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Abstract

Given a group composed of N individuals and given a von Neumann-Morgenstern utility function for each individual in the group, how can these be aggregated to obtain a group von Neumann-Morgenstern utility function. The implications of a set of axioms, analogous to Arrow's, using individual cardinal utilities--rather than Arrow's ordinal rankings--are investigated. The result is a group cardinal utility function which explicitly requires interpersonal comparison of preference. Suggestions for who should make these comparisons and how they might be done are given.

1. Introduction

How should a group of individuals choose among a set of alternatives? Certainly there are a host of possible answers here ranging from formal aggregation schemes to informal discussion until a concensus emerges. The general problem--sometimes referred to as the social welfare problem--has drawn much attention from economists, sociologists, political scientists, etc.

The problem is often formalized along the following lines. A set of N individuals I_i, $i = 1,2,\ldots,N$ must collectively select an alternative $\underline{a}_j$ from the set $A = \{\underline{a}_j, j = 1,2,\ldots,M\}$. It is assumed that each individual I_i can articulate his preferences, denoted by P_i. For instance, P_i could be a ranking of the M alternatives, or it could be a preference structure such as a von Neumann-Morgenstern utility function over the set of possible consequences of the alternatives, or it could be expected utilities associated with the alternatives. The problem is to obtain the group preferences P_G given the individual preferences P_i, $i = 1,2,\ldots,N$. Thus, a function f is needed such that

$$P_G = f(P_1, P_2, \ldots, P_N). \tag{1}$$

The usual approach has been to put reasonable restrictions on the manner in which the P_i are combined, and then derive the implications this places on f. For instance, one such restriction might be if $P_i = P$ for all i, then $P_G = P$, the common individual preference.

There are two versions of the problem formalized by (1) which are of interest in this paper. These will be referred to as the benevolent dictator problem and the participatory group problem. In the former case, the aggregation rule, that is the f in (1), is externally imposed by some individual--the benevolent dictator. In the participatory group, the group itself must internally generate the aggregation rule for selecting a best group alternative. The theoretical development is the same for both of these versions of the "social welfare problem," however, the necessary input assessments needed to implement the results must be obtained in different manners.

In Section 2, we briefly summarize aspects of Arrow's [1] work on the social welfare problem. His formulation assumed P_G and the P_i were rankings of the alternatives. His result is that, in general, there is no f which satisfies five "reasonable" assumptions; and, hence, the assumptions are incompatible. Arrow's formulation, since it used rankings, did not incorporate any concepts of strength of preference nor did it attempt to interpersonally compare preferences. Harsanyi [2] was among the first to investigate assumptions leading to a group von Neumann-Morgenstern utility function. More recently, Sen [7] has shown that formulations with the structure of (1) require interpersonal comparison of utility in order to achieve a group preference for all possible sets of individual preferences.

This paper formulates the group decision problem in a manner analogous to Arrow except that the group preference P_G and the individual preferences P_i are utilities of the alternatives in the von Neumann-Morgenstern sense. In Sections 3 and 4, we successively investigate two models: the certain alternative model, where it is assumed that alternatives have known consequences, and the uncertain alternative model, where there may be uncertainties associated with the consequences of the decision. The uncertainty model allows one to include certain alternatives, that is, each alternative need not involve uncertainty. For both models it is shown that given five assumptions analogous to Arrow's using cardinal utilities rather than rankings, it is always possible to define consistent aggregation rules for a group cardinal utility function. These rules explicitly require interpersonal comparison of preference. Suggestions for obtaining the necessary assessments to utilize these aggregation rules are given in Section 5.

2. Arrow's Impossibility Theorem

Arrow proves that, in general, there is no procedure for obtaining a group ordering (i.e. ranking) of the various alternatives, call this P_G, based on individual group members orderings P_i that is consistent with five seemingly reasonable assumptions. That is there is no f satisfying (1) when the P_i's are rankings that is consistent with these five conditions:

Assumption A1. There are at least two individual members in the group, at least three alternatives, and a group ordering is specified for all possible individual member's orderings.

Assumption A2. If the group ordering indicates alternative a is preferred to alternative b for a certain set of individual orderings, then the group ordering must imply a is preferred to b if:

i) the individual's orderings of alternatives other than a are not changed, and

ii) each individual's ordering between a and any other alternative either remains unchanged or is modified in favor of a.

Assumption A3. If an alternative is eliminated from consideration, the new group ordering for the remaining alternatives should be equivalent (i.e. the same ordering) to the original group ordering for these same alternatives.

Assumption A4. For each pair of alternatives a and b, there is some set of individual orderings such that the group prefers a to b.

Assumption A5. There is no individual with the property that whenever he prefers alternative a to b, the group will also prefer a to b regardless of the other individual's orderings.

Luce and Raiffa [6] examine the reasonableness of these assumptions and suggest that Assumption A3, referred to as Independence of Irrelevant Alternatives Assumption is the weakest of the five. The problem arises from interpreting (or misinterpreting) an individual's strength of preference of one alternative over another based on that individual's "closeness" in ranking of the two alternatives. In what follows, our formulation explicitly utilizes individual's strength of preferences and avoids this particular difficulty.

3. A Cardinal Utility Axiomatization for Certain Alternatives

The specific problem addressed is as follows. For each individual I_i, $i = 1,2,\ldots,N$, we are given the set of cardinal utilities $u_i(\underline{a}_j)$ of the alternatives $\underline{a}_j$, $j = 1,2,\ldots,M$. We wish to obtain group cardinal utilities $u_G(\underline{a}_j)$ for each $\underline{a}_j$ from the $u_i(\underline{a}_j)$ consistent with five assumptions analogous to Arrow's. For decision purposes, the best group alternative is the one associated with the highest group utility. In terms of (1), the problem is to find a function u such that

$$u_G = u(u_1, u_2, \ldots, u_N) \qquad (2)$$

that is consistent with five axioms:

Assumption B1. There are at least two individual members in the group, at least two alternatives, and group utilities are specified for all possible individual member's utilities.

Assumption B2. If the group utilities indicate alternative $\underline{a}$ is preferred to alternative $\underline{b}$ for a certain set of individual utilities, then the group utilities must imply $\underline{a}$ is preferred to $\underline{b}$ if:

i) the individual's utilities of alternatives other than a are not changed, and

ii) each individual's utilities for a either remains unchanged or is increased.

Assumption B3. If an alternative is eliminated from consideration, the new group utilities for the remaining alternatives should be equivalent (i.e. positive linear transformations) to the original group utilities for these same alternatives.

Assumption B4. For each pair of alternatives a and b, there is some set of individual utilities such that the group prefers a to b.

Assumption B5. There is no individual with the property that whenever he prefers alternative a to b, the group will also prefer a to b regardless of the other individual's utilities.

As can be seen, the main distinction--and the only relevant one--between these assumptions and Arrow's is the substitution of group and individual utilities for his group and individual orderings. The interesting result is that for certain alternatives, there are many possible forms of u in (2) which satisfy Assumptions B1 - B5, whereas there were no f's in (1) consistent with Arrow's Assumptions A1 - A5. Let us investigate the properties of such forms to indicate that in fact some do exist.

Theorem 1. A given group cardinal utility function $u_G = u(u_1, u_2, \ldots, u_N)$ over certain alternatives is consistent with Assumptions B1 - B5 if and only if

$$\frac{\partial u}{\partial u_i} \geq 0 \quad , \qquad i = 1,2,\ldots,N \quad , \tag{3}$$

and not equal to zero for at least two u_i's.

Proof. Let us assume u of (2) satisfies condition (3). Then B1 is trivially satisfied. If u_i for an alternative a increases, then u increases if $\partial u/\partial u_i$ in the region of u_i is positive and remains unchanged if $\partial u/\partial u_i$ is zero. Hence B2 is satisfied. Given u, dropping an alternative has no effect on the u values for the remaining alternatives so B3 is met. If each individual prefers a to b, then since $\partial u/\partial u_i$ is positive for at least one u_i and never negative for any u_i, the u assigned to a using (2) must be bigger than that assigned to b. Thus condition B4 is satisfied. Assumption B5 is also satisfied by (2) because there is always some small amount ε such that if individual I_i prefers a to b by a utility margin of ε, and if all other individuals prefer b to a, then since $\partial u/\partial u_i$ must be positive for at least one of the other $N-1$ individuals, alternative b must be assigned a larger u than alternative a implying the group prefers b.

To prove the converse, let us argue by contradiction. Assume assumptions B1 - B5 are satisfied but that condition (3) is not met. Suppose $\partial u/\partial u_1 < 0$. Then an increase in u_1 for some alternative a would imply a decrease in u assigned to a so assumption B2 could be violated. Thus no $\partial u/\partial u_i$ can be negative. If all $\partial u/\partial u_i$ are zero, assumption B4 is violated. If only $\partial u/\partial u_1$ is positive with $\partial u/\partial u_i = 0$, $i = 2,\ldots,N$, then individual I_1 dictates policy violating assumption B5. Thus assumptions B1 - B5 imply condition (3) with at least two $\partial u/\partial u_i$ positive. ◀

It is worthwhile to indicate why cardinal utility functions, the u and the u_i's, might be used when, in fact, there are no uncertainties. All that would be needed is a function which ranks the N-tuples $(u_1,u_2,\ldots,u_N)$ in order to select the best alternative. The reason for using cardinal utility functions is twofold. First u provides an indication of the relative strength of preference for the alternatives. Second, each u_i provides an indication of individual I_i's relative strengths of preferences for the alternatives, which in turn greatly simplifies consistent scaling of the u_i's as shown in Section 5.

4. A Cardinal Utility Axiomatization for Uncertain Alternatives

We have established that group cardinal utility functions consistent with assumptions B1 - B5 do exist when the arguments of these functions are different individual's cardinal utilities of certain alternatives. Let us now expand our problem to include uncertain alternatives. For this interpretation, it may be more convenient to think of the certain alternatives as being tautological to the consequences which they imply. An uncertain alternative indicates which of the $\underline{a}_j$'s may result and their associated probabilities, which will be denoted p_j. In general, the different individuals associated with a particular problem may be in disagreement about the values of the p_j's for any particular situation.

Let us examine the formulation implied by assumptions B1 - B5 when the individual's expected utilities, rather than utilities of certain alternatives, are inputs to the utility function u of (2). Note that in this case u_G will also be an expected utility. We will prove the following strong result.

Theorem 2. A given group cardinal utility function $u_G = u(u_1, u_2, \ldots, u_N)$ over uncertain alternatives is consistent with assumptions B1 - B5 if and only if

$$u(u_1, u_2, \ldots, u_N) = \sum_{i=1}^{n} k_i u_i \qquad (4)$$

where $k_i \geq 0$, $i = 1, \ldots, N$ and $k_i > 0$ for at least two k_i's.

Proof. The power leading to this result is mainly in the formulation (2) itself. The formulation says u is a cardinal utility function and only the expected utility to each individual is important. If the individuals are each indifferent between two uncertain alternatives, then each individual must have the same expected utility for the two alternatives. Since the u_i's in (2) are now expected utilities, the group utility for the two alternatives must be equal since the arguments are identical. Harsanyi [2] proved twenty years ago that if it follows that the group is indifferent between two uncertain alternatives whenever the individuals are indifferent, then u must be additive. Assumption B2 implies the k_i's are non-negative. Assumptions B4 and B5 imply at least two k_i's are positive. The converse follows since (4) implies (3). ◀

The power of the formulation using expected utilities comes about from the fact that the "balance" of utilities among individuals is assumed to be unimportant. To briefly illustrate let us investigate a problem with two individuals and consider two specific alternatives. Alternative A results in either ($u_1 = 1$, $u_2 = 0$) with probability one-half or ($u_1 = 0$, $u_2 = 1$) with probability one-half. Alternative B yields either ($u_1 = 1$, $u_2 = 1$) or ($u_1 = 0$, $u_2 = 0$), each with probability one-half. Note that individual I_1, whose utility is measured by u_1, would be indifferent between alternatives A and B since they both have the same expected utility. For the same reason, individual I_2 would be indifferent. Thus it follows that the group of two must be indifferent if the formulation is accepted. However, note that alternative B might be considered preferable to alternative A because it is "fair" to both individuals. A discussion of such "equity" considerations, as well as forms of cardinal utility functions which promote equity, are found in Kirkwood [5] and Keeney and Kirkwood [4].

5. Interpretation and Assessment of the Group Utility Functions

The assessments necessary to implement the formulation of the last section come from different sources for the two versions--the benevolent dictator and the participatory group--of group decision problems defined at the beginning of the paper. In both cases the cardinal utilities of the alternatives come from the individuals who make up the groups; each individual articulates his own utilities. The more difficult assessments concern obtaining the scaling constants, that is the k's in (4). In the benevolent dictator model, the benevolent dictator himself must make these judgments, whereas the group as a whole must assess the k's in the participatory group model.

Assessing the k's requires interpersonal comparison of preferences. Since (4) is an appropriate utility function for both the certain alternative and uncertain alternative models, let us illustrate the point by considering the benevolent dictator who must assess the k's in (4). Since the individual's utilities can be arbitrarily scaled from zero to one, we can arbitrarily set $u(0,0,\ldots,0) = 0$ and $u(1,1,\ldots,1) = 1$, where u is actually the benevolent dictator's utility function. The benevolent dictator must consider questions like which of $(1,0,\ldots,0)$ or $(0,1,0,\ldots,0)$ he prefers. It is easily to show from (4) that $u(1,0,\ldots,0) = k_1$ and $u(0,1,0,\ldots,0) = k_2$ so if the former is preferred, then $k_1 > k_2$. With similar considerations, a ranking of the k's can be developed. These considerations are not easy since the benevolent dictator must conjur up in his mind what a $u_1 = 0$ and a $u_1 = 1$ means to individual I_1 and what a $u_2 = 0$ and $u_2 = 1$ means to I_2, and then superimpose his own value structure about the relative desirability of the change in u_1 from 0 to 1 versus the change in u_2 from 0 to 1, etc. Suppose k_1 is greater than k_2, then the benevolent dictator must ask himself, how much u_1, call it u_1^* is such that $(u_1^*,0,\ldots,0)$ is indifferent to $(0,1,0,\ldots,0)$. By using (4) and equating utilities of these circumstances, we find $k_1 u_1^* = k_2$. A similar procedure is repeated for each of the u_i's which provides us with a set of N - 1 equations and N unknowns, the k_i's. Because of our scaling convention, the Nth equation is $\sum_{i=1}^{N} k_i = 1$. The values of the k_i's can be found from this set of N equations.

The same type of thinking must be followed in the participatory group decision model by each of the individuals in the group. However, in addition, they must somehow arrive at a concensus for the k's. Sometimes this may not be possible and thus the model could not be used as intended.

In general, to assess the k's, one must find pairs of circumstances $(u_1', u_2', \ldots, u_N')$ and $(u_1'', u_2'', \ldots, u_N'')$ for which the assessor(s) is indifferent. Then naturally $u(u_1', u_2', \ldots, u_N') = u(u_1'', u_2'', \ldots, u_N'')$ gives us one equation with at most N unknown scaling constants. The idea to generate N independent equations involving the scaling constants and then solve for them. Kirkwood [5] discusses an alternative approach for the assessment of the scaling constants.

It is a difficult problem for the decision maker--the benevolent dictator in the benevolent dictator model or the group as a whole in the participatory group model--to make the requisite interpersonal comparisons of utility. An excellent discussion of this issue is found in Harsanyi [3]. We make no pretense that interpersonal utility comparisons are easy, but they are often implicitly made in group decisions.

When one can formalize this aspect of the process, the group utility functions discussed in this paper do provide a means for integrating these preferences which may be reasonable for some problems.

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