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Competition Among Several Gas Pipelines: A Game Model with Exponential Functions

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Abstract

The process of competition of large-scale projects is studied in a setting motivated by real-life problems of optimization of gas and oil transportation networks and optimization of the corresponding investment. The employed mathematical model is a noncooperative game of several players with choice of time moments and payoff functions that contain improper integrals.

It is assumed that the investigated processes are described by exponential functions. This assumption is reasonable because of the economic sense of the problem. Also, this assumption simplifies the mathematical model and the implementation of the corresponding algorithms. The use of exponential functions makes it possible to create effective codes for computer modelling of these problems. The paper contains detailed consideration of the involved mathematical assumptions, description of the algorithms for finding points of Nash equilibrium and the best responses of investors to actions of other investors, and description of the developed software. An illustrative numerical example is also given.

Acknowledgment

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1 Introduction

Mathematical and computer models of the process of competition of investors to large-scale projects, such as construction of gas pipelines, was studied in many publications, see e.g. Brykalov et al., 2005; Klaassen et al., 2002. Klaassen et al., 2004 described Nash equilibrium points in a game of two investors who finance competing projects of gas pipeline construction. Brykalov et al., 2005 showed that if instead of a complete description, we are satisfied with an algorithm that enumerates all the points of Nash equilibrium, then we can consider a game of several investors and significantly relax the imposed mathematical requirements.

The games considered in the above-cited articles provide mathematical models for the following situation. Several gas pipelines are being built by competing investors and are aimed at one and the same regional market of natural gas. When new gas pipelines come into operation, the amount of gas supplied to the market is increased, which obviously can lower the price of gas. The investors that put their pipelines into operation earlier can enjoy a high price of gas for some time. The investor who comes to the market first enjoys some period of monopoly. On the other hand, completing the construction of a pipeline later can be desirable for a number of reasons. In particular, it can reduce the price of construction. This naturally creates a kind of game between the investors. Various aspects of this game were studied in the above cited articles.

In particular, the above described research included mathematical and computer modelling of the Turkish gas market. These considerations are based on the assumption that the price of gas is set by the market itself. Some heuristic algorithms were proposed. A strict mathematical model was developed and published later by Klaassen et al., 2004. This model has initiated further development of computer realizable algorithms and mathematical generalizations in many ways.

In the further research, an attempt was made to apply the developed technique to the Chinese natural gas market. However, many of the assumptions used for modelling the Turkish market appeared to be invalid for the specific Chinese market. From the point of view of economics, the main difference is that in the case of China the prices are fixed not purely by a market mechanism. A mathematical model that takes into account these circumstances was given by Nikonov 2004. Methodically, this article is a continuation of the research by Klaassen et al., 2004, where the basic model is described. However, the assumptions of the model of Chinese market are essentially different and sometimes the opposite.

Below the results and algorithms of Brykalov et al., 2004 are specified for a typical case when the process of construction and exploitation of the gas pipelines is described with the help of exponential functions see Brykalov et al., 2005. This allows us to simplify many of considerations and imposed conditions. On the other hand, this supposition is not too restrictive as exponential functions frequently arise in connection with problems of this type and in research on economics in general. It is convenient to work with exponential functions as each of them is described by two parameters only. The algorithms in Brykalov et al., 2005 require finding the intersection points of the corresponding graphs. In the case of exponential functions, this is reduced to an elementary equation. There is no need to employ numerical methods. Because of that, algorithms from Brykalov et al., 2004 in the case of exponential functions can be effectively realised in the form of computer codes; see Nikonov, 2004. Below we analyse in detail the mathematical requirements that arise in these problems in the case of exponential functions, present algorithms for finding best responses of participants and points of Nash equilibrium, and describe the corresponding software. This paper is a contribution to the Fragility of Critical Infrastructures Project (FCI) which is currently on-going.

2 Basic assumptions and problem statement

We study a mathematical model of the investment process in the form of a game of several players. There are n players, where $n \geq 2$. The players can be treated as investors or managers supervising the construction of several gas pipelines. These projects compete with each other as they are aimed at the same regional gas market. Assume that the construction of the gas pipelines starts at one and the same moment in time $t = 0$. The player number i chooses the commercialization moment t_i of the corresponding project. At this moment, the construction of the gas pipeline number i is finished and its commercial exploitation starts. So the gas supplied by this pipeline is available at any moment in time $t \geq t_i$. Thus the actions of a player are treated as the choice of the commercialization moment (Klaassen et al., 2004). This is a laconic and convenient description, which is informative enough with respect to the stages of both construction and exploitation of the pipeline.

Let $C_i(t_i) = \gamma_i e^{-q_i t_i}$ be the total investment needed for finishing the construction of the pipeline i at the time moment t_i. Here γ_i, q_i are positive parameters. Let us also consider the cost reduction rates

$$a_i(t_i) = -C_i'(t_i) = \alpha_i e^{-q_i t_i},$$

where $\alpha_i = q_i \gamma_i$.

We understand the expression $\{1, ..., i-1, i+1, ..., n\}$ for $i = 1$ as the set $\{2, ..., n\}$, and for $i = n$ as the set $\{1, ..., n-1\}$.

Let for any number $i = 1, ..., n$ and set $H \subset \{1, ..., i-1, i+1, ..., n\}$, positive numbers β_{iH}, p_{iH} be given. For any time moment $t > 0$, the value $b_{iH}(t) = \beta_{iH} e^{-p_{iH} t}$ is assumed to be the benefit rate player i receives by means of sales of gas at the time moment t under the condition that at this time all pipelines $j \in H$ and only they supply gas to the market together with pipeline i. Until player i has made the choice of the commercialization moment t_i, the benefit rate $\beta_{iH}e^{-p_{iH}t}$ can be considered to be 'virtual' because it is not known yet whether t will be larger than t_i, that is, whether the corresponding pipeline will be in operation at the time moment t. The parameters β_{iH}, p_{iH} depend on the price in the regional market and also the cost of extraction of gas and its transportation along a pipeline. In its turn, the price depends on the amount of gas available on the market, and so, on what pipelines are already in operation. This explains the dependence of parameters β_{iH}, p_{iH} on the set H. When construction of new pipelines is completed, the amount of available gas increases on the market. Increase of supply results in decrease of the price. Thus, let us assume the following: If $G \subset H \subset \{1, ..., i-1, i+1, ..., n\}$ and $G \neq H$ then the following inequalities hold

$$\beta_{iG} > \beta_{iH}, \qquad p_{iG} \leq p_{iH}. \tag{1}$$

We also assume that for any number $i = 1, ..., n$ one has

$$p_{i\{1,...,i-1,i+1,...,n\}} < q_i. \tag{2}$$

Note that inequalities (Klaassen et al., 2004) imply: $p_{iG} < q_i$ for any set $G \subset \{1, ..., i-1, i+1, ..., n\}$. In this case, only player i can be present in the market, which corresponds to monopoly. We assume that the parameter α_i in the expression for the cost reduction rate and the value $\beta_{i\emptyset}$, which corresponds to the monopoly case, satisfy the inequality

$$\alpha_i > \beta_{i\emptyset} \tag{3}$$

that should hold for all numbers $i = 1, ..., n$. In connection with inequality see Assumption 4 in Brykalov et al., 2004 and also Assumption 2.2 and Remark 2.1 in Klaassen et al., 2004.

For any $i = 1, ..., n$, denote by A_i the set of all numbers of the form

$$t = \frac{\ln\alpha_i - \ln\beta_{iH}}{q_i - p_{iH}}, \tag{4}$$

$H \subset \{1, ..., i-1, i+1, ..., n\}$. The number of these subsets is 2^{n-1}. So, A_i is a finite set with no more than 2^{n-1} elements. (Values of some of the expressions (Nikonov, 2004) can coincide, which decreases the number of elements.) Note that inequalities (Klaassen et al., 2002 and Brykalov et al., 2004) imply that the numerator of fraction (Nikonov, 2004) is positive, and inequalities (Klaassen et al., 2002, and Klaassen et al., 2004) imply that the denominator is positive as well. Thus, the elements of the set A_i are positive. Note also that equality (Nikonov, 2004) is obtained when the variable t is found from the equation

$$\alpha_i e^{-q_i t} = \beta_{iH} e^{-p_{iH} t}. \tag{5}$$

From a geometrical point of view, A_i is the set of abscissas of the points of intersection of the graphs of functions (Brykalov et al., 2005) for all H.

As $\alpha_i = q_i \gamma_i$, the expression (Nikonov, 2004) can be written in the form

$$\frac{\ln q_i + \ln\gamma_i - \ln\beta_{iH}}{q_i - p_{iH}}.$$

For any given numbers $t_1, ..., t_{i-1}, t_{i+1}, ..., t_n$, denote by

$$G_i(t) = G_i(t|t_1, ..., t_{i-1}, t_{i+1}, ..., t_n) = \{j \neq i \,:\, t_j \leq t\}$$

the set of all rivals of player i who are present in the market at the time moment t. For any $t \geq t_i$, the actual benefit rate $b_i(t)$ of player i at time moment t is defined by

$$b_i(t) = b_i(t|t_1, ..., t_{i-1}, t_{i+1}, ..., t_n) = b_{iG_i(t)}(t) = \beta_{iG_i(t)} e^{-p_{iG_i(t)} t}.$$

The total benefit for player i is

$$B_i(t_1, ..., t_n) = \int\limits_{t_i}^{\infty} b_i(t|t_1, ..., t_{i-1}, t_{i+1}, ..., t_n) dt = \int\limits_{t_i}^{\infty} \beta_{iG_i(t)} e^{-p_{iG_i(t)} t} dt.$$

It should be noted that here the numbers $\beta_{iG_i(t)}$, $p_{iG_i(t)}$ can change with the growth of t. For values of t that exceed all the numbers $t_1, ..., t_{i-1}, t_{i+1}, ..., t_n$, the function to be integrated is an exponent of the form

$$\beta_{i\{1,...,i-1,i+1,...,n\}} e^{-p_{i\{1,...,i-1,i+1,...,n\}} t},$$

where the parameters $\beta_{i\{1,...,i-1,i+1,...,n\}}$, $p_{i\{1,...,i-1,i+1,...,n\}}$ no longer depend on t. As the number $p_{i\{1,...,i-1,i+1,...,n\}}$ is positive, we see that the improper integral is finite. The total profit $P_i(t_1, ..., t_n)$ of player i is the total benefit of this player minus the total investment in the construction of the corresponding pipeline:

$$P_i(t_1, ..., t_n) = -C_i(t_i) + B_i(t_1, ..., t_n) = -\gamma_i e^{-q_i t_i} + \int\limits_{t_i}^{\infty} \beta_{iG_i(t)} e^{-p_{iG_i(t)} t} dt.$$

Thus, we have an n-person game of timing. Strategies t_i of players i in this game are positive numbers. Any collection of strategies $(t_1, ..., t_n)$ of all players determines the payoff $P_i(t_1, ..., t_n)$ to each player. Here the strategy t_i is the commercialization moment, and the payoff $P_i(t_1, ..., t_n)$ is the profit of investor i.

3 Finding best responses and points of Nash equilibrium

Let us recall two widely used definitions of game theory and apply them to the considered case. A strategy t_i of player i is called a best response of this player to strategies $t_1, ..., t_{i-1}, t_{i+1}, ..., t_n$ of other players $1, ..., i-1, i+1, ..., n$ if

$$P_i(t_1, ..., t_{i-1}, t_i, t_{i+1}, ..., t_n) = \max_{s>0} P_i(t_1, ..., t_{i-1}, s, t_{i+1}, ..., t_n).$$

The best response exists if the maximum in the right-hand side is attained at some point. This point might happen to be not unique. So, there might exist several best responses of a player to a fixed collection of strategies of other players. A collection of strategies $t_1, ..., t_n$ of players $1, ..., n$ is called a Nash equilibrium if for every $i = 1, ..., n$, the strategy t_i is a best response of player i to the strategies $t_1, ..., t_{i-1}, t_{i+1}, ..., t_n$ of other players $1, ..., i-1, i+1, ..., n$. A Nash equilibrium corresponds to the case when neither of the players is interested in changing the strategy provided all the other players are not changing their strategies.

Theorem. *For an arbitrary collection of strategies $t_1, ..., t_{i-1}, t_{i+1}, ..., t_n$ of players $1, ..., i-1, i+1, ..., n$, there exists at least one best response t_i of player i to these strategies, and each best response t_i belongs to the set A_i.*

Proof of the existence of the best response is reduced to a direct application of Proposition 2 (Nikonov, 2004), taking into account properties of exponential functions. From Proposition 1 in Nikonov 2004, it directly follows that the best response belongs to the set A_i. In the considered case, the set D_i introduced (Nikonov, 2004) happens to be empty due to the continuity of the functions employed.
Corollary. *If a collection of strategies $t_1, ..., t_n$ is a Nash equilibrium, then $t_i \in A_i$ for every $i = 1, ..., n$.*
Proof of the Corollary consists in application of the second part of Theorem together with the definition of Nash equilibrium.

It was mentioned above that in the considered case the sets A_i are finite. Because of that, the above statements provide the basis for algorithms for direct finding of best responses of a player to strategies of other players and for checking if a given collection of strategies of all players forms a Nash equilibrium. Let us describe these algorithms. We assume that all the above imposed conditions are satisfied, the natural number $n \geq 2$ is fixed.

Best Response Algorithm.

The input data of the algorithm:

(i) an integer i, such that $1 \leq i \leq n$;

(ii) positive numbers γ_i, q_i;

(iii) positive numbers β_{iH}, p_{iH} for all subsets $H \subset \{1, ..., i-1, i+1, ..., n\}$;

(iv) strategies $t_1, ..., t_{i-1}, t_{i+1}, ..., t_n$ of players $1, ..., i-1, i+1, ..., n$. The output of the algorithm is a nonempty finite set S, which consists of positive numbers and contains no more than 2^n elements. Here S is the set of all best responses of player i to the strategies $t_1, ..., t_{i-1}, t_{i+1}, ..., t_n$ of the rest players $1, ..., i-1, i+1, ..., n$.

Sequence of actions of the algorithm:

Step 1 For each subset $H \subset \{1, ..., i-1, i+1, ..., n\}$ find the number $\frac{\ln q_i + \ln \gamma_i - \ln \beta_{iH}}{q_i - p_{iH}}$ and form the set A_i of all these numbers.

Step 2 For all $s \in A_i$ calculate the values

$$v(s) = -\gamma_i e^{-q_i s} + \int_s^\infty \beta_{iG_i(t)} e^{-p_{iG_i(t)} t} dt,$$

where $G_i(t) = \{j \neq i \, : \, t_j \leq t\}$.

Step 3 Find the set S of all points $s \in A_i$ at which the maximum of function $v(s)$ on the finite set A_i is attained.

Indeed, we see from Theorem that the output of this algorithm is the set of all best responses, and that this set is nonempty.

Now we can use Corollary and the definition of Nash equilibrium as the basis for constructing an algorithm for checking this property.

Nash Equilibrium Verification Algorithm.

The input data of the algorithm:

i positive numbers γ_i, q_i, β_{iH}, p_{iH} for all integers $i = 1, ..., n$ and all subsets $H \subset 1, ..., i-1, i+1, ..., n$;

ii a collection of strategies $t_1, ..., t_n$ of all players.

otherwise.

Sequence of actions of the algorithm:

Step 1 Put $i := 1$.

Step 2 For player i and strategies $t_1, ..., t_{i-1}, t_{i+1}, ..., t_n$ of other players $1, ..., i-1, i+1, ..., n$, with the help of the Best Response Algorithm find a nonempty finite set S of best responses of player i to these strategies.

Step 3 If $t_i \notin S$, finish the work of algorithm with the output NO.

Step 4 If $t_i \in S$ and $i < n$, put $i := i+1$ and go to Step 2.

Step 5 If $t_i \in S$ and $i = n$, finish the work of algorithm with the output YES.

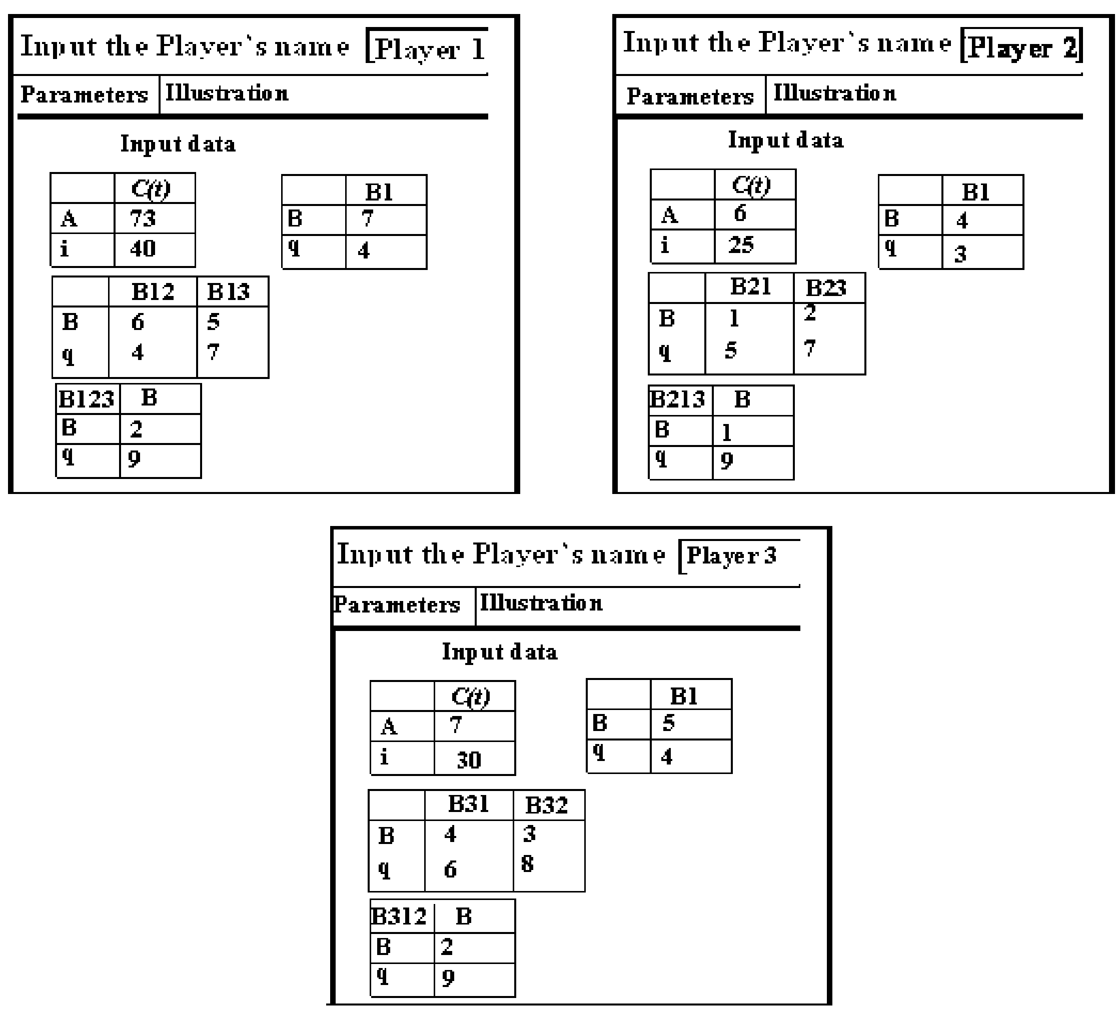
Remark 1. It can be seen from Corollary that all the Nash equilibrium points belong to the set $N = A_1 \times ... \times A_n$. As the set A_i for every number i contains no more than 2^{n-1} elements, we have that the set N is finite and contains no more than $2^{((n-1)^2)}$ elements. Application of Nash Equilibrium Verification Algorithm to all collections of strategies $(t_1, ..., t_n) \in N$ allows one to find all the points of Nash equilibrium in the considered game.
Remark 2. From the point of view of mathematics, the imposed conditions can be somewhat relaxed by discarding inequality (Klaassen et al., 2004) and allowing the parameters γ_i, q_i to equal zero. In this case, a fraction of the form (Nikonov, 2004) can happen to be undefined (can contain zero in denominator or under the sign of logarithm) or can happen to be negative. Only positive numbers should be included into the set A_i, while indefinite and negative fractions should be ignored. Here the set A_i can be empty. Here the assertion of the Theorem about the existence of the best response becomes invalid, however; if the best response exists, then it still belongs to the set A_i. Note that the Corollary also remains valid. Both algorithms need only insignificant changes.
Remark 3. The advantages of the considered model are its simplicity, small number of parameters, and the possibility to work with this model in explicit form without employing any difficult numerical methods. Some drawback can be seen in the finiteness of values $C_i(0) = \gamma_i$. According to the economical sense of the problem, these values should be infinite, because no amount of investment, however large, can permit completiton of construction in a very short time. In order to avoid this drawback, one could take the functions $C_i(t_i)$ in the form employed in Section 5 of Klaassen et al., 2004; however, that would complicate the model and one would have to numerically find the corresponding points of intersection of the graphs. The above-mentioned shows that the values γ_i ideally should be chosen large enough, so that these values $C_i(0) = \gamma_i$ do not interfere in the process of finding the points of Nash equilibrium. In the case when the algorithm gives collections of strategies with the presence of zero components, one should change the parameters of the model (or even use a different model).

4 Computer realization of the algorithm and a numerical example

The above described algorithm for finding points of Nash equilibrium was realized in the form of a Delphi 7 computer code. The code provides a convenient interface including graphical illustrations. We give some results produced by the code for the case of three players ($n = 3$) for illustrative purposes.

The code allows to choose interactively the number of players (investors). After this choice is made, the code asks to fill in tables with characteristics of the participants. An example of the tables with parameters of players in the case $n = 3$ is shown in Fig. 1.

Figure 1: Input parameters of the players.



The first box of each table contains parameters of the function that characterizes the cost of construction ($A = \alpha_i, I = q_i$). The other boxes describe the functions $b_i(t)$ in cases when only one player number i acts on the market, two players in the corresponding combinations are present, and at last, all the three players take part. Here B with indices equals β_{iH} for the corresponding H, and similarly $q = p_{iH}$.

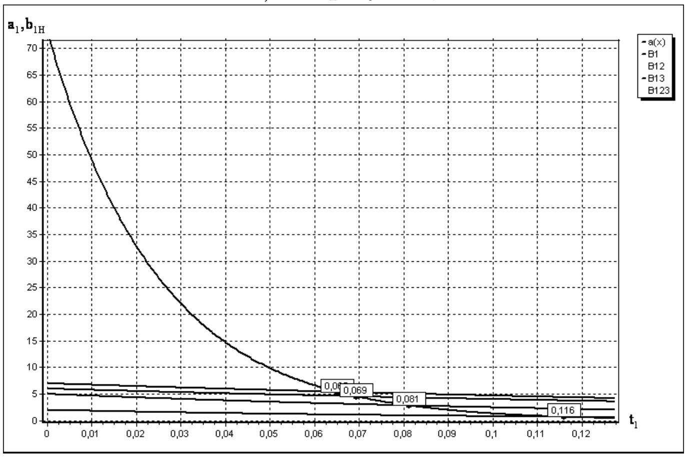
Note that in the case of three players, for each $i = 1, 2, 3$ one has $2^{(3-1)} = 4$ variants of the set H. For each of these variants, the parameters are fixed that describe the benefit function of the player. As an illustration, the code shows the graphs of all the employed functions $a_i(t), b_i(t) = \beta_{iH}e^{-p_{iH}t}$. On these graphs, the intersection points are marked, which are needed to construct the sets A_i, and the corresponding values of abscissa are given. In Figure 2, the graphs are shown of the functions that characterize players with the values of parameters given in Fig. 1.

Here, according to the algorithm, for each player we find the set A_i whose elements are the points that can be expected to give the maximal values of the benefit function of the player. These are the abscissas of the intersection points of the corresponding graphs (Figure 3). Their values can be found as the solutions of equation 5.

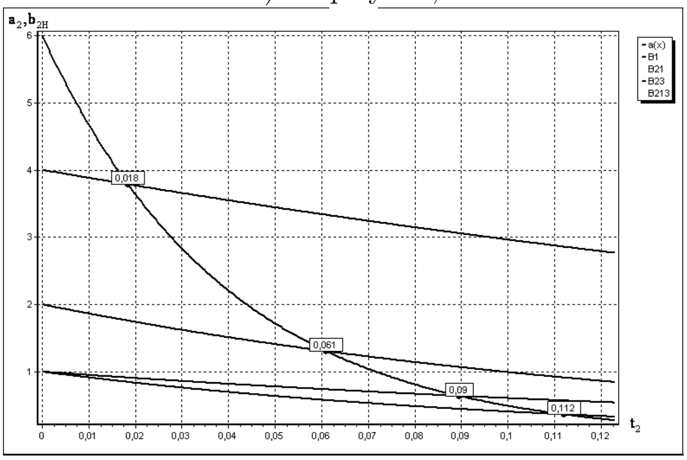
The code forms for the players the sets A_1, A_2, A_3 and the points of Nash equilibrium:

Figure 2: Graphs of functions that characterize the players:

a) for player 1,



b) for player 2,



c) for player 3.

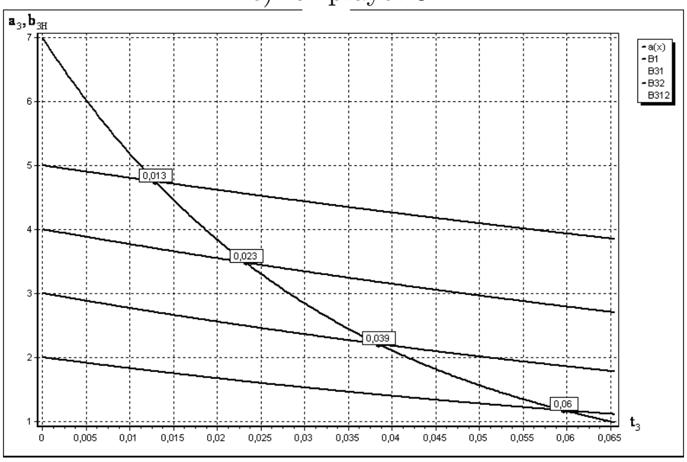


Figure 3: Sets $A_i,\ i = 1, 2, 3$

№1	0,069	0,081	0,116	0,065
№2	0,089	0,061	0,111	0,018
№3	0,023	0,038	0,059	0,012

(0.081, 0.111, 0.012);
(0.116, 0.061, 0.012);
(0.116, 0.018, 0.038).

Let us demonstrate the work of the Nash equilibrium verification algorithm. Consider, e. g., the point (0.116, 0.061, 0.012). Taking into account the inequality $t_2 > t_3$, we see that the benefit function of the first player has the form

$$P_1(t_1|t_2, t_3) =$$

$$= \begin{cases} \int\limits_{t_1}^{\infty} b_{1H_{123}}(t)dt - C_1(t_1), \text{for } t_1 > t_2 > t_3 \\ \int\limits_{t_1}^{t_2} b_{1H_{13}}(t)dt + \int\limits_{t_2}^{\infty} b_{1H_{123}}(t)dt - C_1(t_1), \text{for } t_2 > t_1 > t_3 \\ \int\limits_{t_1}^{t_3} b_{1H_1}(t)dt + \int\limits_{t_3}^{t_2} b_{1H_{13}}(t)dt + \int\limits_{t_2}^{\infty} b_{1H_{123}}(t)dt, \text{for } t_2 > t_3 > t_1. \end{cases}$$

As $t_1 > t_3$, we can construct the benefit function of the second player in the form:

$$P_2(t_1|t_1, t_3) =$$

$$= \begin{cases} \int\limits_{t_2}^{\infty} b_{2H_{213}}(t)dt - C_2(t_2), \text{for } t_2 > t_1 > t_3 \\ \int\limits_{t_2}^{t_1} b_{2H_{23}}(t)dt + \int\limits_{t_1}^{\infty} b_{2H_{213}}(t)dt - C_2(t_2), \text{for } t_1 > t_2 > t_3 \\ \int\limits_{t_2}^{t_3} b_{2H_2}(t)dt + \int\limits_{t_3}^{t_1} b_{2H_{23}}(t)dt + \int\limits_{t_1}^{\infty} b_{2H_{213}}(t) - C_2(t_2)dt, \text{for } t_1 > t_3 > t_2. \end{cases}$$

Similarly, we construct the benefit function of the third player, taking into account that $t_1 > t_2$.

Substituting the considered values of the coefficients, we obtain the functions $a_1(t_1) = 73e^{-40t_1}$, $b_{1H_1}(t_1) = 7e^{-4t_1}$, $b_{1H_{13}}(t_1) = 5e^{-7t_1}$, $b_{1H_{123}}(t_1) = 2e^{-9t_1}$, $a_2(t_2) = 6e^{-25t_2}$, $b_{2H_2}(t_2) = 4e^{-3t_2}$, $b_{2H_{23}}(t_2) = 2e^{-7t_2}$, $b_{2H_{213}}(t_2) = e^{-9t_2}$. Since $C_1(t_1) = \frac{73}{40}e^{-40t_1} + \frac{73}{40}$, one has that

$$P_1(t_1|t_2,t_3) =$$

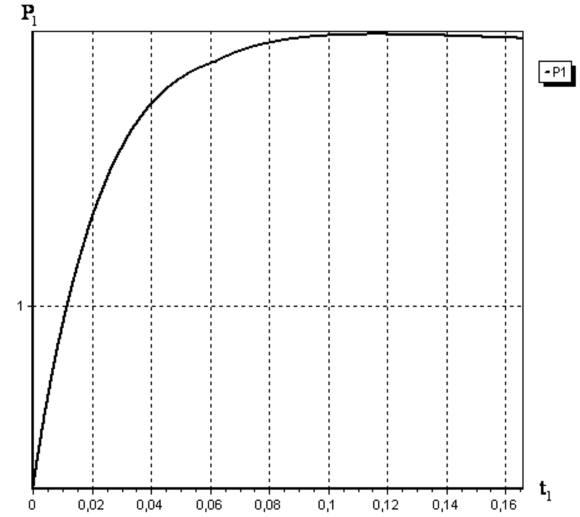
$$= \begin{cases} \frac{2}{9}e^{-9t_1} + \frac{73}{40}e^{-40t_1} - \frac{73}{40}, \text{for } t_1 > t_2 > t_3 \\ -\frac{5}{7}(e^{-7t_2} - e^{-7t_1}) + \frac{2}{9}e^{-9t_2} + \frac{73}{40}e^{-40t_1} - \frac{73}{40}, \text{for } t_2 > t_1 > t_3 \\ -\frac{7}{4}(e^{-4t_3} - e^{-4t_1}) - \frac{5}{7}(e^{-7t_2} - e^{-7t_1}) + \frac{2}{9}e^{-9t_2} + \frac{73}{40}e^{-40t_1} - \frac{73}{40}, \text{for } t_2 > t_3 > t_1. \end{cases}$$

It is easy to see that the number t_1 =0.116 is the best response of the first player to the strategies t_2 =0.061, t_3 =0.012 of the other players, i. e., the maximum point of the function $P_1(t_1|0.061; 0, 012)$ for $t_2 > t_3$. As the considered number is obtained as a solution of the equation $a_1(t) = b_{1H_{123}}(t)$, this number is a stationary point of the function and belongs to the set A_1. It can be checked directly that this number is indeed the point of maximum. Similarly, the number t_2 =0.061 is the best response of the second player to the strategies $t_3 = 0.012$, $t_1 = 0.116$, and the number $t_3 = 0.012$ is the best response of the third player to the strategies of the other players.

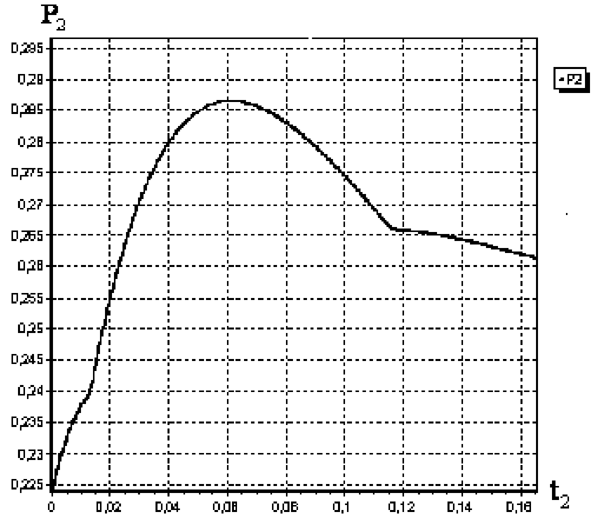
For illustrative purposes, the code gives the graphs of benefit functions of the players and the values of the benefit functions at the points of Nash equilibrium. The graphs for the considered example are presented in Fig. 4. The break points of the curves correspond to the discontinuity points of the benefit function, i. e. the moments when new investors enter the market.

Figure 4: Benefit functions of the players at an equilibrium point:

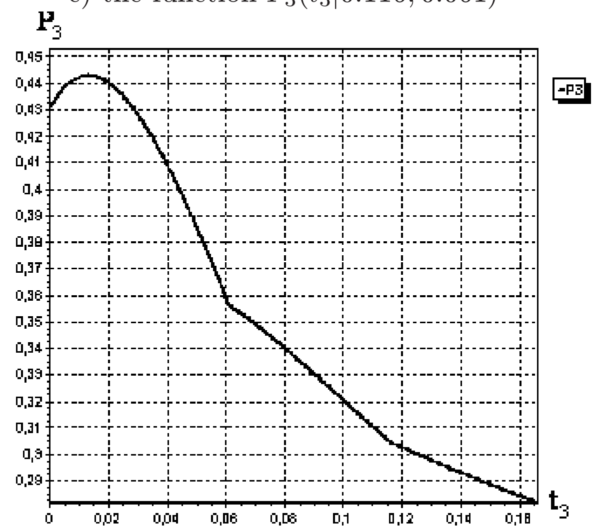
a) the function $P_1(t_1|0.061, 0.012)$,



b) the function $P_2(t_2|0.116, 0.012)$,



c) the function $P_3(t_3|0.116, 0.061)$



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