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PROBLEMS OF DYNAMIC LINEAR PROGRAMMING

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Preface

The paper is devoted to the problems of dynamic linear programming (models and formalizations, theory and computer methods, extensions and applications). It contains a brief survey and discusses the necessities and possibilities for research in the area.

Problems of Dynamic Linear Programming

Abstract

Dynamic linear programming (DLP) can be considered as a new stage of linear programming (LP) development. Nowadays it becomes difficult, maybe even impossible, to make decisions in large systems and not take into account the consequences of the decision over a long-range period. Thus, almost all problems of optimal decision making become dynamic, multi-stage ones.

New problems require new approaches. With DLP it is difficult to exploit only LP ideas and methods: even having found the optimal program, we often do not know how to use it.

This paper represents in some sense the statement of the problem; although it contains a brief survey of DLP, it is focused on the things to be done, rather than on those already being tackled.

1. *Introduction*

The impact of linear programming (LP) [1,2] models and methods on the practice of decision making is well known. However, both the LP theory itself, and the basic range of its application are of a one-stage, static nature; that is in this case the problem of the best allocation of limited resources is considered at some fixed stage of development of a system.

However, when the system to be optimized is developing (and not only in time, but possibly, in space as well), and this development is to be planned, a one-stage solution is inadequate. In this case a decision should be made several stages in advance and the problem of optimization becomes a dynamic, multi-stage one, for example, problems in long- and short-range planning, or generally speaking, in programming of a system development.

In fact, any static LP model may have its own dynamic variant, the latter being of growing importance because of the increasing role of planning in decision making. It leads to the emergence of a general problem of dynamic linear programming (DLP), dynamic transportation and distribution problems, dynamic integer programming, etc.

With a new quality of DLP, new problems arise. While for the static LP the basic question consists of determining the optimal program, the realization of this program (related to the questions of the feedback control of such a program, its stability and sensitivity, etc.) is no less important for the dynamic problem.

Hence, the DLP theory and methods should be both based on the classical methods of linear programming and on the methods of control theory, Pontryagin's maximum principle [3] and its discrete version [4] in particular. One should distinguish in the DLP theory two basic, closely related problems: determination of an optimal program and its realization, i.e., control of the program.

2. *DLP Canonical Form*

In formulating DLP problems it is useful to single out: 1) state (development) equations of the system with the distinct separation of state and control variables; 2) constraints imposed on these variables; 3) planning period (horizon) T, that is the number of stages, during which the system is considered; 4) performance index, which quantifies the quality of control.

State Equations. State equations have the following form:

$$x(t+1) = A(t)x(t) + B(t)u(t) + s(t) \tag{1}$$

where the vector $x(t) = \{x_1(t),\ldots,x_n(t)\}$ defines the state of the system at stage t in the state space X; vector $u(t) = \{u_1(t),\ldots,u_r(t)\}$ specifies the controlling action at stage t; $s(t) = \{s_1(t),\ldots,s_n(t)\}$ is a vector defining the external effect on the system (uncontrolled, but known *a priori* in the deterministic model).

Matrices A(t), B(t) are of dimensions $(n\times n)$, $(n\times r)$ and assumed to be known.

Planning period (horizon) T is supposed to be fixed. Thus in (1): $t = 0,1,\ldots,T-1$.

It is also assumed that the initial state of the system

$$x(0) = x^0 \tag{2}$$

is given.

Constraints. In rather general form constraints imposed on the state and control variables may be written as

$$G(t)x(t) + D(t)u(t) \leq f(t) \tag{3}$$

$$u(t) \geq 0 \quad , \tag{4}$$

where $f(t) = \{f_1(t),\ldots,f_m(t)\}$; $G(t)$ and $D(t)$ are of dimensions $(m\times n)$, $(m\times r)$ and are given.

Performance index (which is to be maximized for certainty) is

$$J_1(u) = (a(T),x(T)) + \sum_{t=0}^{T-1} [(a(t),x(t)+(b(t),u(t))] \quad , \tag{5}$$

where $a(t)(t=0,\ldots,T)$, $b(t)(t=0,\ldots,T-1)$ are known n- and m-vectors; $(\cdot,\cdot)$ denotes the inner product.

Definitions. The vector sequence $u = \{u(0),\ldots,u(T-1)\}$ is a control (program) of the system. The vector sequence $x = \{x^0,x(t),\ldots,x(T)\}$, which corresponds to control u from (1), (2), is the system's trajectory. The process $\{u,x\}$, which satisfies all the constraints of the problem (i.e., (1)-(4)), is feasible. The feasible process $\{u^*,x^*\}$, maximizing (5), is optimal.

Hence, the DLP problem in its canonical form is formulated as follows.

Problem 1. Find a control u and a trajectory x, satisfying the state equations (1) with the initial state (2) and the constraints (3)-(4), which maximize the performance index (5).

The choice of the canonical form of DLP is to some extent arbitrary and there are various possible versions and modifications of Problem 1. In particular, state equations may include time lags;

constraints on state and control variables may be separate, given as equalities or inequalities; the performance index may be defined only, for example, by the terminal state $x(T)$ of the system, etc.

However, such modifications may be either reduced to the canonical Problem 1, or it is possible to use for them the results, stated below for Problem 1 [4].

3. *Discussion*

First of all, it should be noted that if $T = 1$, then Problem 1 becomes a conventional LP problem.

Problem 1 itself can also be considered as a certain "large" LP problem with constraints on variables in the form of equalities (1), (2) and inequalities (3), (4). In this case the optimal control Problem 1 turns out to be an LP problem with the staircase constraint matrix (Table 1). But in the majority of cases dynamic LP problems are formulated now directly in static LP language, as for example Problem 2 (Table 2):

Problem 2. Find vectors $\{x^*(1),\ldots,x^*(T)\}$, which maximize

$$J_2(x) = \sum_{t=1}^{T} (c(t),x(t)) \tag{6}$$

subject to

$$\begin{gathered} A(1)x(1) = d(1) \\ B(t-1)x(t-1) + A(t)x(t) = d(t) \ , \qquad (t=2,\ldots,T) \\ x(t) \geq 0 \ , \qquad (t=1,\ldots,T) \ . \end{gathered} \tag{7}$$

Let us express the state variables $x(t)$ in Problem 1 as an explicit function of control. One can obtain from (1):

$$x(t) = \phi(t-1,0) + \sum_{\tau=0}^{t-1} \phi(t-1,\tau+1)B(\tau)u(\tau) + \sum_{\tau=0}^{t-1} \phi(t-1,\tau+1)s(t) \ , \tag{8}$$

where

$$\Phi(t,\tau) = A(t)\cdots A(\tau) \quad , \qquad 0 \leq \tau \leq t \quad ,$$

$$\Phi(t,t+1) = I \quad , \qquad I \text{ - identity matrix} \quad .$$

Using (8) it is also possible to get directly the constraints (3) imposed on control variables. As a result, we shall get the following LP problem with a block-triangular matrix (Table 3).

Problem 3. Find the control u*, for which

$$\sum_{t=0}^{T-1} (w(t),u(t)) \rightarrow \max \quad ,$$

$$W(t,0)u(0) + \ldots + W(t,t-1)u(t-1) + D(t)u(t) \leq h(t) \quad ,$$

$$u(t) \geq 0 \quad , \qquad (t=0,1,\ldots,T-1) \quad ,$$

where the vectors h(t), w(t) and the matrices $W(t,\tau)$ depend on the known parameters of Problem 1.

Problems 2 and 3 admit their modifications in the same way as Problem 1 (a block diagonal structure with coupling constraints or with both coupling contraints and variables, different types of staircase structure, etc.). They have been studied intensely [1,2,5-15]. But unlike control Problem 1, such formalizations of dynamical problems make no distinction between state and control variables. Therefore this approach makes it difficult to use the ideas and methods of the control theory. This difference will be more significant, when "pure" dynamic problems are considered (stability and sensitivity of DLP systems, control of the optimal programs, etc.).

The DLP problems in the form of Problem 1 were introduced and studied in [4,16-27].

4. *DLP Models*

Dynamic linear models, known in the literature, are usually formalized in static LP language, as Problems 2 and 3. To introduce DLP models in the form of Problem 1, let us consider, as an example, an ecological system.

Ecological Systems. We shall consider, as an example, the problem of optimal species population use within a given planning period [28,29,18].

Let $x_i(t)$ be the quantity of biological type i at stage t $(i=1,\ldots,n)$. There are r ways of using these species, we shall denote as $u_j(t)$ the intensity of way j $(j=1,\ldots,r)$ at stage t. Let the numbers $a_{ij}(t)$ determine the quantity of species of type i, caught (removed from species) per unit intensity of the way j.

If some species is a pest relative to the other, species of type i at stage t will decrease by the $\sum_{j=1}^{n} c_{is}(t)x_s(t)$, where $c_{is}(t)$ determines the number of individuals of type i, devoured by a single individual of type s.

Thus, the dynamic equation for the change of the i-th species quantity will be written as:

$$x_i(t+1) = (1+a_i(t))x_i(t) - \sum_{s=1}^{n} c_{is}(t)x_s(t) - \sum_{j=1}^{r} a_{ij}(t)u_j(t) \quad . \tag{9}$$

Here $a_i(t)$ is the coefficient of natural increase $(a_i(t)>0)$, or mortality $(a_i(t)<0)$ of the species of type i.

It should be noted that in (9) $u_j(t)$ for some j may also determine the quantity of the j-th chemical, used at stage t.

The constraints here may be for example,

$$\sum_{t=0}^{T-1} \sum_{i=1}^{n} g_{ik}(t)x_i(t) \le d_k$$

$$x_i(t) \ge 0 \quad ,$$

where $g_{ik}(t)$ - is the specific requirement of species i for the k-th resource; d_k is the availabe quantity of the k-th resource;

$$0 \le u_j(t) \le \bar{u}_j(t) \quad ,$$

where $\bar{u}_j(t)$ is determined by the technological constraints or the sanitary norms.

The performance index may be the total harvest for the entire planning period

$$J = \sum_{t=0}^{T-1} \sum_{i,j=1}^{n,r} \alpha_i(t) a_{ij}(t) u_j(t)$$

or a specific structure of the ecosystem, desirable at the terminal stage

$$J = \sum_{i=1}^{n} \beta_i(T) x_i(T) \quad ,$$

where the weight $\alpha_i(t)$, $\beta_i(T)$ coefficient is characterized by the importance of the species of type i.

This simple example illustrates the basic idea for formulating DLP models. In what follows models of this type are only mentioned and not written down in detail.

Economic Models. Linear programming is closely related to economic models [1,2]. In fact, transformation of static LP to dynamic ones are stimulated in great degree by transition from static input-output models to dynamic ones. Dynamic types of input-output economic models were considered, for example, in [30, 31]. As the DLP Problem 1, a multisector dynamic economic model was formulated and investigated in [32].

Energy Systems. Many models of short- and long-range development of energy systems are formulated as dynamical linear problems [33-36]. In [33] the energy model was stated as a DLP problem with time lags.

Large Organization Systems. Many problems in large organization systems such as, *manpower planning or educational systems* can be viewed as important applications of DLP. Some dynamic models of such kind were considered, for example, in [37].

Industrial Systems. Many of the short- and long-range planning problems in industry, as well as the production scheduling problems are reduced to DLP. (See, for example, [10,38]).

Regional and Urban Problems. The extensive field of applications of DLP is given by regional and urban planning problems. (See, for example [39] (agricultural model), [40-42] (water resources), [43,48] (transportation systems).)

Reflecting on these short examples and references it should be noted, that many practical problems of control and optimization in energy, water, ecological, regional, and urban systems may be stated in the form of DLP. Therefore the work on the survey of the existing DLP models and on the design of new ones is of essential interest.

5. *Types of DLP Problems*

In the preceding section the DLP problem of a general type was considered. Besides the general one, it is useful to single out dynamic transportation and distribution problems [18,43], integer DLP, convex dynamic programming [19,44]. To illustrate this aspect of the problem, let us consider simple transportation problems of DLP.

The Dynamic Transportation Problem. We shall consider a transportation network with some homogeneous goods of i-th production and j;th consumption points (plants). Let $a_i(t)$ be the production volume of the i-th point $(i=1,\dots,n)$ at stage t $(t=0,1,\dots,T-1)$ and $b_j(t)$ be the demand value of the j-th point $(j=1,\dots,m)$ at stage t.

It is assumed that each production or consumption point has an opportunity to store goods. We shall denote by $y_j(t)$, $z_i(t)$ the quantity of stock goods at the i-th consumption point and j-th production point at stage t; by $c_i(t)$, $d_j(t)$ the storage expenditures of a unit of goods; $u_{ij}(t)$ will be the quantity of goods transported from the i-th to the j-th point at stage t; $c_{ij}(t)$ is the transportation cost of a unit of goods.

Then the dynamics of the change of stocks will be determined by the equation

$$
\begin{aligned}
y_j(t+1) &= y_j(t) + \sum_{i=1}^{n} u_{ij}(t) - b_j(t) \quad , \qquad y_j(0) = y_j^0 \\
z_i(t+1) &= z_i(t) - \sum_{j=1}^{m} u_{ij}(t) + a_i(t) \quad , \qquad z_i(0) = z_i^0
\end{aligned}
\tag{10}
$$

with the constraints

$$y_j(t) \geq 0 \quad , \qquad z_i(t) \geq 0 \quad , \qquad u_{ij}(t) \geq 0 \quad . \tag{11}$$

So, the problem is formulated as follows. To find a transportation plan $\{u^*_{ij}(t)\}$ (control $u^* = \{u^*_{ij}(t)\}$) such that it maximizes the total expenditures

$$J = \sum_{t=0}^{T-1} \{ \sum_{ij} c_{ij}(t) u_{ij}(t) + \sum_i c_i(t) z_i(t) + \sum_j d_j(t) y_j(t) \}$$

subject to constraints (10) and (11).

It should be noted that with $T = 1$ and $y_j(0) = 0$, $z_i(0) = 0$ the problem becomes a conventional LP problem of the transportation type.

6. *Theory of DLP*

The theory of DLP is connected with two main problems:

(i) determination of optimal program;

(ii) realization of this optimal program.

Determination of Optimal Program. This side of the theory is linked with duality relations and optimality conditions for Problem 1, which are the base for building of numerical methods of DLP.

Analysis of the Lagrange function of Problem 1 reveals the following dual DLP problem [16,4,17].

Problem 1D. To find the dual control $\lambda = \{\lambda(T-1), \ldots \lambda(0)\}$ and the associated dual trajectory $p = \{p(T), \ldots, p(0)\}$ satisfying the co-state (dual) equation

$$p(t) = A^T(t) p(t+1) - G^T(t) \lambda(t) + a(t) \quad , \tag{12}$$

with the boundary condition

$$p(T) = a(T) \tag{13}$$

subject to the constraints

$$B^T(t)p(t+1) - D^T(t)\lambda(t) \leq -b(t)$$
$$\lambda(t) \geq 0 \tag{14}$$

and minimizing the performance index

$$J_D(\lambda) = (p(0),x^0) + \sum_{t=0}^{T-1} [(p(t+1),s(t))+(f(t),\lambda(t))] \quad .$$

Here $p(t) \in E^n$; $\lambda(t) \in E^m$; $\lambda(t) \geq 0$ are Lagrange multipliers for constraints (1)-(4).

The dual Problem 1D is a control-type problem as is the primal one 1P. Here the variable $\lambda(t)$ is a dual control and $p(t)$ is a co-state or a dual state at stage t. We have reversed time in the dual Problem 1D: $t = T-1,\ldots,1,0$.

Theorem 1. (The DLP "Global" Duality Theorem). The solvability of either of the **1P** or 1D problems implies the solvability of the other, with

$$J_P(u^*) = J_D(\lambda^*) \quad .$$

If the performance index for any of the pair of dual problems 1P or 1D is not bounded (from above in 1P, from below in 1D), then the other problem has no solution.

Let us introduce Hamilton functions

$$H_P(p(t+1),u(t)) = (b(t),u(t)) + (p(t+1),B(t)u(t))$$

and

$$H_D(x(t),\lambda(t)) = (\lambda(t),f(t)) - (\lambda(t),G(t)x(t))$$

for the primal and dual problems respectively.

Theorem 2. (The DLP "Local" Duality Theorem). The solutions of the primal $\{u^*, x^*\}$ and dual $\{\lambda^*, p^*\}$ problems are optimal if and only if the values of Hamiltonians coincide:

$$H_P(p^*(t+1), u^*(t)) = H_D(x^*(t), \lambda^*(t)) \quad .$$

Thus, the solution of the pair of dynamic dual problems can be reduced to analysis of a pair of static linear programs

$$\begin{aligned} &\max H_P(p(t+1), u(t)) \\ &G(t)x(t) + D(t)u(t) \leq f(t) \\ &u(t) \geq 0 \end{aligned} \tag{15}$$

$$\begin{aligned} &\min H_D(x(t), \lambda(t)) \\ &B^T(t)p(t+1) - D^T(t)\lambda(t) \leq -b(t) \\ &\lambda(t) \geq 0 \end{aligned} \tag{16}$$

linked by the state (1), (2) and co-state (12), (13) equations.

In particular, it can be shown, that there exist the following optimality conditions for problems 1P and 1D [4,17].

Theorem 3. (Maximum Principle of Primal Problem 1P). The control u^* is optimal for Problem 1P if and only if there exists a solution $\{\lambda^*, p^*\}$ to dual Problem 1D, such that for any $t = 0, 1, \ldots, T-1$

$$\max_{u(t)} H_P(p^*(t+1), u(t)) = H_D(p^*(t+1), u^*(t))$$

where the maximization is carried out with respect to all $u(t)$ satisfying constraints (3), (4), and $\lambda(t^*)$ is the optimal dual variable for LP problem (15).

Theorem 4. (Minimum Principle for Dual Problem 1D). The control λ^* is the optimal for the problem 1D if and only if there exists a solution $\{u^*, x^*\}$ to the primal problem 1P such that for any $t = 0, 1, \ldots, T-1$

$$\min_{\lambda(t)} H_D(x^*(t),\lambda(t)) = H_D(x^*(t),\lambda^*(t)) \quad ,$$

where the minimization is carried out with respect to all $\lambda(t)$, satisfying the constraints (14), and $u^*(t)$ is the optimal dual variable for the LP problem (16).

The foregoing optimality conditions define a decomposition principle for solving the pair of dual problems 1P and 1D. These conditions permit replacing the solution of the rT and mT - dimensional dynamic problems with variables $u_j(t)$ and $\lambda_j(t)$ ($i=1,\ldots,r$; $j=1,\ldots,m$; $t=0,1,\ldots,T-1$) by the successive solution of T static LP problems (15), (16), containing r and m variables respectively and linked by the state equations (1), (12) with the boundary conditions (2), (13).

The Control of the Optimal Program. Unlike static LP the realization of optimal solution in dynamic problems has no less importance than its determination. One should mention here the questions of realization of the optimal solution as a program (i.e., in dependence of the numbers of stage: $u^*(t)$ ($t=0,\ldots,T-1$)) or as a feedback control (i.e., in dependence on the current value of states: $u^*(t) = u^*(t,x^*(t))$ ($t=0,\ldots,T-1$); stability and sensitivity of the optimal system, connection of optimal solutions for long- and short-range models, etc. These problems are waiting for their solution. We shall mention only some of them here.

(i) It is often necessary to determine in what way the performance index and/or the optimal control will behave when the parameters of the problem are changing (for example, "prices" $a(t)$, $b(t)$, "resources" $f(t)$, "demand" $s(t)$), (parametric DLP). Solution methods in this case can be developed on the basis of static parametric LP [1,2]. A general approach to parametric problems of linear and quadratic programming is given in [45].

(ii) In computing the optimal program, especially for the large T, it is very important to know, how the inaccuracy in knowledge of matrices $A(t)$, $B(t)$ coefficients and other parameters of the system influences the stability of the optimal program and the quality of control (sensitivity problem).

(iii) Assume that an aggregated DLP problem for a large planning horizon T is solved. How can the information about the optimal dual process $\{p^*(t),\lambda^*(t)\}$ of the aggregate model be of any use to the operative solution (for each current state of the system) of more detailed but having a shorter planning horizon DLP problem?

(iv) How can the local synthesis of the system, i.e., the control of the form

$$\delta u^*(t) = A(t)\delta x^*(t) \quad , \qquad (t=0,1,\ldots,T-1)$$

for small deviations of states $\delta x^*(t)$ from the optimal trajectory $x^*(t)$ be carried out?

7. *Economic Interpretation*

A standard economic interpretation can be given to the pair of dual problems 1P and 1D and relations between them [17,18], analogous to those of the static LP problems [1,2].

8. *DLP Methods*

We shall distinguish finite and iterative methods for solving DLP problems.

The DLP Finite Methods. These methods are the development of large-scale LP methods for the dynamic problems. Now two main approaches begin to be revealed enabling us to build DLP finite methods.

The first approach is based on decomposition methods of LP [1,46,47], especially on Dantzig-Wolfe decomposition [1,46]. For Problems 2 and 3 this technique was used in [10-12,15], for Problem 1 in [19,21,27]. It should be noted here that originally the Dantzig-Wolfe decomposition method was developed for LP problems with block-angular structure such as in Problem 3 [6].

The second approach is based on the factorization of constraint matrix and used for Problems 2 and 3 in [13-15] and for Problem 1 in [26].

Iterative Methods. The application of the LP finite methods to the dynamic problems causes certain difficulties especially for the large planning horizon T. This can be explained by the fact that in these methods the approach of an approximate point to the optimum is fulfilled over the vertices of the feasible polyhedral set (in some space). But the number of vertices of such a set for the dynamic problems increases exponentially with T, so does the volume of calculation.

The iterative LP methods seems to by-pass these difficulties. They are also characterized by low demands to the computer's memory, the simplicity of the computation flowchart, low sensitivity to the disturbances.

We shall differentiate the following iterative methods.

Penalty Functions. This is one of the most universal and simple technique of optimization. But its direct use of the DLP problem is hampered by relatively low convergence rates in the vicinity of solution. The idea of extrapolation of decision was suggested in [22] which remarkably improves the effectiveness of the method for static LP and is developed for DLP in [23].

Generalized Gradients Methods. The other group of methods is based on finding the extremum of function

$$\psi(\lambda,p) = \max_{x,u\geq 0} L(u,x;\lambda,p)$$

or

$$\phi(u,x) = \min_{p,\lambda\geq 0} L(u,x;\lambda,p)$$

where $L(u,x;\lambda,p)$ is the Lagrange function of Problem 1P (Problem 1D).

It can be shown that minimization of $\psi(\lambda,p)$ is equivalent to solution of the dual Problem 1D, while the maximization of $\phi(u,x)$ is equivalent to the solution of primal Problem 1P. But functions ϕ and ψ nondifferentiable by nature, so the generalized gradient technique [48] is needed. Application of the generalized gradients for DLP reduces the solution of large DLP Problem 1P to successive solutions of small LP problems (15), (16).

Modified Lagrange Function Methods. The idea of the method was suggested in [49], although the methods of this group began to be developed only recently [50-51]. This approach combines the accuracy of finite methods with simplicity of iterative ones. The theory of the method for DLP was considered in [24]. One of its realizations based on employment of the Kalman-Bucy filter technique [52] was given in [25].

9. *Some Extensions*

Naturally, all the practical problems cannot be kept within the framework of DLP. Here we shall state the fields of DLP development, which are of the greatest interest.

Nonlinear Dynamic Programming. This is essentially the optimal control theory of the general type of discrete systems with substantial use of nonlinear programming techniques. Some approaches in this direction have been considered in [4,44].

Stochastic DLP. We shall only note [53,54] here the papers on multi-stage stochastic programming.

Maxi-min (mini-max) DLP Problems. The solution of such problems is of considerable practical interest when guaranteed control quality is to be obtained under the conditions of uncertainty, as well as for sensitivity analysis, and game problems of planning.

Let in Problem 1 the values of vectors s(t) be unknown, and only the range of their variations S_t be known, which is assumed to be bounded polyhedrons.

Problem 4. Find control u* and the trajectory x* subject to (1)-(3) and providing

$$\max_u \min_s J_1(u,s) = \omega_{\bar{1}} \quad ,$$

where $s = \{s(t) \varepsilon S_t\}$, the performance index J_1 is determined from (5).

Problem 5. Find control u* and trajectory x* subject to (1)-(3) and providing

$$\max_{u(0)} \min_{s(0)} \ldots \max_{u(T-1)} \min_{s(T-1)} J_1 = \omega_{\bar{2}} \ .$$

The solution of Problems 4, 5 guarantees the values of the performance index J_1 no worse than $\omega_{\bar{1}}$, if the program control u^* is realized and no worse than ω_2 (with $\omega_{\bar{2}} \geq \omega_{\bar{1}}$) if there is a possibility of recalculation of the program for each $x(t)$ (the feedback control $u(t) = u(t,x(t))$ of a system). The solution of Problems 4, 5 is considered in [55].

10. Conclusion

Above a short survey has been given of the contemporary state-of-the-art in dynamic linear programming, reflecting the author's possibilities and point of view. The development of optimization methods for dynamic problems, i.e., planning and control methods for large scale problems (which are of such a necessity in our dynamic world), irrespective of the directions they will take, will, undoubtedly, enrich the practice of decision making in complex systems.

Variables											
u(0)	x(1)	...	x(t)	u(t)	x(t+1)	...	x(T-1)	u(T-1)	x(T)	Constraints	Right-hand side constants
-B(0)	I									=	$+s(0) + A(0)x^0$
D(0)		$\ddots$								$\leq$	$f(0) + G(0)x^0$
											$\vdots$
			-A(t)	-B(t)	-I					=	$+s(t)$
			G(t)	D(t)		$\ddots$				$\leq$	$f(t)$
											$\vdots$
							-A(T-1)	-B(T-1)	I	=	$+s(T-1)$
							G(T-1)	D(T-1)		$\leq$	$f(T-1)$
Performance Index Constants											
b(0)	a(0)	...	a(t)	b(t)	a(t+1)	...	a(T-1)	b(T-1)	a(T)	Max	

Table 1: Staircase Control Structure

Variables										
x(1)	x(2)	...	x(t-1)	x(t)	x(t+1)	...	x(T-1)	x(T)	Constraints	Right-hand Side Constants
A(1)									=	d(1)
B(1)	A(2)	.							=	d(2)
		.								⋮
		.	B(t-1)	A(t)					=	d(t)
				B(t)	A(t+1)	.			=	d(t+1)
						.				⋮
						.	B(T-1)	A(T)	=	d(T)
Performance Index Constants									Max	
c(1)	c(2)	...	c(t-1)	c(t)	c(t+1)	...	c(T-1)	c(T)		

Table 2: Staircase LP Structure

Variables								
u(0)	u(1)	...	u(t-1)	u(t)	...	u(T-1)	Constraints	Right-hand Side Constants
D(0)							$\leq$	h(0)
W(1,0)	D(1)						$\leq$	h(1)
. . .	. . .	. . .	. . . .	. . .	. .	. .	. . . .	. . . . .
W(t,0)	W(t,1)	...	W(t,t-1)	D(t)			$\leq$	h(t)
. . .	. . .	. . .	. . . .	. . .	. .	. .	. . . .	. . . . .
W(T-1,0)	W(T-1,1)	...	W(T-1,t-1)	W(T-1,t)	...	D(T-1)	$\leq$	h(T-1)
Performance Index Constants							Max	
w(0)	w(1)	...	w(t-1)	w(t)	...	w(T-1)		

Table 3: Block Triangular LP Structure

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