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COMPUTING ECONOMIC EQUILIBRIA THROUGH NONSMOOTH OPTIMIZATION

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PREFACE

One of the tasks of the Food and Agriculture Program at IIASA consists of linking together national models through international markets for agricultural products. This requires in particular to find an economic equilibrium, i.e. to solve a system of nonlinear ill-conditioned equations.

On the other hand, the task "Nonsmooth Optimization" has been created in the System and Decision Area to investigate modern methods for solving difficult problems of mathematical programming, and their possible applications to fields of interest at IIASA.

It appeared that these topics are strongly related, one could even say that Nonsmooth Optimization seems to be a fruitful approach for attacking equilibrium problems. Therefore a collaborative research has been carried out, whose result is now a computer code implemented on PDP 11, that finds an equilibrium through Nonsmooth Optimization.

The aim of this paper is to demonstrate the feasibility and the relative efficiency of this approach.

SUMMARY

In an international economy -- i.e. a system of importers-exporters who react to prices -- equilibrium can be defined as a system of prices such that the imports are not larger than the exports. Mathematically, imports and exports are rather complicated nonlinear functions of prices, so that finding an equilibrium amounts to solving a nonlinear system.

To solve this problem, a technique was selected which operates by defining iteratively a sequence of price vectors which aim at reducing the gap between imports and exports, until it is below an acceptable level.

In this paper, we give two examples of an international model, outline the methodology adopted, and report on a substantial set of numerical experiments.

ABSTRACT

In the first section we describe the problem in general terms, setting the precise hypothesis in mathematical language. Section 2 describes the principles of an example which arises in the context of the linkage of national models of food and agriculture. The general methodology is presented in Section 3, where the algorithm of solution is outlined. Section 4 reports on an extensive set of numerical experiments, both on problems known in the literature, and on the example of Section 2. Finally the paper concludes by some remarks about improvements of the algorithm, which motivate further research on the subject.

Computing Economic Equilibria Through Nonsmooth Optimization

1. The Problem

Let $p_1,\ldots,p_n$ be n nonnegative variables which are required to sum up to:

(1) $$p_i \geqq 0 \qquad i = 1,\ldots,n \qquad \sum_i p_i = 1 \quad .$$

These variables are called *prices*, and we denote by S the set of price vectors p which satisfy (1). To these variables are associated n functions $z_1(p),\ldots,z_n(p)$, called *excess demands*, which satisfy the relation

(2) $$\sum_i p_i z_i(p) = 0$$

known as *Walras' law*.

The problem is to find a vector $\bar{p} \in S$ such that

(3) $$z_i(\bar{p}) \leq 0 \qquad i = 1,\ldots,n \quad .$$

A solution to this problem is called an *equilibrium*. From (1) and (2), it is clear that, for any $p \in S$, all the $z_i(p)$ cannot be strictly negative. Therefore it is absolutely impossible to find an exact equilibrium -- at least when the problem is nonlinear -- and we must be content if we find a $p \in S$ such that

(4) $$z_i(p) \leq \varepsilon_i \qquad i = 1,\ldots,n \quad ,$$

for some prescribed small positive tolerances ε_i.

In Section 3, we suggest a methodology which can be applied to solve (4), provided the following fundamental assumptions are satisfied:

(i) *For any* $p \in S$ *such that* $p_i > 0$ $i = 1,\ldots,n$, *each* $z_i(p)$

has a uniquely defined value, which is a continuous function of p.

(ii) *For any* $p \in S$ *such that* $p_i > 0 \quad i = 1,\dots,n$, *there exists a sequence* $p^k \in S$, $p^k \to p$ *such that the gradients* $\nabla z_i(p^k)$ *exist and are bounded.*

We thus require the ability to compute the gradients whenever they exist -- which may not be a trivial task. Moreover, we need the $z_i(p)$ to have a "semismoothness" property which, due to its technical nature, is not defined here, but can be found in [8] and [9].

Hence, (i) rules out problems in which $z(p)$ is a multivalued mapping. For this kind of problem, there exists only one algorithm [11] for finding an equilibrium. Our aim is to compete with this algorithm in the easier case of continuous excess demands. If the excess demand is admitted to be continuous, then the additional differentiability hypotheses are not much more restrictive. In fact, they are only violated by strange (from the point of view of applications) functions such as
$z(p) = p^2 \sin \frac{1}{p}$ for $p \in R$.

The methodology we propose is strongly related to, and comparable with, Newton [1,12] or quasi-Newton [13] methods. However, since we do not require the gradients to be continuous, the field of applicability is significantly larger. Alternatives to the present approach can be found in [2] and [3].

2. An Example

Simple economic examples which fit into the above framework can be found in [11]. However, this study was motivated by a problem which is substantially more complex. We give a simplified description of it here, and refer the interested reader to [4] for a more complete description from the economic point of view. It is a model of international trade, with which is considered the linkage of national models having domestic price policies and quotas on imports and exports.

The vector $z(p)$ of excess demands for n commodities is the sum over $j = 1,\dots,m$ of national excess demand vectors $z^j(p)$. Each country determines its own excess demand $z^j(p)$ as a function of the *international price vector* p, through a rather complicated process. To simplify notation, we drop the superscript j and describe the behavior of one country.

The country contains several *income classes* (indexed by h), each of which is considered as an individual consumer, so that the national excess demand for commodity i is given by

$$z_i = \sum_h z_i^h = \sum_h (x_i^h - y_i^h) \quad .$$

In this notation, x denotes the consumption, whereas y_i^h is the quantity of commodity i owned by the income class h after production. Although economies with production as a function of prices can be considered, we restrict the description to a simpler example in which the availability and ownership of commodities is given, the so-called *exchange economy*. Therefore, the numbers y_i^h are fixed, and it is the behavior of the consumption levels x_i^h, as functions of p, which must be described.

The consumptions depend on p through two kinds of intermediate parameters: those which are national, i.e. common to all income classes, and those which are specific to each income class. We begin with the national parameters.

First there are given *quotas* inside which the national excess demands must lie, which create constraints of the type

$$(5) \qquad L_i \le \sum_h x_i^h \le U_i \quad .$$

The following basic national relation expresses that the *balance of trade condition* is satisfied:

$$(6) \qquad \sum_i p_i \sum_h x_i^h = \sum_i p_i \sum_h y_i^h + k(p_1,\dots,p_n) \quad ,$$

where k is a function (one for each country) continuous and homogenous of degree 1; moreover $\sum_j k^j(p) = 0$. Hence (2) holds.

The income classes are not directly confronted with the international prices p_i, but rather with *effective domestic prices* pe_i. By means of tariffs at the border, the government tries to make pe = pd, a *desired price* vector, which should follow p according to a given *adjustment rule*

(7) $$pd_i = F_i(p_1,\ldots,p_n)$$

each F_i being a differentiable and homothetic function.

However, the equality pe = pd = F(p) may not be realizable, because of quotas. There is a deviation between pe and pd, which can be roughly explained as follows

* if both of the bounds in the ith constraint (5) are inactive, then $pe_i = pd_i$;
* if a bound is active, then the corresponding consumption is not free, and there is a *shadow price difference* λ_i, which represents the marginal "cost" of satisfying the constraint. If the constraint is actually $\sum_h x_i^h = U_i$ (resp. $\sum_h x_i^h = L_i$) then λ_i is positive (resp. negative); and if the constraint is inactive $\lambda_i = 0$. Therefore, we write

(8) $$pe_i = pd_i + \lambda_i \quad .$$

(In the particular case of a single income class, the problem of computing x reduces to an optimization problem. If in addition, pd = p, then the above relations are just the Kuhn-Tucker equations.)

We now turn to the income - class specific aspects. Each income class will have to satisfy an equation similar to (6), relating pe_i and x_i^h. Since pe $\neq$ p, this equation becomes the so-called *budget equation:*

$$(9) \qquad \sum_i pe_i \, x_i^h = \sum_i pe_i \, y_i^h + t^h \quad ,$$

where t^h is the contribution of income class h to the *national income tax* $t = \sum_h t^h$. It is convenient to set

$$t^h = \alpha^h t \qquad \alpha^h \geq 0 \qquad \sum_h \alpha^h = 1 \quad .$$

α^h being determined by a *system of taxation*. Various such systems are possible, for example:

1. α^h depends only on h.

2. $\alpha^h = \sum_i pe_i \, y_i^h / \sum_{i,h} pe_i \, y_i^h$

3. α^h such that $(\sum_i pe_i \, y_i^h + t^h)/(\sum_{i,h} pe_i \, y_i^h + t) = \beta^h$ given.

Finally, when pe and t are fixed, (9) reduces to a system of linear equations in x_i^h, and each income class chooses a consumption vector which maximizes its own *utility function* u^h, i.e. which solves

$$(10) \qquad \begin{cases} \max \; u^h(x_1^h, \ldots, x_n^h) \\ \sum_i pe_i \, x_i^h = \sum_i pe_i \, y_i^h + \alpha^h t \end{cases}$$

We assume that each problem (10) has a unique nonnegative solution, which is the case if the utility function is strictly concave and the *income* $\sum_i pe_i \, y_i^h + \alpha^h t$ is strictly positive.

Suppose now that an index set $I \subset \{1, \ldots, n\}$ is given such that the constraints (5) are required to be active for $i \in I$. Then the government has to find a value of the tax parameter t, which generates a system of prices pe, such that the solutions of (10) satisfy equation (6) together with equalities (5) for $i \in I$. Of course, there is no reason for the constraints (5) to be satisfied for $i \notin I$, unless I has been properly chosen.

In summary, computing z(p) amounts to iterating over I and t such that the equilibrium defined by (6) and (10) satisfies (5). The existence of such an equilibrium is studied in [4] where, for specific utility functions, uniqueness has been shown, and algorithms have been given to compute z(p) and the Jacobian $\frac{\partial z_i}{\partial p_j}$. This computation has been realized as a FORTRAN program (which we will not try to describe here!) We simply point out that the Jacobian is discontinuous at vectors p such that for some country the set of active constraints (5) switches.

3. The Methodology

The idea for solving (3) is to choose a nonnegative *goal function* f(p), which measures how far p is from an equilibrium, and which is zero if and only if p is an equilibrium. Then the problem is reduced to

$$\begin{cases} \min f(p) \\ p \in S \end{cases}$$

i.e. a constrained optimization problem. Since S is particularly simple, and since one is generally interested in problems where the equilibrium prices are all nonzero, we eliminate the constraints by the following technique:

First add to f a *barrier function* Q(p) which is positive for any $p \in S$, and which tends to $+\infty$ when p approaches the boundary of S. It is convenient to set $Q(p) = \sum_i q(p_i)$ where

$$q(t) = \begin{cases} 0 \quad \text{if} \quad t \geq \delta \\ \left[\text{Log}\ \frac{\delta}{t}\right]^2 \text{ if } 0 < t \leq \delta \end{cases} \tag{11}$$

Thus, the actual f(p) is considered only when each p_i is greater than δ (and we can choose for example $\delta = 10^{-4}$.) This eliminates the constraints $p_i \geq 0$. We now change notation and denote by f(p) the function f(p) + Q(p).

To eliminate the constraint $\sum p_i = 1$, we consider the restriction of f to the set $\{p | \sum p_i = 1\}$. This restricted function has a gradient whose components g_i sum up to 0, and are given by the following formula:

$$(12) \qquad g_i = \frac{\partial f(p)}{\partial p_i} - \frac{1}{n} \sum_j \frac{\partial f(p)}{\partial p_j} \qquad i = 1,\ldots,n \quad .$$

It remains to specify the goal function. First, the excess demands should be mutually comparable. This leads us to consider the *scaled excess demands*

$$z_i'(p) = \alpha_i(p) z_i(p)$$

where the coefficients $\alpha_i(p)$ are required to satisfy

$$\alpha_i(p) \geq \alpha > 0 \qquad i = 1,\ldots,n \qquad \forall p \in S \quad .$$

The proper choice of $\alpha_i(p)$ depends on the problem considered. In view of (4), we can choose

$$\alpha_i(p) = 1/\varepsilon_i \qquad \text{i.e. constant coefficients.}$$

It is also possible to choose

$$\alpha_i(p) = \alpha p_i + (1-\alpha)\bar{p}_i$$

where $\alpha \in]0,1[$ and $\bar{p}$ is some strictly positive fixed vector in S, which is hopefully not far from the desired equilibrium.

For the example of Section 2, a particularly suitable choice would be

$$\alpha_i(p) = 1/\sum_j \max\,(0, z_i^j(p))$$

where $z_i^j(p)$ is the excess demand of the j^{th} country. However, we do not study further this choice, which presents some computational difficulties.

Once each excess demand is scaled, various goal functions are possible.

$f(p) = \max_i z'_i(p)$ which is the most natural one

$f(p) = \sum_i \max(0, z'_i(p))$

$f(p) = \sum_i [\max(0, z'_i(p))]^2$ which is differentiable

if every $z'_i(p)$ is differentiable.

Suppose for example we choose

$$f(p) = \max_i (z_i(p)/\varepsilon_i) \quad . \tag{13}$$

(The other choices are simple variants.) This function has a gradient only when p is such that there is exactly *one* i such that $z_i(p)/\varepsilon_i$ is maximum, and the corresponding gradient $\nabla z_i(p)$ exists. Otherwise we speak of a *generalized gradient.* The computation of f and a generalized gradient is simply described by

1. Check that all prices are positive. Compute the excess demands and their (generalized) gradients.

2. Determine some index i such that z_i/ε_i is maximum. This gives f and a (generalized) gradient g.

3. If necessary, add to f and g the barrier terms corresponding to (11).

4. Substract from g the restriction term corresponding to (12).

To minimize f, we just apply some method for nonsmooth optimization described in [9]. Every such method is based on the usual principle of *descent methods* [5], in which a direction d of incrementation of prices is computed from the current iterate p, and then a line-search is performed in this direction, hopefully providing a stepsize t such that $f(p+td) < f(p)$. The common characteristic of these methods is that the direction is computed through a quadratic programming problem involving the gradients accumulated during the previous iterations. We can describe their general scheme briefly as follows.

ALGORITHM 1

Step 1. Suppose k iterates $p^1,\dots,p^k$ have been generated, and the corresponding gradients have been stored in the *bundle* $\{g^1,\dots,g^k\}$.

Step 2. According to some *deletion rule*, select an index set $K \subset \{1,\dots,k\}$.

Step 3. Find the vector d^k of minimal length among the convex combinations

$$d = \sum_{j \in K} \lambda^j g^j \quad , \qquad \lambda^j \geq 0 \qquad \sum_{j \in K} \lambda^j = 1 \quad .$$

Step 4. Perform a line-search along $-d^k$ to find the stepsize $t > 0$ and the next gradient $g^{k+1} = \nabla f(p^k - td^k)$ such that

- either $f(p^k - td^k) < f(p^k)$ (*serious step*). Then set $p^{k+1} = p^k - td^k$
- or t is small enough (*null step*). Then set $p^{k+1} = p^k$.

Step 5. Increase k by 1 and go to Step 1.

In Step 2 the deletion rule can be used in particular to reduce the size of the bundle when it becomes too large for the allocated computer memory.

Under the hypotheses listed in Section 1, the algorithm can be implemented. Whether the sequence p^k converges to an equilibrium is in general a matter of chance, but we can state convergence at least under some classical hypotheses [1].

THEOREM 1

Suppose f is given by (13). Then the above algorithm produces an equilibrium under either of the two following assumptions.

(i) Each $z_i(p)$ is semiconvex [8] (or more simply: convex)

(ii) When p approaches the boundary of S, at least one $z_i(p)$ goes to $+\infty$. Each $z_i(p)$ is a continuously differentiable function, positively homogenous of degree 0: $z_i(\mu p) = z_i(p)\ \forall \mu > 0$. Any subset of at most $n-1$ gradients $\nabla z_i(p)$ is linearly independent.

PROOF. (i) is just Corollary 5.3 of [9].

Now assume (ii). For p and q in R^n, denote $(p,q) = \sum_{i=1}^{n} p_i q_i$ the usual scalar product. Let $p \in S$ be such that $p_i \geq \delta$ $i = 1,\ldots,n$. Let $I(p)$ be the index set such that $z_i(p)/\varepsilon_i$ is maximum for $i \in I(p)$. According to (11), (13) the generalized gradients of f at p are the vectors of the form

$$(14) \qquad [\nabla z_i(p) - \frac{1}{n}(\nabla z_i(p),u)u]/\varepsilon_i \qquad i \in I(p)$$

where $u \in R^n$ is the vector whose all components are 1.

We define a *stationary point* of f as a point such that there exists a convex combination of generalized gradients which vanishes. Theorem 5.2 of [9] states that the algorithm produces such a stationary point $\bar{p}$. Because of (ii) and since $f(p^k)$ is nonincreasing, we may suppose that $\bar{p}_i \geq \delta$. Then from (14) the stationarity of $\bar{p}$ means that there exist $\lambda_i \geq 0, i \in I(\bar{p})$, λ_i not all zero, such that

$$(15) \qquad \sum_{i\in I(\bar{p})} \lambda_i \nabla z_i(\bar{p}) - \frac{1}{n} \sum_{i\in I(\bar{p})} (\lambda_i \nabla z_i(\bar{p}),u)u = 0 \quad .$$

We now show that the coefficient $\frac{1}{n} \sum_{i\in I(\bar{p})} (\lambda_i \nabla z_i(p),u)$ is zero. Call this coefficient α. In (15), we have a vector of R^n which is zero. Therefore, its scalar product with $\bar{p}$ is also zero:

$$\sum \lambda_i (\nabla z_i(\bar{p}), \bar{p}) - \alpha(u, \bar{p}) = 0 \quad .$$

Observe that $(u, \bar{p}) = \sum \bar{p}_i = 1$. On the other hand, the homogeneity of z_i implies that $(\nabla z_i(\bar{p}), \bar{p}) = 0$ (Euler relation), so that we get $\alpha = 0$. Therefore (15) can be written

$$\sum_{i \in I(\bar{p})} \lambda_i \nabla z_i(\bar{p}) = 0 \quad .$$

From the independency assumption in (ii), this implies that $I(\bar{p})$ contains n vectors. Finally we have proved that at $\bar{p}$, all the scaled excess demands are equal. Because of Walras' Law, this means that they are actually zero. Q.E.D.

This is a theorem of global convergence, which requires a global assumption on the independency of the gradients. Of course there are local analogues which state that convergence is ensured provided the starting point p^1 is close enough to an equilibrium. Suppose for example that, for any stationary point $\bar{p}$ which is not an equilibrium, one has $f(\bar{p}) \geq m > 0$. Then the algorithm converges provided that $f(p^1) < m$. Note also that this theorem neglects the influence of the penalty function, which is supposed to be eventually inactive during the algorithm.

The reason why we do not give a theorem of convergence for the general case where the excess demands are not differentiable (such as in Section 2) or with other goal functions, is because the definition of a stationary point is then much more technical. We can just say that in practical examples (as in Section 2 with a significant number of countries), convergence to a wrong point is rather unlikely.

4. Numerical Results

The algorithm of Section 3 can be illustrated with the examples of [11], in which three test problems involving n = 5,8,10 respectively are given. For each test problem, we made 6 runs; with three different goal functions, and two different starting price vectors:

GOAL 1 is $f(p) = \max_i z_i(p)$

GOAL 2 is $f(p) = \max_i (z_i / \sum_j w_{ij})$ (cf. [11] for the meaning of w_{ij})

GOAL 3 is $f(p) = \sum_i p_i |z_i(p)|$.

START 1 is $p_i = \frac{1}{n}$ $i = 1,\ldots,n$

START 2 is close to $(1,0,\ldots,0)$ which is the starting point of Scarf's algorithm.

Each test is then run until the maximum excess demand is reduced to the final maximum excess demand obtained by Scarf [11]. Table 1 gives the comparison in terms of the number of evaluations required (evaluation of the excess demand for Scarf, of the excess demand and its Jacobian for the present algorithm.)

	n = 5		n = 8		n = 10	
Max E.D. Final	.02		.07		.04	
Scarf [11]	158		640		468	
Scarf best [13]	77		104		121	
GOAL 1	27+17 (1)	52	35	65	57+26 (1)	90
GOAL 2	31	52	35	68	50	93
GOAL 3	26	(2)	68	(2)	60	(2)
Max E.D. Initial	20.2	98.	9.8	44.	27.	318.
	START1	START2	START1	START2	START1	START2

Table 1

(1) The algorithm converged to a "critical point" where (n-1) excess demands were equal (the nth being negative). However, the cure was simple: it sufficed to restart -- i.e. to clean up the index set K in Step 2.

(2) It converged to a local optimum near the boundary of S, which we could not jump. This shows that using prices to scale the excess demands us unsafe because the goal function may not go to $+\infty$ near the boundary of S, so that the barrier is active and may introduce local optima.

We turn now to the example of Section 2. We have chosen an economy with 15 commodities, 3 countries, and each country has 10 income classes. The utility functions have the form

$$u^h(x^h) = \sum_{i=1}^{15} e_i^h \operatorname{Log} x_i^h \qquad e_i^h \geq 0 \qquad \sum_i e_i^h = 1 \quad .$$

For each country, the system of taxation is the third one, all the coefficients β^h being 0.1.

The third country is a very small one which, in this exercise, serves as a barrier function (to prevent the prices from going to zero) so that, for this third country, the adjustment rule for the desired domestic prices is in fact $pd_i = p_i$ $i = 1,\ldots,15$. For the first two countries the adjustment rule has the form

$$pd_i = (r\, p_i)^{c_i}\, \bar{p}^{1-c_i}$$

where the exchange rate r is 1 and the vector $\bar{p}$ is [1,2,3,4,2,1,50,100,1,0.2,1,1,1,2,10] for both countries.

We have defined several test problems, taking different values for the *degrees of free trade* c_i

TEST 1 $c_i = 1$ $i = 1,\ldots,15$ for countries 1 and 2

TEST 2 $c_i = 0.5$ " " " " "

TEST 3 $c_i = 0.1$ " " " " "

TEST 4

First country: $c_i = \begin{cases} 0 & i \le 7 \\ 1 & i > 7 \end{cases}$

Second country: $c_i = \begin{cases} 1 & i \le 10 \\ 0 & i > 10 \end{cases}$

The other data y_i^h, e_i^h, L_i, U_i for each country are given on the following pages. For each test we ran the algorithm with three goal functions

GOAL 1: to minimize $f(p) = \max z_i(p)/y_i$

GOAL 2: to minimize $f(p) = \sum_{i=1}^{15} \max\,(0, z_i(p))$

GOAL 3: to minimize $f(p) = \sum_{i=1}^{15} \max\,(0, z_i(p))/y_i$

(where y_i is the total endowment: $y_i = \sum_{j=1}^{3} \sum_{h=1}^{10} y_i^{jh}$).

COUNTRY 1

Endowments y(i,h)

4	3	3	5	6	5	4	2	5	3	2	14	21	23	11
5	4	3	2	1	52	35	2	1	436	21	5	3	25	2
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	1	2	3	2	1	4	2	5	2	5	3	1	3	5
2	3	2	1	4	5	5	2	1	2	4	6	4	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	100	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	100
2	4	6	4	3	2	1	5	7	5	3	1	3	4	43
0	0	0	0	0	0	0	100	0	0	0	1	0	0	0
10	10	101	0	0	0	0	0	0	0	0	0	0	0	0

Budget proportions e(i,h) (X 100)

3	4	7	1	2	6	1	9	12	1	12	10	3	2	27
27	3	4	7	1	2	6	1	9	12	1	12	10	3	2
3	4	7	1	2	6	1	9	12	1	12	1	3	2	37
3	2	4	7	1	2	6	1	9	12	1	12	10	3	27
3	4	7	1	2	6	1	19	2	1	12	10	3	2	27
3	4	7	1	2	6	1	9	12	1	12	10	3	2	27
13	4	7	1	2	6	1	9	2	1	12	10	3	2	27
3	4	10	1	2	9	1	9	9	1	12	7	3	2	27
3	4	7	7	2	6	27	9	12	1	12	4	3	2	1
3	2	7	9	4	6	1	1	12	1	12	10	3	2	27

Minimum national consumptions L(i)

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Maximum national consumptions U(i)

100	30	200	20	30	90	80	200	200	900	200	200	200	400	400

COUNTRY 2

Endowments y(i,h)

10	20	30	40	50	30	40	10	30	1	4	10	3	10	300
100	20	30	40	2	20	23	40	4	55	66	74	75	45	28
10	20	40	39	46	57	58	4	38	45	96	28	38	39	38
10	20	46	40	58	49	49	49	49	49	49	45	10	10	40
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Budget proportions e(i,h) (X 100)

27	3	4	7	1	2	6	1	9	12	1	12	10	3	2
3	4	7	1	2	6	1	9	12	1	12	1	3	2	37
3	2	4	7	1	2	6	1	9	12	1	12	10	3	27
3	4	7	1	2	6	1	19	2	1	12	10	3	2	27
3	4	7	1	2	6	1	9	12	1	12	10	3	2	27
13	4	7	1	2	6	1	9	2	1	12	10	3	2	27
3	4	10	1	2	9	1	9	9	1	12	7	3	2	27
3	4	7	7	2	6	27	9	12	1	12	4	3	2	1
3	2	7	9	4	6	1	1	12	1	12	10	3	2	27
3	2	7	9	4	6	1	1	12	1	12	10	3	2	27

Minimum national consumptions L(i)

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Maximum national consumptions U(i)

1000 2000 800 800 3000 300 40 400 600 500
450 60 600 800 3000

COUNTRY 3

Endowments y(i,h)

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Budget proportions e(i,h) (X 100)

27	3	4	7	1	2	6	1	9	12	1	12	10	3	2
3	4	7	1	2	6	1	9	12	1	12	1	3	2	37
3	2	4	7	1	2	6	1	9	12	1	12	10	3	27
3	4	7	1	2	6	1	19	2	1	12	10	3	2	27
3	4	7	1	2	6	1	9	12	1	12	10	3	2	27
13	4	7	1	2	6	1	9	2	1	12	10	3	2	27
3	4	10	1	2	9	1	9	9	1	12	7	3	2	27
3	4	7	7	2	6	27	9	12	1	12	4	3	2	1
3	2	7	9	4	6	1	1	12	1	12	10	3	2	27
3	2	7	9	4	6	1	1	12	1	12	10	3	2	27

Minimum national consumptions L(i)

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Maximum national consumptions U(i)

900	900	900	900	900	900	900	900	900	900	900	900	900	900	900

In each run, the iterations were stopped as soon as each excess demand z_i was less than 1% of the total corresponding endowment y_i. The starting value for the world vector price was always $p_i = \frac{1}{15}$.

Table 2 below summarizes the results. For each of the 5 tests, we give the maximum initial excess demand, and for each of the 15 runs, we give the final maximum excess demand and the total number of function-gradient evaluations. Note that the excess demands in this model can be shown to be "semi-smooth".

		GOAL 1		GOAL 2		GOAL 3	
	Max E.D. Initial	# e val	Max E.D. Final	# e val	Max E.D. Final	# e val	Max E.D. Final
TEST 1	210.	59	5.28	81	1.25	48	5.81
TEST 2	413.	74	1.85	236	1.18	96	1.8
TEST 3	368.	143	0.28	460	1.02	121	1.3
TEST 4	369.	223 + 69	316.1 2.1[(1)]	138	1.23	213 43	12.3 4.5

TABLE 2

[(1)] Cf. Note (1) below Table 1.

5. A New Technique for Minimax Problems

Minimizing the maximum excess demand consists in solving (3) as a minimax problem, for which a special method exists [4]. Basically, it proceeds as in Algorithm 1 (for the goal function $f(p) = \max_i z'_i(p)$) except that the set $\{g^j | j \in K\}$ at Step 3 is in fact the set $\{\nabla z'_i(p^k) | i = 1,\dots,n\}$. In other words, it uses in the same way (i.e. for computing a direction) all the information computed at p^k (the Jacobian matrix) instead of partial information computed at previous steps (one line of the Jacobian at each p^i).

This method is equivalent to Newton's method for solving the system of inequalities (3), as defined in [10]. Correspondingly, it is shown in [7] to be quadratically convergent, provided the hypotheses of Theorem 1 are satisfied. However, in the present situation, it may fail because the excess demands are not continuously differentiable.

It is therefore attractive to combine the advantages of these two methods, in order to obtain a safe method which converges superlinearly when the problem is smooth enough.

This amounts to modifying Step 2 of Algorithm 1 in the following manner: select an index set K and the corresponding set of gradients $\{g^j, j \in K\}$ by taking

- first some subset of $\{\nabla f(p^j), j = 1,\dots,k\}$
- and then some subset of $\{\nabla z'_i(p^k), i = 1,\dots,n\}$.

We are now studying various strategies for choosing K properly, the problem being to find a good balance between efficiency (rapidity) and safety (avoiding local optima of the goal function).

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