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ABSTRACT

Given n attributes, it is shown that if two subsets of these attributes overlap and are each utility independent of their respective complements, then their union, intersection, symmetric difference, and two differences are each utility independent of their complements. A chaining theorem using this result indicates how to simplify the assessment of a multiattribute utility function to the maximum extent possible, subject to any specific set of utility independence assumptions.

1. INTRODUCTION

This paper presents some general results which permit one to decompose multiattribute utility functions. Given a set of attributes $X \equiv \{X_0, X_1, \ldots, X_n\}$, we illustrate how arbitrary sets of utility independence assumptions among the X_i, $i = 1, \ldots, n$ imply a von Neumann-Morgenstern utility function of the form

$$u(x_0, x_1, \ldots, x_n) = h[x_0, u_1(x_1), u_2(x_2), \ldots, u_n(x_n)] \quad , \quad (1)$$

where x_i is a specific amount of X_i, h is a scalar valued function, and $u_i(x_i)$ is a utility function over X_i. These results relate to forms of (1) which have been derived for specific sets of preference assumptions by Fishburn [1,2,3], Meyer [9], Pollak [10], Raiffa [11] and Keeney [5,6,7]. Note from (1) that x_0 plays a different role than the other x_i.

The organization of the paper is as follows. Section 2 defines terms and specifies our notation. A basic result relating two overlapping utility independence assumptions is given in Section 3. This is the building block for the main result of the paper in Section 4. Section 5 discusses the relevance of the results. The results in this paper are completely analogous to those of Gorman [4], who used the riskless analog to utility independence. He referred to this as separability. In this paper, we will call it preferential independence.

2. NOTATION

Let the consequence space $X_0 \times X_1 \times \ldots \times X_n$ represent a closed and bounded rectangular subset of a finite dimensional Euclidean space. Each X_i may be a vector or scalar attribute, implying that

x_i may be either a vector or a scalar. Then $x \equiv (x_0, x_1, \ldots, x_n)$ is a consequence. We are interested in specifying functional forms of the utility function $u(x)$ that are consistent with various sets of assumptions about the decision maker's preferences. It is assumed that $u(x)$ is continuous in each x_i. Given the complete set of attributes $X = \{X_0, X_1, \ldots, X_n\}$, we will refer to any two subsets Y_1 and Y_2 which partition X as complementary sets of attributes. The complement of Y will be designated as $\bar{Y}$.

Definition. Attribute Y, where $Y \subset X$, is utility independent (UI) of its complement $\bar{Y}$ if the conditional preference order for lotteries involving only changes in the levels of attributes in Y does not depend on the levels at which the attributes in $\bar{Y}$ are held fixed. If Y is utility independent of $\bar{Y}$, then, since utility functions and unique up to positive affine transformations,

$$u(y,\bar{y}) = f(\bar{y}) + g(\bar{y})u(y,\bar{y}') \quad , \text{ for all } y \text{ and } \bar{y} \quad , \qquad (2)$$

where $g(\bar{y}) > 0$ and $\bar{y}'$ is an arbitrarily chosen specific amount of $\bar{Y}$. Rather than repeatedly saying that Y is utility independent of its complement $\bar{Y}$ we will simply write Y is UI.

We will set the origin of the utility function by

$$u(x^0) \equiv u(x_0^0, x_1^0, \ldots, x_n^0) \equiv u(y^0, \bar{y}^0) = 0 \quad , \qquad (3)$$

where y^0 and $\bar{y}^0$ are least preferred levels of Y and $\bar{Y}$. Then, by evaluating (2) at y^0, we find $f(\bar{y}) = u(y^0, \bar{y})$, so condition (2) can be written as

$$u(y,\bar{y}) = u(y^0,\bar{y}) + g(\bar{y})\, u(y,\bar{y}^0) \quad , \qquad (4)$$

where we have chosen to set $\bar{y}'$ in (2) equal to $\bar{y}^0$. Equation (4) will be used in our proofs.

Definition. Attribute Y, where $Y \subset X$, is preferentially independent (PI) of its complement $\bar{Y}$ if the preference order of consequences involving only changes in the levels in Y does not depend on the levels at which attributes in $\bar{Y}$ are held fixed.

Preferential independence implies the conditional indifference curves over Y do not depend on attributes $\bar{Y}$. The concept concerns the decision maker's preferences for consequences where no uncertainty is involved. By definition, it follows that if Y is UI, then Y is PI. The converse is not necessarily true. This relationship can be seen by noting that degenerate lotteries, those involving no uncertainty, are the same things as a consequence. Hence, the preferential independence condition could be stated in terms of the preference order for degenerate lotteries only, and since the utility independence condition holds for all lotteries, the former is implied by the latter. Utility independence is the stronger condition.

A result linking preferential independence and utility independence which we will use is

Lemma 1. Given three attributes $\{X_0, X_1, X_2\}$, if $\{X_1, X_2\}$ is preferentially independent of X_0 and if X_1 is utility independent of $\{X_0, X_2\}$, then $\{X_1, X_2\}$ is utility independent of X_0. A proof of this result is found in Keeney [7].

3. RELATIONSHIPS AMONG UTILITY INDEPENDENCE ASSUMPTIONS

If $Y \subset X$ and Y is UI, the order of the UI condition is defined as the number of X_i's in Y. We are interested in implying higher order utility independence conditions from lower order conditions.

Definition. Let Y_1 and Y_2 be subsets of $X \equiv \{X_0, X_1, X_2, \ldots, X_n\}$. Attributes Y_1 and Y_2 overlap if their intersection is not empty and if neither includes the other.

Theorem 1. Let Y_1 and Y_2 be overlapping attributes included in $X \equiv \{X_0, X_1, \ldots, X_n\}$. If Y_1 and Y_2 are each UI, then

(i) $Y_1 \cup Y_2$, the union of Y_1 and Y_2, is UI,

(ii) $Y_1 \cap Y_2$, the intersection of Y_1 and Y_2, is UI,

(iii) $(Y_1 \cap \bar{Y}_2) \cup (\bar{Y}_1 \cap Y_2)$, the symmetric difference of Y_1 and Y_2, is UI,

(iv) $Y_1 \cap \bar{Y}_2$ and $\bar{Y}_1 \cap Y_2$, the differences, are each UI.

Note before proof. If utility independence is replaced by the weaker preferential independence in both the premise and result of Theorem 1, we have Gorman's theorem [4].

Proof. Since X_i can designate a vector attribute, the general case can be proven by considering the special case where $X = \{X_0, X_1, X_2, X_3\}$, $Y_1 = \{X_1, X_2\}$, and $Y_2 = \{X_2, X_3\}$, and where Y_1 and Y_2 are each assumed to be UI. Since UI implies PI, each set of attributes in (i) through (iv) is PI using Gorman's theorem. We now show that X_1, X_2, and X_3 are each UI and the proof follows from Lemma 1.

From (4), our hypotheses can be written respectively as

$$u(x) = u(x_0,x_1,x_2,x_3) = u(x_0,x_3) + c(x_0,x_3)\ u(x_1,x_2) \quad , \tag{5}$$

and

$$u(x) = u(x_0,x_1,x_2,x_3) = u(x_0,x_1) + d(x_0,x_1)\ u(x_2,x_3) \quad , \tag{6}$$

where we have taken the liberty to delete arguments of u, c, and d when they are at their least preferred levels and no misunderstanding can result; that is, when $x_i = x_i^0$. Hence, for instance, $u(x_1,x_2)$ and $d(x_0)$ will denote $u(x_0^0,x_1,x_2,x_3^0)$ and $d(x_0,x_1^0)$ respectively.

Substituting (6) into (5) and then (5) into (6) gives us, respectively,

$$u(x) = u(x_0) + d(x_0)\ u(x_3) + c(x_0,x_3)[u(x_1) + d(x_1)\ u(x_2)] \quad , \tag{7}$$

and

$$u(x) = u(x_0) + c(x_0)\ u(x_1) + d(x_0,x_1)[u(x_3) + c(x_3)\ u(x_2)] \quad . \tag{8}$$

Equating (8) and (9) with $x_3 = x_3^0$ indicates

$$d(x_0,x_1) = c(x_0)\ d(x_1) \quad . \tag{9}$$

Similarly, equating (5) and (6) with $x_0 = x_0^0$ and $x_2 = x_2^0$ indicates

$$u(x_3) + c(x_3)\ u(x_1) = u(x_1) + d(x_1)\ u(x_3) \quad , \tag{10}$$

which can be rearranged to yield

$$\frac{c(x_3) - 1}{u(x_3)} = \frac{d(x_1) - 1}{u(x_1)} = k \quad , \quad u(x_i) \neq 0 \quad , \quad i = 1,3, \tag{11}$$

where k is a constant since (11) has a function of x_3 equal to a function of x_1. If $u(x_1) = 0$, from (10), it follows that $d(x_1) = 1$, and similarly $c(x_3) = 1$ when $u(x_3) = 0$. Thus, from (11), one sees

$$c(x_3) = ku(x_3) + 1 \quad , \tag{12}$$

and

$$d(x_1) = ku(x_1) + 1 \quad . \tag{13}$$

Substituting (9), (12), and (13) into (8) yields

$$u(x) = u(x_0) + c(x_0)\{u(x_1) + (ku(x_1) + 1)[u(x_3) + (ku(x_3) + 1)u(x_2)]\}, \tag{14}$$

from which one sees that X_1, X_2, and X_3 are each UI, which completes the proof.

4. A CHAINING THEOREM

Roughly speaking, the more utility independence properties we can identify, the simpler the assessment of the utility function becomes. It is important to specify the simplest functional form of the multiattribute utility function consistent with an arbitrary set of utility independence assumptions. With this in mind, we want to generalize the results of Section 3 by constructing a "chaining theorem" using Theorem 1 as the building block.

Definition. A utility independent chain is a collection of sets $\{Y_1, \ldots, Y_R\}$, where (1) Y_j is UI, $j = 1, \ldots, R$, and (2) there is an ordering of Y_1 through Y_R such that each Y_j (other than the first in the ordering) overlaps at least one of its predecessors in the ordering.

We will be interested in finding utility independent chains which consist of as many sets as possible. This will allow us to exploit the utility independence properties to the fullest extent in simplifying the implied functional form of the utility function.

Definition. Let $\{Y_1, \ldots, Y_J\}$ be a set such that Y_j is UI,

$j = 1,\ldots,J$ and let $\{Y_1,\ldots,Y_R\}$, $R \le J$ be a utility independent chain. This chain is a <u>maximal utility independent chain</u> if no Y_j, $j = R + 1,\ldots,J$, overlaps any Y_j, $j = 1,\ldots,R$.

<u>Definition</u>. Let $\{Y_1,Y_2,\ldots,Y_R\}$ be a maximal utility independent chain. Each Y_j, $j \le R$, partitions $X \equiv \{X_1,X_2,\ldots,X_n\}$ into Y_j and $\bar{Y}_j$. There are 2^R possible subsets of X created by taking intersections formed with either Y_j or $\bar{Y}_j$ for each $j \le R$. Each nonempty intersection, except for $\bigcap_{j=1}^{R} \bar{Y}_j$, is defined to be an <u>element</u> of the maximal utility independent chain $\{Y_1,\ldots,Y_R\}$.

An example should help illustrate our definitions.

<u>Example</u>. Consider the set $X = \{X_1,X_2,\ldots,X_8\}$, and suppose Y_j is UI, $j = 1,2,\ldots,5$, where

$$Y_1 \equiv \{X_1,X_2,X_3\},\ Y_2 \equiv \{X_3,X_4,X_5\},\ Y_3 \equiv \{X_2,X_3\},\ Y_4 \equiv \{X_5\},\ Y_5 \equiv \{X_7,X_8\} \ .$$

Note that Y_2 overlaps Y_1 so $\{Y_1,Y_2\}$ is a utility independent chain. Now Y_3 is included in Y_1 but Y_3 does overlap Y_2. Thus, $\{Y_1,Y_2,Y_3\}$ is another utility independent chain. Checking Y_4, we see it is included in Y_2 and distinct from both Y_1 and Y_3. Thus, the attribute Y_4 does not overlap any of Y_1, Y_2, or Y_3, so it does not enter the maximal utility independent chain we are constructing. Also Y_5 does not overlap any of Y_1, Y_2, or Y_3, implying that the collection of sets $\{Y_1,Y_2,Y_3\}$ is a maximal utility independent chain on X. In addition, Y_5 is itself another maximal utility independent chain on X.

To identify the elements of the maximal utility independent chain $\{Y_1,Y_2,Y_3\}$, we note $Y_1Y_2Y_3 = \{X_3\}$, $Y_1\bar{Y}_2Y_3 = \{X_2\}$, $Y_1\bar{Y}_2\bar{Y}_3 = \{X_1\}$, $\bar{Y}_1Y_2\bar{Y}_3 = \{X_4,X_5\}$, and $Y_1Y_2\bar{Y}_3$, $\bar{Y}_1Y_2Y_3$, and $\bar{Y}_1\bar{Y}_2Y_3$ are empty. Thus there are four elements of the chain, namely X_1,X_2,X_3, and

and $\{X_4,X_5\}$. For the maximal utility independent chain Y_5, there is the one element $\{X_7,X_8\}$.

Let us return to the general case and state an important result.

Theorem 2. Each possible union of the elements in each maximal utility independent chain defined on $X = \{X_0,X_1,\ldots,X_n\}$ is utility independent of its complement in X.

Gorman [4] also proved a result analogous to Theorem 2 concerning preferential independence using two overlapping subsets as a building block. The reason each possible union of elements in any maximal utility independent chain is utility independent is that it can be constructed from $\{Y_1,\ldots,Y_R\}$ by taking unions, intersections, and symmetric differences of overlapping UI subsets and using Theorem 1. A proof of Theorem 2 using utility independence assumptions is found in [8].

5. RELEVANCE OF THE RESULTS

We will remark on two issues: verification of utility independence conditions and representation theorems following from Theorem 2.

First, we would often expect that it would be easier to verify lower order utility independence conditions. However, for some problem structures, it may seem convenient to group particular sets of attributes. For instance, if we had several attributes arranged in a matrix, columns may represent time periods and rows may characterize different features (e.g., cost, pollution). If one could justify UI conditions for certain columns and rows, Theorem 2 would be directly relevant.

Second, suppose $C_1,C_2,\ldots,C_m$ are each maximal utility independent chains on $\{X_0, X_1, \ldots, X_n\}$ such that X_0 is not in any C_j and each X_i, $i = 1, \ldots, n$ is in exactly one C_j, $j = 1, \ldots, m$. Then since each C_j is UI, it follows from results in [6,8] that one can assess u from

$$u(x_0, x_1, \ldots, x_n) = \lambda[x_0, u_1(c_1), u_2(c_2), \ldots, u_m(c_m)]$$

where λ is scalar valued, u_j is a utility function over the attributes X_i in C_j, and c_j designates a specific level of the attributes X_i in C_j. Furthermore, given Theorem 2, it follows from a result in [7] that each u_j must be of either the additive or multiplicative form in terms of the component utility functions over the elements in C_j.

In this paper, the implications of arbitrary sets of utility independence assumptions have been investigated. Because of the complexity of considering preferences for various levels of several attributes simultaneously, it is important, if not essential, to exploit such independence properties in structuring utility functions involving multiple attributes. The interested reader will find several applications of decomposition results, such as those in this paper, in Keeney and Raiffa [8].

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