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Network Externalities and the Dynamics of Markets

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Abstract

The evolution of markets on which network externalities prevail can be expected to differ from "classical markets" where no such externalities exist. We suggest a flexible formal model that describes the dynamics of both types of markets. This leads to a stochastic version of the well known replicator dynamics. Based on this approach we analyze the limit behaviour of different market types where consumers use stochastic decision rules. We show that the market shares converge to the set of equilibria with probability one, where, even under network externalities, several technologies can coexist. On the other hand, even if no network externalities prevail it is possible that only one technology stays in the market. This paper contributes to the work on generalized urn schemes and path dependent processes going on at IIASA.

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1 Introduction

The phenomenon of network externalities has received wide attention in the economic literature. Rohlfs [1974] showed that this type of externalities play an important role in the market of telecommunication where the utility of joining a communication network is positively related to the number of its members. This result was confirmed by Callon [1993] and Capello [1994]. However, what has been said for this market holds also for the market of consumption and investment goods of high technological level (henceforth called *complex technologies*). This is due to two interdependent reasons:

First, complex technologies usually require some complementary investment that puts the technology to work. Think of training costs or of some linked product like computer software: once a buyer has chosen a certain technology and realized the corresponding co-investment he very probably sticks to his decision since the co-investment is experienced as *sunk cost*.[1]

Second, through this procedure, the market of complex technologies is linked to some extent to the market of complementary goods. This implies that ease of access to complementary products influences the preference a buyer has for a technology standard as such. If one technology standard dominates the market, its co-products can be expected to be cheaper and easier to obtain. Moreover, it can be expected that the variety among co-products is higher and thus more attractive for a new buyer. Think of access to software or to persons who are trained on a certain technology.[2] Thus, the decisions of buyers are linked through the market of co-products which leads to an investment network. Hence, the market exhibits *network externalities*. As an outcome of this "market failure", the dynamics of a market of such complex technologies can be expected to be fundamentally

[1] See David [1985] and Arthur [1983], Arthur, Ermoliev and Kaniovski [1987].

[2] This topic has been extensively discussed in the literature. See again David [1985] who discusses this relation for typewriters and the "market of secretaries". Cowan [1990] tells a similar story for the market of energy providing systems. Katz and Shapiro [1985, 1986] dealt with the question of network externalities for a market that exhibits investment networks. Their studies are based on a comparative static approach involving the assumption of rational expectations. Church and Gandal [1993] analyzed technology–co-product relations by explicitly considering "hardware-software relations" in a general-equilibrium context.

different to "classical markets". Our aim is to present a simple model that describes the dynamics of markets where network externalities prevail, neither assuming rational expectations nor using a general equilibrium framework. We assume the agents to use a simple stochastic decision rule based on *demand functions* that depend on prices and market shares. This leads to a stochastic version of the well known replicator dynamics.

In the following section we define the model and introduce the market dynamics. In the subsequent section we give some general results on the convergence of market shares. Finally we illustrate this approach for a limited number of goods.

2 The Evolution of Markets with Network Externalities

2.1 A Decision Process under Network Externalities

Consider a market characterized by investment networks. Several types of technologies compete that all fulfil the same task but have different technological characteristics and hence they work with *different standards*[3]. When a potential buyer decides which type of technology to purchase he or she looks at its relative price, at its market share, and at the availability of its co-products.

Suppose at a certain time instant all competing technologies have the same market share and are sold at the same price. We would expect a potential new buyer to be indifferent. Maybe he *is* indifferent and his decision is random. Or he prefers one of the technologies, for which the reasons can be manifold: he may prefer certain special characteristics that are attractive only to a small group of users. Or the outcome of the decision is due to the bounded rationality of consumers: they might not be aware of all prices and market shares, just imitate a friend, or do not like to follow the majority.[4] Due to the manifoldness of influences we do not expect the consumers' initial choice to appear deterministic. Instead, the outcome of the decision process appears random to us[5]. Once the buyer made his first choice, he sticks with this initially chosen system (even in the case of replacing investment) since otherwise the co-investment would be useless and the sunk costs would be lost.

The consequence of a consumers' decision is twofold. First, the market share of the product increases. This increases the market of co-products and thus makes them easier available. This again increases the probability that the next buyer chooses the same technology. Hence, via the market of co-products we can expect a *positive demand feedback* on the market of technologies. We will describe this phenomenon with a parameter called *network-elasticity*. Second, the producers, now confronted with new market shares, might change the product-prices. Several reasons can play a role in this regard. Producers with high market shares are confronted with less costs and might reduce the price or they can

[3]Think e.g. of different computer systems, of digital cassette recorders (DAT and DCC-systems) or – on another scale – different energy providing systems [Cowan, 1990].

[4]Their motivation might be a "search for diversity". See the discussion in Dosi and Kaniovski [1994].

[5]For a similar argument see Arthur [1983]

use their advantage in the market to increase it.[6] Price fluctuations change the consumers' propensity to buy a product. The direction as well as the extent of this change are given by the *price-elasticity*. This elasticity is usually expected to be negative but we do not exclude positive price elasticities from our analysis. Thus, we can model *negative* and *positive demand feedbacks* with respect to prices.[7] Note that on a market with network externalities a negative feedback from price dynamics might be traded off against a positive feedback from market shares (see the discussion below).

2.2 The Model

Consider a market with $K \geq 2$ firms, each producing one technology. We assume a perfect correlation between the market of technologies and the market of co-products.[8] Thus, we can limit the analysis to the market shares of the base-technologies. Let n_k^t be the number of units of technology k in the market at time t. Hence, the market share s_k^t of technology k at time t is given by $s_k^t = n_k^t / \sum_{i=1}^K n_i^t$. We assume that initially all technologies are present in the market, i.e. $n_k^1 > 0$ for all k.

At each time t we assign a demand vector $(D_1^t, \ldots, D_K^t)$ of non-negative numbers to the technologies. The demand D_k^t for technology k is a measure for the confidence the buyers have in this technology.[9] We assume that the demand for each technology depends on its present market share. This maybe due to network externalities as well as pricing policies that rely on market shares. Thus, for each technology k there is a demand function $D_k(\cdot) : (0,1] \to I\!R^+$ such that

$$D_k^t = D_k(s_k^t), \quad t > 0.$$

Note that t is not chronological time but is defined by the sequential moments of buying. The probability that the t-th buyer purchases a certain product is given by the *relative propensity* to buy this product, defined as

$$d_k(\mathbf{s}^t) = \frac{D_k(s_k^t)}{\sum\limits_{i=1}^{K} D_i(s_i^t)}, \quad 1 \leq k \leq K, \tag{1}$$

where $\mathbf{s}^t = (s_1^t, s_2^t, \ldots, s_K^t) \in \Delta = \{\mathbf{s} | \mathbf{s} \in I\!R^K, \, s_i \geq 0, \, \sum_{i=1}^K s_i = 1\}$. Thus, the k-th component of $\mathbf{d}(\mathbf{s}) = (d_1(\mathbf{s}), \ldots, d_K(\mathbf{s})) \in \Delta$, which we call *relative demand* or *preference function*, specifies the conditional probabilities of choosing technology k given the current

[6] See also the discussion in Dosi and Kaniovski [1994] and Dosi, Ermoliev and Kaniovski [1994].

[7] This issue has been discussed in a number of papers. See e.g. Arthur [1983], Arthur *et al.* [1987], David [1985] or Dosi and Kaniovski [1994].

[8] This implies that a certain technology cannot use co-products that fit a different standard. This assumption is straightforward for all types of technical co-products. The correlation can be less than one in the case of human skills.

[9] This corresponds to the concept of *strength* in Arthur [1993].

market shares of all technologies (i.e. given the vector $\mathbf{s}^t$).[10] This formalizes the decision process discussed in section 2.1.[11] Let $\mathbf{n}^t = (n_t^1, \ldots, n_K^t)$. Then the evolution of the market is given by

$$\mathbf{n}^{t+1} = \mathbf{n}^t + \boldsymbol{\beta}^t(\mathbf{s}^t), \tag{2}$$

where $\boldsymbol{\beta}^t(\mathbf{s})$ denotes a sequence of K-dimensional independent random vectors whose distribution depends on $\mathbf{s}$ in such a way that $\mathbf{P}\{\boldsymbol{\beta}^{\mathbf{t}}(\mathbf{s}) = \mathbf{e}_k\} = d_k(\mathbf{s})$, $1 \le k \le K$, where $\mathbf{e}_k$, $1 \le k \le K$ denotes the k-th unit vector.

If network externalities are present the demand of the consumer that buys at time $t+1$ depends not only on the price of the product at time t but also on the present market share. A quite general class of demand functions that depend on the market share and on price is given by

$$D_k(s_k^t) = (s_k^t)^{\sigma_k} \cdot [p_k(s_k^t)]^{\rho_k}, \tag{3}$$

where ρ_k denotes the elasticity of demand for technology k with respect to its price and σ_k stands for its elasticity with respect to its market share. In the remainder of this paper we will refer to σ_k as *network elasticity.* Let us assume that the pricing policy of firm k can be described through a *share-response function* that is denoted by $p_k(s_k^t)$.[12]

It should be noted that the number of variables that can have an influence on the choice – like different technological characteristics or influences of friends – can be implicitly included in this demand function. Note that by equation (3) the demand does not shift if the price rises (falls) but the market share falls (rises) simultaneously.

3 Market Dynamics and the Replicator Equation

The dynamics of market shares can be interpreted as an urn scheme of the type studied in Arthur [1983], Arthur *et al.* [1987], Dosi and Kaniovski [1994], Dosi *et al.* [1994]: Consider an urn of infinite capacity with balls of K different colors. At each time step a ball is added. The color is chosen randomly and the probability for each color is given by a so called *urn function* $\mathbf{q}(\cdot) : \Delta \to \Delta$ which is a function of the present distribution of balls in the urn. The application to the market dynamics is straightforward. The urn is associated with the market. Consumers choice among technologies corresponds to adding of a ball. The market shares are identified with proportions of balls in the urn. Finally, the urn

[10] The concept of function (1) is very closely related to the notion of *allocation function* used by Arthur *et al.* [1987], Dosi and Kaniovski [1994], Dosi *et al.* [1994].

[11] Another interpretation is the following: assume that D_k^t gives the number of potential buyers that prefer technology k at time t. At every time t, indicating a moment of buying, a buyer is randomly chosen from the set of potential buyers and buys the preferred technology. Thus, the probability that at time t technology k is chosen is again given by $d_k(\mathbf{s}^t)$.

[12] For a further discussion of this behaviour see section 4.

function is just the relative demand function $\mathbf{d}(\mathbf{s})$. In the following analysis we extend some standard results on urn processes, i.e. on the limit distribution of balls when the number of additions goes to infinity.[13]

First we formulate the market dynamics for shares and establish the connection to replicator dynamics. Then we prove that the market shares converge almost surely to a random vector living on the fixed points (Theorem 1) of the dynamics. In Theorem 2 we distinguish attainable and unattainable fixed points, i.e. fixed points to which the process converges with positive resp. zero probability. Finally, we handle two special cases, where the elasticity of the demand functions with respect to market shares is always greater (resp. less) than 1. We give here only sketches of the proofs. The exact proofs are given in the appendix. Writing equation (2) in terms of market shares the evolution is given by[14]

$$\mathbf{s}^{t+1} = \mathbf{s}^t + \frac{1}{n+t}\left[\boldsymbol{\beta}^t(\mathbf{s}^t) - \mathbf{s}^t\right], \tag{4}$$

where $n = n_1^1 + n_2^1 + \ldots + n_K^1$ denotes the initial number of goods in the market. Adding and subtracting the term $\frac{1}{n+t}\,\mathbf{d}(\mathbf{s}) = \frac{1}{n+t}\,(d_1(\mathbf{s}), d_2(\mathbf{s}), \ldots, d_K(\mathbf{s}))$ to equation (4) yields

$$\mathbf{s}^{t+1} = \mathbf{s}^t + \frac{1}{n+t}\left[\mathbf{d}(\mathbf{s}^t) - \mathbf{s}^t\right] + \frac{1}{n+t}\left[\boldsymbol{\beta}^t(\mathbf{s}^t) - \mathbf{d}(\mathbf{s}^t)\right]. \tag{5}$$

Since $E\left(\boldsymbol{\beta}(\mathbf{s})\right) = \mathbf{d}(\mathbf{s})$ we have $E(\mathbf{s}^{t+1}|\mathbf{s}^t) = \mathbf{s}^t + \frac{1}{n+t}\left[\mathbf{d}(\mathbf{s}^t) - \mathbf{s}^t\right]$ and, consequently, on average system (5) shifts from a point $\mathbf{s}$ at time t by $\frac{1}{n+t}\left[\mathbf{d}(\mathbf{s}) - \mathbf{s}\right]$. Hence, the limit points of the system (if any) belong to the set of zeros of $\mathbf{d}(\mathbf{s}) - \mathbf{s}$. At these points the expected motion is 0. We call these points the *fixed points* of the system. The limit dynamics of the stochastic process (5) is closely related to the asymptotic behaviour of the differential equation

$$\dot{\mathbf{s}} = \mathbf{d}(\mathbf{s}) - \mathbf{s},$$

which can be written as

$$\dot{s}_k = \frac{D_k(s_k)}{\sum_{j=1}^K D_j(s_j)} - s_k,\ k = 1, \ldots, K. \tag{6}$$

To prove convergence of the market dynamics we have to introduce some smoothness conditions for the demand functions. For all $s > 0$ they have to be positive and twice continuosly differentiable. For technical reasons we assume additionally that $D(0) := \lim_{s\to 0} D_k(s)$ exists or $D_k(s) \to \infty$ for $s \to 0$ (then we set $D_k(0) = \infty$) and that $\lim_{s\to 0} D_k(s)/s$ exists or $D_k(s)/s \to \infty$ for $s \to 0$ (in the latter case we write $D_k'(0) = \infty$). The fixed points of (6) are points $\bar{\mathbf{s}} \in \Delta$ such that $D_k(\bar{s}_k) < \infty$ and $\frac{D_k(\bar{s}_k)}{\sum_{i=1}^K D_i(\bar{s}_i)} = \bar{s}_k$ for all k.

[13] See Arthur [1983], Arthur *et al.* [1987], Arthur, Ermoliev and Kaniovski [1988a, 1994], Pemantle [1990] and Posch [1994].

[14] See Arthur *et al.* [1987], Dosi and Kaniovski [1994], Dosi *et al.* [1994].

Replicator Dynamics By Hofbauer and Sigmund [1988, p. 92] the phase portrait of differential equation (6) does not change if we multiply it by a positive factor. Thus, multiplying by $\sum_{j=1}^{K} D_j(s_j)$, we get the new differential equation

$$\dot{s}_k = D_k(s_k) - s_k \sum_{j=1}^{K} D_j(s_j), \quad k = 1, \ldots, K. \tag{7}$$

We see that the market share of technology k increases (decreases, remains constant) if $D_k(s_k) > s_k \sum_{j=1}^{K} D_j(s_j)$ (respectively "$<$" or "$=$"). We now define the *fitness* of technology k by

$$G_k(s_k) := \frac{D_k(s_k)}{s_k} \quad \text{for all } s_k \in (0,1] \text{ and } k = 1, \ldots, K. \tag{8}$$

For the boundary we set $G_k(0) = \lim_{s_k \to 0} G_k(s_k) \in \mathbb{R} \cup \{\infty\}$. In the interior of Δ and all boundary faces where the fitnesses are finite, equation (7) becomes

$$\dot{s}_k = s_k \left(G_k(s_k) - \bar{G}(\mathbf{s})\right), \quad k = 1, \ldots, K, \tag{9}$$

where $\bar{G}(\mathbf{s}) = \sum_{j=1}^{K} s_j \, G(s_j)$ gives the average fitness. (9) restricted to Δ is a well studied replicator equation (see Hofbauer and Sigmund [1988] and Hofbauer, Schuster and Sigmund [1981]).

Hence, the dynamics of the market corresponds to a replicator dynamics, where the "fitness" of a product is given by the ratio of demand and market share. We see from equation (9) that, if the fitness of a technology is greater (smaller) than the average fitness (which is equal to the sum of absolute demand), its market share increases (decreases). Thus, the fitness of a product is its capacity to stay in the market or even to take over the whole market.

Note that even a technology with a low absolute demand can have a high fitness. Thus, a low absolute demand does not automatically lead to extinction of the technology. Also a high absolute demand for a technology does not assure that it will survive. Additionally, even if the relative demand for a product increases with its market share, its fitness might *de*crease, if the absolute demand grows slower than the market share. Also, if $\sigma_k = 1$ for all k (i.e. if network externalities are present!), the fitness of a technology depends only on its price level (see (3) and (8)).

Stochastic processes of the type studied here may not converge but exhibit a cyclic behaviour.[15] However, for the market dynamics we can show the following Theorem:

Theorem 1 *Let Z denote the set of fixed points of (7) and assume that Z is finite. Then the market shares converge almost surely and* $P(\lim_{t\to\infty} \bar{\mathbf{s}}^t \in Z) = 1$.

[15] See Posch [1997], Barucci and Posch [1996], Benaïm [1996].

The argument exploits that (7) is a Shashahani gradient system [Hofbauer and Sigmund, 1988] and thus, all solutions of the differential equation converge to fixed points. To reformulate this result to the stochastic process we need an extension of standard stochastic approximation results to handle demand functions that go to infinity at the boundary (as e.g. the demand function given by (3)).

Not all fixed points of (7) are attained in the limit with positive probability. Denote the *share elasticity* of D_k with respect to s, by $\delta_k(s) = \frac{D'_k(s)}{D_k(s)}\, s$. If, at a fixed point $\bar{\mathbf{s}}$ in the interior of Δ, all $\delta_k(\bar{s}_k)$ are less than 1 then the fixed point is attained with positive probability. If two or more of these elasticities are larger than 1 then the fixed point is attained with probability 0. If exactly one elasticity is less than one then the fixed point can be attainable or unattainable depending on a more complicated condition. This is summarized in the following theorem.

Theorem 2 *Let $\bar{\mathbf{s}} \in Z$ be a fixed point and set supp$(\mathbf{s}) = \{\, k \,|\, s_k > 0, 1 \le k \le K\,\}$. Assume that:*

- *(a) $\delta_k(\bar{s}_k) \neq 1$, for all $k \in$ supp$(\bar{\mathbf{s}})$,*
- *(b) $D'_k(\bar{s}_k) \neq \sum_{l=1}^{K} D_l(\bar{s}_l)$, for all $k \notin$ supp$(\bar{\mathbf{s}})$,*
- *(c) the term in (10) is not zero.*

Then the necessary and sufficient conditions that $P(\lim_{t\to\infty} \mathbf{s}^t = \bar{\mathbf{s}}) > 0$ are $D'_k(0) < \sum_{l=1}^{K} D_j(\bar{s}_l)$, for all $k \notin$ supp$(\bar{\mathbf{s}})$, and

- *1) $\delta_k(\bar{s}_k) < 1$ for all $k \in$ supp$(\bar{\mathbf{s}})$, or*
- *2) there exists exactly one $l \in$ supp$(\bar{\mathbf{s}})$ such that $\delta_l(\bar{s}_l) > 1$ and*

$$\sum_{k \in supp(\bar{s})} \frac{\bar{s}_k^2}{D_k(\bar{s}_k)\,(\delta_k(\bar{s}_k) - 1)} > 0. \tag{10}$$

The argument exploits that the sinks of the differential equation correspond to the maxima of the potential. By theorem 8 in (Posch [1994]) sinks are attained with positive probability.For the non-convergence part, that is saddles and sources, we extend results of Arthur *et al.* [1988a] and Pemantle [1990]. Namely, we prove that unstable fixed points on the boundary of the simplex, i.e. where the share of one color is zero, are attained in the limit with probability zero (Arthur *et al.* [1988a] and Pemantle [1990] looked at interior points only).

In the next two statements we discuss the special case where the share-elasticity of all demand functions is greater (resp. less) than 1 on the whole interval $(0, 1]$. Then there exists at most one fixed point in the interior of Δ. If these elasticities are all less than 1 this fixed point is attained with probability one. Thus the outcome of the market dynamics is deterministic in this case. If the demand functions for all technologies are the same, the fixed point is in the interior of Δ. Hence, in the limit all technologies coexist.

Theorem 3 *If for all* $\mathbf{s}$ *in the interior of* Δ, $\delta_k(s_k) < 1$, $k = 1, \ldots, K$ *then there exists an* $\bar{\mathbf{s}} \in \Delta$ *such that* $P(\lim_{t\to\infty} \mathbf{s}^t = \bar{\mathbf{s}}) = 1$.
If $D'_k(0) = D'_l(0)$, $0 \le k, l \le K$ *then* $\bar{\mathbf{s}}$ *is in the interior of* Δ.

In the opposite case where all share-elasticities are greater than 1 the market dynamics converges to one of the vertices of the simplex Δ. Hence, in the limit only one technology survives. If the demand functions for all technologies are identical, each vertex is attained with positive probability. Thus, the market outcome is random and path dependent.

Theorem 4 *Assume that for all* $\mathbf{s}$ *in the interior of* Δ, $\delta_k(\mathbf{s}) > 1$, $k = 1, \ldots, K$.
Then $P(\lim_{t\to\infty} \mathbf{s}^t \in E) = 1$, *where* $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_K\}$ *denotes the set of vertices.*
If $D'_k(0) = D'_l(0)$, $0 \le k, l \le K$, *then* $P(\lim_{t\to\infty} \mathbf{s}^t = \mathbf{e}_k) > 0$ *for all* $k = 1, \ldots, K$.

In the proofs of theorems 3 and 4 we adapt a result on replicator dynamics in Hofbauer *et al.* [1981] and show that the potential of (7) has a unique extremum.

4 Dynamics of Markets Under Different Scenarios

So far the model has been formulated for K goods. Let us now illustrate the dynamic behaviour of a market with network externalities where three commodities compete ($K = 3$). In Section 2.1 we argued that firms change the price of their products as market shares change. That is we assume that the pricing policy of firm k can be described by a *share-response function* that we denote by $p_k(s_k^t)$.[16] This implies the assumption that firms base their price settings on their average costs (which includes "normal profit", i.e. the opportunity costs of production) such that the minimum price of the product equals its average costs. Moreover, we assume that with increasing market share firms can extend their production capacity and hence they experience a sinking long-term average cost function[17]. A simple specification of this behaviour is that the price is just the reciprocal value of the market share:[18]

$$p_k(s_k) = \frac{a_k}{s_k}, \tag{11}$$

where a_k are constants. We call function (11) a *share-response function* and assume for simplicity that this function is identical for all K firms, i.e. $a_k = a$. Fig. 1 gives a graphical representation of the domain Δ projected on the plane.

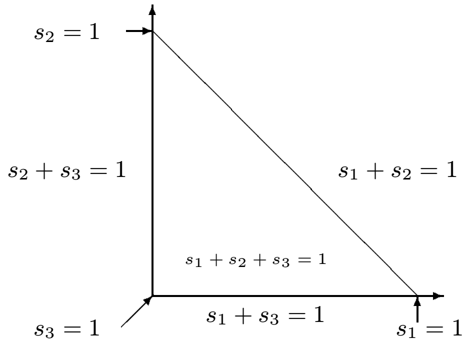

Figure 1: Graphical representation of the domain Δ (projected on the plane) where the dynamics of shares evolves.

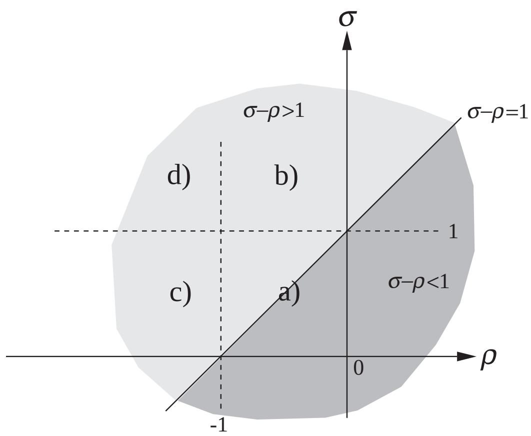

Figure 2: Graphical representation of the three scenarios in the ρ-σ-space. For parameters in the bright region only one technology survives in the limit. For parameters in the dark region all technologies coexist in the limit with equal market shares. On the line separating the regions the technologies coexist in the limit – their limit market shares are however random.

4.1 The Scenarios

Let us analyze the market behaviour under different elasticity parameters ρ and σ. Basically, four areas of the (ρ, σ)-space are conceptually interesting (see Figure 2)[19]. We consider the demand to be

a) price- and network-inelastic: $\rho \in (-1, 0]$, $\sigma \in [0, 1)$,

b) price-inelastic and network-elastic: $\rho \in (-1, 0]$, $\sigma > 1$,

c) price-elastic and network-inelastic: $\rho < -1$, $\sigma \in [0, 1)$,

d) price- and network-elastic: $\rho < -1$, $\sigma > 1$.

The cases where $\rho = -1$ (resp. $\sigma = 1$) are of special interest since then demand is neither price- (resp. network-) elastic nor inelastic. We therefore will refer to *intermediate* elasticity.[20] Additionally, the cases where one of the elasticities vanishes is of special interest: if $\sigma = 0$ this corresponds to a "classical" market without network externalities and if $\rho = 0$ to a market where consumers are indifferent with respect to prices. To investigate the dynamics for different elasticities we insert function (11) into (3) to obtain

$$D_k(s_k) = a^{\rho}\, s_k^{\sigma-\rho}. \tag{12}$$

4.2 Emerging Market Dynamics

Since the elasticity of (12) with respect to market share is $\sigma - \rho$, the above mentioned four cases can be analyzed by considering the following *three scenarios*: $\sigma - \rho = 1$, $\sigma - \rho > 1$ and $\sigma - \rho < 1$. We show in which of these scenarios all three technologies coexist in the time limit and in which only one technology survives. A summary of the results is given in Figure 2.

I. Let us start with $\sigma - \rho = 1$, which encompasses the two *reference cases* $\rho = -1$, $\sigma = 0$, i.e. a "classical" market with intermediate price elasticity and $\sigma = 1$, $\rho = 0$, i.e. intermediate network externality and no price elasticity. The demand functions simplify to $D_k(s_k) = a^{\rho} s_k$ and by (1) the relative demand for k equals the existing proportion of k in the market (i.e. $d_k(\mathbf{s}) = s_k$). The *fitnesses* of the technologies are constants and independent of market shares.

[16] We could also assume that the price depends on the market shares of the other technologies or is also stochastic. To simplify the subsequent analysis we assume that it depends only on its market share. A similar approach was chosen by Dosi and Kaniovski [1994] and Dosi *et al.* [1994].

[17] The model studied here is intrinsically dynamic. Since the production structure, hence costs, is subject to change with time we consider the *long-term average cost function* (see e.g. Varian, 1995).

[18] A more complex price function is analyzed in Keilbach and Posch [1997]. See section 5 for a brief discussion of the results.

[19] In the following discussion we consider neither positive price elasticities nor negative network elasticities explicitly. Note however that these cases may be derived from the analysis given in section 4.2.

[20] Some economic textbooks refer to *unitary* elasticity. See e.g. Pinola and Sher [1981].

In this case the shares converge with probability one.[21] Moreover, the limit of shares is (conditional on the initial condition) *Dirichlet*-distributed with the density function

$$f_D(\mathbf{s}) = \begin{cases} c \cdot s_1^{n_1^1-1} s_2^{n_2^1-1} s_3^{n_3^1-1} & \text{for } \mathbf{s} \in \Delta \\ 0 & \text{else} \end{cases} \tag{13}$$

where $n_1^1, n_2^1, n_3^1 \geq 1$ are the initial numbers of products of each technology in the market and $c = \frac{\Gamma(n_1+n_2+n_3)}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)}$. Thus, we cannot predict to which point in Δ the market shares converge. This implies, that *even under positive network externalities* (namely in all cases where $\sigma = 1 + \rho$), *all three technologies coexist in the limit.* If $n_1^1 = n_2^1 = n_3^1 = 1$ the limit distribution is uniform on Δ (see Figure 3.*I* for an illustration). Note that this case is not generic, i.e. small deviations from the condition $\sigma - \rho = 1$ will lead to different market behaviours. This will be discussed in turn.

II. $\sigma - \rho > 1$. This scenario encompasses the scenarios b), c), d) and half of the parameter space of scenario a) (see Figure 2). In these cases the share elasticity $\delta_k(\cdot)$ of the demand functions is greater than one. Thus, by theorem 4 the process converges with probability one to one of the vertices and each vertex, i.e. the point (1,0,0) and its permutations, is attained with positive probability. Thus, the market locks into one technology but we cannot predict into which one. To this case (illustrated in Figure 3.*II*) the discussion on path-dependence and "lock-in" usually refers. However, network externalities are not a necessary condition for lock-in and the emergence of a monopoly. It can also be a result of the price dynamics if the price elasticity is less than -1.[22]

III. $\sigma - \rho < 1$. This encompasses the other half of region *a*) where both parameters are inelastic. It also includes conceptually less plausible cases where $\sigma < 0$ and $\rho > 0$. Now the share elasticities $\delta_k(s)$ are less than one. Thus, by Proposition 3 there is an interior fixed point $\bar{s}$ such that $P(\lim_{t\to\infty} \mathbf{s}^t = \bar{\mathbf{s}}) = 1$. By symmetry we have $\bar{\mathbf{s}} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. See Figure 3.*III* for an illustration.

This result is somewhat counterintuitive since it encompasses the cases $\sigma \in [0, 1), \rho = 0$. That is technologies will coexist although positive network externalities prevail on the market. This is due to the fact that the demand for a technology increases slower than its market share. Hence, its *fitness* actually *decreases* with market share. Thus, *positive network externalities do not automatically imply lock-in effects.*

[21] See Arthreya [1969].

[22] One might argue that a firm will modify its pricing behaviour to stay in the market. However, as firms base their pricing behaviour on average costs, small firms may be driven out of the market due to higher cost of production. Thus, a monopoly emerges with probability one.

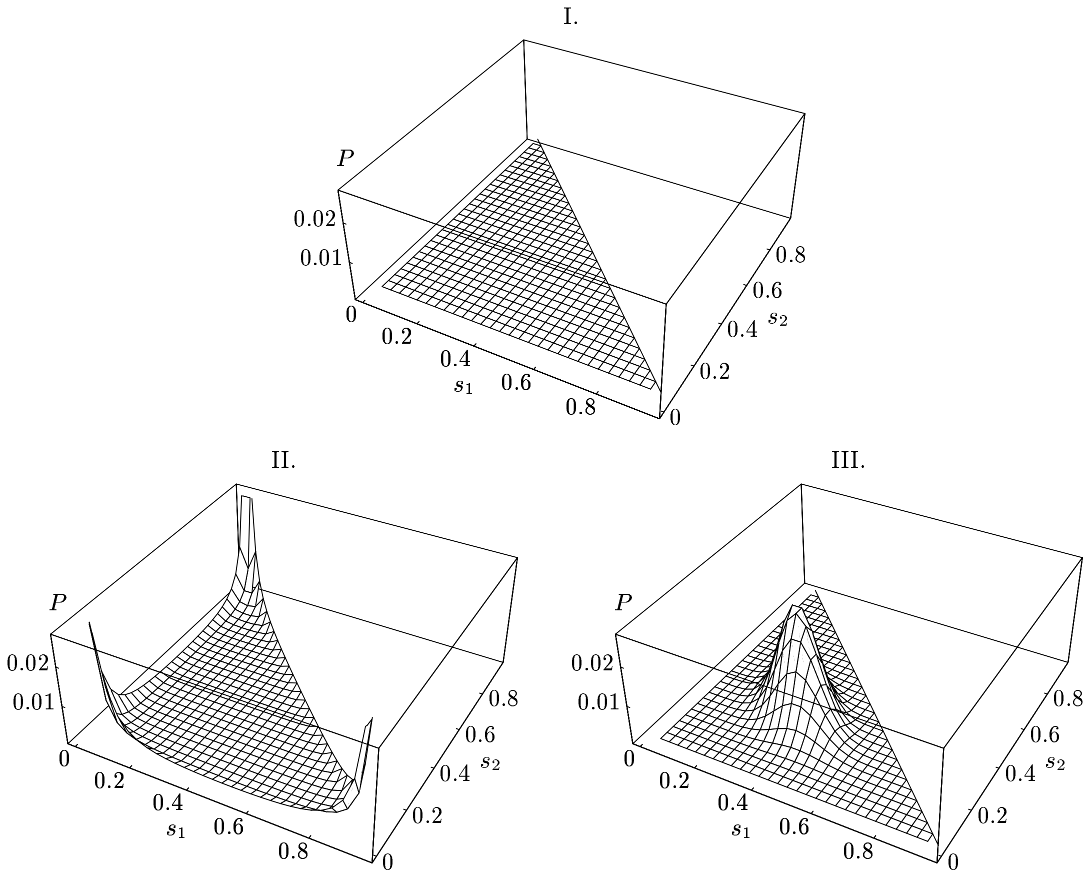

Figure 3: Probability distribution of market shares at $t = 30$ given share response function (11), $n_1^1 = n_2^1 = n_3^1 = 1$ and the three scenarios: (I) $\sigma - \rho = 1$, (II) $\sigma - \rho > 1$, (III) $\sigma - \rho < 1$.

Also if $\sigma = 0$ and the price response is inelastic ($\rho > -1$) no monopoly emerges in our model. Note finally that scenario III includes also the case $\rho = \sigma = 0$, i.e. where consumers do not respond to any signal from the market. Although admittedly implausible it is interesting to observe that such a behaviour will equally lead to coexistence of all technologies.

5 Summary and Outlook

This paper deals with markets where different technologies compete that all fulfill the same task but have different characteristics. We analyze the behaviour of such markets for different levels of price- and network-elasticities. To this purpose we specify demand functions that depend not only on the price of a technology but also on its market share. Based on these functions we define conditional probabilities of buying a certain technology. Assuming that firms decrease their prices if their market share increases we can identify the dynamics and limit states of these markets.

We illustrate this for a market where three goods compete under several constellations of elasticity parameters. The model thus encompasses "classical" markets, i.e. markets where no network externalities prevail. Several interesting results are obtained. First, if no network externalities exist and demand is inelastic, none of the technologies is pushed out of the market and all technologies have in the limit identical market shares. If however demand is elastic only one of the technologies survives, i.e. monopoly emerges. If demand is "intermediate" (i.e. neither elastic nor inelastic) all technologies coexist but it is not possible to predict the distribution of market shares.

Second, even if network externalities prevail all technologies may coexist in the market if the demand is network inelastic. Thus, a market dynamic as we know it from the story of the QWERTY-keyboard or the market of video recorders (as discussed in section 2.1) can only emerge in a relatively limited area of the parameter space. Thus – to come back to the question asked in the title – it is not certain that Bill Gates' operating system will oust the others from the market.

In Keilbach and Posch [1997] we analyze the model for a different share response function. Here, firms first decrease their prices with market share but increase it once a certain critical market share is passed. Here we obtain a whole spectrum of possible market outcomes. We show that even if network externalities prevail on a market it is possible that several (but not necessarily all) technologies coexist. Our approach is not restricted to simple continuous share-response functions. On the contrary, the flexibility of the chosen approach allows for integration of arbitrary (or even stochastic) share-response behaviour. The choice of this function is of course of decisive influence on the market dynamics.

At present, we run an empirical investigation of the model. Once econometric estimates of the demand functions and of the share response behaviour are obtained the approach we suggest here should allow for empirical investigation (i.e. prediction) of markets where network externalities prevail.

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Appendix

A note on the notation Since the dynamics given by (9) leaves the simplex Δ invariant, we have $s^t_K = 1 - \sum_{k=1}^{K-1} s^t_k$, $\forall t > 0$. Thus, the system can be reduced to the first $K-1$ dimensions. Then we get a dynamics on $\bar{\Delta} = \{\mathbf{s} \in I\!R^{K-1} \mid s_i \geq 0, \sum_{i=1}^{K-1} s_i \leq 1\}$. In the literature on urn processes mainly the $K-1$-dimensional notation is used, while the analysis of the replicator equation was mainly done in the $K-$dimensional setting. To apply the results on urn processes (including proposition 1) we have to translate the results between the two setups. While the characterization of fixed points is the same in both setups, the stability conditions have to be adapted. Let $\dot{\mathbf{s}} = \mathbf{g}(\mathbf{s})$ denote a vector field on Δ that leaves Δ invariant. Hence, $\sum_{k=1}^{K} g_k(\mathbf{s}) = 0$, $\forall \mathbf{s} \in \Delta$. The equivalent $K-1$ dimensional system is then given by $\dot{s}_k = g_k(s_1, \ldots, s_{K-1}, 1-\sum_{l=1}^{K-1} s_l)$, $k = 1, \ldots, K-1$ and is defined on $\bar{\Delta}$. Thus, the tangent space of each interior point in $\bar{\Delta}$ is given by $I\!R^{K-1}$. The tangent space for interior points in Δ for $\dot{\mathbf{s}} = \mathbf{g}(\mathbf{s})$ restricted to Δ is $\{\boldsymbol{\xi} \in I\!R^K \mid \sum_{k=1}^{K} \xi_k = 0\}$. Denote the Jacobian of $\mathbf{g}(\cdot)$ by $\mathbf{Dg}$, the one of $\left(g_k(s_1, \ldots, \mathbf{s}_{K-1}, 1 - \sum_{l=1}^{K-1} s_l)\right)_{k=1}^{K-1}$ by $\mathbf{D\bar{g}}$. Hence, a fixed point $\mathbf{s} \in \Delta$ is a linearly stable fixed point of $\dot{\mathbf{s}} = \mathbf{g}(\mathbf{s})$ restricted to Δ if

$$\langle \mathbf{Dg}(\mathbf{s}), \boldsymbol{\xi}, \boldsymbol{\xi} \rangle < 0, \quad \boldsymbol{\xi} \in \{\boldsymbol{\zeta} \in I\!R^K \mid \sum_{k=1}^{K} \zeta_k = 0\}$$

which holds if and only if

$$\langle \mathbf{D\bar{g}}(s_1, \ldots, \mathbf{s}_{K-1}, 1 - \sum_{l=1}^{K-1} s_l)\, \boldsymbol{\xi}, \boldsymbol{\xi} \rangle < 0, \quad \boldsymbol{\xi} \in R^{k-1},$$

i.e. that $(s_1, \ldots, \mathbf{s}_{K-1})$ is hyperbolically stable for $\dot{s}_k = g_k(s_1, \ldots, \mathbf{s}_{K-1}, 1-\sum_{l=1}^{K-1} s_l)$, $k = 1, \ldots, K-1$.

Proposition 1 *Let $\mathbf{s}^t$ be an urn process with a C^1 urn function $\mathbf{d}$ as specified in (2) and let $\bar{\mathbf{s}} \in \Delta$ be a hyperbolically unstable fixed point of the vector field $\dot{\mathbf{s}} = \mathbf{d}(\mathbf{s}) - \mathbf{s}$ restricted to Δ.*

Then

$$P(\lim_{t\to\infty} \mathbf{s}^t = \bar{\mathbf{s}}) = 0.$$

Proof *Let $I = \{\, i \,|\bar{\mathbf{s}}_i = 0\}$. By (if necessary) relabeling technologies we can assume that $I \subset \{1, \ldots, K-1\}$. Hence,*

$$\sum_{k=1}^{K-1} \bar{s}_k < 1. \tag{14}$$

For the proof we use the $K-1$ dimensional notation. For simplicity we denote the $K-1$-dimensional vectors again by $\bar{\mathbf{s}} = (\bar{s}_k)_{k=1}^{K-1}$ and the $K-1$ dimensional vector field by $\dot{\mathbf{s}} = \mathbf{f}(\mathbf{s})$. Now, $\bar{\mathbf{s}} \in \bar{\Delta}$ is a hyperbolically unstable fixed point of the $K-1$ dimensional vector *field* $\mathbf{f}$. *If* $\bar{\mathbf{s}}$ *lies in the interior of* $\bar{\Delta}$ *then the non-convergence follows from proposition 1 in Pemantle [1990]. Thus, we assume in the following that* $\bar{\mathbf{s}}$ *lies on the boundary of* $\bar{\Delta}$.

For the proof we have to consider two cases. First, we assume that at least one of the eigenvectors corresponding to an eigenvalue with positive real part points away from a boundary surface of $\bar{\Delta}$ *(lemma 2). Second we consider the remaining case where all such eigenvectors lie in one of the boundary surfaces (lemma 3). To make this exact we use the notion of* saturated *fixed points, introduced by Hofbauer (1990).*

Definition: $\bar{\mathbf{s}}$ *is said to be a* saturated fixed point *if all eigenvalues of the matrix* $A := \left(\frac{\partial f_i}{\partial s_j}(\bar{\mathbf{s}})\right)_{i,j \in I}$ *have non-positive real part.*

Lemma 1 *If* $\bar{\mathbf{s}}$ *is not saturated then there is a* $d > 0$, *a neighbourhood* U *of* $\bar{\mathbf{s}}$, *and a vector* $\mathbf{v} \in I\!R^{K-1}$, $v_i \geq 0\, i = 1, \ldots, K-1, v_i = 0\, \forall i \notin I$, $||\mathbf{v}|| = 1$ *such that for all* $\bar{\mathbf{s}} \in U$

$$\langle \mathbf{f}(\mathbf{s}), v \rangle > d \, \langle \mathbf{s}, \mathbf{v} \rangle \tag{15}$$

This is shown in the proof of proposition 1 of Hofbauer (1990).

Lemma 2 *Assume* $\bar{\mathbf{s}}$ *is not saturated. Then* $P(\lim_{t\to\infty} \mathbf{s}^t = \bar{\mathbf{s}}) = 0$.

Proof: *Assume to the contrary that* $P(\lim_{t\to\infty} \mathbf{s}^t = \bar{\mathbf{s}}) > 0$. *Choose a neighbourhood* U *of* $\bar{\mathbf{s}}$, *a vector* $\mathbf{v}$, *and a* $d > 0$ *as specified in lemma 1. Then there exists a* $T > 1$ *such that* $P(\{\lim_{t\to\infty} \mathbf{s}^t = \bar{\mathbf{s}}\} \cap \{\mathbf{s}^t \in U,\, t > T\}) > 0$ *Let* τ *be the following stopping time*

$$\tau = \begin{cases} \min t > T : \mathbf{s}^t \notin U, & \textit{if there is a finite } t > T \textit{ s.t. } \mathbf{s}^t \notin U; \\ \infty & \textit{otherwise} \end{cases} .$$

We show that

$$P(\lim_{t\to\infty} \langle \mathbf{s}^t, \mathbf{v} \rangle = 0 \,|\, \tau = \infty) = 0. \tag{16}$$

Since $v_i > 0$ *for at least one* i *such that* $\bar{s}_i = 0$ *this proves the proposition. We first show that*

$$P(\lim_{n\to\infty} \langle \mathbf{n}^t, \mathbf{v} \rangle = \infty \,|\, \tau = \infty) = 1. \tag{17}$$

Let $J_v = \{\, i \,|\, v_i > 0\}$. *Then (17) holds if* $P(\lim_{t\to\infty} \sum_{i\in J_v} n_i^t = \infty \,|\tau = \infty) = 1$. *Since the* n_i^t *are monotonically increasing, we get*

$$s_i^t = \frac{n_i^t}{n+t} \geq \frac{n_i^1}{n+t} \geq \frac{1}{n+t}, \quad i = 1, \ldots, K-1. \tag{18}$$

By (15), (18), and since $||\mathbf{v}|| = 1$ *we get for all* $t > T$

$$\begin{aligned} P(\sum_{i\in J_v} \beta_i^t(\mathbf{s}^t) = 1) &\geq E\Big(\sum_{i\in J_v} \beta_i^t(\mathbf{s}^t) \,\Big|\, \mathcal{F}^{\mathbf{t}}\Big) \geq E\Big(\langle \boldsymbol{\beta}^t(\mathbf{s}^t), \mathbf{v}\rangle \,\Big|\, \mathcal{F}^{\mathbf{t}}\Big) \\ &\geq E\Big(\langle \boldsymbol{\beta}^t(\mathbf{s}^t) - \mathbf{s}^t), \mathbf{v}\rangle \,\Big|\, \mathcal{F}^{\mathbf{t}}\Big) \geq d \, \langle \mathbf{s}^t, \mathbf{v} \rangle \geq \frac{d}{n+t} \end{aligned}$$

By construction of $\boldsymbol{\beta}^t$ *there exists a sequence of independent random variables* χ^t, $t \geq 1$ *such that* $\sum_{i\in J_v} \beta_i^t(\bar{\mathbf{y}}^t) \geq \chi^t \geq 0$ *and* $P(\chi^t \geq 1) \geq \frac{d}{n+t}$ *for all sequences* $\bar{\mathbf{y}}^t \in U$ *satisfying* $\bar{y}_i^t > \frac{1}{n+t}$, $i = 1, \ldots, K-1$. *Since* $\sum_{t=1}^{\infty} P(\chi^t \geq 1) = \infty$, *by the Borel-Cantelli lemma we get* $\sum_{t=1}^{\infty} \chi^t = \infty$ *a.s. Hence (17) holds true.*

For a $t > 0$ let E^t denote the set of all paths where $\langle \mathbf{n}^t, \mathbf{v}\rangle > \frac{1}{d}$, i.e. $E^t = \{\langle \mathbf{n}^t, \mathbf{v}\rangle > \frac{1}{d}\}$. Note, that E^t is $\mathcal{F}^t$-measurable and since $\mathbf{n}^t$ is increasing in t, for all $t' > t$ we have $E^{t'} \supseteq E^t$. Hence, by (17) we have $\lim_{t\to\infty} P(E^t \mid \tau = \infty) = 1$.

Now we prove (16) by contradiction. Assume that $P(\lim_{t\to\infty}\langle \mathbf{s}^t, \mathbf{v}\rangle = 0 \mid \tau = \infty) > 0$. Then there is a $T' > T$ such that

$$P(E^{T'} \cap \{\tau = \infty\} \cap \{\lim_{t\to\infty} \langle \mathbf{s}^t, \mathbf{v}\rangle = 0\}) > 0. \tag{19}$$

Let $G^t = \{t < \tau\} \cap E^{T'}$. We claim that $\mathbf{1}_{G^t} \frac{1}{\langle \mathbf{s}^t, \mathbf{v}\rangle}$ is a supermartingale. By (18) we have $s_i^t > 0$ for all t and thus, $\frac{1}{\langle \mathbf{s}^t, \mathbf{v}\rangle}$ is finite for all t. Let $t > T'$ be arbitrary but fixed. On the $\mathcal{F}^t$-measurable set G^t the process is not equal to zero and thus, by (2)

$$\begin{aligned}
\mathbf{1}_{G^t}\, E\Big(\frac{1}{\langle \mathbf{s}^{t+1}, \mathbf{v}\rangle} - \frac{1}{\langle \mathbf{s}^t, \mathbf{v}\rangle} \,\Big|\, \mathcal{F}^{\mathbf{t}}\Big) &= \mathbf{1}_{G^t}\, E\Big(\frac{n+t+1}{\langle \mathbf{n}^{t+1}, \mathbf{v}\rangle} - \frac{n+t}{\langle \mathbf{n}^t, \mathbf{v}\rangle} \,\Big|\, \mathcal{F}^{\mathbf{t}}\Big) \\
&= \mathbf{1}_{G^t}\, E\Big(\frac{\langle \mathbf{s}^t, \mathbf{v}\rangle - (n+t)\,\langle \boldsymbol{\beta}^t, \mathbf{v}\rangle}{\langle \mathbf{s}^t, \mathbf{v}\rangle\, \langle \mathbf{s}^t + \boldsymbol{\beta}^t, \mathbf{v}\rangle} \,\Big|\, \mathcal{F}^{\mathbf{t}}\Big) \\
&\le \mathbf{1}_{G^t}\, \frac{1}{\langle \mathbf{n}^t, \mathbf{v}\rangle}\left(1 - \frac{n+t}{\langle \mathbf{s}^t, \mathbf{v}\rangle + 1} E\Big(\langle \boldsymbol{\beta}^t, \mathbf{v}\rangle \,\Big|\, \mathcal{F}^{\mathbf{t}}\Big)\right)
\end{aligned} \tag{20}$$

On G^t we have $\mathbf{s}^t \in U$, and thus, by (15) we get $\mathbf{1}_{G^t}\, E\Big(\langle \boldsymbol{\beta}^t, \mathbf{v}\rangle \,\Big|\, \mathcal{F}^{\mathbf{t}}\Big) \ge \mathbf{1}_{G^t}\, d\,\langle \mathbf{s}_t, \mathbf{v}\rangle$. Substituting $\mathbf{s}^t$ by $\frac{\mathbf{n}^t}{n+t}$ we get by (20)

$$\mathbf{1}_{G^t}\, E\Big(\frac{1}{\langle \mathbf{s}^{t+1}, \mathbf{v}\rangle} - \frac{1}{\langle \mathbf{s}^t, \mathbf{v}\rangle} \,\Big|\, \mathcal{F}^{\mathbf{t}}\Big) \le \mathbf{1}_{G^t}\, \frac{1}{\langle \mathbf{n}^t, \mathbf{v}\rangle}\left[1 - \frac{(d+1)\,\langle \mathbf{n}^t, \mathbf{v}\rangle}{\langle \mathbf{n}^t, \mathbf{v}\rangle + 1}\right] \le 0.$$

Since G^t is $\mathcal{F}^t$-measurable and $G^{t+1} \subset G^t$, we proved that

$$E\Big(\mathbf{1}_{G^{t+1}}\, \frac{1}{\langle \mathbf{s}^{t+1}, \mathbf{v}\rangle} - \mathbf{1}_{G^t}\, \frac{1}{\langle \mathbf{s}^t, \mathbf{v}\rangle} \,\Big|\, \mathcal{F}^{\mathbf{t}}\Big) \le 0,$$

and hence $\mathbf{1}_{G^t} \frac{1}{\langle \mathbf{s}^t, \mathbf{v}\rangle}$ is a non-negative supermartingale. Thus, the limit $\mathbf{s}^* = \lim_{t\to\infty} \frac{1}{\langle \mathbf{s}^t, \mathbf{v}\rangle} \cdot \mathbf{1}_{G^t}$ exists a.s. Now, by the Fatou lemma we have

$$E(\mathbf{s}^*) \le \liminf_{n\to\infty} E\Big(\mathbf{1}_{G^t}\, \frac{1}{\langle \mathbf{s}^t, \mathbf{v}\rangle} \,\Big|\, \mathcal{F}^{\mathbf{t}}\Big) \le \frac{1}{\langle \mathbf{s}^{T'}, \mathbf{v}\rangle} < \infty.$$

Hence, $\mathbf{s}^* < \infty$ a.s. and thus, $P(E^{T'} \cap \{\tau = \infty\} \cap \{\lim_{t\to\infty}\langle \mathbf{s}^t, \mathbf{v}\rangle = 0\}) = 0$, which contradicts with (19). □

Lemma 3 *Assume $\bar{\mathbf{s}}$ is saturated. Then $P(\lim_{t\to\infty} \mathbf{s}^t = \bar{\mathbf{s}}) = 0$.*

Proof: *Let E^+ denote the eigenspace corresponding to the eigenvalues of A with positive real part. We apply proposition 1 in Pemantle [1990], checking conditions (i)- (iv). Conditions (i), (ii) and (iv) are straightforward. Condition (iii) reads:*

> *There exists a neighbourhood U of $\bar{\mathbf{s}}$, a $c > 0$ such that for all unit vectors $\boldsymbol{\theta} \in E^+$, all $\mathbf{s} \in U$ and all $t > 1$*
>
> $$E\Big(\langle (\beta_k^t(\mathbf{s}) - s_k)_{k=1}^{K-1}, \boldsymbol{\theta}\rangle^+ \,\Big|\, \mathcal{F}^{\mathbf{t}}\Big) \ge c,$$
>
> *where $\langle\, .,\, .\rangle^+ = \max(0, \langle\, .,\, .\rangle)$.*

(Pemantle actually requires this inequality to hold for all $\boldsymbol{\theta}$ in the tangent space of $\bar{\Delta}$. However, in the proof he uses only the above weaker condition). Choose a unit vector $\boldsymbol{\theta}$ in E^+ and set $\epsilon = \min_{i \notin I} \bar{s}_i$. Since $\bar{\mathbf{s}}$ is a fixed point, for all $i = 1, \dots, K-1$, $P(\beta_i(\bar{\mathbf{s}}) = 1) = \bar{s}_i$. Thus,

$$\begin{aligned} & E\left(\langle \boldsymbol{\beta}^t(\bar{\mathbf{s}}) - \bar{\mathbf{s}}, \boldsymbol{\theta}\rangle^+ \,\middle|\, \mathcal{F}^{\mathbf{t}}\right) \\ = & \sum_{i=1}^{K-1} P(\beta_i(\bar{\mathbf{s}}) = 1) \left(\theta_i - \sum_{j=1}^{K-1} \bar{s}_j\theta_j\right)^+ + \left(1 - \sum_{i=1}^{K-1} \bar{s}_i\right)\left(-\sum_{j=1}^{K-1} \bar{s}_j\theta_j\right)^+ \\ \geq & \ \epsilon \sum_{i \notin I}\left(\theta_i - \sum_{j \notin I} \bar{s}_j\theta_j\right)^+ + \left(1 - \sum_{i \notin I} \bar{s}_i\right)\left(-\sum_{j \notin I} \bar{s}_j\theta_j\right)^+. \end{aligned} \tag{21}$$

Since $\boldsymbol{\theta}$ lies in the tangent space of E^+ we have $\theta_i = 0$, $\forall i \in I$ and $\theta^* := \max_{i \notin I} \theta_i \neq 0$. If $\theta^* > 0$ then the first term in (21) is positive, since by (14) $\sum_{j \notin I} \bar{s}_j < 1$. If $\theta^* < 0$ then the second term is positive.

By continuity there is a neighbourhood U of $\bar{\mathbf{s}}$ and a $c > 0$ such that for all $\mathbf{s} \in U$

$$E\left(\langle (\beta_k^t(\mathbf{s}) - s_k)_{k=1}^{K-1}, \boldsymbol{\theta}\rangle^+ \,\middle|\, \mathcal{F}^{\mathbf{t}}\right) > c.$$

□

This completes the proof of proposition 1. In the following we use again the K-dimensional notation. Let intΔ denote the interior of Δ in the relative topology.

Proposition 2 *Let $\bar{\mathbf{s}} \in int\Delta$ be a fixed point of (9) such that $\sum_{k=1}^{K} \frac{1}{G_k'(\bar{s}_k)} > 0$ and $G'(\bar{s}_i) \neq 0$ for all i.*

1. *If $G'(\bar{s}_k) < 0$, $k = 1, \dots, K$ or*
2. *if there is exactly one l s.t. $G_l'(\bar{s}_l) > 0$ and $\sum_{k=1}^{K} \frac{1}{G_k'(\bar{s}_k)} > 0$,*

then $\bar{\mathbf{s}}$ is a sink. Otherwise $\bar{\mathbf{s}}$ is a saddle or source.

Proof: *By Hofbauer* and *Sigmund [1988] (9) is a Shashahani gradient system for the potential $V(\mathbf{s}) = -\int_{s_1}^1 G_1(t)\,dt - \int_{s_2}^1 G_2(t)\,dt - \cdots - \int_{s_K}^1 G_K(t)\,dt$. Thus, the fixed points of (9) in intΔ are the critical values of $V(\cdot)$ restricted to Δ. To see this, let $\Xi = \{\boldsymbol{\xi} \in I\!R^K \mid \sum_{k=1}^K \xi_k = 0\}$ denote the tangent vector space of Δ. Then for all fixed points $\bar{\mathbf{s}} \in int\Delta$*

$$\langle \nabla V(\bar{\mathbf{s}}), \boldsymbol{\xi}\rangle = \sum_{k=1}^{K} G_k(\bar{s}_k)\,\xi_k = G_1(\bar{s}_1)\sum_{k=1}^{K} \xi_k = 0, \quad \forall \boldsymbol{\xi} \in \Xi,$$

holds. The maxima of $V(\cdot)$ restricted to Δ are the asymptotically stable points of (9). Since (9) is a gradient system $\bar{\mathbf{s}}$ is a sink, if and only if for the Jacobian $\mathbf{DV}(\cdot)$ of $V(\cdot)$ we have

$$\langle \mathbf{DV}(\bar{\mathbf{s}})\,\boldsymbol{\xi}, \boldsymbol{\xi}\rangle < 0, \quad \forall \boldsymbol{\xi} \in \Xi. \tag{22}$$

If there is a $\boldsymbol{\xi} \in \Xi$ such that the inequality in the other direction holds then $\bar{\mathbf{s}}$ is a saddle or source. $\mathbf{DV}(\cdot)$ is a diagonal matrix given by $\mathbf{DV}(\bar{\mathbf{s}}) = (\delta_{ij}G_i'(\bar{s}_i))_{ij}$. Thus, if condition 1 of the proposition holds (22) follows immediately.

If condition 2 holds then assume w.l.o.g. that $G_1'(\bar{s}_1) > 0$ *and* $G_k'(\bar{s}_k) < 0$ *for* $k > 1$. *We give a proof by contradiction. Assume there exists a* $\boldsymbol{\xi} \in \Xi$ *such that* $\langle \mathbf{DV}(\bar{\mathbf{s}})\,\boldsymbol{\xi}, \boldsymbol{\xi}\rangle > 0$ *we have* $|\xi_1| > 0$ *and hence*

$$G_1'(\bar{s}_1) + \sum_{k=2}^{K} G_k'(\bar{s}_k)\,\frac{\xi_k^2}{\xi_1^2} > 0.$$

Set $\xi_k' = \frac{\xi_k}{\xi_1}$ *and note that* $|\sum_{k=2}^K \xi_k'| = 1$. *Hence,*

$$\max_{\mathbf{x}\in \mathbb{R}^K,\,|\sum_{k=2}^K x_k|=1} G_1'(\bar{s}_1) + \sum_{k=1}^{K} G_k'(\bar{s}_k) x_k^2 = G_1'(\bar{s}_1) + \frac{1}{\sum_{k=2}^K \frac{1}{G_k'(\bar{s}_k)}} > 0 \tag{23}$$

A straightforward calculation shows, that (23) holds if and only if $\sum_{k=1}^K \frac{1}{G_k'(\bar{s}_k)} < 0$. *This is a contradiction to condition 2. Hence* $\bar{\mathbf{s}}$ *is a sink. With the same argument we see that if* $\sum_{k=1}^K \frac{1}{G_k'(\bar{s}_k)} > 0$, $\bar{\mathbf{s}}$ *is not a local maximum of* $V(\cdot)$ *restricted to* Δ *but a saddle or minimum, and hence its a saddle or source for (7).*

Finally, assume there exist $k, l,\ k \neq l$ *such that* $G_k'(\bar{s}_k) > 0$ *and* $G_l'(\bar{s}_l) > 0$. *For* $\boldsymbol{\xi} \in \Xi$ *such that* $\xi_l = 1$, $\xi_k = -1$ *and* $\xi_j = 0,\ j \neq k,\ j \neq l$ *we have* $\langle \mathbf{DV}(\bar{\mathbf{s}})\,\boldsymbol{\xi}, \boldsymbol{\xi}\rangle > 0$, *and hence the fixed point is a saddle or source.*

□

Proposition 3 *Let* $I \subset \{1,\dots,K\}$ *and* $\Delta_I = \{\,\mathbf{s}\,|\,s_i = 0, \forall\, i \in I\,\}$ *denote a boundary face. Assume that* $G_i(0) < \infty$ *for all* $i \in I$. *Then* Δ_I *is an invariant set. If* $\bar{\mathbf{s}} \in \Delta_I$ *is a sink for (9) restricted to* Δ_I *and*

1. *for all* $i \in I,\ G_i(0) < \sum_{j=1}^K \bar{s}_j\, G_j(\bar{s}_j) =: \bar{G}(\bar{\mathbf{s}})$ *then* $\bar{\mathbf{s}}$ *is a sink for (7) on* Δ.

2. *there exists an* $i \in I$ *such that* $G_i(0) > \sum_{j=1}^K \bar{s}_j\, G_j(\bar{s}_j)$ *then* $\bar{\mathbf{s}}$ *is a saddle.*

Proof: *1. By (if necessary) relabeling technologies we can assume that* $s_1 > 0, \dots, s_m > 0,\ s_{m+1} = \dots = s_K = 0$ *for some* m. *Then the gradient* ∇V *is given by* $\nabla V = (\bar{G}, ..., \bar{G}, G_{m+1}(0), ..., G_K(0))^T$. *The tangent vector space at the boundary face* Δ_I *is given by* $\Xi_I = \{\boldsymbol{\xi} \in \mathbb{R}^K \mid \sum_{k=1}^K \xi_k = 0,\ \xi_i \geq 0, \forall i \in I\}$. *Thus, if* $G_i(0) < \bar{G}, \forall i \in I$ *then* $\langle \nabla V, \boldsymbol{\xi}\rangle \leq 0$ *for all* $\boldsymbol{\xi} \in \Xi_I$. *The inequality is strict for all* $\boldsymbol{\xi} \in \Xi_I$ *such that* $\exists i \in I,\ \xi_i > 0$. *By our assumptions* $\bar{\mathbf{s}} \in \Delta_I$ *is a sink for (9) restricted to* Δ_I *and hence a maximum of* V *restricted to* Δ_I. *It follows, that* $\bar{\mathbf{s}}$ *is a local maximum of* V *restricted to* Δ. *It remains to show that* $\bar{\mathbf{s}}$ *is a hyperbolic fixed point. Since (9) is a gradient vector field, all eigenvalues of its Jacobian are real. Thus, it suffices to show that the determinant is not zero. This follows straightforward. The proof of statement 2. is analogous and thus omitted.* □

For every set $A \subset \mathbb{R}^K$ *and* $\epsilon > 0$ *set* $U_\epsilon(A) := \{\bar{\mathbf{s}} \in \Delta \,|\, \mathcal{D}(\bar{\mathbf{s}}, A) < \epsilon\}$, *where* $\mathcal{D}(\bar{\mathbf{s}}, A) = \inf_{\bar{\mathbf{s}}' \in A} ||s - s'||$. *The following lemma is a modification of proposition 7.3 in Nevel'son* and *Has'minskii [1973].*

Lemma 4 *Let* $D \subset \Delta$ *and consider the sequence*

$$\mathbf{s}^{t+1} = \mathbf{s}^t + \frac{1}{n+t}\,\mathbf{f}(\mathbf{s}^t) + \frac{1}{n+t}\,\boldsymbol{\xi}^t \tag{24}$$

where $\mathbf{s}^t \in \Delta$, $\forall t > 0$. *Assume there is a random time instant* τ_1 *such that a.s.* $\mathbf{s}^t \in D$ *for all* $t > \tau_1$ *and* $\tau_1 < \infty$ *a.s., that* $\mathbf{f} : int\Delta \to I\!R^K$ *is bounded and continuous, and* $\boldsymbol{\xi}^t$ *is a sequence of uniformly bounded random vectors such that* $E\left(\boldsymbol{\xi}^t \,\middle|\, \mathbf{s^t}\right) = \mathbf{0}$. *Let* $c > 0$ *and assume that* V *is a strict* C^2 *Liapunov function on* $A = \{\mathbf{s} \,|\, 0 < V(\mathbf{s}) < c\,\} \cap D$. *Let* $E = D/A$.

Then, for all $\epsilon > 0$ *there is a random time* τ_2 *such that* $\mathbf{s}^t \in U_\epsilon(E)$, $\forall t > \tau_2$ *and* $\tau_2 < \infty$ *a.s.*

Proof: *We extend the definition of* V *by setting* $V(\mathbf{s}) = c$, $\forall \mathbf{s} \in \Delta/\{\mathbf{s}|V(\mathbf{s}) < c\}$. *Let* $\phi(x)$ *be a positive monotone* C^2 *function such that* $\phi(x) = c$, $\forall x \geq c$ *and such that* ϕ *is strictly monotone for all* $x < c$. *Then* $\phi(V(\mathbf{s}))$ *is again a* C^1 *Liapunov function. To simplify notation we set* $V(\mathbf{s}) = \phi(V(\mathbf{s}))$ *in the following. Note that* $\langle \nabla V(\mathbf{s}), \mathbf{f}(\mathbf{s})\rangle \geq 0$ *for all* $\mathbf{s} \in D$. *A Taylor expansion gives for* $t > 0$

$$V(\mathbf{s}^{t+1}) \geq V(\mathbf{s}^t) + \frac{\mathbf{1}_D(\mathbf{s}^t)}{n+t}\,\left[\langle \nabla V(\mathbf{s}^t), \mathbf{f}(\mathbf{s}^t)\rangle + \langle \nabla V(\mathbf{s}^t), \boldsymbol{\xi}^t\rangle\right] - \frac{L\,\mathbf{1}_{\Delta/D}(\mathbf{s}^t)}{n+t} - \frac{L}{t^2},$$

where $\mathbf{1}_D(\mathbf{s}^t)$ *is the indicator function of* D, *and* L *is an upper bound for* $|\langle \nabla V(\mathbf{s}^t), \mathbf{f}(\mathbf{s}^t) + \boldsymbol{\xi}^t\rangle|$ *and the absolute value of the second order terms in the Taylor series. Then, for all* $T_2 > T_1$ *we have*

$$\begin{aligned} V(\mathbf{s}^{T_2}) \quad \geq \quad & V(\mathbf{s}^{T_1}) + \sum_{t=T_1}^{T_2-1} \frac{\mathbf{1}_D(\mathbf{s}^t)}{n+t}\,\langle \nabla V(\mathbf{s}^t), \mathbf{f}(\mathbf{s}^t)\rangle + \sum_{t=T_1}^{T_2-1} \frac{\mathbf{1}_D(\mathbf{s}^t)}{n+t}\,\langle \nabla V(\mathbf{s}^t), \boldsymbol{\xi}^t\rangle \\ & - \sum_{t=T_1}^{T_2-1} \frac{L\,\mathbf{1}_{\Delta/D}(\mathbf{s}^t)}{n+t} + \frac{L}{t^2}. \end{aligned}$$

The process $\sum_{t=T_1}^{T_2-1} \frac{\mathbf{1}_D(\mathbf{s}^t)}{n+t}\,\boldsymbol{\xi}^t$ *is an* L^2 *martingale and converges a.s. for* $T_2 \to \infty$. *Also* $\sum_{t=T_1}^{T_2-1} \frac{L\,\mathbf{1}_{\Delta/D}(\mathbf{s}^t)}{n+t} + \frac{L}{t^2}$ *converges almost surely, since from some time on the process stays in* D *a.s. Since* $V(\mathbf{s})$ *is bounded*

$$\left|\sum_{t=T_1}^{\infty} \frac{\mathbf{1}_D(\mathbf{s}^t)}{n+t}\,\langle \nabla V(\mathbf{s}^t), \mathbf{f}(\mathbf{s}^t)\rangle\right| < \infty, \quad a.s.$$

Hence, almost surely the paths of the process (24) are of the form

$$\mathbf{s}^{t+1} = \mathbf{s}^t + \frac{1}{n+t}\,\mathbf{f}(\mathbf{s}^t) + \boldsymbol{\epsilon}^t, \tag{25}$$

where $\boldsymbol{\epsilon}^t$ *is a deterministic vector sequence such that* $\left|\sum_{t=1}^{\infty} \boldsymbol{\epsilon}^t\right| < \infty$,

$$\left|\sum_{t=T_1}^{\infty} \frac{\mathbf{1}_D(\mathbf{s}^t)}{n+t}\,\langle \nabla V(\mathbf{s}^t), \mathbf{f}(\mathbf{s}^t)\rangle\right| < \infty, \tag{26}$$

and for each path there exists a T_3 *such that* $\mathbf{s}^t \in D$ *for all* $t > T_3$.

Now let $\epsilon > 0$ *and set* $U_\epsilon^c := D/U_\epsilon(E)$. *Assume that a path stays from some time* T_1 *onward in* U_ϵ^c. *Let* $d = \min_{\mathbf{s} \in U_\epsilon^c} \langle \nabla V(\mathbf{s}), \mathbf{f}(\mathbf{s})\rangle$. *Then,* $d > 0$ *and*

$$\left|\sum_{t=T_1}^{\infty} \frac{\mathbf{1}_{U_\epsilon^c}(\mathbf{s}^t)}{n+t}\,\langle \nabla V(\mathbf{s}^t), \mathbf{f}(\mathbf{s}^t)\rangle\right| \geq \left|\sum_{t=T_1}^{\infty} \frac{d}{n+t}\right| = \infty,$$

which is a contradiction with (26) since $U_\epsilon^c \subset D$. Hence, after a finite random time the path leaves the set U_ϵ^c. Now we prove that the path cannot enter the set $U_{2\epsilon}^c$ infinitely often.

Assume it would. Then there are times $T_3 < t_l < \bar{t}_l < t_{l+1}$ such that $\mathbf{s}^{\bar{t}_l} \in U_\epsilon$, $\mathbf{s}^{t_l} \in U_{2\epsilon}^c$ and $\mathbf{s}^t \in U_\epsilon^c$ for all $\bar{t}_l \leq t \leq t_{l+1}$. Choose an $l_0 > 0$ such that for all $t > \bar{t}_{l_0}$ we have $|\sum_{l=t}^\infty \epsilon^l| \leq \frac{\epsilon}{2}$. Thus, there is a $C_1 > 0$ such that for all $l > l_0$

$$\epsilon \leq |\mathbf{s}^{t_l} - \mathbf{s}^{\bar{t}_l}| \leq \sum_{t=\bar{t}_l}^{t_{l+1}-1} \frac{1}{n+t} |\langle \nabla V(\mathbf{s}^t), \mathbf{f}(\mathbf{s}^t)\rangle| + \frac{\epsilon}{2} \leq C_1 \sum_{t=\bar{t}_l+1}^{t_{l+1}-1} \frac{1}{n+t} + \frac{\epsilon}{2}$$

Hence, $\frac{\epsilon}{2\,C_1} \leq \sum_{t=\bar{t}_l+1}^{t_{l+1}-1} \frac{1}{n+t}$. Thus, we obtain

$$\begin{aligned}\sum_{t=T_3}^{\infty} \frac{\mathbf{1}_{U_\epsilon^c}(\mathbf{s}^t)}{t+n} \langle \nabla V(\mathbf{s}^t), \mathbf{f}(\mathbf{s}^t)\rangle &\geq \sum_{l=l_0}^{\infty} \sum_{t=\bar{t}_l+1}^{t_{l+1}-1} \frac{1}{n+t} \langle \nabla V(\mathbf{s}^t), \mathbf{f}(\mathbf{s}^t)\rangle \\ &\geq \sum_{l=l_0}^{\infty} d \sum_{t=\bar{t}_l+1}^{t_{l+1}-1} \frac{1}{n+t} \geq \sum_{l=l_0}^{\infty} d\, \frac{\epsilon_0}{2\,C_1} = \infty.\end{aligned}$$

This gives again a contradiction with (26). □

Let $J = \{i \,|\, G_i(0) = \infty\}$.

Lemma 5 *Consider the vector field (6) and let $I \subseteq J$. Then for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $\mathbf{s} \in U_I(\epsilon, \delta) = \{\,\mathbf{s}\,|\, \sum_{i\in I} s_i < \delta,\ s_j > \epsilon,\ \forall\, j \in J/I\,\} \cap \Delta$*

$$\sum_{i\in I} \dot{s}_i \geq \sum_{i\in I} s_i.$$

Proof: *Let $\epsilon > 0$. We distinguish two cases. First, assume that there is an $i^* \in I$ such that $D_{i^*}(0) > 0$. Let $L_1 := \sup_{\mathbf{s}\in\Delta} \sum_{i\notin J} D_j(s_j)$. Then, $L_1 < \infty$ and for $\mathbf{s} \in \Delta$ we get by (7)*

$$\begin{aligned}\sum_{i\in I} \dot{s}_i &= \left(\sum_{i\in I} s_i\right) \left[\frac{\sum_{i\in I} D_i(s_i)}{\left(\sum_{i\in I} s_i\right) \sum_{i=1}^K D_i(s_i)} - 1\right] \\ &\geq \left(\sum_{i\in I} s_i\right) \left[\frac{\sum_{i\in I} D_i(s_i)}{\left(\sum_{i\in I} s_i\right) \left(\sum_{i\in I} D_i(s_i) + L_1\right)} - 1\right] \qquad (27)\end{aligned}$$

Set $L_2 = \inf_{\mathbf{s}\in\Delta} \frac{\sum_{i\in I} D_i(s_i)}{\sum_{i\in I} D_i(s_i) + L1}$. Since $D_{i^*}(0) > 0$ we get $L_2 > 0$. Now, let $\delta = \frac{L_2}{2}$. Then, continuing (27), we get for all $\mathbf{s} \in U_I(\epsilon, \delta)$

$$\sum_{i\in I} \dot{s}_i \geq \left(\sum_{i\in I} s_i\right) \left[\frac{L_2}{\sum_{i\in I} s_i} - 1\right] \geq \sum_{i\in I} s_i$$

Now consider the case where $D_i(0) = 0$ for all $i \in I$. Let $L_3 = sup_{\mathbf{s}\in U_I(\epsilon,1)} \sum_{i=1}^K D_i(s_i)$. We have $L_3 < \infty$. Choose a $\delta > 0$ such that $\frac{D_i(s)}{s} > 2\,L_3$ for all $s < \delta$ and all $i \in I$. Then, for $\mathbf{s} \in U_I(\epsilon, \delta)$ we get by (7)

$$\sum_{i\in I}\dot{s}_i = \left(\sum_{i\in I} s_i\right)\left[\frac{\sum_{i\in I} D_i(s_i)}{\left(\sum_{i\in I} s_i\right)\sum_{i=1}^{K} D_i(s_i)} - 1\right]$$
$$\geq \left(\sum_{i\in I} s_i\right)\left[\frac{2\,L_3}{\sum_{i=1}^{K} D_i(s_i)} - 1\right] \geq \sum_{i\in I} s_i$$

□

For $I \subset \{1,\dots,K\}$ define $Bd_I = \{\,\mathbf{s}\in\Delta \mid \exists i \in I \text{ s.t. } s_i = 0\,\}$ and let $J = \{1 \leq k \leq K \mid G_k(0) = \infty\}$. Thus, Bd_J is the union of all boundary faces where at least one product has infinite fitness. For all $\epsilon > 0$ and $I \subseteq J$ let $\delta(I,\epsilon) > 0$ be a number such that on $U_I(\epsilon, \delta(I,\epsilon))$ we have $\sum_{j\in I}\dot{s}_j \geq \sum_{j\in I} s_j$. Set $\Delta_I = \{\mathbf{s}\in\Delta \mid s_i = 0\,\forall i \in I\}$.

Lemma 6 *Let $I \subseteq J$, and let $V \subseteq Bd_{J/I}$. Assume there exists a neighbourhood U of V such that on U, $\sum_{i\in J/I} s_i$ is a Liapunov function. Then*

$$P(\lim_{t\to\infty}\langle \mathbf{s}^t, \mathbf{v}\rangle = 0) = 0,$$

where $\mathbf{v}$ is a vector such that $v_i = 1$ for all $i \in J/I$ and zero otherwise.

Proof: *This follows by the arguments for step (16) in lemma 2.* □

Proposition 4 *Let $\mathcal{D}(\mathbf{s}, Bd_J)$ denote the distance between $\mathbf{s}$ and Bd_J. Then*

$$P\{\liminf_{t\to\infty}\mathcal{D}(\mathbf{s}^t, Bd_J) = 0\} = 0$$

Proof: *For simplicity we assume that $J = \{1,\dots,K\}$. The other case follows by analogy. In a first step we cover Bd_J with sets of the form $U_I(\epsilon,\delta)$, defined in the above lemma.*

Let $0 < \epsilon_1 < 1$ and set $\delta_1 = \min_{i\in J}(\delta(J/\{i\},\epsilon_1),\epsilon_1)$. Note, that

$$W_{J/\{i\}} := U_{J/\{i\}}(\epsilon_1,\delta_1)$$

is a neighbourhood of the i-th vertex $\Delta_{J/\{i\}}$. For a finite set H, denote by $|H|$ the cardinality of H.

Let $k > 1$. Assume we have covered the edges $\Delta_{J/I}$, $I \subset J$, $|I| = k$ with the neighbourhoods $W_{J/I} := \bigcup_{H\subseteq I,|H|\leq k} U_{J/H}(\epsilon_{|H|},\delta_{|H|})$. Now choose an $\epsilon_{k+1} < \delta_k$ and set $\delta_{k+1} = \frac{1}{2}\min_{I\subseteq J,|I|=k+1}\left(\delta(I,\epsilon_{k+1}),(\delta_k-\epsilon_{k+1})\right)$. Then for all sets $I \subseteq J$ such that $|I| = k+1$ we have

$$\Delta_{J/I} \subseteq W_{J/I} := \bigcup_{H\subseteq I,|H|\leq k+1} U_{J/H}(\epsilon_{|H|},\delta_{|H|}).$$

Thus, using this procedure iteratively we get neighbourhoods $W_{\{i\}}$ covering $\Delta_{\{i\}}$. The union $W = \cup_{i\in J} W_{\{i\}}$ covers Bd_J.

To prove that the process does not approach the boundary Bd_I, we show by induction in k, that from some time onwards the process does not belong to the sets $W_{J/I}$ for all $I \subseteq J$, $|I| = k$.

$k = 1$. Let $i \in J$ be fixed. By lemma 5 $V(\mathbf{s}) = \sum_{j\in J/\{i\}} s_j$ is a linear Liapunov function on the set $W_{J/\{i\}} = \{\mathbf{s} \mid \sum_{j\in J/\{i\}} s_j < \delta_1\}$. There is an $\epsilon > 0$ such that $\mathbf{f}$ is also a Liapunov

function on $A := \{\mathbf{s} \in \Delta \mid \sum_{j \in J/\{i\}} s_j < \delta_1 + \epsilon\}$. *Hence, by lemma 4 the process a.s. either converges to the edge* $\mathbf{e}_i$ *or it leaves* $W_{J/\{i\}}$ *and does not return from some time onward. By lemma 6 the process converges to* $\mathbf{e}_i$ *with probability 0. Thus, a.s. from some time onward the process does not belong to* $W_{J/\{i\}}$.

Induction step. Let $k > 1$, *Assume that for all* $I \subset J$, $|I| = k-1$ *from some time onward the process is not in* $W_{J/I}$. *Let* $I \subseteq J$, $|I| = k$.

By the induction assumption from some time onward the process belongs to $D = \Delta / \cup_{I \subset J, |I|=k-1} W_{J/I}$. *Since* $\delta_k < \frac{1}{2}(\delta_{k-1} - \epsilon_k)$ *there is an* $\epsilon > 0$ *such that on* $A := D \cap U_{J/I}(\epsilon_k, \delta_k + \epsilon) = D \cap \{\bar{\mathbf{s}} \mid V(\bar{\mathbf{s}}) \leq \delta_k + \epsilon\}$ *the function* $V(\mathbf{s}) = \sum_{j \in J/I} s_j$ *is a strict Liapunov function. By lemma 4 the process converges either to the set* $Bd_{J/I}$ *or it leaves the set* $W_{J/I}$. *By Corollary 6 the former occurs with probability 0. This rights the result.* □

Proof of theorem 1: *Assume first that all* G_k, $k = 1, \ldots, K$ *can be extended to continuous functions on the whole interval* $[0, 1]$. *Then, by Hofbauer* and *Sigmund [1988] (9) is a Shashahani gradient system for the potential* $V(\mathbf{s}) = -\int_{s_1}^{1} G_1(t)\,dt - \int_{s_2}^{1} G_2(t)\,dt - \cdots - \int_{s_K} G_K(t)\,dt$. *Thus, V is a strict* C^1 *Liapunov function for (7). Now, the result follows by proposition 1 in Arthur* et al. *[1988b].*

If some G_k *cannot be continuosly extended to the whole interval by the conditions on the demand functions it follows that* $\lim_{s \to 0} G_k(s) = \infty$. *By proposition 4* $P(\lim_{t \to \infty} d(\mathbf{s}^t, Bd_J) = 0) = 0$. *Thus, there is an* Ω' *with* $P(\Omega') = 1$ *such that for every elementary outcome* $\omega \in \Omega'$ *there exists an* $\epsilon > 0$ *and a* $T > 0$ *such that for all* $t > T$ *we have* $\mathcal{D}(\mathbf{s}^t(\omega), Bd_J) > \epsilon$. *Additionally, since* $\left(\sum_{t=1}^{T_2} \frac{1}{n+t}\left[\boldsymbol{\beta}^t(\mathbf{s^t}) - \mathbf{d}(\mathbf{s}^t)\right]\right)$ *is an* L^2 *martingale we can choose* Ω' *such that every path in* Ω' *can be written as* $\mathbf{s}^{t+1}(\omega) = \mathbf{s}^t + \frac{1}{n+t}\mathbf{f}(\mathbf{s}^t(\omega)) + \boldsymbol{\epsilon}^t(\omega)$, *where* $\mathbf{f}(\mathbf{s}) := \mathbf{d}(\mathbf{s}) - \mathbf{s}$ *and* $\boldsymbol{\epsilon}^t(\omega)$ *is a deterministic sequence of vectors such that* $||\sum_{t=1}^{\infty} \boldsymbol{\epsilon}^t|| < \infty$. *Let* $U_\epsilon(Bd_J)$ *denote an* ϵ*-neighbourhood of* Bd_J. *Since the vector field* $\mathbf{f}$ *is* C^2 *in* $\Delta - U_\epsilon(Bd_J)$ *we can apply proposition 1 in Benaïm [1993] to deduce that the limit sets are chain recurrent. Since* $\mathbf{f}$ *is a gradient vector field on* $\Delta - U_\epsilon(Bd_J)$ *the only chain recurrent sets are the fixed points.* □

Proof of theorem 2: *Since* $G_k'(s_k) = \frac{s_k^2}{D_k(s_k)\,(\delta_k(s_k)-1)}$ *by propositions 2 and 3* $\bar{\mathbf{s}}$ *is a sink for (7) if and only if the above conditions hold, otherwise it is a source or saddle. In the former case by proposition 8 in Posch [1994] we have* $P(\lim_{t \to \infty} \mathbf{s}^t = \bar{\mathbf{s}}) > 0$. *In the latter cases we apply proposition 1 and get* $P(\lim_{t \to \infty} \mathbf{s}^t = \bar{\mathbf{s}}) = 0$ □

Proof of theorem 3: *Adapting the proof of proposition A in Hofbauer* et al. *[1981] we prove the result. If the elasticity of* D_k *is less than one, then the fitnesses* G_k *are strictly decreasing functions. Without restricting generality we assume that* $G_1(0) \geq G_2(0) \ldots \geq G_K(0) \geq 0$. *We first compute the fixed point* $\bar{\mathbf{s}}$ *and show that there exists a unique* $C \leq G_1(0)$ *and a unique* $\bar{\mathbf{s}} \in \Delta$ *such that*

$$G_1(p_1) = \cdots = G_m(p_m) = K$$

and $p_1 > 0, \ldots, p_m > 0$, $p_{m+1} = 0, \ldots, p_K = 0$, *where m is the largest integer k with* $G_k(p_k) > C$.

Let $G_k^{-1}(\cdot)$ *be the inverse of* $G_k(\cdot)$ *defined on* $[G_k(1), G_k(0))$. *For* $s > G_k(0)$ *we set* $G_k^{-1}(s) = 0$. *The function*

$$H(c) = \sum_{k=1}^{K} G_k^{-1}(c)$$

is defined for $c \in [\max_{1 \leq k \leq K} G_k(1), G_1(0))$ *and strictly decreases from some* $a \geq 1$ *to* 0. *Thus, there exists a unique constant* $C \leq G_1(0)$ *such that* $H(C) = 1$. *Let*

$$\bar{s}_k = G_k^{-1}(C), \quad k = 1, \ldots, K.$$

Then $\sum_{k=1}^{K} \bar{s}_k = 1$*. If* $G_k(0) \leq C$ *then* $\bar{s}_k = 0$*, if* $G_k(0) > C$ *then* $G_k(0) = K$ *and* $\bar{s}_k > 0$*. It follows straightforward that* $\bar{\mathbf{s}}$ *is a fixed point. It is the unique fixed point on the set* $\{\mathbf{s} \in \Delta \mid s_1 > 0, \ldots, s_m > 0\}$*. In particular, if all* $G_k(0)$ *are equal,* $\bar{\mathbf{s}}$ *is in the interior of* Δ*.*

Since (9) is a gradient system all solutions of the differential equation converge to a fixed point. It remains to study the stability of all fixed points. We show that $\bar{\mathbf{s}}$ *is a sink and all other fixed points are saddles or sources.*

Since $G_k(0) < K$ *for all* $k \leq m$ *and* $G_k(\bar{s}_k) = K$ *for all* $k > m$ *we have*

$$G_k(0) < \sum_{l=m+1}^{K} s_l\, G_l(\bar{s}_l) = \sum_{l=1}^{K} \bar{s}_l\, G_l(\bar{s}_l).$$

Since additionally $\delta_k(\bar{s}_k) < 1$*, by theorem 2* $\bar{\mathbf{s}}$ *is a sink.*

Now let $\mathbf{s}'$ *be another fixed point. If* $G_k(s'_k) = \infty$ *for some* k*, by proposition 4* $\mathbf{s}'$ *is attained with probability 0. Assume now that all* $G'_k(s'_k)$ *are finite. Note that* $\mathbf{s}'$ *lies in some boundary face where* $s_l = 0$ *for some* $l \leq m$*. Now we consider the system restricted to this boundary face, and set*

$$H'(c) = \sum_{k \in \{l \mid s'_l > 0\}} G_k^{-1}(c).$$

We have $H'(\cdot) \leq H(\cdot)$*. There is a unique* C' *such that* $s'_k = G_k^{-1}(C')$*,* $k \in \{l \mid s'_l > 0\}$*. Since* $C' \leq C$*,*

$$G_l(0) > C > C' > \sum_{k=1}^{K} s'_k G(s'_k)$$

and by theorem 2 *it follows that* s'_k *is a saddle.* □

Proof of theorem 4 *By theorem 1 the process converges a.s. to a fixed point of (7). Since* $\delta_k(s_k) > 1$*, by theorem 2 all interior fixed points are attained with probability 0. This argument also holds for fixed points in the interior of a boundary face. (Here we consider the dynamics of (7) restricted to that boundary face.) Thus, the process converges a.s. to one of the vertices.*

If $D'_k(0) = D'_l(0)$*,* $0 \leq k, l \leq K$ *then all vertices are sinks. Let* $1 \leq k \leq K$ *be arbitrary but fixed. Since the elasticities are greater than 1, we have* $D_l(0) = 0$*,* $1 \leq l \leq K$ *and that* $G_k(s_k)$ *is monotonically increasing. Hence,*

$$D'_l(0) = D'_k(0) = G_k(0) \leq G_k(1) = D_k(1), \qquad l = 1, \ldots, K.$$

Thus, by proposition 3 $\mathbf{e}_k$ *is a sink, and by proposition 8 in Posch[1994] it is attained with positive probability.* □