. On the one hand, the movement approach may lead to the duplication of gross moves but not to that of consolidated moves. On the other hand, the transition approach allows for a "reduced-form" duplication of transitions or consolidated moves, which fortunately is sufficient to calculate most multiregional life table columns.

A further consequence is that the movement migration rates which were derived earlier in this section, in agreement with the transition migration rates, were not true movement rates, but were those which led to the same multiregional life table as the transition migration rates.

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*The use of the Markovian assumption keeps multiple moves (and especially return moves) at a low level. For example, in the case of a two-region system, the ratio of return moves to the gross outmigration flow can be obtained as

$$r^{i}_{x} = \frac{{}^{j}m^{i}_{x}\ {}_{ix}L^{j}_{x}}{{}^{i}m^{j}_{x}\ {}_{ix}L^{i}_{x}} \quad .$$

CONCLUSION

The most important feature of combined life tables that allow entries as well as withdrawals, is the existence of more than one stationary population in the multi-radix case. If individuals are born in at least two states of the population system at hand, the solution of the differential equation (17) underlying such life tables shows that all are linear combinations of the r independent stationary populations generated by the survivors of each state-specific group of the initial cohorts. Consequently, in the case of more than one radix, life table functions characteristic of age groups depend on the relative weight accorded to the r independent stationary populations, i.e. depend on the state allocation of the initial cohort.

The consequence is that the construction of a coherent increment-decrement life table requires the additional assumption of the independence of life table rates vis-a-vis the allocation of the initial cohort. It is not correct to state that the definition (35) of movement rates

$$ {}^{i}m^{j}_{x} = \frac{{}^{i}d^{j}_{x}}{L^{i}_{x}} \tag{148} $$

is equivalent to (61)

$$ \underset{\sim}{m}_{x} = ({}_{0}\underset{\sim}{l}_{x} - {}_{0}\underset{\sim}{l}_{x+T})\ {}_{0}\underset{\sim}{L}_{x}^{-1} \quad . \tag{149} $$

(149) implies (148), but (148) does not imply (149). In fact, there is equivalence only if (148) holds not only for the whole stationary population but also for the independent stationary populations generated by each radix of the initial cohort:

$${}_{0,k}^{\;i}m_x^j = \frac{{}^{i}d_x^j}{L_x^i} \qquad \forall k = 1,\ldots,n \Longleftrightarrow \underset{\sim}{m}_x = ({}_0\underset{\sim}{l}_x - {}_0\underset{\sim}{l}_{x+T})\,{}_0\underset{\sim}{L}_x^{-1}$$

Another striking feature of the increment-decrement life tables and their associated multistate life table functions is the appropriateness of matrix notation that permits the derivation of multistate life table functions as simple extensions of the scalar life table functions of the single-state case. As shown earlier, the matrix format used by Rogers/Ledent (1974, 1976) and Rogers (1975) makes it possible to derive additional multistate life table functions with regard to the vector notation suggested by Schoen (1975). Note that this statement applies to the multiradix case as well as to the single-radix case.

Basically, there are two main approaches to constructing increment-decrement life table functions:

- The first approach emphasizes the movements of individuals between intercommunicating states (movement approach).
- The alternative approach focuses on the net movements of individuals determined by a simple comparison of the states of presence at the beginning and end of the period considered (transition and mixed approaches*).

A priori, the former approach appears more desirable since the latter is characterized by a certain loss of information in that transitions represent the net balances of the corresponding movements. However, since the methodology underlying the construction of increment-decrement life tables focuses on age-specific survival probabilities that are nothing but transition probabilities, the movement approach reduces to the consideration of transitions. The difference between the movement and the transition approaches is that the reduction in scope from movements to transitions occurs within the model rather than outside of it.

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*The mixed approach which emphasized deaths as moves and migratory movements as transitions is a slight variation of the transition approach.

Which approach is most suitable in practice? Earlier, in Section V, we suggested that the utility of an increment-decrement life table depended on its ability to replicate the actual processes of the demographic events at hand. We then showed that the transition (mixed) approach, in the context of interregional migration, was more appropriate than the movement approach. In fact, the less the hypothesis of independence of moves holds vis-a-vis the past history of individuals, the less desirable is the utilization of the movement approach. In any case, whenever the necessary data are available, the transition (mixed) approach is the more desirable, since it permits to avoid the problems associated with multiple moves.

Among the two alternative options of increment-decrement life table construction, Option 1 - based on equating life table and observed mortality and mobility rates is a more reliable method than Option 2. Nevertheless, Option 2 can be used when there is a lack of data. However, the results will not necessarily be accurate. The numerical estimates of the multistate life table functions depend on the choice of the integration method for deriving $\{L_x\}$. Two alternative variations have mainly been explored in the course of this paper: a linear integration method and an interative-interpolative method.

Note that, in the linear case, explicit expressions of the life table functions can be obtained as shown in Sections II and III. In fact, we have established the existence of a general formula for estimating the age-specific survival probabilities:

$$\mathbf{p}_x = [\mathbf{I} - \frac{T}{2}\mathbf{u}_x][\mathbf{I} + \frac{T}{2}\mathbf{v}_x]^{-1}$$

in which $\mathbf{u}_x$ and $\mathbf{v}_x$ are to be taken as follows, according to the approach chosen:

a) movement approach: $\mathbf{u}_x = \mathbf{v}_x = \mathbf{m}_x$

b) transition approach: $\mathbf{u}_x = \hat{\mathbf{m}}_x - \overset{m}{\hat{\mathbf{m}}}_x + \frac{T}{2}\overset{\delta}{\hat{\mathbf{m}}}_x \overset{mt}{\hat{\mathbf{m}}}_x$

$$\mathbf{v}_x = \hat{\mathbf{m}}_x + \overset{m}{\hat{\mathbf{m}}}_x + \frac{T}{2}\overset{\delta}{\hat{\mathbf{m}}}_x \overset{mt}{\hat{\mathbf{m}}}_x$$

c) mixed approach: (emphasizing deaths as moves and migratory movements as transitions)

$$u_x = \underset{\sim}{\hat{m}}_x - \underset{\sim}{\overset{m}{\hat{m}}}_x + \frac{T}{2}\,\underset{\sim}{\overset{\delta}{\hat{m}}}_x\,\underset{\sim}{\overset{mt}{\hat{m}}}_x$$

$$v_x = \underset{\sim}{\hat{m}}_x + \underset{\sim}{\overset{m}{\hat{m}}}_x + \frac{T}{2}\,\underset{\sim}{\overset{\delta}{\hat{m}}}_x\,\underset{\sim}{\overset{mt}{\hat{m}}}_x$$

in which $\underset{\sim}{\overset{\delta}{\hat{m}}}_x$ is a diagonal matrix whose elements are to be obtained from $\{\overset{\delta}{\hat{m}}_x\} = (\underset{\sim}{I} + \frac{T}{2}\,\underset{\sim}{\overset{mt}{\hat{m}}}_x)^{-1}(\underset{\sim}{I} + \frac{T}{2}\,\underset{\sim}{\overset{m}{\hat{m}}}{}'_x)\{\overset{\delta}{m}_x\}$.

Finally, for future research, we may conclude that the concept of multiregional life table as defined above does not constitute as strong a starting point of multiregional mathematical demography as does the single-state life table. This is because the underlying (Markovian) assumption does not hold in observed populations as well as does the corresponding assumption in the single-state case.

Therefore, one direction of future research is to introduce more reality into the migration process underlying the life table. However, this can only be obtained at the expense of additional complexity and data requirements. Consequently, such a direction of research appears to be not very useful because multiregional life table functions do not necessarily require a focus on moves. As just shown, they can be adequately estimated from the transition approach first developed by Rogers (1973a, 1975a) and expanded in the present paper. A more rewarding direction of research is the further development of the transition approach, especially the estimation of transition migration rates, as suggested in Section V.

Appendix 1: The Aggregation of a Multiradix Increment-Decrement Life Table into a National Single-State Life Table

As a first step, we characterize the difference exhibited by the aggregate and disaggregate life table by examining the relationship between the survivorship probabilities of the two tables.

A national single-state life table is generally derived by ignoring internal migration between subregions. From estimates of age-specific death rates ${}_{T}m_{x}^{\delta}$ for the nation, survival probabilities at exact age x are obtained in the case of a uniform distribution of deaths over time, from:

$$p_x = \frac{1 - \frac{T}{2} m_x^{\delta}}{1 + \frac{T}{2} m_x^{\delta}}$$

in which m_x^{δ} has been generally derived from:

$$m_x^{\delta} = \frac{d_x^{\delta}}{L_x}$$

where d_x^{δ} is the number of deaths occurring to those aged x to x + T and L_x the mid-period population aged x to x + T. Note that if ${}^{i}m_x^{\delta}$, ${}^{i}d_x^{\delta}$ and L_x^{i} are the region-specific counterparts of m_x^{δ}, d_x^{δ} and L_x, we have the following:

$$m_x^{\delta} = \frac{d_x^{\delta}}{L_x} = \frac{\sum_{i=1}^{n} {}^{i}d_x^{\delta}}{L_x} = \frac{\sum_{i=1}^{n} {}^{i}m_x^{\delta} L_x^{i}}{L_x} = \sum_{i=1}^{n} \frac{L_x^{i}}{L_x} {}^{i}m_x^{\delta} \; .$$

Clearly:

$$m_x^{\delta} = \{i\}'\overset{\delta}{\underset{\sim}{m}}_x\{\alpha_x\}$$

where

$\{i\}'$ is a row vector of ones

$\overset{\delta}{\underset{\sim}{m}}_x$ is a diagonal matrix of observed death rates, the general diagonal element of which is ${}^{i}m_x^{\delta}$

$\{\alpha_x\}$ is a column vector whose general element is

$$\alpha_x^i = \frac{L_x^i}{L_x} \quad .$$

Note that, since the multiregional population considered is a closed system, the aggregation of $\overset{m}{\underset{\sim}{m}}_x$ yields a zero scalar:

$$\{i\}'\overset{m}{\underset{\sim}{m}}_x\{\alpha_x\} = 0$$

then, we may express $\overset{\delta}{m}_x$ as

$$m_x^{\delta} = \{i\}'\underset{\sim}{m}_x\{\alpha_x\}$$

in which $\underset{\sim}{m}_x$ is the full matrix of observed death and migration rates. To establish a relationship between p_x and its multiregional counterpart $\underset{\sim}{p}_x$ we start from (58) rewritten as:

$$[\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x] \quad \underset{\sim}{p}_x = [\underset{\sim}{I} - \frac{T}{2}\underset{\sim}{m}_x] \quad .$$

Premultiplying by $\{i\}'$ and post multiplying by $\{\alpha_x\}$ yields:

$$\{i\}'[\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x] \quad \underset{\sim}{p}_x \ \{\alpha_x\} = \{i\}'[\underset{\sim}{I} - \frac{T}{2}\underset{\sim}{m}_x]\{\alpha_x\}$$

Dividing both sides by $1 + \frac{T}{2}\,{}_T m_x^{\delta}$ finally leads to:

$$p_x = \{j\}'\underset{\sim}{p}_x\{\alpha_x\}$$

in which

$$\{j\}' = \frac{\{i\}'[\underset{\sim}{I} + \frac{T}{2}\underset{\sim}{m}_x]}{1 + \frac{T}{2}m_x^{\delta}} = \frac{1}{1 + \frac{T}{2}m_x^{\delta}}\left\{\begin{matrix} 1 + \frac{T}{2}\,{}^1m_x^{\delta} \\ \cdot \\ \cdot \\ \cdot \\ 1 + \frac{T}{2}\,{}^n m_x^{\delta} \end{matrix}\right\}'$$

the relationship linking p_x and $\underset{\sim}{p}_x$ is thus similar to that linking m_x and $\underset{\sim}{m}_x$, $\{j\}'$being substituted for $\{i\}'$.

Note that $\{j\}' \neq \{i\}'$unless ${}^1m_x^{\delta} = \cdots = {}^n m_x^{\delta} = m_x^{\delta}$, that is, age-specific death rates are identical across regions. The result is that a national life table can be interpreted as the aggregate life table of a multiregional system, in which death rates are identical in all regions.

Furthermore, *the aggregation problem* does not really stem from the consideration of internal migration, but from the *existence of differing age-specific mortality rates across regions*.

The question is then how to carry out the aggregation of the n linearly independent stationary populations into a national life table accounting for differing mortality patterns across regions. A priori two alternatives are possible. The first possibility is to derive the age-specific survival probabilities of the national life table from those of the multiregional life table. This can be done, for example, by setting the survival probabilities p_x of the national life table equal to the eigenvalue of $\underset{\sim}{p}_x$.* However, this would result in a particular multiregional

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*It can be shown that this is equivalent to picking a value of m_x^{δ} equal to the eigenvalue of $\underset{\sim}{m}_x$.

system in which the age-specific net quit (absence) rates would be identical in all regions and equal to the national death rate.*

Alternatively, we can pick a particular regional allocation of the initial cohort. We can then build a national life table recognizing differential mortality rates by assigning to the ith radix a share of the initial cohort such as

$$p_i = \frac{B^i}{\sum_i B^i}$$

in which B^i is the total number of births in region i. If regional birth data are not available, a good substitute can be, assuming that each region of a multiregional life table is characterized by a number of births equal to the number of departures (i.e., the number of deaths minus the number of net-(in)migrants),

$$p_i = \frac{D^i + O^i - I^i}{\sum_{i=1}^{n} D^i}$$

in which D^i is the total number of deaths in region i,
in which O^i is the total number of migrants out of region i,
in which I^i is the total number of migrants into region i.

Over the observation period.

This alternative clearly presents the advantage of imposing no additional assumptions on the regional patterns of mortality and mobility and should therefore be preferred.

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* The demonstration of this feature follows from the previously mentioned equivalence.

Interpretation of Life Table Symbols Used

Scalar Notation	Vector Notation	Matrix Notation	Interpretation
			A. Continuous Statistics
${}^{i}(ad)^{j}_{x}(y)$			The net change between ages y and y + dy in the number of individuals, members of l^{i}_{x}, who are present in state j.
${}^{i}(ad)^{\delta}_{x}(y)$			The number of deaths occurring between ages y and y + dy to the survivors of l^{i}_{x}
${}^{i}d^{j}(y)$			The number of moves from i to j made between ages y and y + dy.
${}^{k}_{ix}d^{l}(y)$			The number of moves from k to l made between ages y and y + dy by the survivors of l^{i}_{x}.
$l^{i}(y)$	$\{l(y)\}$		The number of persons in state i at age y.
${}_{ix}l^{k}(y)$	$\{{}_{ix}l(y)\}$	${}_{x}\underset{\sim}{l}(y)$	The number of individuals, members of l^{i}_{x}, who survive to age y in region k.

Scalar Notation	Vector Notation	Matrix Notation	Interpretation
${}^i\Pi^j_x(y)$		$\underset{\sim}{\Pi}_x(y)$	The probability that an individual present in state i at age x will be in state j at age y (transition approach).
${}_{y1}^{\,i}\Omega^j(y2)$ $({}_{y1}^{\,i}\hat{\Omega}^j(y2))$		${}_{y1}\underset{\sim}{\Omega}(y2)$ $({}_{y1}\hat{\underset{\sim}{\ell}}(y2))$	The probability that an individual present in state i at age y_1 will become present in state j at age y_2 in the movement (transition) approach.*
${}^i\mu^j(y)$		$\underset{\sim}{\mu}(y)$	The force of transition from state i to state j at exact age y (movement) approach.
${}^i\hat{\mu}^j_x(y)$		$\hat{\underset{\sim}{\mu}}_x(y)$	The force of transition from state i to state j at exact age y for the survivors of l^i_x.
		B. Discrete Statistics	
${}^i\overset{o}{a}{}^j_x$		$\overset{o}{\underset{\sim}{a}}_x$	$(= {}^ia^j_x\ {}^im^j_x$ if $j \neq i)$
${}^ia^j_x$		$\underset{\sim}{a}_x$	Mean duration of transfers form i to j within a T-year interval.

*Symbols with a caret refer to the transition approach.

Scalar Notation	Vector Notation	Matrix Notation	Interpretation
${}^{i}{}_{(ad)}{}^{j}_{x}$			The number of transitions from state i to state j between ages x and x + T (obtained by comparison of states of presence at the start and end of the interval).
${}^{i}{}_{d}{}^{j}_{x}$			The number of moves from state i to state j between ages x and x + T.
${}_{iy}e^{j}_{x}$		${}_{y}\underset{\sim}{e}_{x}$	The expected time to be spent in state j beyond age x by those present at age y in state i.
l^{i}_{x}	$\{l_{x}\}$	${}^{mt}\underset{\sim}{l}_{x}$	The number of persons in region i at exact age x.
${}_{jy}l^{i}_{x}$	$\{{}_{jy}l_{x}\}$	${}_{y}\underset{\sim}{l}_{x}$	The number of persons in region i at exact age y who were in region j at exact age y.
L^{i}_{x}	$\{L_{x}\}$		The number of person-years lived in state i between ages x and x + T.
${}_{jy}L^{i}_{x}$	$\{{}_{jy}L_{x}\}$	${}_{y}\underset{\sim}{L}_{x}$	The number of person-years lived in state i between ages x and x + T by the survivors of l^{j}_{y}.

Scalar Notation	Vector Notation	Matrix Notation	Interpretation
${}^{i}m_{x}^{j}$		$\underset{\sim}{m}_{x}$	The life table rate of transfer from state i to state j between the ages of x and x + T (movement approach).
${}^{i}_{ky}m_{x}^{j}$			The life table rate of transfer from state i to state j between the ages of x and x + T relating to the survivors of l_{y}^{k}.
$\sum_{\substack{j=1 \\ j\neq i}}^{n} {}^{i}m_{x}^{j}$		$\underset{\sim}{m}_{x}^{mt}$	The life table migration rate out of state i between ages x and x + T.
${}^{i}m_{x}^{\delta}$ $({}^{\hat{i}}m_{x}^{\delta})$		$\underset{\sim}{m}_{x}^{\delta}$ $({}^{\hat{i}}\underset{\sim}{m}_{x}^{\delta})$	The life table rate of mortality in state i between the ages of x and x + T in the movement (transition) approach.
${}^{\hat{i}}m_{x}^{j}$		${}^{\hat{i}}\underset{\sim}{m}_{x}$	The life table rate of mobility from state i to state j between the ages of x and x + T relating to the survivors of l_{x}^{i}.
${}_{iy}n_{x}^{j}$		${}_{y}\underset{\sim}{n}_{x}$	The number of moves out of state j made beyond age x by the survivors of l_{y}^{i}.

Scalar Notation	Vector Notation	Matrix Notation	Interpretation
${}^{i}p^{j}_{x}$ $({}^{i}\hat{p}^{j}_{x})$		$\underset{\sim}{p}_{x}$ $(\hat{\underset{\sim}{p}}_{x})$	The probability of being in state i at age x and in state j at age x + T in the movement (transition) approach.
${}^{i}p^{\delta}_{x}$ $({}^{i}\hat{p}^{\delta}_{x})$			The probability of an individual alive in state i at age x to die before reaching age x + T in the movement (transition) approach.
${}^{i}q^{j}_{x}$		$\underset{\sim}{q}_{x}$	The probability for an individual alive in state i at age x to die in state j before reaching age x + T.
${}^{i}s^{j}_{x}$		$\underset{\sim}{s}_{x}$	The probability of individuals aged x to x + T in state i who survive in state j, T years later.
${}^{i}_{ky}s^{j}_{x}$			The proportion of individuals aged x to x + T in state i (but present at age y in state k) who survive in state j, T years later.
T^{i}_{x}	$\{T_{x}\}$		The number of person-years lived in state i beyond age x.

Scalar Notation	Vector Notation	Matrix Notation	Interpretation
${}_{j}{}_{y}T_{x}^{i}$		$\underset{\sim}{T}_{y\sim x}$	The number of person-years lived in state i beyond age x by the survivors of l_{y}^{j}.

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