International Institute for
Applied Systems Analysis
IIASA www.iiasa.ac.at

Generic Phase Transition and Profit Singularities in Arnold's Model

Davydov, A.A. and Mena-Matos, H.

IIASA Interim Report
December 2005

International Institute for
Applied Systems Analysis
Schlossplatz 1
A-2361 Laxenburg, Austria

Tel: +43 2236 807 342
Fax: +43 2236 71313
E-mail: publications@iiasa.ac.at
Web: www.iiasa.ac.at

Interim Report **IR-05-058**

Generic Phase Transition and Profit Singularities in Arnold's Model

Alexey Davydov (davydov@iiasa.ac.at, davydov@vpti.vladimir.ru)
Helena Mena-Matos (mmmatos@fc.up.pt)

Approved by

Arkady Kryazhimskiy (kryazhim@iiasa.ac.at, kryazhim@mi.ras.ru)
Program Leader, Dynamic Systems

December 2005

Contents

1 Introduction **1**

2 Notions and Theorems **2**

2.1 Optimal motions 2

2.2 Singularities of the profit for optimal level cycles 3

2.3 Singularities of the profit for optimal stationary strategies 5

2.4 Transition between optimal strategies 6

3 Selected strategies: proof of Theorem 2.1 **7**

4 Proof of Theorem 2.5 **8**

5 Strategy Transition: Proof of Theorem 2.6 **10**

5.1 Cyclic and transition values 10

5.2 Proof of Theorem 2.6 15

Abstract

For one parametric smooth family of pairs of control systems and profit densities on the circle, the transition between optimal strategies of cyclic and stationary types in the problem of maximization of infinite horizon averaged profit is studied. We show that only two types of such transition can be observed for a generic pair, study the corresponding singularities of the averaged profit as a function of the family parameter and prove their stability to small perturbations of such a family. We also complete the classification of generic singularities of the maximum averaged profit for such families.

About the Authors

Alexey Davydov
Dynamic Systems
International Institute for Applied Systems Analysis
Schlossplatz 1, A-2361 Laxenburg, Austria
and
Vladimir State University
Vladimir, Russia

Helena Mena-Matos
Centro de Matemática da Universidade do Porto
Porto, Portugal

Acknowledgements

This work was partially performed during the visit of the first author to Departamento de Matemática Aplicada e Centro de Matemática da Universidade do Porto. This author is very thankful to the department staff for a creative scientific atmosphere and excellent conditions for work.

Generic Phase Transition and Profit Singularities in Arnold's Model

*Alexey Davydov**

*Helena Mena-Matos***

1 Introduction

A control system on a smooth ($=C^\infty$) manifold is defined by a smooth family of vector fields on it, parameterized by a control parameter. We assume that the control parameter space U is a smooth closed manifold or a disjoint union of ones with at least two different points.

An *admissible motion* of a system is an absolutely continuous map $x : t \mapsto x(t)$ from the time interval to the system's phase space such that its velocity (at each point of differentiability) belongs to the convex hull of the admissible velocities of the system. Taking into account the autonomous character of the control system, the start time of an admissible motion can be placed to zero without loss of generality.

In the presence of a smooth (or even continuous) profit density f on the phase space of the control system, any admissible motion on the interval $[0, T]$, $T > 0$, provides the *profit*

$$P(T) = \int_0^T f(x(t))dt$$

and the *averaged profit* $A(T) = P(T)/T$. If the phase space is compact any admissible motion can be extended to the whole time axis. For the noncompact case this is not always so, like for the solutions of the equation $\dot{x} = x^2$ on the real line.

The problem of maximizing the infinite horizon averaged profit over all admissible motions is an important problem of control theory. If there is no limit of the profit $A(T)$ when $T \to \infty$ one needs to take its upper limit. In particular this problem includes optimization of the averaged profit of cyclic processes, arising in the case where the phase space is a circle. Due to that, problems of such type are well known and were treated in different ways [14], [16], [17].

A new approach in the investigation of such problems, based on the achievements of the singularity theory, was proposed by V.I. Arnold in papers [1]-[3]. V.I.Arnold showed that for control systems and profit densities on the circle, among the motions maximizing the infinite horizon averaged profit there can appear *level cycle* which uses the maximum and minimum velocities when the profit density is less or greater, respectively, then the maximum of infinite horizon averaged profit, or *a stationary strategy* (*equilibrium point*)

*Partially supported by grants RFBR 03-01-00140 and FCT XXI/BBC/22225/99

**Partially supported by Centro de Matemática da Universidade do Porto (CMUP) financed by FCT (Portugal) through the programmes POCTI and POSI with national and European Community (FEDER) structural funds

which is the staying at a equilibrium point (with the admissible zero velocity). V.I.Arnold also studied some of the corresponding generic singularities of the maximum averaged profit as a function of a parameter in the one parametric case [2]. Here the analysis of Arnold's one-parameter model is continued. We prove that a strategy providing the maximum infinite horizon averaged profit can always be found within these two classes of motions. However we use a more general definition of an equilibrium point, namely, we understand an equilibrium point as a point, at which the convex hull of the system's admissible velocities contains zero.

Additionally we analyze the transitions between the optimal strategies of these two types and prove that generically only two forms of them can appear under the change of the one-dimensional parameter. We find the corresponding singularities of the maximum infinite horizon averaged profit as a function of the parameter, and prove their stability to the small perturbation of a generic families of pairs of control systems and profit densities is proved.

In the case of one-dimensional parameter, the obtained results hold true for a generic family of control systems (or profit densities) provided a generic family of profit densities (control systems, respectively) is fixed. The formulation and the proof of a corresponding theorem are the same as in [10].

2 Notions and Theorems

Here the main results are formulated. We denote by x and p the phase variable and the family's parameter, respectively. In what follows a *generic* object (a family of control systems or profit densities, or else pairs of these, ...) will always be understood an an element of an open and dense subset in the space of objects endowed with smooth or sufficiently smooth fine topology. A property or a statement is said to hold *generically* if it holds for a generic object.

2.1 Optimal motions

An admissible motion is called *optimal* if it provides the maximum infinite horizon averaged profit, which we for simplicity will call the *maximum averaged profit* or the *best averaged profit*, provided it is defined for a selected class of strategies, for example, the equilibrium points.

For a value c of the profit density we define the *c-level motion* as the one keeping the maximum and minimum admissible velocities at points, at which the profit density is not greater and greater then c respectively. A value of the profit density is called *cyclic,* if for all nearby values, the motions of the corresponding levels rotate over the circle. For a cyclic value c its level motion is called a *c-level cycle* or just a *level cycle*. The *period* of a level cycle is defined as its smallest period.

For a control system with positive velocities only, the maximum infinite horizon averaged profit A is provided by an A-level cycle. Along this periodic motion the averaged profit gained in one cycle equals the maximum one gained on the infinite horizon. Indeed, as we mentioned above, for such a system an optimal motion keeps the maximum and minimum admissible velocities at points at which the profit density is less and greater than the maximum averaged profit A, respectively [2]. Besides the motion kept on the density level A does not have any influence on the averaged profit [10], thus it can be selected as an A-level cycle.

As said before an *equilibrium point* is a point at which the convex hull of the admissible velocities contains zero. This point is stationary in the sense that for a control system with a one dimensional phase space there exists an admissible motion circulating close to that point, which is generated by a piecewise continuous control and tends to the point as time goes to infinity. It is clear that the infinite horizon averaged profit provided by such motion equals the profit density value at this point, that is, the profit value gained through the permanent staying at the point. Our definition of an admissible motion permits such staying (=stationary strategy) directly.

Theorem 2.1 *For continuous control system and profit density on the circle, the maximum infinite horizon averaged profit can always be provided by a level cycle or by a stationary strategy.*

The theorem is proved in Section 3. Note that there can exist a number of different optimal motions. For example, the change of an optimal motion on any finite interval of time preserves its optimality. Besides if there are both an optimal stationary strategy and an optimal (cyclic) level motion, then the "combined" motion designed as the cyclic motion which stops periodically at the corresponding equilibrium point for arbitrary long duration is also optimal. Such a situation is possible as we will see later.

When the control system and profit density depend on parameters, the optimal strategy can vary depending on the parameters and the best averaged profit, as a function of the parameters, can have singularities. For example, this profit can be discontinuous, even when the families of control systems and densities are smooth [2]. Theorem 2.1 gives us the possibility to subdivide these singularities into three groups in order to analyze them, namely, the singularities for optimal stationary strategies, for optimal level cycles and for the transitions between optimal stationary strategies and level cycles. The respective results are formulated in the following subsections.

2.2 Singularities of the profit for optimal level cycles

Theorem 2.2 [10] *On a circle for a generic smooth one parameter family of pairs of profit densities and control systems with positive velocities only, the germ of the maximum averaged profit at any parameter value is the germ at the origin of one of the seven functions listed in the second column of Table 1 up to the equivalence pointed out in the third column and under the conditions given in the fourth one. Besides, for a generic pair and any one sufficiently close to it, the graphs of their maximum averaged profits can be transformed one to another by a smooth* Γ*-equivalence close to the identity.*

To make this statement more clear we must give a few explanations. Γ-*equivalence* admits diffeomorphisms of the graph space preserving the natural foliation over the function's domain. R^+-*equivalence* which permits diffeomophisms of the domain space and the adding of smooth functions is a particular case of it.

The maximum (minimum) velocity of a family of control systems on the circle is the maximum (minimum, respectively) of the respective family of vector fields with respect to the control parameter. When the family's parameter is one dimensional such a maximum is either smooth or can have generic singularities of three types, namely its germ is R^+-equivalent to the germ at the origin of one of the following three functions

$$1)\ |u|, \quad 2)\ \max\{v, |u|\}, \quad 3)\ \max\{-w^4 + uw^2 + vw \mid w \in R\}, \tag{2.1}$$

for the minimum we have to change the sign of these functions [4], [6], [7]. Here u and v are local coordinates in the space of the phase variable and the parameter. Note that these functions are not the normal forms for the germs of the vector fields themselves. A

Table 1:

$N sup0$	Singularity	Equiv	Conditions
1	0	R^+	$\#U \geq 2$
2	$(\lvert p\rvert^{3/2} + p^2)(1 + \mathrm{sign} p)$	Γ_a	$\#U \geq 2$, transition through a local minimum of the profit density
3	$(\lvert p\rvert^{3/2} - p^2)(1 + \mathrm{sign} p)$	Γ_a	$\#U \geq 2$, transition through a local maximum of the profit density
4	$\lvert p\rvert^{3/2}(1 + \mathrm{sign} p)$	Γ	$\#U \geq 2$, transition through a tangent double point of the velocity used
5	$p\lvert p\rvert$	R^+	$\#U \geq 3$, transition through a triple point
6	$\lvert p\rvert^3$	R^+	$\#U \geq 2$, switching at a double point
7	$\lvert p\rvert^{7/2}(1 + \mathrm{sign} p)$	Γ	$\dim U > 0$, transition through a swallow point

point with singularity 1), 2) or 3) we will call *double, triple* or *swallow* point, respectively. Transversality theorems imply that for a one dimensional parameter in a generic case the closure of the set where the minimum or maximum of a family of functions has such singularities (=*Maxwell set*) is either empty or is

– a smooth curve when the number $\#U$ of different control parameter values is equal to 2, or

– a smooth curve with triple points when $\#U = 3$, the triple points for minimum and maximum velocities are the same and Maxwell set near each of them is the union of three pairwise transversal smooth curve, or else

– a smooth curve with triple (and swallow) points with transversal self-intersection outside them when $\#U > 3$ ($\dim U > 0$, respectively), and triple and swallow points for minimum and maximum velocities are different.

Besides, this set is stable, namely, for a generic family of control systems and any one sufficiently close to it such sets can be reduced one to another by a smooth diffeomorphism close to the identity (see [4], [6], [7], [9]). Hence in a generic case this set has a typical replacement with respect to the natural foliation of the ambient space over the family's parameter space. Consequently the Maxwell set of a generic family of control systems can be tangent to fibers of this foliation only at points where it is smooth and exactly with first tangency order. Besides for any value of the parameter, at the phase space there is no more then one double point with such tangency (=*tangent double point*), triple or swallow point, or else a point of self-intersection of this set. A point where the Maxwell set is smooth and which is not a tangent double point, is called *regular.* At last in Table 1

– the notation Γ_a is used for a Γ-equivalence which is affine along the profit axis,

– transition through a local extremum of the profit density means the coincidence of the maximum averaged profit with a local value of the density at the corresponding parameter value,

– transition through a tangent double (triple or swallow) point means , for the respective parameter value, the using of a velocity with such a singularity on some interval of the optimal motion, and

– switching at a double point means that the optimal motion switches between maximum and minimum velocities at a regular point of the Maxwell set.

Denote by T and P the functions of the period of a c-level cycle and the profit along it, respectively, $T = T(c)$ and $P = P(c)$. The proof of Theorem 2.2 is mainly based on the following statement.

Theorem 2.3 ([10]) *For a differentiable profit density with a finite number of critical points and a control system with maximum and minimum velocities coinciding at isolated points only, the best averaged profit is the unique solution c_0 of equation*

$$c - P(c)/T(c) = 0, \tag{2.2}$$

if this profit is provided by a level cycle. Besides the left hand side of equation (2.2) *is a differentiable function of c near this solution.*

2.3 Singularities of the profit for optimal stationary strategies

The *stationary domain S* is the union of all equilibrium points. For a continuous system this domain is closed. The maximum averaged profit A_s over the stationary strategies is the solution of the following extremal problem $A_s(p) = \max\{f(x,p) \mid x \in S_p\}$, where $S_p = \{x \mid (x,p) \in S\}$.

Thus to classify the generic singularities of this profit one can firstly study the generic singularities for the stationary domain and prove their stability up to small perturbations of a generic family of systems, and then analyze the generic profit singularities themselves as solutions of this extremal problem.

Theorem 2.4 *For a generic smooth one-parameter family of control systems on the circle, the germ of the stationary domain at any of its boundary points is the germ at the origin of one of the seven sets from Table 2 in an appropriate smooth coordinate system foliated over the parameter. Besides,*

• *the number of different values of the control parameter must be no less then 2 for singularities 1 and 2, no less then 3 for singularities 3 and 4 and equal to 2 for singularity 5 and*

• *the stationary domains for a generic family of systems and any one sufficiently close to it can be carried one to another by a C^∞-diffeomorphism that is close to the identity and preserves the natural foliation over the parameter.*

Table 2:

1	$2_\pm$	3	4	$5_\pm$
$x \le 0$	$p \ge \pm x^2$	$p \le \lvert x \rvert$	$x \ge -\lvert p \rvert$	$\pm(p^2 - x^2) \le 0$

We omit the proof of this theorem. It is based on the transversality theorems and is simple enough.

Remark 1 *In a generic case the maximum and minimum admissible velocities have different signs inside the stationary domain, and on the boundary of this domain at least one of them vanishes. Besides, outside singularities of type $2_\pm$ the vanishing velocity always* *has both one sided derivatives and they are different from zero. For the case of polidynamical systems (when the number of different values of the control parameter is finite) Theorem 2.4 was proved by Célia Moreira [8].*

Theorem 2.5 *For a generic smooth one-parameter family of pairs of control systems and profit densities on the circle and any value of the parameter admitting equilibrium points, the germ of the best averaged profit over the stationary strategies at such a value is the germ at the origin of one of the five functions in the second row of Table 3 up to the equivalence from the third one. Besides, the graphs of the best averaged profits provided by stationary strategies for a generic pair and any one sufficiently close to it can be reduced one to another by a* Γ*-equivalence which is close to the identity.*

Table 3:

Type	1	2	3	4	5
Singularity	0	$\vert p\vert$	$p\vert p\vert$	$\sqrt{p}, p \geq 0$	$\max\{0, 1+\sqrt{p}\}$
Equivalence	R^+	R^+	R^+	R^+	Γ

The proof of Theorem 2.5 is done in Section 4.

Remark 2 *The singularity 1 from Table 3 appears when the profit* A_s *is attained at an unique equilibrium point, which is either a nondegenerate local maximum of the profit density inside the stationary domain, or a boundary point of this domain with a type 1 singularity from Theorem 2.4 where* $f_x \neq 0$*. The singularity 2 corresponds either to the nondegenerate competition of two singularities of type 1 from Table 3 or to the case of the profit* A_s *being attained at an unique point which is a boundary point of the stationary domain with the singularity 4 or* 5_- *from Theorem 2.4 and where* $f_x \neq 0$*. The singularity 3 arises with the outgoing of the optimal stationary strategy from the interior of this domain to its boundary. In a generic case that takes place at a point with a type 1 singularity from Theorem 2.4 and with* $f_x = 0 \neq f_{xx}$*. Finally, the singularities 4 and 5 from Table 3 occur when the (unique) optimal strategy is a boundary point of the stationary domain with singularity* 2_+ *from Theorem 2.4: when locally on one side of the respective parameter value there are no stationary strategies we get singularity 4, otherwise we have singularity 5.*

All singularities from Table 3 are well known in the parametric optimization theory [6], [9], [12], [13]. Relating to the time averaged optimization they were found by V.I. Arnold [2] as generic singularities of the profit for equilibrium points and level cycles. Here all of them appear as generic singularities of the maximum profit for stationary strategies only. All of them appear generically already in the case of bidynamical systems. In this case they were found by Célia Moreira [8] who used the same definition of equilibrium point as in this paper.

2.4 Transition between optimal strategies

A parameter value is called a *transition value* if in any neighborhood of it, the maximum averaged profit can not be provided by one and only one type of strategy, namely, either by level cycles or by equilibrium points.

Theorem 2.6 *For a generic smooth one-parameter family of pairs of control systems and profit densities on the circle, the germ of the maximum averaged profit at a transition parameter value is the germ at the origin of one of the two functions from the second column of Table 4 up to R^+-equivalence. Besides, the sets of transition parameter values for a generic pair and any one sufficiently close to it can be carried one to another by a diffeomorphism of the parameter space which is close to the identity and preserves the type of the profit singularities at those values.*

Table 4:

N sup 0	Singularity	Type
1	$\lvert p\rvert$	Stop at an interior point of the stationary domain, $f_x(.) = 0$
2	$\max\left\{0, \frac{p}{\lvert\ln p\rvert}(1+H)\right\}$	Stop at a boundary point of the stationary domain with the singularity 1 from Table 2, $f_x(.) \neq 0$

Remark 3 *In this table $H = h(p, \frac{p}{|\ln p|}, \frac{\ln|\ln p|}{|\ln p|})$ with a smooth function h, $h(p,0,0) \equiv 0$. The first singularity corresponds to the nondegenerate competition between the best averaged profit among level cycles and a local maximum of the profit density inside the stationary domain which provides the profit A_s. The second one corresponds to the forthcoming of a switching point between maximum and minimum velocities on the optimal level cycle to the boundary of the stationary domain under the change of the parameter.*

Theorem 2.6 is proved in Section 5.

3 Selected strategies: proof of Theorem 2.1

We begin to prove two useful statements.

Proposition 3.1 *If an admissible motion of a differentiable control system comes in (or outcomes from) a point in finite time, then at that point the maximum velocity in the direction of the motion is positive.*

Proof. It is clear that at such a point N_0 the maximum velocity v in the direction of the entrance or outcome of the motion has to be nonnegative, due to continuity of the maximum velocity. But the maximum (minimum) velocity V of a differentiable control system, having a closed smooth manifold or a disjoint union of closed smoothed manifolds as the control parameter's set, is a Lipschitz function. So if v vanishes at the point x_0 then near this point $|v(x) - v(x_0)| \leq C|x - x_0|$ with some positive constant C. Therefore it is not possible to come in or walk out the point in finite time, because the function $1/x$ has a nonintegrable singularity at zero. That contradicts the assumptions, and so the value $v(x_0)$ must be positive. □

Proposition 3.2 *If for a continuous control system and profit density, an admissible motion $x(t)$, $0 \leq t < \infty$, provides an profit A on the infinite horizon and some arc of its trajectory between points $x(t_1)$, $x(t_2)$, with $f(x(t))$ less (greater) then A for all $t \in [t_1, t_2]$, can be crossed faster (slower, respectively) then the modified motion with the faster or lower crossing, respectively, provides no worse averaged profit on the infinite horizon.*

Proof. We consider only the case of the faster crossing (the other case can be treated in an analogous manner). Our proof is a modification of that of [11] for an analogous statement. Without loss of generality we can assume that the profit density is positive, because the addition of any constant to it leads to the addition of the same constant to the averaged profit.

Denote $C = \max\{f(x(t)),\ t \in [t_1, t_2]\}$. Due to the definition of averaged profit on the infinite horizon there exists a sequence of times $\{T_i\}$ such that $T_i \to \infty$ and $A(T_i) \to A$ when $i \to \infty$. Let us take i so big that $T_i > t_2$ and $A(T_i) > C$. Denote by Δ, $0 < \Delta < T$, the economy of time when we pass the arc between $x(t_1)$ and $x(t_2)$ faster. Calculating now the difference D between the averaged profits of the initial motion on the interval $[0, T_i]$ and the modified one on the interval $[0, T_i - \Delta]$ we get

$$D \leq A(T_i) - \frac{T_i A(T_i) - C\Delta}{T_i - \Delta} = \frac{\Delta}{T_i - \Delta}(C - A(T_i)) < 0.$$

So the motion with the faster crossing of the arc between $x(t_1)$ and $x(t_2)$ provides no worse averaged profit on the infinite horizon then the initial one. □

Now let us prove Theorem 2.1 itself. Let f and A be the profit density and the maximum averaged profit on the infinite horizon, respectively. If the domain $\{x : f(x) \geq A\}$ contains equilibrium points then the maximum averaged profit can be provided by the respective stationary strategy.

If it does not, then for sufficiently small $\epsilon > 0$ the closed domain $D = \{x : f(x) \geq A - \epsilon\}$ also does not contain equilibrium points due to continuity of the profit density and of the maximum (minimum) velocity. In particular, in this domain these velocities have the same sign and are separated from zero. So, any admissible motion on a connected component of the domain D has to leave it in finite time and can income back to it only after a complete rotation along the circle.

Hence the number of rotations along the circle of an admissible motion providing the maximum averaged profit has to go to infinity as the time goes to infinity. Otherwise this motion would have to spend a finite time in the domain D and an infinite one in the rest part of the circle, where the density is less then $A - \epsilon$. Consequently, the averaged profit of the motion on the infinite horizon would be not greater then $A - \epsilon$, what contradicts the optimality of the motion.

On each cycle of the rotation the motion under consideration can be improved in the stream of Proposition 3.2. In fact, the existence of such rotation permits us to conclude by Proposition 3.1 that the maximum velocity in the direction of its motion is always positive. In particular, it is separated from zero because of its continuity and compactness of the circle. The A-level cycle uses it on the set $f(x) \leq A$. On the rest part $f(x) > A$ of the circle the improved motion has to use the minimum velocity which is also separated from zero on this part due to the absence of equilibrium points in its closure.

Thus the modified motion is the A-level cycle. Its averaged profit on the horizon is not less than the one for the initial motion, e.c. no less than the value A. But it is also not greater than this value due to the optimality of A.

Theorem 2.1 is proved. □

4 Proof of Theorem 2.5

To simplify language we say that a boundary point of the stationary domain is *regular* if at that point the stationary domain has singularity 1 of Theorem 2.4. Otherwise it is called *singular*. The following statement is useful.

Proposition 4.1 *For a generic smooth one parametric family of pairs of control systems and profit densities on the circle*

(a) *at any point at least one of the derivatives f_x, f_{xx} and f_{xxx} of the profit density family f does not vanish,*

(b) *at any singular boundary point of the stationary domain, the derivative f_x does not vanish, and at a regular one at least one of the derivatives f_x and f_{xx} does not vanish,*

(c) *for any parameter value the maximum averaged profit over the stationary strategies can be attained at most at two different points.*

We omit the proof of these statements because they follow easily from Thom transversality theorem in the case ***(a)*** and ***(b)*** and from multijet transversality theorem in the case ***(c)***.

Now let us prove Theorem 2.5. Suppose firstly that at a parameter value p_0 the maximum $A_s(p_0)$ is provided by an unique equilibrium point $Q = (x_0, p_0)$.

If Q is an interior point of the stationary domain, then $f_x(Q) = 0$ and ***(a)*** implies that in a generic case $f_{xx}(Q) < 0$ because Q provides a local maximum of the profit density $f(., p_0)$. Consequently for p near p_0 we get $A_s(p) = f(x(p), p)$ where $x = x(p)$ is the curve of local maxima of the densities $f\cdot, p)$ near (x_0, p_0), namely, the solution of equation $f_x = 0$ near this point. This solution exists and it is unique and smooth due to the implicit function theorem because $f_{xx}(Q) < 0$. Thus the maximum averaged profit is a smooth function near the point p_0 and so it has the first singularity of Table 3 at it.

If Q is a regular boundary point of the stationary domain, then at this point the derivative f_x either vanishes or not. It is clear that in the second case near the point p_0 the function A_s coincides with the restriction of the family f to the boundary of the stationary domain near the point Q and so is smooth. Thus, as above we get the first singularity of Table 3. In the first case, when $f_x(Q) = 0$, the derivative f_{xx} is negative at Q by ***(b)*** and due to the fact that Q provides a maximum of $f(., p_0)$. Besides, due to Thom transversality theorem, generically the differential of the restriction of the derivative f_x to the boundary of the stationary domain does not vanish at Q. Hence at the point Q this boundary is not tangent to the curve of local maxima of the family f with respect to x. In particular, near the point Q this curve and the stationary domain take forms $x = p$ and $x \leq 0$ near the origin, respectively, in appropriate smooth local coordinates foliated over the parameter. In such coordinates, due to Hadamard lemma, the family of profit densities can be written locally near the origin ($= (0,0) = Q$) in the form $f(x,p) = f(p,p) + (x-p)^2 h(x,p)$ with some smooth function h; $h_{xx}(0,0) < 0$ because $f_{xx}(Q) < 0$. Now it is easy to see that near the origin the profit A_s is the maximum of the restrictions of the family f to the stationary's domain boundary $x = 0$ and to the part of the curve $x = p$ of local maxima of this family with respect to x which is inside the stationary domain. This maximum near the point p_0 coincides with the function $f(p,p) + p^2 h(p,p)$ when $p \geq 0$ and with the function $f(p,p)$ when $p \leq 0$. It is smooth near the origin except at the origin itself where it is differentiable but has different one side second derivatives. Consequently the profit A_s has the third singularity of Table 3 at the point p_0.

If the point Q is a singular boundary point of the stationary domain, then in a generic case $f_x(Q) \neq 0$ due to statement ***(b)*** of Proposition 4.1 . Consequently, at such a point the stationary domain can have only one of the singularities 2_+, 4 or 5_- of Theorem 2.4. Indeed in the case of singularities 2_- and 5_+ the point Q is an interior point of S_{p_0} and it can not provide the maximum of the averaged profit because $f_x(Q) \neq 0$. Now simple calculations imply that if at (x_0, p_0) the stationary domain has singularities 4 or 5_-, then at p_0, the maximum averaged profit A_s has the second singularity of Table 3. If at the point P the boundary domain has singularity 2_+, then the singularity type of A_s at the point p_0 depends on whether S_{p_0} contains points different from Q or not.

If there are no such points, then at p_0 the profit A_s has the fourth singularity of Table 3. If there are points of this kind, then generically they must be either interior points or regular boundary points of the stationary domain at which the derivative f_x does not vanish. In this case we conclude that the maximum averaged profit has the fifth singularity of Table 3 at the point p_0.

To complete the classification of singularities we only have to consider the case when the best averaged profit $A_s(p_0)$ provided by two different equilibrium points. Multijet transversality theorem implies that each of these points is either an interior point of the stationary domain with $f_x = 0 < f_{xx}$ or a regular boundary point with $f_x \neq 0$. So the best local stationary strategies near each of these two optimal strategies provide an averaged profit which is smooth at the point p_0. The respective two functions have the same value at p_0 but in a generic case, due to the multijet transversality theorem, they have different derivatives at this point. Hence the profit A_s which is, near the point p_0, the maximum of these two functions has at the point p_0 the second singularity from Table 3 in the case considered.

Now it is easy to see that for a generic pair a singularity from Table 3 is defined by the transversality of its jets or multijet extensions to submanifolds in the jet or multijet spaces, respectively. That implies the stability of singularities, namely, the last statement of Theorem 2.5. □

5 Strategy Transition: Proof of Theorem 2.6

Firstly we prove some useful statements and then study transitions.

5.1 Cyclic and transition values

Proposition 5.1 *For a cyclic level c_0 of the density at a value p_0 of the parameter, the close levels of the density are also cyclic for all sufficiently close values of the parameter, if the families of profit densities and control systems are differentiable and for parameter value p_0 the density has only a finite number of critical points.*

Proof. Due to Proposition 3.1 at the value p_0 of the parameter the maximum velocity V in the direction of the c_0-level cycle is positive everywhere. That is also true for any sufficiently close value of the parameter due to continuity of the maximum velocity for a continuous family of control systems. Hence it is sufficient to show that for any pair (c, p) near the pair (c_0, p_0) the domain $\{x : \quad f(x, p) \geq c\}$ does not contain equilibrium points.

If it does then there exist sequences of equilibrium points $\{(x_i, p_i)\}$ and of levels $\{c_i\}$ with $\lim\limits_{i\to\infty}\{(p_i, c_i)\} = (p_0, c_0)$ and $f(x_i, p_i) \geq c_i$. But the circle is compact. So due to Bolzano-Weierstrass theorem there exists a subsequence $\{x_{i_j}\}$ of points which converges to some point Q of the circle. (Q, p_0) is the equilibrium point due to continuity of the family of systems, and the value of the profit density at this point is not less then c_0 due to continuity of the family of densities. Due to positiveness of the maximum velocity $V(Q, p_0)$ the minimum one has to be non-positive at this point.

Now if the level c_0 is not the minimum of the density $f(\cdot, p_0)$ along the circle then the $(c_0 - \epsilon)$-level motion is not cyclic for a positive ϵ small enough. Really such a motion has to use the minimum velocity near the point (Q, p_0). But this velocity is either negative or it vanishes at this point and satisfies the Lipschitz condition due to differentiability of the control system. In both these cases the motion can not cross the point Q. Really it can not do that with negative velocity and can not income to the point in finite time with the vanishing velocity satisfying the Lipschitz condition because the function $1/x$ has a non-integrable singularity at zero. That implies that the value c_0 is not cyclic, and we get a contradiction to the proposition assumptions.

When the level c_0 is the minimum of the density $f(\cdot, p_0)$ along the circle then it is a strong local minimum because the density has only a finite number of critical points. Consequently, the c_0-level motion uses the minimum velocity near the point (Q, p_0) except eventually in the point itself. Hence this motion can not cross the point as it was shown just above, and again we arrive to a contradiction to the proposition assumptions.

Thus the set $\{x : \quad f(\cdot, p) \geq c\}$ does not contain equilibrium points for any point (c, p) near the point (c_0, p_0), and the level c is cyclic for the parameter value p. $\square$

Proposition 5.2 *Suppose c_0 is a cyclic level for the parameter value p_0. Then for a point (c, p) near the point (c_0, p_0) the measure of the symmetric difference between the domains where the level cycles corresponding to these points keep the maximum (minimum) velocities goes to zero when $(c, p) \to (c_0, p_0)$, if the families of profit densities and control systems are differentiable and for parameter value p_0 the density has only a finite number of critical points.*

Proof. By Proposition 5.1 for (c, p) near (c_0, p_0) the c-level cycle for a value p of the parameter is well defined. Only such pairs (c, p) are considered in this proof.

The level c_0 of the density function $f(\cdot, p_0)$ has only a finite number of different points. Otherwise this function would have an infinite number of critical points due to Rolle's theorem. Thus for any positive ϵ this level can be covered by a finite number of open intervals with total length less then ϵ. Deleting these intervals from the circle we get some compact K. On this compact the continuous function $|f(\cdot, p_0) - c_0|$ is positive and achieves a minimum which we denote by m. Due to continuity of the family of densities, in the space of variables x, p the compact $\{(x, p_0) : \quad x \in K\}$ has a neighborhood in which $|f(x, p) - c_0|$ is not less then $2m/3$. It is clear that this neighborhood contains the product $K \times]p_0 - \delta, p_0 + \delta[$ for a positive δ small enough. For any point (x, p) of this product we have

$$|f(x, p) - c| \geq |f(x, p) - c_0| - |c - c_0| > m/3$$

if $|c - c_0| < m/3$. That implies that for a point (c, p) near the point (c_0, p_0) satisfying the conditions $|c - c_0| < m/3$ and $|p - p_0| < \delta$ the measure of the symmetric difference between the domains where the level cycles corresponding to these points keep the maximum (minimum) velocities goes to zero when $(c, p) \to (c_0, p_0)$. That proves the proposition. $\square$

Proposition 5.3 *The period of a level cycle and the (averaged) profit along it are continuous functions of the level and the parameter at any point (c_0, p_0) such that c_0 is a cyclic value for the parameter value p_0, if the families of profit densities and control systems are differentiable and for the parameter value p_0 the density has only a finite number of critical points.*

Proof. Due to Proposition 5.1, any point (c, p) close enough to the point (c_0, p_0) provides a level cycle. Besides, the measure of the symmetric difference between domains where the level cycles provided by these points use the maximum (minimum) velocities goes to zero when $(c, p) \to (c_0, p_0)$, due to Proposition 5.2. All this, together with the continuity of maximum (minimum) velocities and of family of profit densities, implies the statement of Proposition 5.3 as it is easy to see. □

Corollary 5.4 *For a differentiable family of pairs of control systems and profit densities and a transition parameter value p_0 the best averaged profit $A_s(p_0)$ provided by stationary strategies is not less then the upper limit $A_l(p_0)$ of the best averaged profit provided by level cycles under $p \to p_0$, if the density $f(., p_0)$ has only a finite number of critical points.*

Proof. Assume that $A_s(p_0) < A_l(p_0)$. It is easy to see that the level $c_0 = A_l(p_0)$ is cyclic for the parameter value p_0. Hence due to Proposition 5.1 any point (c, p) close enough to the point (c_0, p_0) also provides a level cycle.

Due to Proposition 5.3 the averaged profit A along level cycles is a continuous function of (c, p) at the point (c_0, p_0). Consequently c_0 is the maximum averaged profit for the parameter value p_0. Also due to this continuity the profit $A(c, p)$ is bigger then $A_s(p_0)$ for a pair (c, p) close enough to (c_0, p_0). Hence near the parameter value p_0 the maximum averaged profit is always provided by level cycles because the function A_s is upper semi-continuous. Thus p_0 is not a transition value of the parameter. We arrive to a contradiction and so $A_s(p_0) \geq A_l(p_0)$. □

Proposition 5.5 *For a generic smooth one parametric family of pairs of control systems and profit densities on the circle $A_s(p_0) < A_l(p_0)$ if the profit A_s has at the point p_0 one of the two last singularities from Table 3.*

Proof. In a generic case both these singularities are caused by the attainment of the best averaged profit for stationary strategies at a point (x_0, p_0) where $f_x \neq 0$ and at which the stationary domain has a singularity 2_+ from the list of Theorem 2.4. Let us take this point for the origin of ne smooth coordinates with coordinate x increasing in the direction of the motion along c-level cycle where c is a little bit greater then $f(x_0, p_0)$. Such a level c is really cyclic because $f_x(x_0, p_0) \neq 0$, and so the profit density has values being bigger then $A_s(p_0)$.

Due to the implicit function theorem, equation $c = f(x, p)$ has a unique smooth solution $x = X(c, p)$ near the origin because $f_x(x_0, p_0) \neq 0$. Selecting up to the sign this solution as new coordinate along profit axis we reduce the function f to the form $\pm x$ near the origin and the profit $A_s(p_0)$ to zero.

The singularity 2_+ corresponds to the vanishing of the smooth minimum velocity v at the point together with its first derivative v_x when the second derivative v_{xx} is not zero. Due to Hadamard Lemma, on the fiber $p = p_0$ the minimum velocity takes the form $x^2 h(x)$ near the origin with some smooth function h, $h(0) > 0$.

Now let us show that in a generic case, for the parameter value p_0, a cyclic motion with a small positive level provides a positive averaged profit. It is clear that this implies the proposition statement. We consider the case "$+x$" of the profit density near the origin. The case "$-x$" gives the same result and can be treated analogously. The following statement is useful.

Lemma 5.6 *For a generic smooth one parametric family of pairs of control systems and profit densities, if (x_0, p_0) is a singular boundary point of the stationary domain, then $f(x_0, p_0)$ is not a critical value of the density $f(\cdot, p_0)$, and except this point the level $f(x_0, p_0)$ of this density does not contains points of the Maxwell set and, additionally, on the fiber $p = p_0$ all other points of this set and boundary points of the stationary domain are regular.*

We omit the proof of this lemma which follows easily from transversality theorems.

Now by Lemma 5.6 we conclude that the value $f(x_0, p_0)$ is not critical of the function $f(., p_0)$ in a generic case. Hence there exists a positive a such that all values of this function from the interval $[0, a]$ are not critical, because the set of critical values of a differentiable function on the circle is closed.

Let us take a so small that for $x \in [-a, a]$ the profit density and the minimum velocity have the forms x and $x^2h(x)$, respectively. Consider now a c-level cycle with $c \in (0, a)$. Both the period of the cycle and the profit along it we subdivide into two parts corresponding to the motion outside and inside the interval $[-a, a]$. The second ones have the forms

$$\int_{-a}^{c} \frac{dx}{V(x)} + \int_{c}^{a} \frac{dx}{x^2h(x)} \quad \text{and} \quad \int_{-a}^{c} \frac{xdx}{V(x)} + \int_{c}^{a} \frac{xdx}{x^2h(x)},$$

respectively, where V is the maximum velocity near the origin. The first ones, $T(a, c)$ and $P(a, c)$, are differentiable functions near zero because in a generic case all other points of Maxwell set and for the function $f(., p_0)$ its $f(x_0, p_0)$ is not critical and its level $f(x_0, p_0)$ does not contain points of this set [2], [10]. Consequently, the averaged profit for a c-level cycle is provided by the expression

$$\left[P(a, c) + \int_{-a}^{c} \frac{xdx}{V(x)} + \int_{c}^{a} \frac{xdx}{x^2h(x)}\right] \left[T(a, c) + \int_{-a}^{c} \frac{dx}{V(x)} + \int_{c}^{a} \frac{dx}{x^2h(x)}\right]^{-1}$$

It goes to zero and its derivative with respect to c goes to $+\infty$, when positive c goes to zero, as it is easy to verify by direct calculations. Hence, at the point p_0 the best averaged profit for level cycles is positive, and so $A_l(p_0) > A_s(p_0)$. □

Corollary 5.7 *For a generic smooth one parametric family of pairs of control systems and profit densities on the circle $A_s(p_0) = A_l(p_0)$ at any transition value p_0 of the parameter.*

Proof. At p_0 we have $A_s(p_0) \geq A_l(p_0)$ due to Corollary 5.4. If $A_s(p_0) > A_l(p_0)$ then the profit A_s is not continuous at the parameter value p_0, otherwise this value would not be a transition value. Hence in a generic case this profit has at the point p_0 one of the two last singularities from Table 3 due to Theorem 2.5. But in such a case $A_s(p_0) < A_l(p_0)$ due to Proposition 5.5, and we arrive to a contradiction.

The *transition level* is the maximum averaged profit at a transition parameter value. Denote by m, $m = m(p)$, the maximum of the family of densities with respect to x.

Proposition 5.8 *For a generic smooth one parametric family of pairs of control systems and profit densities on the circle and a transition parameter value p_0, the averaged profit provided by level cycles can be extended as a continuous function to the domain $A_s(p) \leq c \leq m(p)$ for parameter values p close enough to p_0. In particular, for the extended function A we have $[c - A](A_s(p_0), p_0) = 0$.*

Proof. Generically the profit A_s has at the point p_0 one of the three first singularities from Table 3 due to Proposition 5.5. So this profit is a smooth function in each sufficiently small semi-neighborhood V of p_0. Due to Remark 2 in a generic case the profit $A_s(p)$ for $p \in V$ and close enough to p_0 is attained at a point of the curve $(X(p), p)$ either of local maxima of the family of profit densities inside the stationary domain (except eventually the point $(X(p_0), p_0)$ itself) or of boundary points of this domain. Here X is a smooth function.

It is sufficient to show that the needed continuous extension near p_0 exists "over" the semi-neighborhood V. Selecting new local coordinates $\tilde{c}$ and x such that $\tilde{c} = c - f(X(p), p)$ and $x(X(p), p) = 0$, we reduce the profit A_s in V to zero and the curve $(X(p), p)$ to the p-axis.

Now near this axis the density function can be presented in the form $f = xh(x, p), p \in V$, due to Hadamard lemma, but the velocities used locally near this axis by the c-level motion with $c \geq 0$ can have a zero at most of first order on this axis. That implies that the profit P is continuous near $(A_s(p_0) = 0, p_0)$ for $c \geq 0$ while the period T when $c \to 0+$ can have the same singularity as $\ln c$ at zero. Consequently for the averaged profit for level motions can be continuously extended to the closure of the epigraph of A_s over V.

In the same manner these arguments work in the small semi-neighborhood V including the point p_0, when the profit $A_s(p_0)$ is attained at an unique point, namely, in the case of singularities 1 and 3 from Table 3 or in the case of singularity 2 if it is caused by the respective stationary domain's singularity but not by the competition of different stationary strategies.

For the case of singularity 2 from Table 3 when it is caused by the competition of two stationary strategies, it is necessary to apply the same arguments locally near each of these two strategies.

To end the proof we need to establish the equality $[c - A](A_s(p_0), p_0) = 0$ for the extended profit. Because p_0 is a transition parameter value there exists a sequence of parameter values p_i, $p_i \to p_0$ when $i \to \infty$ such that at each of this values the maximum averaged profit $A_l(p_i)$ provided by level cycles is bigger then $A_s(p_i)$. Thus this profit is provided by a c_i-level cycle, $c_i = A_l(p_i)$, and the equality $[c - A](A_l(p_i), p_i) = 0$ takes place due to Theorem 2.3. But $A_l(p_i) \to A_s(p_0)$ when $i \to \infty$ by Corollary 5.7. Consequently due to the continuity proved above we have the equality $[c - A](A_s(p_0), p_0) = 0$ in the limit for the extended profit. □

Proposition 5.9 *For a generic smooth one parametric family of pairs of control systems and profit densities on the circle and any transition parameter value p_0 the profit A_s is attained at one equilibrium point only. This point Q is either a regular boundary point of the stationary domain with $f_x(Q) \neq 0$ or a nondegenerate local maximum of the density lying inside the stationary domain with $f(Q) < m(p_0)$.*

Proof. As we saw in the proof of the previous proposition after subtraction of A_s, the profit P is well defined near $(A_s(p_0) = 0, p_0)$ for $c \geq 0$, while the period T can have logarithmic singularities on c when $c \to 0+$. Consequently now the equality from Proposition 5.8 can be rewritten in the form $[cT - P](A_s(p_0), p_0) = 0$. In such a form the equality is valid at a point of the graph of A_s exactly if and only if the profit P vanishes at this point.

By small perturbations of the family of profit densities outside the local maxima of the family of densities with respect to x and also outside the Maxwell set and the stationary domain boundary one can remove the points of such vanishing from the set of singular points of this graph, as it is easy to see. Note that such sufficiently small perturbations do not change the profit A_s. New sufficiently small perturbations of the family of pairs of systems and profit densities preserve the obtained situation because in a generic case the averaged profit A_s and its singularities, as well as the profit P after subtracting A_s, continuously depend on such a pair near singularities of type 2 and 3 from Table 3.

In addition, in a generic case, a parameter value where A_s has singularity 4 or 5 from Table 3 can not be a transition one due to Proposition 5.5.

Thus in a generic case the profit A_s is smooth near a transition parameter value. That implies that the profit $A_s(p_0)$ is attained at one equilibrium point only, and this point is either a regular boundary point of the stationary domain with nonzero derivative f_x or a nondegenerate local maximum of the density lying inside the stationary domain (see Remark 2). In particular $A_s(p_0) < m(p_0)$ otherwise $A_s \equiv m$ near p_0, and so p_0 could not be a transition parameter value. □

Proposition 5.10 *For a generic smooth one parametric family of pairs of control systems and profit densities on the circle and any transition parameter value p_0, in the fiber $p = p_0$ there is no tangent double, triple or swallow points, or selfintersection points of the Maxwell set, or else double point in the level $A_s(p_0)$ of the profit density.*

Proposition 5.11 *For a generic smooth one parametric family of pairs of control systems and profit densities on the circle and any transition parameter value p_0 the density $f(.,p_0)$ has no critical points with the value $A_s(p_0)$ and has only one such a point if the profit $A_s(p_0)$ is attained at the boundary or interior point of the stationary domain, respectively.*

Indeed for a generic family of pairs both the Maxwell set and the set of critical values of the densities are stable up to small perturbations of the family. Consequently the results stated in Propositions 5.10 and 5.11 can be obtained by small perturbations in the same manner as the one in Proposition 5.9. Due to that we omit the proofs.

5.2 Proof of Theorem 2.6

Now we are ready to proof Theorem 2.6. Due to Proposition 5.9 for a transition parameter value p_0 the profit A_s is attained at one point only, say Q, which is either a nondegenerate local maximum with respect to x of the family of densities inside the stationary domain with $f(Q) < m(p_0)$ or a regular boundary point of the stationary domain with $f_x(Q) \neq 0$. Consider these two cases subsequently.

In the first case, minimum and maximum velocities have different signs near the point Q (see Remark 1). Consequently the c-level motion with $c \geq A_s(p)$ is well defined for a value p close enough to p_0 because it uses the maximum velocity near the point Q. So if we do not permit any switching between maximum and minimum velocities near this point we can extend the computation of functions of the period T and the profit P to all (c,p) close enough to the point $R = (A_s(0), 0)$. These functions and the respective averaged profit A_l are smooth near R due to Theorem 2.2 and Propositions 5.10, 5.11. Besides $A_l(R) = A_s(p_0)$ due to Corollary 5.7, $(cT - P)(R) = 0$ in the strength of Proposition 5.8 and also $(P/T)_c(R) = 0$ due to the optimality of the level cycle provided by the point R.

Consequently equation (2.2) has a unique and smooth solution $c, c = C(p)$ near the point R due to the implicit function theorem. So near the point p_0 the maximum averaged profit is defined as the maximum of the functions A_s and C. The coincidence of derivatives of these functions at the point p_0 is excessive independent condition on the transition parameter value and so it does not take place in a generic case. Consequently, these derivatives are different and the maximum averaged profit has at the point p_0 the singularity 1 from Table 4 up to R^+-equivalence.

In the second case the stationary domain boundary near the point Q has the form either $x \leq 0$ or $x \geq 0$ near the origin in an appropriate smooth local coordinate near the point Q which are foliated over the parameter and preserve the direction of of the growth of x. Consider the first of these two subcases. The another is studied analogously.

In the first subcase near the origin the minimum velocity v is smooth, negative in the domain $x < 0$ (see Remark 1) and has a non-degenerate singular point at zero. Consequently this velocity is reduced to the form $x\gamma(p)$ with some smooth function γ by a new choice of smooth x-coordinate near the origin [5] with the preservation of the form $x \le 0$ of the stationary domain; $\gamma(0) > 0$ as it is easy to see.

Further subtracting the profit A_s from the profit density near $p = 0$ we reduce to zero this profit and the restriction of the family f to the stationary domain boundary $x = 0$. Now due to Hadamard lemma this family can be presented near the origin in the form $f(x,p) = xh(x,p)$ with some smooth function h; $h(0,0) \ne 0$ because $f_x(Q) \ne 0$. Besides $h(0,0) > 0$ due to the existence of c-level cycles for small positive c for zero parameter value.

Due to the implicit function theorem, equation $c = xh(x,p)$ has a unique smooth solution $x, x = X(c,p)$ near the origin with $X(0,p) \equiv 0$ and $X_c(0,0) > 0$. In new smooth coordinate along the profit axis which depends on c and p and coincides with this solution near the origin the profit density takes the form $f(x,p) = x$.

Now we take the positive values a and ϵ such that in the rectangle $[-a,a] \times [-\epsilon,\epsilon]$ the stationary domain, the family of densities and minimal velocity have the forms $x \le 0$, x and $x\gamma(p)$, respectively. Then the functions of the period of a c-level cycle with $0 < c < a$ and the profit along it for the values p, $|p| < \epsilon$, have the forms

$$T^*(c,p) + \int_c^a \frac{dx}{x\gamma(p)} = T^*(c,p) + \frac{1}{\gamma(p)} \ln(\frac{a}{c}),$$

$$P^*(c,p) + \int_c^a \frac{dx}{\gamma(p)} = P^*(c,p) + \frac{a-c}{\gamma(p)},$$

respectively, where T^* and P^* correspond to the time of motion and the profit outside the interval $[-a,a]$. Due to Proposition 5.11 the transition level $A_s(0)$ is not a critical value of the function $f(.,p_0)$, and by Proposition 5.10 in the fiber $p = p_0$ there is no double tangent, triple or swallow points or selfintersection points of the Maxwell set, or else double point in the level $A_s(p_0)$ of the profit density. Consequently the functions P^* and T^* are smooth near the origin [2], [10], and so for selected values of levels and parameter if the values a and ϵ are sufficiently small. Now we just have to analyse for parameter values near p_0 which level cycles provide a positive averaged profit. The respective levels of best ones must be the solutions of equation (2.2) , that is

$$c - [P^*(c,p) + \frac{a-c}{\gamma(p)}]/[T^*(c,p) + \frac{1}{\gamma(p)} \ln(a/c)] = 0$$

in our case. Simple transformations reduce it to the form

$$c \ln c = F(c,p), \tag{5.1}$$

where $F(c,p) = c\gamma(p)T^*(c,p) + c\ln a - \gamma(p)P(c,p)$ is smooth near the origin and $F(0,p) = -\gamma(p)P(0,p)$. Using Hadamard lemma we write the function F in the form $F(c,p) = cH(c,p) + pB(p)$ near the origin with some smooth functions H and B, $B(0) = -\gamma(0)P_p(0,0)$. The vanishing of the derivative P_p at the origin gives an excessive independent condition on the transition and so it does not take place in a generic case. Consequently $B(0) \ne 0$ in a generic case. Changing eventually the sign of p we always can get $B(0) < 0$.

Now near the origin equation (5.1) has no solution for negative parameter values but for positive ones its unique solution can be found in the form $c = \frac{p}{|\ln p|} z$ with some function

z of p. Substituting this form and the form of the function F in equation (5.1) we get the equation

$$\frac{p}{|\ln p|}z\ln(\frac{p}{|\ln p|}z) = \frac{p}{|\ln p|}zH(\frac{p}{|\ln p|}z,p) + pB(p).$$

Dividing by p and making simple transformations we reduce it to the form

$$z\left(-1 - \frac{\ln|\ln p|}{|\ln p|} + \frac{\ln|z|}{|\ln p|}\right) - \frac{1}{|\ln p|}zH(\frac{p}{|\ln p|}z,p) - B(p) = 0.$$

Now it is easy to see that the value $z(0)$ must be equal to $-B(0)$, and so it is positive. Introducing new variables r and s such that $r = 1/|\ln p|$ and $s = \ln|\ln p|/|\ln p|$ we get that the left hand side of last equation is a smooth function of p, z, r and s near the point $(0, -B(0), 0, 0)$. Besides its derivative at this point with respect to z is equal to -1. Consequently near this point this equation has a unique smooth solution $z = Z(p, r, s)$ with $Z(0,0,0) = -B(0)$.

That implies that equation (5.1) has the solution $c = \frac{p}{|\ln p|}Z\left(p, \frac{1}{|\ln p|}, \frac{|\ln p|}{\ln|\ln p|}\right)$. The change of c, $\tilde{c} = c/Z(p,0,0)$, reduces the function Z to the form $Z = 1+H\left(p, \frac{1}{|\ln p|}, \frac{|\ln p|}{\ln|\ln p|}\right)$ with $H\left(0, \frac{1}{|\ln p|}, \frac{|\ln p|}{\ln|\ln p|}\right) \equiv 0$. Hence the maximum averaged profit has at the point p_0 singularity 2 from Table 4 up to R^+-equivalence.

Theorem 2.6 is proved.

References

[1] V.I. Arnol'd, *Convex hulls and the increase of the efficiency of systems under pulsating loading*// Sib. Math. J. **28**, No.4, 540-542 (1987).

[2] V.I. Arnol'd - *Optimization in mean and phase transitions in controlled dynamical systems*// Funct. Anal. and its Appl. **36**, No.2, 83-93 (2002).

[3] V.I. Arnol'd - *On a Variational Problem Connected with Phase Transitions of Means in Controllable Dynamical Systems*//in M.Birman (ed) et al., Nonlinear Problems in Mathematical Physics and Related Topics I, Kluwer/Plenum Publishers, ISBN 0-306-47333-X, July 2002.

[4] V.I. Arnol'd; A.N. Varchenko; S.M. Gusein-Sade -*Singularities of differentiable maps, vol.1*// Monographs in Mathematics, vol. 82, Birkhauser, Boston, 1985, 382 pp.

[5] V.I. Arnol'd; V.S. Afrajmovich; Yu.S. Il'yashenko; L.P. Shil'nikov - *Bifurcation theory and catastrophe theory*// Encyclopaedia of Mathematical Sciences (Dynamical systems V), 5 (1994). Berlin: Springer. 271 p.

[6] L.N. Bryzgalova -*Maximum functions of a family of functions depending on parameters*// Funct. Anal. Appl. **12**, 50-51 (1978).

[7] L.N. Bryzgalova -*Singularities of the maximum of a parametrically dependent function* // Funct. Anal. Appl. **11**, 49-51 (1977).

[8] C. Moreira - *Singularidades do Proveito Médio Óptimo para Estratégias Estacionárias*// Tese de Mestrado, Universidade do Porto, 2005.

[9] A.A. Davydov - *Singularities of the maximum function over a preimage*// Geometry in nonlinear control and differential inclusions, Banach Center Publications, **32**, 167-181, (1995).

[10] A.A. Davydov - *Generic profit singularities in Arnold's model of cyclic processes*// Proceedings of the Steklov Institute of mathematics, V.250 , 70-84, (2005).

[11] A.A. Davydov - *Generic singularities of optimal strategy velocities in one parametric cyclic processes* //Chebyshevskie chteniia V. 5 , No.1, 87-94, (2004) .

[12] A.A. Davydov, V.M. Zakalyukin - *Coincidence of generic relative minimum singularities in problems with explicit and implicit constraints*// J. Math. Sci., New York 103, No.6, 709-724 (2001).

[13] H.Th. Jongen, P. Jonker, E. Twilt - *Critical sets in parametric optimization*// Mathematical Programming **34**, 333-353 (1986).

[14] H. Maurer, CH. Büskens, G. Feichtinger-*Solution techniques for periodic control problems: a case study in production planning*// Optim. Control Appl. Meth. **19**, 185-203 (1998).

[15] J.N.Mather - *Generic projections*// Ann. of Math., II. Ser. 98, 226-245 (1973).

[16] A.A. Zevin, - *Optimal control of periodic processes*// Autom. Remote Control **41**, 304-308 (1980); translation from Avtom. Telemekh. No.3, 20-25 (1980).

[17] A.M. Tsirlin, -*Methods of averaging optimization and their applications*// M.: Nauka. Fizmatlit.304 pp. (1997), ISBN 5-02-0150991-6, (in Russian).