Spatial Zero Population Growth

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SPATIAL ZERO POPULATION GROWTH[*]

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April 1976

Preface

Interest in human settlement systems and policies has been a critical part of urban-related work at IIASA since its inception. Recently this interest has given rise to a concentrated research effort focusing on migration dynamics and settlement patterns. Four sub-tasks form the core of this research effort:

    I.   the study of spatial population <u>dynamics</u>;

   II.   the definition and elaboration of a new research area called <u>demometrics</u> and its application to migration analysis and spatial population forecasting;

  III.   the analysis and design of migration and settlement <u>policy</u>;

  IV.   a <u>comparative study</u> of national migration and <u>settlement patterns</u> and policies.

This paper, the seventh in the spatial population dynamics series, examines the spatial evolution of multiregional population systems that converge to a zero rate of growth. It considers how stabilization of a national population might affect migration and local growth, demonstrating that where people choose to live in the future presents issues and problems that are potentially as serious as those posed by the number of children they choose to have. Related papers in the spatial population dynamics series, and other publications of the migration and settlement study, are listed on the back page of this report.

April 1976

Summary

Increasing concern about the sizes and growth rates of
national populations has generated a vast literature dealing
with the socioeconomic and environmental consequences of a
reduction of fertility to replacement levels and the consequent
evolution of national populations to a zero growth condition
called stationarity. But where people choose to live in the
future presents issues and problems that are potentially as
serious as those posed by the number of children they choose
to have. Yet the spatial implications of reduced fertility
have received very little attention and we are, in consequence,
ill-equipped to develop adequate responses to questions about
the ways in which stabilization of a national population is
likely to affect migration and local growth.

This paper considers some of the redistributional conse-
quences of an immediate reduction of fertility levels to bare
replacement levels. It adopts the mathematical apparatus
that has been used by demographers to analyze the evolution of
national populations to zero growth and extends it by intro-
ducing the spatial impact of internal migration. Such an
extension shows that stabilization of the regional populations
in a multiregional system will alter the relative contributions
of natural increase and migration to regional growth. Regional
age compositions will also be affected, and in ways that are
strongly influenced by the age patterns of migration. Retire-
ment havens, for example, will receive proportionately higher
flows of inmigrants as a national population increases in aver-
age age, whereas destinations that previously attracted mostly
younger migrants will receive proportionately fewer inmigrants.
Finally, the redistributional effects of stabilization will
depend in a very direct way on the redistribution of
total births that is occasioned by fertility reduction. A
numerical illustration using data for India vividly illustrates
the staggering population congestion that lies ahead for that
country's urban areas, even if fertility there were to decline
to replacement level immediately.

"Sense of Helplessness"

by

Jonathan Power

International Herald Tribune

April 6, 1976, p. 4

Bombay — The doors of the jumbo jet swing open and the night
air rushes in. The warm stench of industrial waste and human
excrement overpowers the smell of jet exhaust. This is Bom-
bay -- city of six million, industrial giant, metropolis of
the western seaboard, ringed by forests of chemical plants,
textile mills and engineering factories. Inside are the
people, crushed together, man on man, woman on woman, child
on child. Squeezed between humanity run the open sewers, full
of the putrid outpourings of an overwrought civilization.

Bombay at the end of the 17th century had only 10,000 inhabi-
tants. By 1872 it was 644,000. Today, the density of popula-
tion is higher than Manhattan's and growing steadily. Only
economic recession keeps the numbers down. For once the wheels
of industrial society move, the people come in their hordes,
leaving the economic insecurity of the villages for this city
where they think there must be hope.

SPATIAL ZERO POPULATION GROWTH

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and

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Abstract

If age-specific death rates and replacement level birth rates both remain fixed, a population that is undisturbed by migration will ultimately evolve into a stationary "zero-growth" population. This paper explores the spatial consequences of such zero growth by examining how a sudden reduction of fertility to replacement levels affects the spatial evolution of a multiregional population whose constituent regional populations experience the redistributional effects of internal migration.

1. Introduction

The growing public concern about the sizes and growth rates of national populations has generated a vast literature on the social, economic, and environmental impacts of a reduction of fertility to replacement levels and the consequent evolution of national populations to a zero growth condition. But where people choose to live in the future presents issues and problems that are potentially as serious as those posed by the number of children they choose to have. Yet the spatial implications of reduced fertility have received relatively little attention and we are, in consequence, ill-equipped to develop adequate responses to questions such as the one recently posed by the Commission on Population Growth and the American Future[1]:

> "How would stabilization of the national population affect migration and local growth"? (C.P.G.A.F., 1972, p. 13).

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[1] A notable exception is the work of Peter Morrison, who concludes: "...demographic processes interact in subtle and often complex ways, and the mechanisms by which declining fertility would influence population redistribution are only partially understood. Their elucidation can furnish a clearer picture of how and under what circumstances population redistribution can be influenced by public policy," (Morrison, 1972, p. 547).

The Commission observes that zero growth for the nation
will mean an <u>average</u> of zero growth for local areas. This, of
course, still allows for the possibility of nonzero growth in
particular localities. Thus <u>spatial</u> zero growth, like <u>temporal</u>
zero growth, may be viewed either as a condition that ultimately
prevails uniformly or one that exists only because of a fortui-
tous balancing of regional rates of positive growth, of zero
growth, and of decline. Since no obvious advantages arise from
the latter case, demographers quite naturally have viewed the
attainment of temporal zero growth in the long-run in terms of
an indefinite continuation of temporal zero growth in the short-
run. We follow this tradition in this paper and view the attain-
ment of spatial zero growth in the long-run in terms of temporal
zero growth within each region of a closed multiregional popula-
tion system whose long-run behavior is defined by the multiregional
Lotka equation (Rogers, 1975, Ch. 4):

$$\{ \underset{\sim}{B}(t) \} = \int_{\alpha}^{\beta} \underset{\sim}{m}(a) \, \underset{\sim}{\ell}(a) \, \{ \underset{\sim}{B}(t-a) \} da \quad .$$

In consequence, we confine our attention here to the evolution
of a particular subset of stationary populations, called
<u>spatial zero growth populations</u>, i.e., stable multiregional
populations that have a zero growth rate. Thus we augment
the usual twin assumptions of a fixed mortality schedule and
a fixed fertility schedule, set at replacement level, with
the assumption of a fixed migration schedule. Multiregional
populations subjected to such regional growth regimes ulti-
mately assume a persisting zero rate of growth in every region and
exhibit zero growth both over time and over space.

2. Characteristics of Spatial Zero Growth Populations

If age-specific death rates are fixed and replacement
level birth rates remain unchanged, a population that is closed
to migration will ultimately evolve into a stationary popula-
tion. The characteristics of such a population are well known.
The number of individuals at any age would remain fixed, and
the total number of deaths would exactly equal the total number
of births. Because mortality risks would be relatively low from
just after birth through middle age, the age composition of such
a population would be nearly rectangular until ages 50 or 60,
tapering much more rapidly thereafter with the increase in death
rates among the older population.

The maintenance of a stationary population requires that
parents have only as many children as are needed to maintain a
fixed number of births every year. This means, for example,
that a 1000 women must on the average produce a 1000 baby girls
during their lifetime. Moreover, since some girls will not
survive to become mothers, those who do must have slightly more
than 1000 daughters in order to compensate for those who don't.
Hence the gross reproduction rate (GRR) must be greater than
unity by an amount just sufficient to maintain a unit level of
net reproduction. For example, about 97 to 98 percent of women
in the United States today survive to the principal ages of
childbearing. Consequently, those who do must have approximately
1.027 daughters, on the average, as they pass through the child-
bearing ages. In other words, the GRR must be 1.027 in order
for the NRR to be unity[2].

The net reproduction rate, like the total fertility rate
and the gross reproduction rate, summarizes the fertility ex-
perience of a population of all ages during a single year as if
it were the experience of a single cohort that passed through
all ages of childbearing. It is a hypothetical value that
treats cross sectional data as if it were longitudinal data, in

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[2]Because there are usually about 105 baby boys born for
every 100 baby girls, mothers in a stationary population of males
and females would need to have a total rate of reproduction about
three percent more than twice 1.027. In this way we obtain the
total fertility rate of 2.11 used, for example, in the United
States Census Bureau projections, (U.S. Bureau of the Census, 1972).

order to estimate the number of daughters that would be born
per woman subjected to specified age-specific risks of fertil-
ity and mortality. A commonly used procedure for obtaining
NRR is to multiply each female age-specific fertility rate, m(a)
say, by the corresponding probability of surviving from birth
to that age $\ell(a)$, integrating the product over all ages of child-
bearing:

$$NRR = \int_{\alpha}^{\beta} \ell(a)m(a)da \equiv R(0) \quad .$$

Since NRR may be viewed as the zeroth moment of the net matern-
ity function, it usually is denoted by R(0), a notation which
we shall now adopt.

Total births in a stationary _multiregional_ population must,
of course, equal total deaths. However, because of the redistri-
butive effects of migration, total births in any particular
region need not equal total deaths in that region. This can be
readily demonstrated by means of the accounting identity con-
necting regional stable intrinsic rates:

$$r_j = b_j - d_j - o_j + i_j = b_j - d_j + n_j \quad .$$

Setting $r_j = 0$ gives the fundamental relationship that must
hold in every region of a spatial zero growth population:

$$\hat{b}_j = \hat{d}_j - \hat{n}_j \quad ,$$

where the caret is introduced to designate a stationary popu-
lation. Thus only if the net migration rate is zero will
regional births equal regional deaths in a spatial zero growth
population.

The maintenance of a spatial zero growth population requires
that the total number of births in every region remain constant
over time. Thus we may substitute the trial solution vector
$\{B(t)\} = \{\hat{Q}\}$ into the multiregional Lotka equation to find

$$\{\hat{Q}\} = \left[\int_{\alpha}^{\beta} \hat{\underset{\sim}{m}}(a)\underset{\sim}{\ell}(a)da \right] \{\hat{\underset{\sim}{Q}}\} = \hat{\underset{\sim}{R}}(0)\{\hat{\underset{\sim}{Q}}\} \quad , \tag{1}$$

where carets are once again used to distinguish stationary population measures, and the element in the ith row and the jth column of $\hat{\underset{\sim}{R}}(0)$ is the stationary regional net reproduction rate <u>in region i</u> of women born in region j:

$$_{j}\hat{R}_{i}(0) = \int_{\alpha}^{\beta} {}_{j}\ell_{i}(a)\hat{m}_{i}(a)da \quad .$$

Equation (1) shows that for a spatial zero growth population to be maintained, the dominant characteristic root of the matrix $\hat{\underset{\sim}{R}}(0)$ must be unity. Consequently a reduction of fertility to replacement level may be interpreted as a reduction of the elements of $\underset{\sim}{m}(a)$ to a level that reduces the dominant characteristic root of a given net reproduction matrix $\underset{\sim}{R}(0)$ to unity. Such an operation transforms $\underset{\sim}{m}(a)$ to $\hat{\underset{\sim}{m}}(a)$ and $\underset{\sim}{R}(0)$ to $\hat{\underset{\sim}{R}}(0)$.

Stabilization of the regional populations in a multi-regional system will alter the relative contributions of natural increase and migration to regional growth. Regional age compositions will also be affected, and in ways that are strongly influenced by the age patterns of migration. Retirement havens such as San Diego and Miami, for example, will receive proportionately higher flows of inmigrants as the national population increases in average age; destinations that previously attracted mostly younger migrants, on the other hand, will receive proportionately fewer inmigrants. Finally, as we demonstrate in the next section, the redistributional effects of stabilization depend in a very direct way on the redistribution of total births that is occasioned by the reduction in fertility.

3. Alternative Spatial Paths to Zero Population Growth

In his paper for the Commission on Population Growth and the American Future, Ansley Coale (1972) considers three alternative paths to a stationary population: (1) maintaining births constant at the levels recorded in 1970; (2) moving to a replacement level of fertility either immediately or in the very near future; and (3) reducing childbearing levels such that total population size is held fixed beginning immediately. He finds only slight differences between the first two alternatives and rejects the third as infeasible since it would require an immediate decline in the birth rate of almost 50 percent. We shall therefore confine our attention to Coale's second alternative path and will explore a few of its spatial ramifications.

Imagine a multiregional population system growing at some positive rate of growth, i.e., exhibiting a net reproduction matrix $\underset{\sim}{R}(0)$ with a dominant characteristic root $\lambda_1[R(0)]$ that is greater than unity. If the rate of childbearing in each region of this multiregional population system were immediately reduced such that every woman born in that region would now have a net reproduction rate of unity, then

$$_i\hat{R}(0) = \sum_{j=1}^{m} {}_i\hat{R}_j(0) = 1$$

or, in matrix form,

$$\hat{\underset{\sim}{R}}(0)'\{\underset{\sim}{1}\} = \{\underset{\sim}{1}\} \quad , \tag{2}$$

where the prime denotes transposition.

Following the normal practice in single-region exercises of this kind, assume that the reduction of the fertility of each <u>regional</u> cohort of women is achieved by reducing each region's age-specific fertility rates by the same fixed proportion, γ_i say. Then

$$_i\hat{R}(0) = \sum_{j=1}^{m} {}_i\hat{R}_j(0) = \sum_{j=1}^{m} \int_{\alpha}^{\beta} {}_i\ell_j(a)\gamma_j m_j(a)da = \sum_{j=1}^{m} \gamma_j {}_iR_j(0) = 1$$

and

$$\hat{\underset{\sim}{R}}(0) = \underset{\sim}{\gamma}\underset{\sim}{R}(0) \quad , \tag{3}$$

where $\underset{\sim}{\gamma}$ is a diagonal matrix of fertility adjustment factors. Substituting (3) into (2) gives

$$\underset{\sim}{R}(0)'\underset{\sim}{\gamma}\{\underset{\sim}{1}\} = \{\underset{\sim}{1}\} \quad ,$$

whence

$$\{\underset{\sim}{\gamma}\} = [\underset{\sim}{R}(0)']^{-1}\{\underset{\sim}{1}\} \quad . \tag{4}$$

The adjustment factor γ_i may be re-expressed in a way that offers additional insights into its properties. According to (1)

$$\hat{Q}_i = \sum_{j=1}^{m} {}_j\hat{R}_i(0)\hat{Q}_j \quad .$$

Dividing both sides of the equation by $\hat{Q}_i$ gives

$$1 = \sum_{j=1}^{m} \frac{\hat{Q}_j}{\hat{Q}_i} \ {}_j\hat{R}_i(0) = \hat{R}_i(0) \quad , \text{ say,}$$

where $\hat{R}_i(0)$ may be defined to be the net reproduction rate of women _living_ in region i (as distinguished from the net reproduction rate of those _born_ in region i). But

$$\hat{R}_i(0) = \sum_{j=1}^{m} \frac{\hat{Q}_j}{\hat{Q}_i} \gamma_i \ {}_jR_i(0) = \gamma_i R_i(0) = 1 \quad ,$$

where we define $R_i(0) = \sum_{j=1}^{m} \frac{\hat{Q}_j}{\hat{Q}_i} \ {}_jR_i(0)$. Hence

$$\gamma_i = \frac{1}{R_i(0)} \quad ,$$

and

$$\{R(0)\} = \gamma^{-1} \{1\} = \hat{Q}^{-1} R(0) \{\hat{Q}\} \quad , \tag{5}$$

where $\{\hat{Q}\} = \hat{Q} \{1\}$ and $\hat{Q}$ is a diagonal matrix.

The vector $\{\hat{Q}\}$ in (5) is the characteristic vector associated with the unit dominant characteristic root of $\hat{R}(0)$ and denotes the total number of births in each region of a spatial zero growth population. The proportional allocation of total births that it defines is directly dependent on the transformation that is applied to change $R(0)$ to $\hat{R}(0)$, a particular example of which is given by (3). Since in a spatial zero growth population the regional stationary equivalent population $\hat{Y}_i$ is equal to the quotient $\hat{Q}_i/\hat{b}_i$, we see that the different ways in which $R(0)$ is transformed into $\hat{R}(0)$ become, in fact, alternative "spatial paths" to a stationary multiregional population.

A numerical illustration may be instructive at this point. The net reproduction behavior of the urban and rural female populations of the United States in 1968 is crudely approximated by the net reproduction matrix

$$R(0) = \begin{bmatrix} {}_uR_u(0) & {}_rR_u(0) \\ {}_uR_r(0) & {}_rR_r(0) \end{bmatrix} = \begin{bmatrix} 0.85 & 0.45 \\ 0.25 & 0.90 \end{bmatrix} \quad , \tag{6}$$

where, for example, ${}_rR_u(0) = 0.45$ denotes the net reproduction rate in urban areas of rural-born women. In other words, under the regime of growth observed in 1968, each woman born in rural areas will, on the average, replace herself in the succeeding generation by 1.35 daughters, one third of whom will be born in urban areas. Urban-born women, on the other hand, have a lower net reproduction rate; i.e., ${}_uR(0) = 1.10 < {}_rR(0)$, which when combined with the net reproduction rate of rural-born women gives the United States female population an overall net reproduction rate of $\lambda_1[R(0)] = 1.21$, where $\lambda_1[R(0)]$ is the dominant characteristic root of the net reproduction matrix $R(0)$.

About 73 percent of the 1968 United States female population lived in urban areas. A projection to stable growth under the 1968 growth regime reduces that allocation to approximately 2/3 of the stable population and yields an intrinsic growth rate of slightly under one percent per annum. What would be the spatial allocation under a similar projection, but one in which fertility was immediately reduced to a level of one daughter per urban- or rural-born woman? To obtain an estimate of the regional shares in the stationary population that would evolve out of such a projection we need first to derive the fertility adjustment factors γ_u and γ_r, respectively. Calculations carried out using (4) give $\gamma_u = 1$ and $\gamma_r = 3/5$, whence

$$\hat{\underset{\sim}{R}}(0) = \begin{bmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{bmatrix} \tag{7}$$

Note that both groups of women now exhibit unit rates of net reproduction, and observe that the dominant characteristic root of $\hat{\underset{\sim}{R}}(0)$ is unity.

The characteristic vector associated with the unit dominant characteristic root of $\hat{\underset{\sim}{R}}(0)$ indicates that 3/4 of the total births in the spatial zero growth population will occur in urban areas. Since $\hat{Q}_i = \hat{b}_i \hat{Y}_i$,

$$\frac{\hat{Y}_i}{\hat{Y}_j} = \frac{\hat{Q}_i}{\hat{Q}_j} \cdot \frac{\hat{b}_j}{\hat{b}_i} , \tag{8}$$

a result that equates the ratio of stationary regional shares to the corresponding stationary birth ratio times the reciprocal of the corresponding ratio of regional intrinsic birth rates. Since the stationary birth ratio of urban to rural births is given by $\hat{\underset{\sim}{R}}(0)$ to be 3 (i.e., 3/4 to 1/4) and because the ratio of rural to urban intrinsic birth rates is likely to be close to unity (it comes out to be 1.07 in the projection) we conclude

that about 3/4 of the spatial zero growth population will reside in urban areas[3].

We have observed earlier that the proportional allocation of total births in a spatial zero growth population depends directly on the transformation by which $R(0)$ is changed to $\hat{R}(0)$. Alternative transformations are in effect alternative spatial paths to such a population inasmuch as they lead to alternative spatial allocations of the total multiregional population. This can be easily illustrated by considering an alternative fertility reduction program: one which reduces the aggregate net reproduction rate to unity by reducing each regional fertility schedule by the same proportion, γ say. That is, let

$$\hat{\underset{\sim}{R}}(0) = \gamma \underset{\sim}{R}(0) \quad , \text{ where} \tag{9}$$

$$\gamma = \frac{1}{\lambda_1 [\underset{\sim}{R}(0)]} \quad .$$

In the context of our numerical illustration this means that the fertility of urban-born women would now be reduced to below replacement levels whereas that of rural-born women would be permitted to exceed replacement fertility levels. That is,

$$\hat{\underset{\sim}{R}}(0) = \frac{1}{1.211} \underset{\sim}{R}(0) = \begin{bmatrix} 0.70 & 0.37 \\ 0.21 & 0.74 \end{bmatrix} \tag{10}$$

and $_u\hat{R}(0) = 0.91$, $_r\hat{R}(0) = 1.11$.

The spatial implications of this alternative path to a spatial zero growth population are quite different, as can be seen by calculating the characteristic vector associated with the unit dominant characteristic root of $\hat{R}(0)$. The characteristic vector in this case allocates approximately 55 percent of total multiregional births to urban areas. Since the ratio

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[3]This result of course refers to regional designations that existed in 1968. In light of the continuing urbanization of rural regions it is probably a conservative estimate.

of rural to urban intrinsic birth rates would now be somewhat
higher than unity, however, we should expect a correspondingly
higher concentration in urban areas than is indicated by this
allocation of total births.

Table 1 summarizes the principal numerical results of
this section, augmenting them with a hypothetical example whose
numerical values more clearly illustrate the two alternative
spatial paths described above.

Table 1:   Examples of Two Alternative Spatial
Paths to Zero Population Growth

I.   Data: USA, 1968 (urban-rural)

$$R(0) = \begin{bmatrix} _uR_u(0) & _rR_u(0) \\ _uR_r(0) & _rR_r(0) \end{bmatrix} = \begin{bmatrix} 0.85 & 0.45 \\ 0.25 & 0.90 \end{bmatrix} \qquad \lambda_1[R(0)] = 1.211$$

Alternative A: $\{\gamma\} = [R(0)']^{-1}\{1\}$	Alternative B: $\gamma = \dfrac{1}{\lambda_1[R(0)]}$
$= [1 \quad 3/5]$	$= \dfrac{1}{1.211}$
$\hat{R}(0) = \gamma R(0) = \begin{bmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{bmatrix}$ $\lambda_1[\hat{R}(0)] = 1$ $\{\hat{Q}\} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$	$\hat{\hat{R}}(0) = \gamma R(0) = \begin{bmatrix} 0.70 & 0.37 \\ 0.21 & 0.74 \end{bmatrix}$ $\lambda_1[\hat{\hat{R}}(0)] = 1$ $\{\hat{\hat{Q}}\} = \begin{bmatrix} 1.22 \\ 1 \end{bmatrix}$

II.   Data: Hypothetical

$$R(0) = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1 \end{bmatrix} \qquad \lambda_1[R(0)] = 1\frac{1}{4} = \frac{5}{4}$$

Alternative A: $\{\gamma\}' = [6/5 \quad 2/5]$	Alternative B: $\gamma = 4/5$
$\hat{R}(0) = \gamma R(0) = \begin{bmatrix} 9/10 & 3/5 \\ 1/10 & 2/5 \end{bmatrix}$ $\lambda_1[\hat{R}(0)] = 1$ $\{\hat{Q}\} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$	$\hat{\hat{R}}(0) = \gamma R(0) = \begin{bmatrix} 3/5 & 2/5 \\ 1/5 & 4/5 \end{bmatrix}$ $\lambda_1[\hat{\hat{R}}(0)] = 1$ $\{\hat{\hat{Q}}\} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

4. The Momentum of Spatial Zero Population Growth

Differences between most observed population age composi-
tions and those of stationary populations make it virtually
impossible to attain zero growth in the near future. A closed
population's birth rate and growth rate depend on its fertility
schedule and its age composition. Consequently whether and how
long a population continues to grow after achieving a net re-
production rate of unity depends on that population's age
composition and its degree of divergence from that of a sta-
tionary population. The ratio by which the ultimate stationary
population exceeds a current population is the "momentum" of
that population, a quantity that recently has been given ana-
lytical content by Keyfitz (1971) who shows that the momentum
of a population numbering K individuals and having an age com-
position close to stable may be approximated by the expression

$$\frac{\hat{Y}}{K} = \frac{b\,e(0)}{r\,\mu} \left(\frac{R(0) - 1}{R(0)} \right) , \qquad (11)$$

where b is the birth rate, r the rate of growth, e(0) the ex-
pectation of life, and R(0) the net reproduction rate, all
measured before the drop in fertility, and μ is the mean age
of childbearing afterward[4]. The derivation assumes that the
population is approximately stable before the decline in fer-
tility so that b and r are intrinsic stable rates of the ini-
tial (nonstationary) regime of growth.

Straightforward population projection calculations may be
used to obtain the future population that evolves from any

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[4]Observe that (11) also may be expressed as

$$\hat{Y} = e(0)\hat{Q} ,$$

where

$$\hat{Q} = \frac{bK}{r\mu} \left(\frac{R(0) - 1}{R(0)} \right) . \qquad (12)$$

particular observed or hypothetical regime of growth. There-
fore (11) is not needed to obtain a numerical estimate of an
ultimate stationary population. However Keyfitz's simple
momentum formula gives us an understanding of the population
dynamics that are hidden in the arithmetical computations of
a population projection. It identifies in an unambiguous way
the five parameters of a current population that determine the
size of the ultimate stationary population.

In order to evaluate the accuracy of Keyfitz's momentum
formula we have carried out a two-region projection of the 1968
United States female population on the assumption that age-
specific fertility rates in each region drop immediately to
replacement levels. Table 2 shows that the ultimate total
stationary multiregional population exceeds its 1968 level by
about a third. Equation (11) estimates the momentum to be
about the same[5].

$$\frac{\hat{Y}}{K(1968)} = \frac{0.01878}{0.00432} \cdot \frac{74.3}{26.3} \frac{(1.12-1)}{1.12} = 1.31 \quad .$$

A multiregional generalization of Keyfitz's momentum for-
mula may be shown to be

$$\left\{\frac{\hat{Y}}{K}\right\} = \frac{1}{r} \, \underset{\sim}{e}(0) \, \underset{\sim}{R}(1)^{-1} \left[\underset{\sim}{R}(0) - \underset{\sim}{\Psi}(r)\right] \left\{\underset{\sim}{b}\right\} \quad , \tag{13}$$

where $\underset{\sim}{R}(1)$ is a matrix with elements ${}_j R_i(1) = {}_j \mu_i \cdot {}_j R_i(0)$, $\underset{\sim}{e}(0)$
is a matrix of regional expectations of life at birth, and where
total multiregional stable births Q are allocated to regions
according to the stationary proportions defined by the character-
istic vector associated with the unit characteristic root of $\underset{\sim}{\hat{R}}(0)$.
Evaluating (13) with the same two-region data that produced the

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[5]Unlike Keyfitz we do not use the observed birth rate but
divide total stable births Q by the current population, i.e.,
b = Q/K(1968) = 1,920,961/102,276,992 = 0.01878. That is why
our approximation is more accurate than similar ones reported
by Keyfitz.

Table 2:   Relations Under Stability and Spatial Zero Growth:
United States Females, 1968*

Variables	1. West	2. Rest of U.S.	Total
K(1968)	17,264,114	85,012,878	102,276,992
r		0.00432	
Y	26,989,870	94,302,612	121,292,482
Q	410,412	1,510,549	1,920,961
$\hat{r}$		0.00000	
$\hat{Y}$	31,013,997	103,625,370	134,639,366
$\hat{Q}$	406,374	1,399,361	1,805,735

*Spatial zero growth projection carried out under "Alternative A,"
i.e., one baby girl per woman.

above momentum of 1.31 gives:

$$
\begin{bmatrix} \dfrac{\hat{Y}_1}{K_1(1968)} \\[2ex] \dfrac{\hat{Y}_2}{K_2(1968)} \end{bmatrix}
= \dfrac{1}{0.00432}
\begin{bmatrix} 52.39 & 6.95 \\[2ex] 23.10 & 67.34 \end{bmatrix}
\begin{bmatrix} 0.0509 & -0.0045 \\[2ex] -0.0161 & 0.0382 \end{bmatrix}
\begin{bmatrix} 0.0833 & 0.0098 \\[2ex] 0.0350 & 0.1108 \end{bmatrix}
$$

$$
\times \begin{bmatrix} 0.02377 \\[2ex] 0.01777 \end{bmatrix}
= \begin{bmatrix} 1.74 \\[2ex] 1.23 \end{bmatrix} .
\tag{14}
$$

A comparison of these regional momenta with those found by the population projection summarized in Table 2 reveals that the quality of approximation afforded by (12) is adequate (1.74 and 1.23 as approximations of 1.80 and 1.22, respectively).

Equation (13) is not as practically useful as its single-region counterpart because it is much more difficult to come up with accurate guesses or estimates of the values taken on by the many parameters. Thus a more effective procedure may be to first estimate the ultimate size of the total stationary multiregional population, $\hat{Y}$, using Keyfitz's formula and then rely on (8) to allocate that total to the various regions of the multiregional system. But such a procedure requires estimates of the stationary birth rate ratios, and these are hard to come by. A more feasible alternative is to first estimate the ultimate size of the total stationary births, $\hat{Q}$, by means of (12); next, distribute that total among the various regions according to the allocation defined by the characteristic vector associated with the unit root of $\hat{R}(0)$; and then premultiply the resulting vector by $\underset{\sim}{e}(0)$ to find $\{\underset{\sim}{\hat{Y}}\}$, i.e.,

$$
\{\underset{\sim}{\hat{Y}}\} = \underset{\sim}{e}(0)\{\underset{\sim}{\hat{Q}}\} ,
\tag{15}
$$

where $\{\underset{\sim}{\hat{Q}}\}$ is scaled to sum to $\hat{Q}$, the latter coming from the single-region calculation in (12).

Returning to the example used in (14) we may establish
that

$$
\begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{bmatrix} = \begin{bmatrix} 52.39 & 6.95 \\ 23.10 & 67.34 \end{bmatrix} \begin{bmatrix} 406,374 \\ 1,399,361 \end{bmatrix} = \begin{bmatrix} 31,013,997 \\ 103,625,370 \end{bmatrix}
$$

and with this result obtain the same regional momenta as before.

Equation (15) shows that the geographical distribution of a
spatial zero growth population depends very fundamentally on
three matrices: $\underset{\sim}{e}(0)$, $\underset{\sim}{R}(0)$, and $\underset{\sim}{\gamma}$. The first describes the
multiregional levels of mortality and migration; the second
sets out the multiregional net reproduction patterns before the
decline in fertility; and the third defines the particular
"spatial path" by which fertility is reduced. The product $\underset{\sim}{\gamma}\underset{\sim}{R}(0)$
gives $\hat{\underset{\sim}{R}}(0)$, whose characteristic vector associated with the unit
root and scaled to sum to $\hat{Q}$ is $\{\hat{\underset{\sim}{Q}}\}$.

Equation (15) also may be used to dramatically underscore
our earlier assertion that "where people choose to live in the
future presents issues and problems that are potentially as
serious as those posed by the number of children they choose
to have." Consider, for example, the projection to zero growth
of India's population that was recently carried out by Norman
Ryder on the basis of the following assumptions:

> "To simplify the task of projecting the population
> of India, we make the following assumptions: it is a
> stable population with a growth rate r = +0.025 and
> survival functions corresponding to those labelled "West,
> level 13" (for which the female and male expectations of
> life at birth are 50 and 47.114, respectively) in the
> Coale/Demeny collection; the mean age of (gross) maternity
> m = 29; the ratio of male to female births k = 1.05; and
> the current population size is 600 million." (Ryder, 1974,
> p. 6)

From these assumptions it follows that the initial number
of female births per annum B(t) = 12.156 million, R(0) = 2.019,
and μ = 28.672. Applying (12), Ryder finds a $\hat{Q}$ of 8.558 million
and a zero growth population of 851 million. He then shows that
if India's survival level rises to e(0) = 70 years for females and
e(0) = 66.023 for males, and

"if replacement level fertility takes 40 years to achieve and the mean age of gross reproduction declines from 29 to 27, the ultimate female birth cohort size will be...15.029 million. Given that value,...the consequent ultimate population size is 2.094 billion." (Ryder, 1974, p. 7)

Ryder concludes that "the thought of a population of 2.1 billion for India is staggering" and goes on to examine in what respects the components of his projection may be modifiable.

There is no question but that a 2.1 billion population for India is staggering. What is even more mind-boggling, however, is that approximately 70 to 80 percent of this total is likely to be found in that nation's already teeming and over-congested urban areas (the current figure is 20 percent). To show this, we need only to introduce a few additional assumptions and then apply Equation (15). Specifically, assume that life expectancy is 55 years in urban areas and 45 in rural areas, with the migration pattern being such that

$$
\underset{\sim}{e}(0) = \begin{bmatrix} {}_u e_u(0) & {}_r e_u(0) \\ {}_u e_r(0) & {}_r e_r(0) \end{bmatrix} = \begin{bmatrix} 50 & 20 \\ 5 & 25 \end{bmatrix} .
$$

Assume, further, that the spatial pattern of net reproduction after the drop to replacement fertility is that of the U.S. example in Table 1. Then, under Alternative A,

$$
\{\hat{\underset{\sim}{Q}}\} = \hat{Q} \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}
$$

and

$$
\{\hat{\underset{\sim}{Y}}\} = \begin{bmatrix} \hat{Y}_u \\ \hat{Y}_r \end{bmatrix} = \hat{Q} \begin{bmatrix} 50 & 20 \\ 5 & 25 \end{bmatrix} X \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} = \hat{Y} \begin{bmatrix} 0.80 \\ 0.20 \end{bmatrix} \quad \text{million.}
$$

Thus, under our assumptions, a fertility reduction of the first kind, puts roughly 80 percent of India's ultimate zero growth population into urban areas. A similar calculation using the second scheme of fertility reduction (i.e., Alternative B) gives

about 72 percent for the same figure.

Table 3 summarizes the above numerical calculations. Note that the "momentum" for urban areas varies from 5 to 14, depending on the time that it takes for the fertility decline to occur and on the spatial pattern of that fertility reduction.

Table 3: Spatial Zero Population Growth in India

I. The Single-Region (Nonspatial) Model

Assumptions

Mortality: Females, e(0) = 50 ; Males, e(0) = 47.114

(Coale and Demeny,
1966)

Fertility: r = 0.025 ; mean age of gross maternity,
m = 29 ; k = 1.05

Current Population: K(t) = 600 million

Females

Current Births: B(t) = 12.156 million

Net Reproduction Rate: R(0) = 2.019

Mean Age of Childbearing in Stationary Population:
$\mu = R(1)R(0)^{-1} = 28.672$

Equation (12)

$$\hat{Q} = \frac{12.156}{0.025(28.672)} \left(\frac{2.019-1}{2.019} \right) = 8.558 \text{ million}$$

$\hat{Y} = [50 + 1.05(47.114)]\ 8.558 = 851$ million females <u>and</u> males

"Immediate Decline" Momentum = $\frac{851}{600} = 1.42$

"Gradual Decline" Momentum = $\frac{2,094}{600} = 3.49$

II. The Multiregional (Spatial) Model

Assumptions

Mortality and Migration: $\underset{\sim}{e}(0) = \begin{bmatrix} 50 & 20 \\ 5 & 25 \end{bmatrix}$

Net Reproduction After Drop to Replacement Level Fertility
(Table 1):

Table 3 (Continued):   Spatial Zero Population Growth in India

Alternative A

$$\hat{\underset{\sim}{R}}(0) \; = \; \begin{bmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{bmatrix}$$

Equation (15)

$$\left\{ \hat{\underset{\sim}{Y}} \right\} \; = \; \begin{bmatrix} 50 & 20 \\ 5 & 25 \end{bmatrix} \; \times \; \hat{Q} \; \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} \; = \; \hat{Y} \; \begin{bmatrix} 0.80 \\ 0.20 \end{bmatrix}$$

"Immediate Decline" Urban Momentum $= 1.42 \; \times \; \dfrac{0.80}{0.20} = 5.68$

"Gradual Decline" Urban Momentum $= 3.49 \; \times \; \dfrac{0.80}{0.20} = 13.96$

Alternative B

$$\hat{\underset{\sim}{R}}(0) \; = \; \begin{bmatrix} 0.70 & 0.37 \\ 0.21 & 0.74 \end{bmatrix}$$

$$\left\{ \hat{\underset{\sim}{Y}} \right\} \; = \; \begin{bmatrix} 50 & 20 \\ 5 & 25 \end{bmatrix} \; \times \; \hat{Q} \; \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix} \; = \; \hat{Y} \; \begin{bmatrix} 0.72 \\ 0.28 \end{bmatrix}$$

"Immediate Decline" Urban Momentum $= 1.42 \; \times \; \dfrac{0.72}{0.20} = 5.11$

"Gradual Decline" Urban Momentum $= 3.49 \; \times \; \dfrac{0.72}{0.20} = 12.56$

5. The Spatial Reproductive Value and Zero Population Growth

The concept of reproductive value, as developed by R.A.
Fisher (1929), revolves around the notion of regarding the off-
spring of a child as the repayment of a debt. Specifically, if
the birth of a baby is viewed as a loan of a life and if the
future offspring of this child are viewed as the subsequent
repayment of this loan, suitably discounted at the annual rate
r and compounded momently, then the present value of the repay-
ment may be taken to be

$$\int_{\alpha}^{\beta} e^{-ra} \ell(a) m(a) da \quad .$$

Equating the loan with the discounted repayment gives

$$1 = \int_{\alpha}^{\beta} e^{-ra} \ell(a) m(a) da \quad ,$$

which is recognizable as the characteristic equation used to
solve for r, the intrinsic rate of growth. Thus, as Keyfitz
points out:

> "the equation can now be seen in a new light:
> the equating of loan and discounted repayment is
> what determines r, r being interpretable either
> as the rate of interest of an average loan or as
> Lotka's intrinsic rate of natural increase." (Keyfitz,
> 1975, p. 588)

In the above cited paper, Keyfitz considers how much of
the debt is outstanding by the time the child has reached age
x. He defines this quantity to be v(x), the reproductive value
at age x, where

$$v(x) = \int_{x}^{\beta} e^{-r(a-x)} \frac{\ell(a)}{\ell(x)} m(a) da \quad , \tag{16}$$

and v(0) is scaled to equal unity. He then goes on to show how

the notion of reproductive value may be used to derive the momentum equation previously set out in (11).

Keyfitz begins by observing that if fertility were to drop immediately to replacement level, the ultimate stationary number of births in the resulting zero growth population would be

$$\hat{Q} = \frac{1}{\mu} \int_0^\beta K(x)\hat{V}(x)\,dx = \frac{\hat{V}}{\mu} \quad,$$

a quantity he evaluates numerically by means of the approximation

$$\hat{Q} = R(0)R(1)^{-1} \sum_{x=0}^{\beta-5} K(x)\hat{V}(x) = R(0)R(1)^{-1}\hat{V} \quad,$$

in which

$$\hat{V}(x) = \frac{5}{2}\hat{F}(x) + 5 \sum_{t=5}^{\beta-x-5} \hat{F}(x+t)L(x+t)L(x)^{-1} \quad,$$

and

$$\hat{F}(x) = \gamma F(x) \quad.$$

He then notes that the corresponding ultimate stationary population, calculated as

$$\hat{Y} = e(0)\hat{Q} = e(0)R(0)R(1)^{-1}\hat{V}, \tag{17}$$

gives the same result as a full population projection with the new $\hat{F}(x)$. Finally, he demonstrates that if $K(x)$, the initial age-disaggregated population, can be taken to be stable, then the even simpler form for $\hat{Y}$ given earlier in (12) holds true.

Keyfitz's arguments have their multiregional counterparts. To develop these it is convenient first to reexpress (16) for arbitrary values of $v(0)$, namely:

$$v(x) = v(0) \int_x^\beta e^{-r(a-x)} \ell(a) \ell(x)^{-1} m(a) da \quad ,$$

where

$$v(0) = v(0) \int_0^\beta e^{-ra} \ell(a) m(a) da = v(0) \psi(r) \quad .$$

This form of the equation immediately suggests the multiregional analog

$$\{\underset{\sim}{v}(x)\}' = \{\underset{\sim}{v}(0)\}' \int_x^\beta e^{-r(a-x)} \underset{\sim}{m}(a) \underset{\sim}{\ell}(a) \underset{\sim}{\ell}(x)^{-1} da \quad , \tag{18}$$

where

$$\{\underset{\sim}{v}(0)\}' = \{\underset{\sim}{v}(0)\}' \int_0^\beta e^{-ra} \underset{\sim}{m}(a) \underset{\sim}{\ell}(a) da = \{\underset{\sim}{v}(0)\}' \underset{\sim}{\psi}(r) \quad . \tag{19}$$

Equations (18) and (19) have the following demographic interpretation. If lives are loaned to regions according to the (row) vector $\{\underset{\sim}{v}(0)\}'$ then the amount of "debt" outstanding x years later is given by the (row) vector $\{\underset{\sim}{v}(x)\}'$, the regional expected number of subsequent offspring discounted back to age x. The elements of this vector, therefore, may be viewed as regional (or spatial) reproductive values at age x.

Observe that the vector $\{\underset{\sim}{v}(0)\}'$ is the <u>left</u> characteristic vector associated with the unit characteristic root of $\underset{\sim}{\psi}(r)$. Since the elements of this vector are determined only up to a scalar multiple, it becomes convenient to adopt a scaling that ensures consistency with the single-region reproductive value, namely that $v(0) = 1$. Hence $\{\underset{\sim}{v}(0)\}'$ will henceforth be scaled such that its elements add up to unity, and $\{\underset{\sim}{v}_1(0)\}'$ will denote the same vector with an arbitrary scaling. An analogous

distinction will be made between $\{Q\}$ and $\{Q_1\}$, with the former being scaled to sum to Q and the latter assuming an arbitrary scaling.

Using the multiregional notion of reproductive value, one can establish that the multiregional version of (17) is

$$\{\hat{Y}\} = \underset{\sim}{e}(0) \; \frac{\{\hat{\underset{\sim}{v}}_1(0)\}'\{\hat{\underset{\sim}{V}}\}}{\{\hat{\underset{\sim}{v}}_1(0)\}'\hat{\underset{\sim}{R}}(1)\hat{\underset{\sim}{R}}(0)^{-1}\{\hat{\underset{\sim}{Q}}_1\}} \; \{\hat{\underset{\sim}{Q}}_1\} \quad , \qquad (20)$$

where

$$\{\hat{\underset{\sim}{V}}\} = \sum_{x=0}^{\beta-5} \hat{\underset{\sim}{V}}(x)\{\underset{\sim}{K}(x)\} \quad ,$$

$$\hat{\underset{\sim}{V}}(x) = \frac{5}{2}\hat{\underset{\sim}{F}}(x) + 5 \sum_{t=5}^{\beta-x-5} \hat{\underset{\sim}{F}}(x+t)\underset{\sim}{L}(x+t)\underset{\sim}{L}(x)^{-1} \quad ,$$

and

$$\hat{\underset{\sim}{F}}(x) = \gamma\underset{\sim}{F}(x) \quad .$$

As in the single-region case, if the initial population can be taken to be stable then $\{\hat{Y}\}$ may be expressed in the somewhat simpler form of (13).

Table 4 illustrates the numerical evaluation of (20) with the two-region example of Table 2, in this instance assuming that fertility reduction is achieved via Alternative B. Note that under this particular "spatial path," the West receives a slightly smaller allocation of the total U.S. female population.

Table 4: The Spatial Reproductive Value and Spatial Zero
Population Growth Calculations: Alternative B
for United States Females, 1968

$$\{\hat{V}\} = \begin{bmatrix} 8,541,377 \\ 39,015,836 \end{bmatrix} \qquad \underset{\sim}{e}(0) = \begin{bmatrix} 52.39 & 6.95 \\ 23.10 & 67.34 \end{bmatrix}$$

$$\{\hat{v}_1(0)\}' = [1 \quad 1.0447] \qquad \{Q_1\} = \begin{bmatrix} 1 \\ 3.6879 \end{bmatrix}$$

$$\underset{\sim}{R}(1) = \begin{bmatrix} 20.401 & 2.419 \\ 8.613 & 27.183 \end{bmatrix} \qquad \underset{\sim}{R}(0)^{-1} = \begin{bmatrix} 1.305 & -0.112 \\ -0.395 & 1.000 \end{bmatrix}$$

$$\underset{\sim}{\gamma} = \gamma \underset{\sim}{I} \text{ where } \gamma = \frac{1}{\lambda_1[\underset{\sim}{R}(0)]} = \frac{1}{1.1201}$$

$$\{\hat{Y}\} = \begin{bmatrix} 52.39 & 6.95 \\ 23.10 & 67.34 \end{bmatrix} \frac{49,301,220}{127.717} \begin{bmatrix} 1 \\ 3.6879 \end{bmatrix} = \begin{bmatrix} 30,118,924 \\ 104,789,706 \end{bmatrix}$$

Alternative A (from Table 2):

$$\{\hat{Y}\} = \begin{bmatrix} 31,013,997 \\ 103,625,370 \end{bmatrix}$$

6.   Concluding Remarks

      In this paper we have argued that where people choose to
live may present issues and problems that are potentially as
serious as those posed by the number of children they choose to
have. This troublesome feature of spatial population dynamics
appears even in zero growth populations, where the redistribu-
tional consequences of an immediate reduction of fertility
levels can be of considerable importance.

      With respect to methodological issues, this paper has
demonstrated that the mathematical apparatus commonly used by
demographers to examine the evolution of national populations
to zero growth may be extended for application in spatial pop-
ulation analysis. The principal role in this extension is played
by the characteristic matrix $\psi(r)$ and its right and left char-
acteristic vectors, $\{Q\}$ and $\{v(0)\}'$, respectively. The former
defines the regional allocation of stable equivalent births; the
latter gives the regional distribution of the national reproduc-
tive value at birth. This distinction is hidden in the single-
region model, where stable equivalent births and the reproduc-
tive value at birth are cancelled out in each of their respective
definitional equations to give $1 = \psi(r)$.

APPENDIX

Summary of Results

Nonspatial (1-region)	Spatial (m-regions)

1. Introduction

The Lotka Model:

$$B(t) = \int_\alpha^\beta B(t-a)\,\ell(a)\,m(a)\,da$$

Stable Growth: $B(t) = Qe^{rt}$

$$Q = Q\int_\alpha^\beta e^{-ra}\,\ell(a)m(a)\,da = \Omega\,\psi(r)$$

r is such that $\psi(r) = 1$

$r = b - d$

$c(a) = be^{-ra}\ell(a)$

$$\dot{K}(t) \doteq \left(\frac{Q}{b}\right)e^{rt} = Ye^{rt}$$

where

$Q = bY$

The Lotka Model:

$$\{B(t)\} = \int_\alpha^\beta \underset{\sim}{m}(a)\,\underset{\sim}{\ell}(a)\,\{B(t-a)\}\,da$$

Stable Growth: $\{B(t)\} = \{Q\}e^{rt}$

$$\{Q\} = \left[\int_\alpha^\beta e^{-ra}\underset{\sim}{m}(a)\,\underset{\sim}{\ell}(a)\,da\right]\{Q\} = \underset{\sim}{\psi}(r)\{Q\}$$

r is such that $\lambda_1\left[\underset{\sim}{\psi}(r)\right] = 1$

$r_j = b_j - d_j - o_j + i_j = b_j - d_j + n_j$

$\underset{\sim}{c}(a) = \underset{\sim}{b}e^{-ra}\underset{\sim}{\ell}(a)$

$$\{\dot{K}(t)\} \doteq \{Y\}e^{rt}$$

where

$$\{Q\} = \underset{\sim}{b}\{Y\}$$

2. Spatial Zero Population Growth

$$B(t) = \hat{Q} \quad ; \quad \hat{r} = 0 \text{ and } \hat{b} = \hat{d}$$

$$\left\{ \underset{\sim}{B}(t) \right\} = \left\{ \underset{\sim}{\hat{Q}} \right\} \quad ; \quad \hat{r}_j = 0 \text{ and } \hat{b}_j = \hat{d}_j - \hat{n}_j$$

$$\boxed{\hat{Q} = \hat{Q} \int_{\alpha}^{\beta} \ell(a)\, \hat{m}(a)\, da = \hat{Q}\hat{R}(0)}$$

$$\boxed{\left\{ \underset{\sim}{\hat{Q}} \right\} = \left[\int_{\alpha}^{\beta} \underset{\sim}{\hat{m}}(a)\, \underset{\sim}{\ell}(a)\, da \right] \left\{ \underset{\sim}{\hat{Q}} \right\} = \underset{\sim}{\hat{R}}(0) \left\{ \underset{\sim}{\hat{Q}} \right\}}$$

where $\hat{m}(a) = \gamma m(a)$,

where $\underset{\sim}{\hat{m}}(a) = \underset{\sim}{\gamma} m(a)$,

$$\hat{R}(0) = \int_{\alpha}^{\beta} \ell(a)\gamma m(a)\, da = \gamma R(0)$$

$$\underset{\sim}{\hat{R}}(0) = \int_{\alpha}^{\beta} \underset{\sim}{\gamma} m(a)\, \underset{\sim}{\ell}(a)\, da = \underset{\sim}{\gamma} \underset{\sim}{R}(0)$$

and

and

$$R(0) = \int_{\alpha}^{\beta} \ell(a) m(a)\, da$$

$$\underset{\sim}{R}(0) = \int_{\alpha}^{\beta} \underset{\sim}{m}(a)\, \underset{\sim}{\ell}(a)\, da$$

3. Alternative Paths to SZPG

$$\hat{R}(0) = 1 \quad \text{for} \quad \boxed{\gamma = \frac{1}{R(0)}}$$

$$\lambda_1 \left[\underset{\sim}{\hat{R}}(0) \right] = 1 \quad \text{for feasible } \underset{\sim}{\gamma}$$

e.g.

$$\boxed{\begin{array}{l} \left\{ \underset{\sim}{\gamma} \right\} = \left[\underset{\sim}{R}(0)^{\cdot} \right]^{-1} \left\{ \underset{\sim}{1} \right\} \\[2mm] \text{or} \quad \gamma = \frac{1}{\lambda_1 [\underset{\sim}{R}(0)]} \end{array}}$$

(Table 1)

4. <u>Momentum of ZPG</u>

$$\left[\frac{\hat{Y}}{K}\right] = \frac{b\ e(0)}{r\mu}\ \left(\frac{R(0)-1}{R(0)}\right)$$

$$= \frac{1}{r}\ e(0)R(1)^{-1}\left[R(0)-1\right]b$$

$$\left\{\frac{\hat{Y}}{K}\right\} = \frac{1}{r}\ \underset{\sim}{e}(0)\ \underset{\sim}{R}(1)^{-1}\left[\underset{\sim}{R}(0)-\psi(r)\right]\left\{\underset{\sim}{b}\right\}$$

(Table 2)

since

$$\mu = \frac{R(1)}{R(0)}$$

$$\hat{Y} = e(0)\hat{Q}$$

where

$$\hat{Q} = \frac{bK}{r\mu}\ \left(\frac{R(0)-1}{R(0)}\right)$$

$$\left\{\hat{\underset{\sim}{Y}}\right\} = \underset{\sim}{e}(0)\ \left\{\hat{\underset{\sim}{Q}}\right\}$$

where $\left\{\hat{\underset{\sim}{Q}}\right\}$ is the <u>right</u> characteristic vector of $\underset{\sim}{R}(0)$, scaled to sum to $\hat{Q}$.

(Table 3)

5. <u>Reproductive Value and ZPG</u>

Definition:

$$v(x) = \int_{x}^{\beta} e^{-r(a-x)}\frac{\ell(a)m(a)da}{\ell(x)}$$

where
$$v(0) = 1\ .$$

Definition:

$$\left\{\underset{\sim}{v}(x)\right\}' = \left\{\underset{\sim}{v}(0)\right\}'\left[\int_{x}^{\beta} e^{-r(a-x)}\underset{\sim}{m}(a)\underset{\sim}{\ell}(a)\underset{\sim}{\ell}(x)^{-1}da\right]$$

where
$$\left\{\underset{\sim}{v}(0)\right\}'\text{ is the }\underline{\text{left}}\text{ characteristic}$$
vector of the characteristic
matrix $\underset{\sim}{\psi}(r)$, scaled to sum to unity.

ZPG: Numerical Evaluation

$$\hat{Y} = e(0)\ \frac{\hat{V}}{\mu} = e(0)R(0)\ R(1)^{-1}\hat{V}$$

where

$$\hat{V} = \sum_{x=0}^{\beta-5} K(x)\hat{V}(x)\ ,$$

$$\hat{V}(x) = \frac{5}{2}\hat{F}(x)+5\sum_{t=5}^{\beta-x-5}\hat{F}(x+t)L(x+t)L(x)^{-1}\ ,$$

and

$$\hat{F}(x) = \gamma F(x)$$

SZPG: Numerical Evaluation

$$\left\{\hat{\underset{\sim}{Y}}\right\} = \underset{\sim}{e}(0)\ \frac{\{\hat{\underset{\sim}{v}_1}(0)\}'\{\hat{\underset{\sim}{V}}\}}{\{\hat{\underset{\sim}{v}_1}(0)\}'\hat{\underset{\sim}{R}}(1)\hat{\underset{\sim}{R}}(0)^{-1}\{\hat{\underset{\sim}{Q}_1}\}}\ \left\{\hat{\underset{\sim}{Q}_1}\right\}$$

where

$$\left\{\hat{\underset{\sim}{V}}\right\} = \sum_{x=0}^{\beta-5}\hat{\underset{\sim}{V}}(x)\ \left\{\underset{\sim}{K}(x)\right\}\ ,$$

$$\hat{\underset{\sim}{V}}(x) = \frac{5}{2}\hat{\underset{\sim}{F}}(x)+5\sum_{t=5}^{\beta-x-5}\hat{\underset{\sim}{F}}(x+t)\underset{\sim}{L}(x+t)\underset{\sim}{L}(x)^{-1}\ ,$$

and

$$\hat{\underset{\sim}{F}}(x) = \underset{\sim}{\gamma}\underset{\sim}{F}(x)\ ,$$

6. Conclusion

The key roles are played by the right and left characteristic vectors associated with the unit characteristic root of $\underset{\sim}{\psi}(r)$. The right vector is $\{\underset{\sim}{Q}\}$, the vector of stable equivalent births. The left vector is $\{\underset{\sim}{v}(0)\}'$, the vector of reproductive values at birth. That is, let

$$\underset{\sim}{\psi}(r) = \int_{\alpha}^{\beta} e^{-ra} \underset{\sim}{m}(a)\underset{\sim}{\ell}(a)da \quad ,$$

then

$$\boxed{\begin{array}{l} \{\underset{\sim}{Q}\} \;\;=\;\; \underset{\sim}{\psi}(r)\{\underset{\sim}{Q}\} \\[2ex] \{\underset{\sim}{v}(0)\}' = \{\underset{\sim}{v}(0)\}'\, \underset{\sim}{\psi}(r) \end{array}} \qquad .$$

At spatial zero population growth r=0, whence

$$\underset{\sim}{\hat{\psi}}(0) = \int_{\alpha}^{\beta} \underset{\sim}{\hat{m}}(a)\underset{\sim}{\ell}(a)da = \underset{\sim}{\hat{R}}(0) \quad ,$$

and

$$\boxed{\begin{array}{l} \{\underset{\sim}{\hat{Q}}\} \;\;=\;\; \underset{\sim}{\hat{R}}(0)\{\underset{\sim}{\hat{Q}}\} \\[2ex] \{\underset{\sim}{\hat{v}}(0)\}' = \{\underset{\sim}{\hat{v}}(0)\}'\underset{\sim}{\hat{R}}(0) \end{array}} \qquad .$$

Note that in the single-region (nonspatial) model, stable equivalent births and the reproductive value at birth are cancelled out in each of the two definitional equations, thereby giving the identical result

$$1 = \quad \psi(r) \qquad .$$

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