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OPTIMAL FLOOD LEVEE DESIGNS BY DYNAMIC PROGRAMMING

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PREFACE

As human interventions increase into natural hydrological processes and the world water demands continue to grow, tools of systems analysis are required to design and operate economically and socially efficient water resources systems. The IIASA Water Resources Project, therefore, seeks to integrate experts of different disciplines to attack water problems of world-wide concern. Floods are still menacing a great number of countries all over the world and the immense economic and social losses caused by floods call for protective measures which seem to be in a sense (e.g. economic) at least - optimal.

This interim report is a joint effort of the IIASA methodological and water resources projects as well as the Hungarian National Water Authority. Hungary is a flat land country where half of the population and one third of the country's area must be protected against floods. Methods elaborated to cope with flood problems there, however, can be transferred to the situation in a number of other countries, e.g. India, Pakistan, Japan, Rumania, etc.

ABSTRACT

An economic optimal development of a levee system along a river is investigated and a dynamic programming (DP) approach is used to find the optima under various conditions. The system consists of a number of levee reaches or stages. A random input of flood wave is regarded at the upstream point of the system. There are two failure modes considered and, consequently, two parameters of the flood wave (state variables) to trigger failure modes in every stages. Stochastic DP is used since the state transition functions (flood routing along the stages) are random functions. Three methods are discussed. In Method I, the expected value of the objective function is taken first, then DP is used as a numerical technique. In Method II, a fixed design flood is chosen as an input under which both optimum cost and policy is determined. In Method III, the value of the expected optimum objective function is calculated. It is shown that the full power of DP cannot be used if Method I is applied. Future research involves comparing the solutions of the three methods.

I. INTRODUCTION

The purpose of this paper is to discuss various methodologies for solving levee design problems under random flood input. Flood protection by means of levees is just one alternative for flood protection measures (Yevjevich, 1974) but it is very commonly used in flat-land rivers such as the Tisza in Hungary (Szidarovszky et al, 1976), the Vistula in Poland, the Loire in France, or the Mississippi in the USA.

Generally, a river section is divided into stages or reaches, then flood protection is examined stage by stage. The flood input into the upstream stage is to be routed so as to satisfy continuity equations. One stage consists generally of a levee stretch of 10-60 km; one gaging station within the stage characterizes flood conditions for that stage.

Flood protection, that is, the reliability of the levee within the reach, can be analyzed by regarding the stage itself as a stochastic system (Bogárdi et al, 1975). The systems approach is warranted since:

- there are various failure modes (overtopping, boiling, slope sliding, wind-wave attack) along the stage;
- the resistances against failure modes are uncertain; and
- there are different flood-wave parameters (peak flow, volume, duration, etc.) of random character that may trigger failure modes.

A present limitation of the above approach is that subsequent stages along the river are considered separately while in reality the reliability of subsequent stages are not independent; e.g. an upstream levee failure results in greater safety for a downstream stage.

In this paper, several stages along a river form a system where safeties of the individual stages are not independent. For each stage, various failure modes are considered which can be triggered by different random flood parameters, but resistances against failure modes are assumed to be known with certainty.

The methodologies examined herein are based on economics: construction costs and flood losses are traded off. Furthermore, since the problem becomes unwieldy by calculus when more than a few stages are considered, dynamic programming is used to decompose the problem.

In the next section, the problem is described in mathematical terms; that is, flood routing equations and loss functions are given. Method I, which minimizes the expected loss, may be solved by standard dynamic programming. Then, a dynamic programming formulation with stochastic input is described, leading to methods II and III. Finally, a numerical example of method II is presented and the necessity of developing further the methodology to solve levee design problems is pointed out.

II. Problem Statement

Physically, the situation we consider is that of a river levee system composed of M reaches. The system is to be designed to balance optimally the cost of construction against the losses due to flooding, should a flow of sufficient magnitude breech some part of the system. Since the prospect of flooding is a direct consequence of the unknown rainfall input to the system, the situation is treated as a control process with stochastic input in which the resistances of each reach against different failure modes are the design (control) variables, selected to minimize the total expected loss. In the following, two common failure modes--overtopping and slope sliding--are considered; the flood parameters triggering these modes are the height, h of the flood and the so-called flood exposure, w (the area of the stage hydrograph above bankful capacity) (Bogárdi, 1968).

The elements of this problem, cast in system theoretical terms so that a dynamic programming approach may be used, are as follows:

<u>Stage:</u> river reach $k \in (1, M)$, where 1 is the initial (downstream) stage and M the final one; a distinction is made between left (L) and right (R) banks.

State (at stage k): a vector $x_k = (h_k, w_k)$, in which h_k is the flood stage height and w_k is the flood exposure. In this problem the final (upstream) state x_M is random.

Decision variable: a vector $v_k = (H_k, W_k)$ in which H_k is the design levee height against overtopping and W_k is the design flood exposure against slope sliding. Since right and left banks are assumed to be different, H_k and W_k are decomposed into HR_k, HL_k and WR_k, WL_k, respectively.

State transition function: a flood routing equation to be specified later, written in a general form as

$$x_{k+1} = \phi_k(x_k, v_k) \quad . \tag{1}$$

Loss function: (1) at stage k: $G_k(x_k, v_k)$, which is the sum of construction costs and flood losses.

(2) overall: $Z = \sum_K G_k(x_k, v_k) \quad , \; k = 1, \ldots, M \quad . \qquad (2)$

The problem is to minimize the expected value of Z; the expectation is to be taken with respect to the random variable x_M. Throughout the remainder of the levee system, there are no external inputs from either underground sources or rainfall.

To describe the evolution of the variables h,w, as we pass through the levee system, equations of "motion", corresponding to a linear routing of flood waves are postulated (Lengyel and Horkai, 1974). These equations give the vector (h_{k+1}, w_{k+1}) as a function of (h_k, w_k). Two cases are distinguished: either the levee holds or it fails.

Let $H3R_k$, $H3L_k$ be the river stages below which flood exposure cannot cause damage. Then the levee does not fail either if the river stage h_k is below the smallest of $H3R_k$, $H3L_k$ (for any value of flood exposure) or if both h_k and w_k are below their threshold values. Using the logical symbols $\vee$= "or" and $\wedge$ = "and", we can write:

$$h_k < \min\{H3R_k, H3L_k\} \vee [h_k < \min\{HR_k, HL_k\} \wedge \wedge w_k < \min\{WR_k, WL_k\}] \quad (3)$$

The <u>levee fails</u> if (3) does not hold, that is, if an event AR_k occurs on the right side (or AL_k on the left bank) such that

$$AR_k = (h_k > H3R_k) \wedge (w_k > WR_k) \vee (h_k > HR_k)$$

Note that AR_k and AL_k are truth-valued logic variables, and not numerical quantities. The equations of motion or state transition equations may thus be written as:

$$h_{k+1} = \begin{cases} a_k h_k + b_k & \text{, if (1) holds} \\ c_k h_k + d_k & \text{, otherwise, if } (AR_k) \vee (AL_k) \text{ occurs} \end{cases} \quad (4)$$

$$w_{k+1} = \begin{cases} e_k w_k + f_k & \text{, if (1) holds} \\ g_k w_k \text{ ,} & \text{otherwise.} \end{cases} \quad (5)$$

Here a_k, b_k, c_k, d_k, e_k, f_k, g_k are parameters characterizing the system.

Equations (4) and (5) describe the manner in which the flood height and flood exposure are influenced by a choice of the decision variables HR_k, HL_k, WR_k, and WL_k. Simple linear relationships can be used to express state transitions between neighboring stages (Linsley <u>et al</u>, 1958).

The final ingredient needed to characterize the control version of the levee design problem is the specification of a cost function. As mentioned, costs are incurred in two separate ways: i) the losses associated with flooding land behind the levee, ii) costs associated with building a reach of a given height and strength. Clearly, the optimal design is a balance between these two costs.

As a measure of loss due to flooding at reach k, we use the function (Horkai, 1975):

$$L_k(h_k, w_k, HR_k, WR_k, HL_k, WL_k) = \begin{cases} 0 & , \bar{A}_{k,R} \wedge \bar{A}_{k,L} , \\ w_k IR_k & , A_{k,R} \wedge \bar{A}_{k,L} , \\ w_k IL_k & , \bar{A}_{k,R} \wedge A_{k,L} , \\ \max\{w_k IL_k, w_k IR_k\} & , A_{k,R} \wedge A_{k,L} \end{cases} \tag{6}$$

where IR_k, IL_k are given parameters. It is assumed that there is no simultaneous loss on both sides of the reach, but rather the loss is taken to be the greater of the two losses in such instances; also, that loss in one reach is independent of losses in adjacent reaches.

The reinforcement (building) costs on the right side of reach k are given by the function (Horkai, 1975):

$$C_k^R(HR_k, WR_k) = \begin{cases} 0 , & \{HR_k \le HRO_k\} \wedge \{WR_k \le WRO_k\} \\ (JR_k)\sqrt{WR_k} + KR_k , & \{HR_k \le HRO_k\} \wedge \{WR_k \le WRO_k\} \\ (LR_k)(HR_k)^2 - (MR_k)(HR_k) + NR_k , & \{HR_k \le HRO_k\} \wedge \{WR_k > WRO_k\} \\ (JR_k)\sqrt{WR_k} + KR_k + (LR_k)(HR_k)^2 \\ \quad - (MR_k)(HR_k) + NR_k , & \{HR_k > HRO_k\} \wedge \{WR_k > WRO_k\} \end{cases} \tag{7}$$

As above, the quantities JR_k, LR_k, KR_k, MR_k, NR_k are parameters, while HRO_k and WRO_k represent the current levee configurations ($HRO_k = WRO_k = 0$ for the design case). An expression completely analagous to C_k^R holds for the left side of reach k upon substitution of R by L in all quantities. The objective function G_k is the sum of (6) and (7).

As described in Eq. (2), the total loss ($\mathfrak{X}$) for the systems is taken to be the sum of all losses;

$$\mathfrak{X} = \sum_{k=1}^{M} \left[L_k(h_k,w_k,HR_k,WR_k,HL_k,WL_k) + C_k^R(HR_k,WR_k) + C_k^L(HL_k,WL_k)\right] \quad (8)$$

In this expression h_M and w_M are random variables. For example, let the final objective be to minimize the expected value of ,
i.e., the objective function is chosen as

$$J = \underset{(h_M,w_M)}{E} (\mathfrak{X}) \quad (9)$$

where E denotes the mathematical expectation; then method I, described next can be used.

III. Methodology

By substituting equation (1) for $k = 1,2,\ldots,M$, into equation (2), the objective function becomes

$$J = E(\mathfrak{X}) \doteq E\{\sum_k G_k(x_M,v_k)\} = \sum_k E[G_k(x_M,v_k)] \quad (10)$$

which may be solved by standard dynamic programming utilized as a numerical technique, which becomes more efficient than calculus whenever M 3, or 4; note that the number of independent variables is **2M.**

However, should a methodology be desired that can accommodate random transition functions, formulation (10) would be inadequate; in such a case, the dynamic programming (DP) formulation should be carried out before taking expectations as it is done when Markov transitions occur from one stage to the next. Let the optimal value function be

$F_k(h_k,w_k)$ = loss for a system which begins at reach k in state (h_k,w_k) when an optimal decision policy is employed throughout the remaining reaches of the system, $k = 1,2,...,M$.

With a self evident deletion of indices, the recursion equation is (Bellman, 1957)

$$F_k(x_M) = \min_v \{G_k(x_M,v) + F_{k+1}[\phi_k(x_M,v)]\} \quad . \qquad (11)$$

$$F_M(x_M) = 0$$

As in Eq. (10), the objective function depends upon the random initial state x_M, but in (11) it is carried throughout calculations by means of the recurrence relationship. In the specific terms of our case study, Eq. (11) is written as

$$F_k(h_k,w_k) = \min_{HR_k,HL_k,WR_k,WL_k} \begin{cases} L_k(h_k,w_k,HR_k,WR_k,HL_k,WL_k) + \\ C_k^R(HR_k,WR_k) + C_k^L(HL_k,WL_k) + \\ F_{k+1}(h_{k+1},w_{k+1}) \quad , \quad k < M \quad , \end{cases} \qquad (12)$$

$$F_M(h_M,w_M) = \min_{HR_M,HL_M,WR_M,WL_M} \begin{cases} [L_M(h_M,w_M,HR_M,WR_M,HL_M,WL_M) + \\ C_M^R(HR_M,WR_M) + C_M^L(HL_M,WL_M)] \quad . \end{cases} \qquad (13)$$

By iterating relation (12) using the initial function (13), an optimal control for each state of each reach is produced.

For computational purposes, a DP table is constructed as a function of the realization of x_M using Eq. (11), or (12) and (13). The columns of this table are as follows (Larson and Casti, 1976):

	Stage M		Stage M-1		....	Stage 1	
x_M	$F_M(x_M)$	$V_M(x_M)$	$F_{M-1}(x_M)$	$V_{M-1}(x_M)$		$F_1(x_M)$	$V_1(x_M)$

This DP table is the basis for two algorithms for solving our levee design problem, which are labelled methods II and III, respectively.

Method II uses the fact that the table contains the optimum policy corresponding to any flood input; thus a design flood x_M^*, say the 99% one, is chosen and both optimum cost $f_a^*(x_M)$ and optimum policy can be determined. Note that no new computation is necessary if one decides to change the design flood. Also, the optimum economic design (or control) is found, not just the levee height that corresponds to the regulation flood; in other works, method II provides a mean to handle design floods within an economic framework.

Method III consists of computing an expected optimum objective function value

$$\bar{f}_a^*(x_M) = \iint f_a(x)\, d\, G\,(x)$$

where G(x) is the joint distribution function of the state x = (h,w). Then, tracing back through the DP Table, the value x_M^* that pertains to $\bar{f}_a^*(x_M)$ is calculated; and another tracing through the table starting with $x_M^{\ddagger}$ leads to the corresponding policy. This policy may thus be labelled "expected value of minimum objective function" policy (EVMOF).

IV. Numerical Experiments

A levee system consisting of three reaches was chosen for the numerical experiments using method II. Hypothetical but realistic values of the parameters in the cost, loss, and transition functions are given in Table I. The 99% design flood parameters are: h = 20.0 and w = 36.5.

There is insufficient space to reproduce the entire set of optimal control tables produced from the dynamic programming here; however, they are all following the same pattern. Keeping h or w constant and increasing the other, the configuration was kept at the original level until conditions for flooding were reached. Then the levee was reinforced up to the level of the flow. Finally, a point was reached where the building costs out-weighed the losses. At that point the optimum fell back to the original level.

In most cases, building was even on right and left sides of the reach. However, if the losses due to flooding were much lower on one side of the reach and the cost of building high, the optimal control was to reinforce the more costly side to force any flooding to occur on the other side.

The levee configurations to minimize the expected loss are given in Table II. It should be noted that the results should not be taken at face value because the input data were in part hypothetical.

DISCUSSION and CONCLUSIONS

The three methods proposed in this paper lead almost certainly to different values of the goal function and different optimal policies.

Method I is akin to standard benefit-risk analysis; the expected value of the objective function is taken first; then, dynamic programming is used as a numerical technique for solution instead of calculus, due to the potential dimensions of the decision space and the non-differentiability of the goal functions. However, in a sense the dynamics of the problem have disappeared. It is an open-loop approach since the random flood is routed through the levee system before starting the optimization procedure. In contrast with method I, methods II and III correspond to a closed-loop approach, since the random flood is present at least implicitly at every stage during the optimization. Method II enables the decision-maker to mix the traditional concept of design flood with the minimization of an economic goal function. It may be noted that high uncertainties may be present in the design flood (Davis et al, 1976) and/or the goal function (Szidarovszky et al, 1976). Method III corresponds to choosing policy for the expected value of the optimum goal function; its implementation would necessitate a finer grid and better computing capabilities than the ones available for the present study. Also, the policy found by this method may not be unique in the case when the goal function is not monotone. This would not be a problem in the present case, since g_k is an increasing function of $v_k = (h_k, w_k)$.

To summarize, the distinct advantages of DP to solve a levee design problem are:

(1) Calculations can be made once and for all input values.

(2) Non-differentiable functions must be used to describe loss functions for non-computational problems.

(3) A choice of approaches is given, i.e. optimize under a design flood constraint or find an EVMOF policy; both approaches are of a closed-loop nature.

The full power of DP as described in (1),(2), and (3) above is not utilized if method I is used, since the expectation is taken before the DP algorithm is applied.

Future research involves comparing the solutions provided by methods I, II, and III.

Table I. Levee Parameters

	3R	3L	2R	2L	1R	1L
a			.96		.77	
b			.64		1.56	
c			.475		.448	
d			3.45		3.53	
e			1.21		.63	
f			2.23		.30	
g			.475		.448	
I	12.6	17.35	5.67	27.57	14.81	23.98
J	2.69	2.04	2.84	2.77	2.12	1.26
K	-9.88	-7.07	-10.43	-10.18	-7.93	-4.18
L	6.05	3.07	9.08	6.24	4.88	3.59
M	121.0	58.3	192.4	124.8	97.6	67.5
N	605.0	276.78	1,019.21	624.00	488.00	317.29
H3	8.5	8.4	8.5	8.5	9.3	8.2
HO	10.0	9.5	10.6	10.0	10.0	9.4
WO	13.5	12.0	13.5	13.5	14.0	11.0

Table II.

METHOD II - Building for 99% Flood

			OPTIMAL CONTROL			
	h	w	HR	WR	HL	WL
Reach 3	20.0	36.5	10.0	13.5	9.5	12.0
Reach 2	12.9	17.3	12.9	17.3	12.9	17.3
Reach 1	9.3	7.8	10.0	14.0	9.4	11.0

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